Binomial Option Pricing: Basic Principles

Financial 5350: Computational Finance

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Section 10.1: A One-Period Binomial Tree

Section 10.2: Constructing A Binomial Tree

Section 10.3: Two Or More Binomial Periods

Section 10.4: Put Options

Section 10.5: American Options

Section 10.6: Options on Other Assets

Introduction to Binomial Option Pricing

$t = 0$

C_0 call option

P_0 put option

- The binomial option pricing model enables us to determine the price of an option, given the characteristics of the stock or other underlying asset
- The binomial option pricing model assumes that the price of the underlying asset follows a binomial distribution - that is, the asset price in each Period can move only up or down by a specified amount
- The binomial model is often referred to as the "Cox-Ross-Rubinstein pricing model"

Bill Sharpe (CAPM)

CRR

1979 73

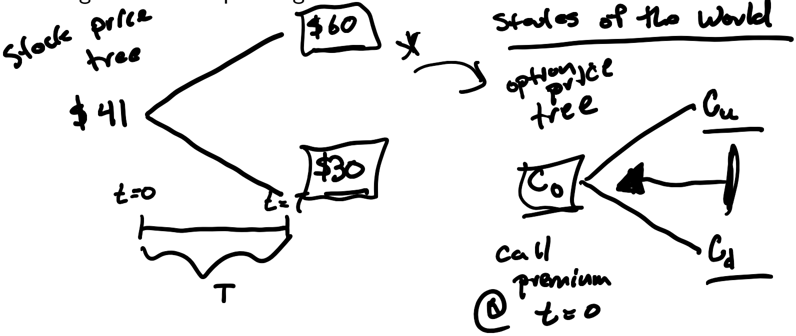
BSM 1973

A One-Period Binomial Tree

δ = Greek delta (lower)

$K = \$40$

- Example
  - Consider a European call option on the stock of XYZ, with a \$40 strike and 1 year to expiration $T=1$
  - XYZ does not pay dividends (i.e. $\delta = 0$), and its current price is $\boxed{S_0 = \$41}$
  - The continuously compounded risk-free interest rate is 8% (i.e. $r = 0.08$)
  - The following is the corresponding binomial tree:



Computing the Option Price

- Next, consider two portfolios:
  - *Portfolio A:* buy one call option
  - *Portfolio B:* buy 2/3 shares of XYZ and borrow \$18.462 at the risk-free rate, r

- Portfolio Costs:
  - *Portfolio A:* the call premium, **which is unknown and what we are solving for** (c_0)
  - ~~*Portfolio*~~ *B:* $\underbrace{2/3 \times \$41}_{27.33} - \$18.462 = \boxed{\$8.871}$

? =

No-arbitrage

Applying the Law of one price (LOOP)

Computing the Option Price Continued

$K = 40$

States of the World $(t = T)$

Payoffs	Down state \$30	Up state \$60
Portfolio A (call option)	$\max\{30-40, 0\} = \underline{0}$	$\max\{60-40, 0\} = 20$
Portfolio B	\$30	\$60
2/3 shares of XYZ	$2/3\ 30 = \underline{20}$	$2/3\ 60 = \underline{40}$
Borrow 18.462	-20	-20
	$\varnothing$	20

$\boxed{\text{LOOP argument}}$

No-arbitrage

$FV(18.462) = e^{(r).08/4}\, 20$

Computing the Option Price Continued

Notes:
- does not price the option directly
- relative pricing model
- LOOP argument (No-arb arg)

Rational market ≠ arbitrage

- Portfolios A and B have the same payoff. Therefore:
  - Portfolios A and B should have the same cost. Since Portfolio B costs \$8.871, the price of on option must also be \$8.871 $\boxed{C_0 = 8.871}$ (LOOP)
  - There is a way to create the payoff to a call by buying shares and borrowing. Portfolio B is a *synthetic call* (replicates the option's payoff)
  - One option has the risk of 2/3 shares. The value 2/3 is the *delta* (Δ) of the option: the number of shares that replicates the option payoff

Δ

$\boxed{\Delta = 2/3 ? \quad B = \$8.462 ?}$

→ Δ ?
→ B ? } two unknown

Other names for Port B:
- replicating portfolio
- hedge portfolio

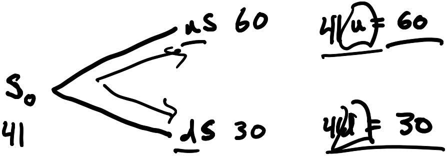
The Bimonial Solution

$\boxed{\begin{matrix}\Delta \\ B\end{matrix}}$

- How do we find a replicating portfolio consisting of Δ shares of stock and a dollar amount B in lending, such that the portfolio imitates the option whether the stock or falls?
  - Suppose that the stock has a continuous dividend yield of δ, which is reinvested in the stock. Thus, if you buy one share at time t, at time $t+h$ you will have $\boxed{e^{\delta h}}$ shares
  - If the length of a period is $\boxed{h}$, the interest factor per period is $\boxed{e^{rh}}$
  - uS denotes the stock price when the price goes up, and dS denotes the stock price when the price goes down

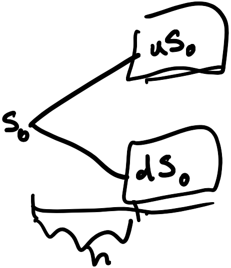
continuous discounting

$\boxed{u, d}$

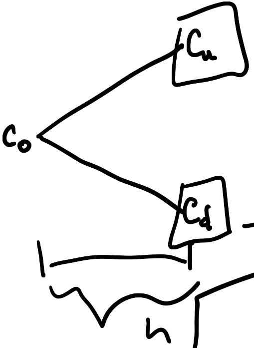


The Binomial Solution Continued

stock price tree



option price tree



$$\boxed{\Delta} S_h e^{\delta h} + e^{rh} B$$

Rep. Port: $\Delta S_h e^{\delta h} + e^{rh} B$

LOOP

$$C_h = \Delta S_h e^{\delta h} + e^{rh} B$$

The Binomial Solution Continued

- $S_h = uS$, $S_h = dS$

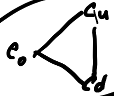
$\boxed{\text{Derivation of the Binomial model}}$

$\cancel{u, d}$

$$(\Delta * uS e^{\delta h}) + (B * e^{rh}) = C_u$$

$$(\Delta * dS e^{\delta h}) + (B e^{rh}) = C_d$$

Two eqns
Two unknowns

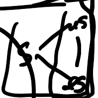


- Solving for Δ, B:

$$\boxed{\Delta = \underbrace{e^{-\delta h}}_{1} \boxed{\frac{C_u - C_d}{S(u-d)}}}$$

$\delta = 0$

$$\boxed{B = e^{-rh} \frac{uC_d - dC_u}{S(u-d)}}$$



$\boxed{\text{Std Dev}}$

$S(u-d) = uS - dS$

The Bimonial Solution Continued

- $\boxed{\Delta S + B} = e^{-rh}\left[\boxed{C_u}\,\frac{e^{(r-\delta)h} - d}{u-d} + \boxed{C_d}\,\frac{u - e^{(r-\delta)h}}{u-d} \right]$

- No-arbitrage condition

$$\boxed{u > e^{(r-\delta)h} > d}$$

LOOP

money pump

Forward tree

$$S \to uS, \quad S \to dS$$

Boundary condition

$\boxed{C_0}$

$\max\{uS-K, 0\} = \boxed{C_u}$

$\max\{dS-K, 0\} = \boxed{C_d}$

Arbitraging a Mispriced Option

$C^{observed} \neq \underline{C_0}$

Buy low

Sell hi

Q = quoted

- If the observed option price differs from its theoretical price, arbitrage is possible

sell buy

$\underline{\boxed{C^Q} > C_0}$: — Sell the option for C^Q

— Buy replicating portfolio for C_0

$\underline{30}$ $\underline{60}$

buy sell

$\underline{C^Q < C_0}$: — buy the option for C^Q

— Sell the replicating portfolio for C_0

$S = 41$
$K = 40$
$r = .08$
$T = 1$
$\delta = 0$

comp stat Arb

Arbitrage Thinking

Risk-Neutral Pricing

$$\Delta S + B = e^{-rh}\left(C_u \frac{e^{(r-\delta)h} - d}{u-d} + C_d \frac{u - e^{(r-\delta)h}}{u-d}\right)$$

- Interpret $\underline{\left(\frac{e^{(r-\delta)h} - d}{u-d}\right)}$, $\underline{\left(\frac{u - e^{(r-\delta)h}}{u-d}\right)}$ as $\underline{\text{probabilities}}$

- Let $p^* = \frac{e^{(r-\delta)h} - d}{u-d}$ up state

$0 < p^* < 1$

$1 - p^* =$ down state

$p^* + (1-p^*) = 1$

- NOT the prob of up/down in the actual real world
- ARE the prob of <u>up/down</u> in FICTIONAL Risk-neutral

Summary

$$\boxed{\boxed{C} = \boxed{e^{-rh}}\left[p^* \boxed{C_u} + (1-p^*)\boxed{C_d}\right]}$$

Expected value

- In order to price an option, we need to know the following:
  - Stock price (S)
  - Strike price (K)
  - The expiration date of the options $(T) = 1$
  - Standard deviation of returns on the stock $\boxed{(\sigma)}$
  - Divided yield $\boxed{\delta}$
  - Risk-free rate $\boxed{(r)}$
- Using the risk-free rate and σ, we can approximate the future distribution of the stock by creating a binomial tree using equation (10.9)
- Once we have the binomial tree, it is possible to price the option using equation (10.3)

–Expected option payoff in a RN world

$h = 1 \quad n = 1$

$h = \frac{T}{n}$

C_u & uS

C_d dS

u, d B

Δ, B

$h = T/n$

$n =$

S uS dS

T

multiperiod

Continuously Compounded Returns

Volatility

The Standard Deviation of Continuously Compounded Returns Continued

Constructing u and d

Constructing u and d Continued

Estimating Historical Volatility

Estimating Historical Volatility Continued

One-Period Example with a Forward Tree

One-Period Example with a Forward Tree Continued

A Two-Period European Call

A Two-Period European Call Continued

Pricing the Call Option

Pricing the Call Option Continued

Pricing the Call Option Continued

Many Binomial Periods

Many Bimonial Periods Continued

Many Bimonial Periods Continued

Put Options

Put Options Continued

American Options

American Options Continued

American Options Continued

American Options Continued

Options on Other Assets

Options on a Stock Index

Options on a Stock Index Continued

Options on Futures Contracts

Options on Futures Contracts Continued

Options on Futures Contracts Continued

Options on Futures Contracts Continued

Options on Commodities

Options on Bonds

Summary