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The influence of tracking error on volatility risk premium estimation

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I investigate whether the volatility risk premium is negative in energy and equity markets by examining the statistical properties of delta–gamma hedged option portfolios (selling the option, hedging with the underlying contract, and correcting for tracking error with an additional option). By correcting for gamma, these hedged portfolios are not subject to the same discretization and model misspecification problems as traditional delta-hedged portfolios. Within a stochastic volatility framework, I demonstrate that ignoring an option's gamma can lead to incorrect inference on the magnitude of the volatility risk premium. Using a sample of S&P100 Index and natural gas contracts, empirical tests reveal that the delta–gamma hedged strategy outperforms zero and the degree of overperformance is proportional to the level of volatility.

1 INTRODUCTION

Why do option traders on average find it beneficial selling option contracts (going short) versus buying option contracts (going long)? The answer may lie in the size of the volatility premium, as owning options helps to hedge the negative correlation between volatility and price changes in the equity markets. The current state of the literature suggests that the volatility risk premium is negative, but due to estimation problems there is no clear sense as to the extent of the magnitude or significance of the premium.

Within this work, I estimate the value of the volatility premium as a percent of the call price and underlying asset non-parametrically. The empirical design is similar to that of Bakshi and Kapadia (2003a,b), but with key improvements that significantly enhance the results. A delta-hedged position is created by forming a portfolio of a short call that is hedged with a long position in the stock. While Figlewski (1989), Bertsimas, Kogan and Lo (2000), and Bakshi and Kapadia (2003a) do not control for gamma, and argue that it will have limited impact on the results, not controlling for an option's gamma leads to an estimation with an incorrect interpretation of the volatility risk premium due to model misspecification and discretization error. In addition, as Robins and Schachter (1994) point

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out, delta hedges are not minimum-variance hedges. This may result in higher standard errors in estimation and an insignificant relationship between volatility risk premium and positive delta-hedged errors. Consequently, the delta portfolio is amended by holding an additional option and forming a delta–gamma portfolio which can overcome these problems in estimation. The gains from the delta–gamma portfolio can be attributed to the volatility risk premium. This gain is demonstrated both theoretically and through quasi-Monte Carlo simulation. The inclusion of the additional option eliminates "tracking error" and improves upon the Branger and Schlag (2004) analysis by eliminating the additional term dependent only on the drift of the underlying.

As a result of this formation, the estimation does not depend on the option's Vega, but on the cross-partial derivative with respect to the underlying asset and volatility. As such, this requires an additional control prior to the estimation of the volatility premium to preserve the linear regression framework. This additional control allows for a side-by-side comparison of the coefficient estimates generated from the regression on the delta- and delta–gamma hedged portfolios. The results provide strong evidence for why gamma must be controlled for in inferring the size and direction of the market price of volatility risk.

The estimation is conducted on options on both the S&P100 and natural gas contracts. I update the S&P estimation for two reasons. First, the sample used in Bakshi and Kapadia (2003b) had an average volatility almost half the level experienced between 1996 and 2002. This difference in level of volatility can result in large differences in daily hedged positions. Second, and more importantly, the S&P options are used as a test case to demonstrate the impact of ignoring gamma on the estimation of the market price of volatility risk. The findings show that the more prices fluctuate, the greater the variability in the estimation of volatility risk premium when gamma is ignored.

This is further demonstrated for the natural gas contracts. The natural gas market provides a unique sample to estimate volatility premiums, as the market dynamics are quite distinct from the traditional equity markets. As shown by Doran and Ronn (2006), price and volatility innovations in natural gas markets tend to be positively correlated and volatility levels can exceed 100% as a contract approaches maturity. Thus, power suppliers who are short power will hedge exposure with long calls to prevent buying power at spiked prices due to excessive demand. Irrespective of this fact, the implied volatility for natural gas is still greater than contemporaneous realized volatility counterpart, and similar to the finding in Jackwerth and Rubinstein (1996) for equity options. This finding is consistent with the notion of a negative market price of volatility risk. Thus, the finding of a negative volatility premium may imply a negative price premium for this energy commodity due to the positive correlation between price and volatility.[1] As of today, this is a finding that has yet to be documented.

[1] Doran and Ronn (2006) document a correlation of 0.377 using a stochastic volatility model to determine option model fit in the natural gas markets.

The advantage of estimating the volatility risk premium in this manner is that the estimation does not rely on the implicit function form of the data-generating process. The findings of Pan (2002) and Chernov and Ghysels (2000) attempt to estimate multiple risk premium using GMM and EMM methodologies respectively. In particular, Pan (2002) suggests that a model which does not incorporate a volatility risk premium best fits the data for the S&P, which is contrary to the findings presented here and inconsistent with the Jackwerth and Rubinstein (1996) findings. There are two plausible reasons for this difference. First, that the dependence on a given parametric form results in too tight a restriction on the estimation procedure, resulting in assigning more weight to the jump premiums where short-term volatility skews are most pronounced. Second, the model may be specified incorrectly. In estimating volatility premium in this manner, there is no reliance on function form, and the volatility premium can be measured without error.

The empirical findings support a negative market price of volatility risk and show that the volatility risk premium is proportional to the level of volatility. However, if gamma is not controlled for, a delta-hedged portfolio can demonstrate a negative volatility premium, but the magnitude is severely muted and may not be considered different from zero. These results are consistent in sign with Bakshi and Kapadia (2003b) and with the conclusions in Coval and Shumway (2001) and Buraschi and Jackwerth (2001) but demonstrate the flaw in using a delta-hedged portfolio to estimate the volatility risk premium, especially in periods of high volatility. By comparison, the delta–gamma hedged portfolios demonstrate significant and negative results, suggesting that the volatility risk premium is higher on average than previously found. The findings for the natural gas markets corroborate the findings of Doran and Ronn (2006) and further support the claim of a negative price and volatility risk premium in energy markets.

The rest of the article is organized as follows. Section 2 addresses the methodology implemented and presents the theoretical findings. Section 3 discusses the simulation. Section 4 explains the data and the empirical controls. In Section 5, the delta- and delta–gamma hedged gains are reported along with the inference on the volatility premium. Section 6 concludes.

2 METHODOLOGY

The goal of this section is to correct the problem associated with ignoring tracking error over discrete time intervals. This correction is accomplished by including an additional option in the traditional delta-hedged portfolio and adjusting the underlying stock position. By including an additional option, I demonstrate how the problems of model misspecification and discretization are reduced in both the Black–Scholes (1973) and stochastic volatility framework.

For completeness, I will first engage in a review of the delta-hedged position. Consider $\Pi^{\delta}_{t,t+\tau}$ to be the gain or loss of a delta-hedged position at time t,

$$\Pi_{t,t+\tau}^{\delta} = C_{it} - C_{t,t+\tau} + \int_t^{t+\tau} \Delta_{iu} \mathrm{d}S_u + \int_t^{t+\tau} r\left(C_{it} - \Delta_{iu} S_u\right) \mathrm{d}u \tag{1}$$

where C_{it} represents the call price of an option on the underlying asset S_t at strike price K_i at time t, maturing at $t+\tau$. Δ_{it} is a defined as the hedge ratio at time t and is synonymous with $\partial C/\partial S$. To account for the option's gamma, where $\Gamma = \partial^2 C/\partial S^2$, Equation (1) is modified by including an additional option C_j at strike price K_j:

$$\Pi_{t,t+\tau}^{\delta\gamma} = C_{it} - C_{t,t+\tau} + \int_t^{t+\tau} \Delta_{iu} \mathrm{d}S_u + \int_t^{t+\tau} r\left(C_{it} - \Delta_{iu} S_u\right) \mathrm{d}u$$
$$+ \int_t^{t+\tau} \frac{\Gamma_{iu}}{\Gamma_{ju}} \mathrm{d}C_{jt} - \int_t^{t+\tau} \frac{\Gamma_{iu}}{\Gamma_{ju}} \Delta_{ju} \mathrm{d}S_u - \int_t^{t+\tau} \frac{\Gamma_{iu}}{\Gamma_{ju}} r\left(C_{ju} - \Delta_{ju} S_u\right) \mathrm{d}u \tag{2}$$

The second line of Equation (2) is the adjustment from the inclusion of buying option C_{jt} and hedging with Δ_{jt} shares of S_t. Buying Γ_i/Γ_j of option C_j produces a gamma-neutral portfolio; to attain delta–gamma neutrality, it is then necessary to adjust the underlying stock position by $(\Gamma_i/\Gamma_j)\Delta_j$. The expected $\Pi_{t,t+\tau}^{\delta\gamma}$ can be interpreted as the excess rate of return on the delta–gamma hedged option portfolio. With the delta–gamma hedge in place, the objective now is to demonstrate how this portfolio controls the model misspecification and discretization error that can arise when an option's gamma is ignored.

2.1 The Black–Scholes model

Initially assume that the stock price follows the classic dynamics of the Black–Scholes geometric Brownian motion under the real-world measure:

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}Z_t \tag{3}$$

where μ is the drift of the stock, σ is the volatility, and $\mathrm{d}Z_t$ represents the Brownian motion. By Itô's lemma, the call price is given by

$$\mathrm{d}C_t = \int_t^{t+\tau} \frac{\partial C}{\partial t} \mathrm{d}u + \int_t^{t+\tau} \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 C}{\partial S^2} \mathrm{d}u + \int_t^{t+\tau} \frac{\partial C}{\partial S} \mathrm{d}S_u \tag{4}$$

and, following the traditional arguments and assumptions, the price of the call must equal the Black–Scholes partial differential equation

$$rC = \frac{\partial C}{\partial t} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial S} rS \tag{5}$$

Plugging Equation (5) into Equation (4) results in

$$\begin{aligned} \mathrm{d}C_t &= rC\,\mathrm{d}t + \frac{\partial C}{\partial S}\left(\mathrm{d}S_t - rS_t\,\mathrm{d}t\right) \\ &= \int_t^{t+\tau} rC_u\,\mathrm{d}u + \int_t^{t+\tau} \frac{\partial C}{\partial S}\left(\mathrm{d}S_u - rS_u\right)\mathrm{d}u \end{aligned}$$

$$C_{t+\tau} = C_t + \int_t^{t+\tau} \frac{\partial C}{\partial S}\,\mathrm{d}S_u + \int_t^{t+\tau} r\left(C_t - \frac{\partial C}{\partial S}S_u\right)\mathrm{d}u \qquad (6)$$

Equation (6) reveals how the call option can be replicated by using $\partial C/\partial S$ shares of the underlying and a risk-free asset. With continuous updating of the hedged shares, the delta-hedged portfolio, $\Pi^{\delta}_{t,t+\tau}$, will be equal to zero if $\partial C/\partial S$ is equal to the hedge ratio Δ. However, as shown by Branger and Schlag (2004), even without stochastic volatility, $\Pi^{\delta}_{t,t+\tau} \neq 0$ when the model moves from continuous to discrete time. Consequently, the remaining focus shifts to demonstrating how model misspecification and discretization error are minimized when controlling for an option's gamma. Thus, it is first necessary to show that $\Pi^{\delta\gamma}_{t,t+\tau} = 0$ for both continuous and discrete trading in a Black–Scholes world.

By substituting (6) into (2), the continuous delta–gamma hedged portfolio should equal

$$\begin{aligned} \Pi^{\delta\gamma}_{t,t+\tau} = &- \int_t^{t+\tau} \frac{\partial C_i}{\partial S}\,\mathrm{d}S_u - \int_t^{t+\tau} r\left(C_{it} - \frac{\partial C_i}{\partial S}S_u\right)\mathrm{d}u \\ &+ \int_t^{t+\tau} \Delta_{iu}\,\mathrm{d}S_u + \int_t^{t+\tau} r\left(C_{it} - \Delta_{iu}S_u\right)\mathrm{d}u \\ &+ \int_t^{t+\tau} \frac{\Gamma_{iu}}{\Gamma_{ju}}\frac{\partial C_j}{\partial S}\,\mathrm{d}S_u + \int_t^{t+\tau} \frac{\Gamma_{iu}}{\Gamma_{ju}} r\left(C_{jt} - \frac{\partial C_j}{\partial S}S_u\right)\mathrm{d}u \\ &- \int_t^{t+\tau} \frac{\Gamma_{iu}}{\Gamma_{ju}}\Delta_{ju}\,\mathrm{d}S_u - \int_t^{t+\tau} \frac{\Gamma_{iu}}{\Gamma_{ju}} r\left(C_{ju} - \Delta_{ju}S_u\right)\mathrm{d}u \end{aligned} \qquad (7)$$

If $\Pi^{\delta\gamma}_{t,t+\tau} = 0$, it must be the case that for each call at strike price K_z, where $z = i, j$, the $\partial C_z/\partial S = \Delta_{zt}$. For continuous trading, the delta–gamma hedged portfolio is equal to the delta-hedged portfolio. In discrete time this is not the case, which will lead to uninformative results when drawing statistical conclusions on the sign and magnitude of $\Pi^{\delta}_{t,t+\tau}$. As shown in Branger and Schlag (2004), a discreetly delta-hedged portfolio will result in a quadratic error term which is always positive if the drift of the stock is greater than the risk-free rate.[2] As a result, if the portfolio is long (short) the option and hedged daily with the underlying asset, the delta-hedged portfolio will be upward (downward)-biased toward zero. If the portfolio initially controls for gamma, this discretization error can be eliminated.

PROPOSITION 1 *For trading that takes place on days $n = t, 1, 2, 3, \ldots, t+\tau$, a portfolio $\Pi^{\delta\gamma}_{t,t+\tau}$ that sells a call C_i that expires on day $t+\tau$ and is delta-hedged with stock S and gamma-hedged with call C_j will have an expected delta–gamma hedged portfolio $\mathrm{E}[\Pi^{\delta\gamma}_{t,t+\tau}] = 0$ if prices follow the Black–Scholes price process and $\partial C_i/\partial S = \Delta_i$ and $\partial C_j/\partial S = \Delta_j$.*

The proof is shown in Appendix A. The introduction of the additional option leads to the following equation:

$$\mathrm{E}^P\left[\Pi^{\delta\gamma}_{t,t+1}\,\middle|\,\Psi_t\right] = -(\mu - r)\left(\frac{\partial C_i}{\partial S} - \Delta_{iu}\right)\int_t^{t+1} \mathrm{E}^P[S_t \mid \Psi_t]\,\mathrm{d}u$$
$$+ (\mu - r)\left(\frac{\partial C_j}{\partial S} - \Delta_{ju}\right)\int_t^{t+1} \mathrm{E}^{P^*}\left[\frac{\dfrac{\partial^2 C_i}{\partial S^2}}{\dfrac{\partial^2 C_j}{\partial S^2}}\,\middle|\,\Psi_t\right] \mathrm{E}^P[S_t \mid \Psi_t]\,\mathrm{d}u \tag{8}$$

When $\partial C_i/\partial S = \Delta_i$ and $\partial C_j/\partial S = \Delta_j$, $\mathrm{E}[\Pi^{\delta\gamma}_{t,t+\tau} = 0]$. Consequently, the introduction of the additional option eliminates the discretization error while achieving a zero-hedged error portfolio.

2.2 Stochastic volatility

Since the main goal of this work is to make inference on the volatility risk premium, I now assume that volatility is non-constant and that prices follow a two-dimensional price process with correlation ρ, as given in Heston (1993):

$$\frac{\mathrm{d}S_t}{S_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}Z^S_t \tag{9}$$

$$\mathrm{d}\sigma^2_t = \left(\kappa(\theta - \sigma^2_t) + \lambda_\sigma \xi \sigma^2\right)\mathrm{d}t + \xi\sigma\,\mathrm{d}Z^{\sigma^2}_t \tag{10}$$

where λ_σ represents the market price of volatility/variance risk, κ is the speed of mean-reversion, ξ is volatility of the variance process and θ is the level to which variance reverts. From this point on, I will use volatility and variance synonymously. The transformed price and volatility process under the risk-neutral

[2] Refer to corollary 1 of Branger and Schlag (2004).

measure are

$$\frac{\mathrm{d}S_t}{S_t} = r\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{Z}_t^S \tag{11}$$

$$\mathrm{d}\sigma_t^2 = \kappa\left(\theta - \sigma_t^2\right)\mathrm{d}t + \xi\sigma\,\mathrm{d}\tilde{Z}_t^{\sigma^2} \tag{12}$$

To derive the relationship between the volatility premium and the gain to a delta–gamma portfolio, I engage in similar steps as done before for the one-dimensional process.

From Itô's lemma, the price of a call is given by

$$\begin{aligned}\mathrm{d}Ct_t = &\int_t^{t+\tau} \frac{\partial C}{\partial t}\,\mathrm{d}u + \int_t^{t+\tau} \frac{1}{2}S^2\sigma^2\frac{\partial^2 C}{\partial S^2}\,\mathrm{d}u + \int_t^{t+\tau} \frac{\partial C}{\partial S}\,\mathrm{d}S_u + \int_t^{t+\tau} \frac{\partial C}{\partial \sigma}\,\mathrm{d}\sigma_u \\ &+ \int_t^{t+\tau} \frac{1}{2}\xi^2\frac{\partial^2 C}{\partial \sigma^2}\,\mathrm{d}u + \int_t^{t+\tau} \rho\xi\sigma S\frac{\partial^2 C}{\partial S\partial\sigma}\,\mathrm{d}u\end{aligned} \tag{13}$$

The price of a call option with stochastic volatility must equal

$$\begin{aligned}rC = &\frac{\partial C}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial S}rS + \frac{1}{2}\xi^2\frac{\partial^2 C}{\partial \sigma^2} + \rho\xi\sigma S\frac{\partial^2 C}{\partial S\partial\sigma^2} \\ &+ \left(\kappa\left(\theta - \sigma_t^2\right) + \lambda_\sigma\xi\sigma^2\right)\frac{\partial C}{\partial \sigma}\end{aligned} \tag{14}$$

Substituting Equation (14) into Equation (13), the call price can be simplified to

$$\begin{aligned}C_{t+\tau} = &C_t + \int_t^{t+\tau} \frac{\partial C}{\partial S}\,\mathrm{d}S_u + \int_t^{t+\tau} r\left(C_t - \frac{\partial C}{\partial S}S_u\right)\mathrm{d}u + \int_t^{t+\tau} \lambda_\sigma\xi\sigma^2\frac{\partial C}{\partial \sigma}\,\mathrm{d}u \\ &+ \int_t^{t+\tau} \xi\frac{\partial C}{\partial \sigma}\,\mathrm{d}Z_u^{\sigma^2}\end{aligned} \tag{15}$$

After combining Equations (15) and (2), the resulting delta–gamma portfolio will be

$$\begin{aligned}\Pi_{t,t+\tau}^{\delta\gamma} = &-\int_t^{t+\tau} \frac{\partial C_i}{\partial S}\,\mathrm{d}S_u + \int_t^{t+\tau} r\left(C_{it} - \frac{\partial C_i}{\partial S}S_u\right)\mathrm{d}u \\ &+ \int_t^{t+\tau} \Delta_{iu}\,\mathrm{d}S_u + \int_t^{t+\tau} r\left(C_{it} - \Delta_{iu}S_u\right)\mathrm{d}u\end{aligned}$$

$$
+\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}\frac{\partial C_j}{\partial S}\,\mathrm{d}S_u+\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}r\left(C_{jt}-\frac{\partial C_j}{\partial S}S_u\right)\mathrm{d}u
$$

$$
-\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}\Delta_{ju}\,\mathrm{d}S_u-\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}r\left(C_{ju}-\Delta_{ju}S_u\right)\mathrm{d}u
$$

$$
-\int_t^{t+\tau}\lambda_\sigma\xi\sigma^2\frac{\partial C_i}{\partial\sigma}\,\mathrm{d}u-\int_t^{t+\tau}\xi\frac{\partial C_i}{\partial\sigma}\,\mathrm{d}Z_u^{\sigma^2}
$$

$$
+\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}\lambda_\sigma\xi\sigma^2\frac{\partial C_j}{\partial\sigma}\,\mathrm{d}u+\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}\xi\frac{\partial C_j}{\partial\sigma}\,\mathrm{d}Z_u^{\sigma^2} \tag{16}
$$

With the inclusion of stochastic volatility, the $\Pi_{t,t+\tau}^{\delta\gamma}$ will only equal zero if the market price of volatility risk, λ_σ, is equal to zero. If λ_σ is non-zero, then gains to $\Pi_{t,t+\tau}^{\delta\gamma}$ could be attributed to a volatility premium if hedging errors are zero. Consequently, as in the case of the one-dimensional price process, it is necessary to demonstrate that the gamma control eliminates the additional terms that arise from the implementation of a continuous-time process for a discrete trading interval for the stochastic volatility process.

PROPOSITION 2 *For trading that takes place on days $n = t, 1, 2, 3, \ldots, t+\tau$, a portfolio $\Pi_{t,t+\tau}^{\delta\gamma}$ that sells a call C_i that expires on day $t+\tau$ and is delta-hedged with stock S and gamma-hedged with call C_j will have an expected delta–gamma hedged portfolio $\mathrm{E}[\Pi_{t,t+\tau}^{\delta\gamma}] = 0$ if $\lambda_\sigma = 0$, $\partial C_i/\partial S = \Delta_i$, $\partial C_j/\partial S = \Delta_j$, and prices follow the stochastic volatility model outlined in Equations (9)–(12).*

The proof is demonstrated in Appendix B. As shown in the appendix, the inclusion of stochastic volatility incorporates two additional terms per option, each dependent on the market price of volatility risk. Combining the expressions, the delta–gamma gains are then related to two terms, both including the volatility risk premium:

$$
-\lambda_\sigma\int_t^{t+\tau}\left(1-\mathrm{E}^*\left[\left.\frac{\dfrac{\partial^2 C_i}{\partial S^2}}{\dfrac{\partial^2 C_j}{\partial S^2}}\right|\Psi_t\right]\right)\mathrm{E}^{P^*}\left[\left.\xi\sigma_u^2\frac{\partial C}{\partial\sigma}\right|\Psi_t\right]\mathrm{d}u \tag{17}
$$

$$
-\lambda_\sigma(\mu-r)\int_t^{t+\tau}\mathrm{E}^{P^*}[\Omega]\,\mathrm{E}^P\left[S_t\middle|\Psi_t\right]\mathrm{d}u \tag{18}
$$

where

$$\Omega = \left(\left[\xi \sigma_u^2 \frac{\partial^2 C_i}{\partial S \partial \sigma_i} \middle| \Psi_t \right] - \left[\frac{\dfrac{\partial^2 C_i}{\partial S^2}}{\dfrac{\partial^2 C_j}{\partial S^2}} \middle| \Psi_t \right] \left[\xi \sigma_u^2 \frac{\partial^2 C_j}{\partial S \partial \sigma_j} \middle| \Psi_t \right] \right)$$

If $\lambda_\sigma = 0$, then the expected delta–gamma hedged gains should be equal to zero; if $\lambda_\sigma = < 0$, then gains should be positive. It is clear from the first term that the delta–gamma gains are directly proportional to the market price of volatility risk. However, the relationship between

$$\left(\frac{\partial^2 C_i}{\partial S^2} \middle/ \frac{\partial^2 C_j}{\partial S^2} \right) \frac{\partial C_i}{\partial \sigma}$$

and $\partial C_i / \partial \sigma$ is almost identical, resulting in a negligible impact on delta–gamma hedged gains.[3] Consequently, the value of Equation (18) determines the sign and magnitude of the delta–gamma gains.

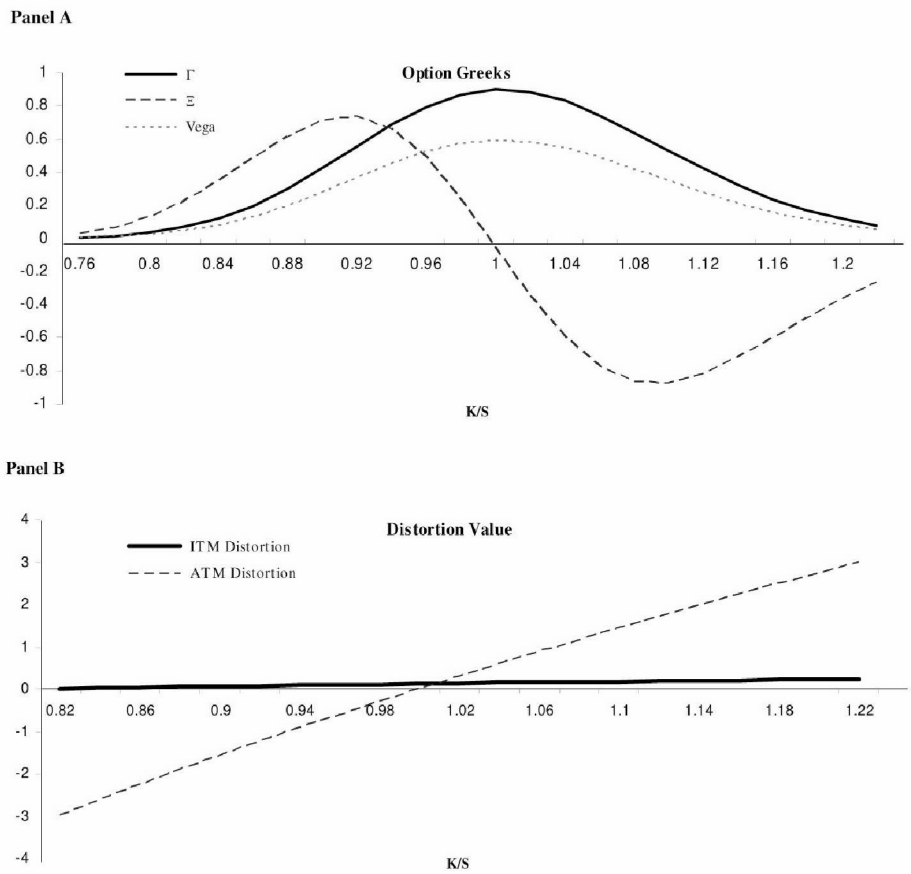
The impact of Equation (18) is unclear due to the cross-partial with respect to the stock price and volatility. What is clear is that the delta–gamma hedged gains are proportional to the cross-partial and not to Vega. Figure 1 documents the behavior of $\partial C/\partial\sigma$, $\partial^2 C/\partial S^2$ and $\partial^2 C/\partial S \partial\sigma$, denoted Vega, Γ and Ξ, respectively, for a given maturity and volatility level. Ξ is positive for in-the-money (ITM) options, close to zero for at-the-money (ATM) options and negative for out-of-the-money (OTM) options. For a fixed time interval, the value of the Ξ tends to decrease and become more muted over the cross-section of strike prices as volatility increases. This could result in positive values for Equation (18), and subsequently a misleading negative delta–gamma gain, with a negative λ_σ for OTM options.

However, since the delta–gamma portfolio holds two options simultaneously, the combined impact of the cross-partial is quite transparent. As long as C_i has a lower strike price than C_j, the value of the "distortion", $(\Xi_i - (\Gamma_i/\Gamma_j)\Xi_j)$, is always positive. The resulting effect on delta–gamma hedged gains with a volatility of 30% and 22 days to expiration is shown in Panel B of Figure 1. The ITM distortion documents the value of C_i with a strike/spot ratio of 0.8 and strike/spot values of (0.82–1.22) for C_j. The ATM distortion has a C_i with a strike/spot ratio of 1 and (0.82–1.22) for C_j. In each case, with $K_i < K_j$, the distortion is positive. The effect of a volatility increase results in a decrease of the absolute value of the distortion. For longer maturities, the value of the distortion also decreases and becomes more muted across K_j. Thus, if the delta–gamma gains are positive, they must be associated with a negative market price of volatility risk if the risk premium is positive.[4] However, even in eliminating the problems associated with discretization, inferring the magnitude of the volatility premium requires

[3] In a Black–Scholes world these two values are exactly equal.

[4] For options on futures, the drift should equal the risk-free rate and should always be positive.

FIGURE 1 Higher-order Greeks.



Panel A demonstrates the value of higher-order greeks across moneyness. Γ is the second partial derivative of the option prices with respect to the stock, $\partial^2 C/\partial S^2$. Ξ is equal to $\partial^2 C/\partial S \partial \sigma$. Vega is equal to $\partial C/\partial \sigma$. The greeks were calculated with a volatility level of 30% and 22 business days to expiration. Panel B demonstrates the distortion value, defined as $(\Xi_i - (\Gamma_i/\Gamma_j)\Xi_j)$. The strike/price ratio for C_i for the ATM distortion has a value of 1; for the ITM distortion the value is equal to 0.8. The lines are generated by calculating the distortion values using different strike/price ratios of C_j.

controlling for the degree of distortion since the distortion is not equal across strike prices. This will be addressed in the empirical estimation.

A question that arises in estimation is whether the added benefit from adding the option may make the estimation of the volatility risk premium more difficult. In other words, the level of the "distortion" may be too variable through time to make correct inferences on the sign and magnitude of the market price of volatility risk, such that it is worth estimating the risk premium with error. However, if the correct option is used to hedge gamma, the value of the distortion is quite stable. This will be demonstrated in the following section.

3 SIMULATION EVIDENCE

To confirm the results of the preceding section a quasi-Monte Carlo simulation is conducted for a 30-day option. In addition to demonstrating how a delta–gamma portfolio is not prone to discretization error, there is the added benefit of measuring model misspecification error since the hedge ratios will be calculated using Black–Scholes. For $\Pi^{\delta}_{t,t+\tau}$, daily hedging gains/errors are initially calculated using the discrete analog of Equation (1) assuming prices follow the traditional Black–Scholes (BS) one-dimensional process. The Black–Scholes process is then augmented to account for stochastic volatility (SV). The formation of $\Pi^{\delta\gamma}_{t,t+\tau}$ errors, calculated using Equation (2), requires the use of an additional option. For this option, the value of the strike price is set at a value 5% greater than the strike price of the initial option sold.

For each portfolio, 40,000 30-day price paths are run on Equation (3), and its risk-neutral counterpart, for a 10% ITM, ATM, and 10% OTM option using one-day time intervals. The values found from Equations (1)–(2) are then plotted to demonstrate the distribution. The price of the underlying asset is given at \$50, with a volatility of 30%, risk-free rate of 3% and one month to expiration. The delta and gamma of the option are calculated as the traditional partial derivatives from Black–Scholes, $\Delta_t = N[d_1(F_t, t)]$ and $\Gamma_t = n[d_1]/(S\sigma\sqrt{T})$, where $N[\cdot]$ and $n[\cdot]$ are the cumulative normal distribution and normal density respectively. The value for d_1 is calculated as

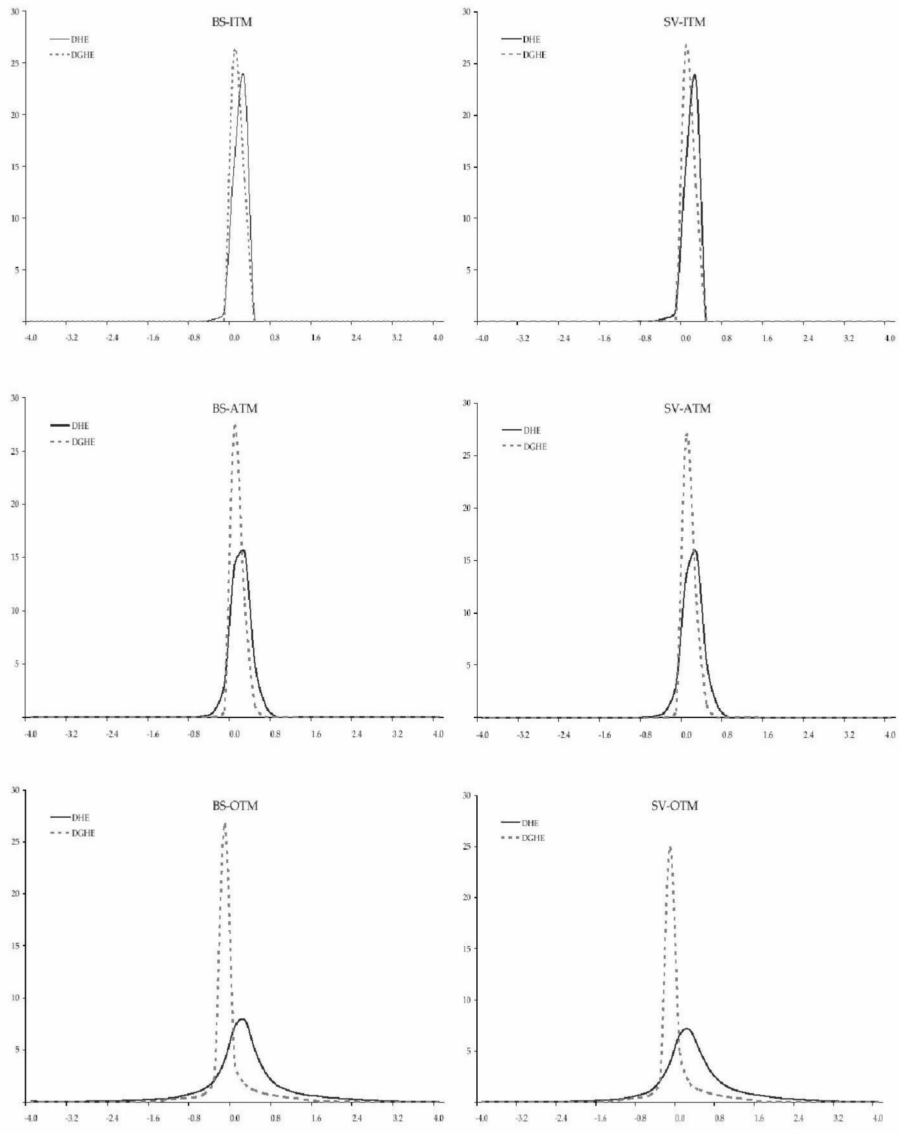
$$d_1 = \frac{\ln(S/X) + \left(r + \dfrac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \tag{19}$$

The distributions of average discrete hedging errors over the 30-business day period are shown in Figure 2 for both the BS and SV data-generating processes. The $\Pi^{\delta}_{30,1}$ are mostly centered around zero, it is clearly more variable, and less centered than $\Pi^{\delta\gamma}_{30,1}$. This is a direct result of not controlling for gamma and the result demonstrated in Equation (33) in Appendix A. The mean error for $\Pi^{\delta}_{30,1}$ is increasing from zero for out-of-the money options as well as a large increase in the variability. While gamma tends to be highest for ATM options, the greatest proportional effect on the delta-hedged portfolio occurs for OTM options because of smaller call values. By contrast, the mean error for $\Pi^{\delta\gamma}_{30,1}$ is centered on zero regardless of moneyness since the portfolio initially controls for tracking error and has a smaller proportional decrease in the precision of the estimate.

To test for the effect of model misspecification, stochastic volatility is incorporated but Black–Scholes assumptions are used to price the option and corresponding delta- and gamma-hedge ratios. This test requires an additional simulation conducted over Equations (9)–(12). For the simulation, $\kappa = 7$, $\theta = 0.09$, $\xi = 0.3$ and $\rho = -0.53$.[5] In each case the market price of volatility risk is set equal

[5] These parameters were chosen to be consistent with prior works; however, alternative values were used but did not alter the general conclusion of the simulation.

FIGURE 2 Black–Scholes and stochastic volatility-hedging errors.



The figure shows the distribution of simulated dollar-hedging errors of 40,000 price paths for a 30-day option for both a delta-hedged ($\Pi^{\delta}_{30,1}$) and a delta–gamma hedged portfolio ($\Pi^{\delta\gamma}_{30,1}$). Each portfolio is tested using one of two data-generating processes: (1) Black–Scholes (BS) using stock price and strike price equal to 50, a risk-free rate equal to 3%, a volatility equal to 30%, and a market price of risk equal to 0.41; or (2) stochastic volatility (SV) using the BS inputs and a speed of mean-reversion equal to 7, a level of mean-reversion equal to 0.09, volatility of variance process equal to 0.3, a correlation of −0.53, and market price of volatility risk, λ_{σ}, equal to 0. The ITM option has a strike price of $45, the OTM option had a strike of $55, and the second option strike price was 5% greater than that of the first option. DHE is the delta-hedged error defined as the discrete form of Equation (1) from day 30 to day one. DGHE is the delta–gamma hedged error defined as the discrete form of Equation (2) from day 30 to day one.

to zero. With $\lambda_\sigma = 0$, the stochastic volatility hedging errors should demonstrate similar distributions to the findings for the Black–Scholes price process if there is no model misspecification. The resulting distributions are also shown in Figure 2.

While there is small measurement error, there is no significant difference between the distributions shown in Figure 2 for either $\Pi^{\delta}_{30,1}$ or $\Pi^{\delta\gamma}_{30,1}$. For example, a test for the difference in means between BS and SV models for $\Pi^{\delta}_{30,1}$ resulted in a t-value of 1.20 for ATM options. This suggests that discretization is of greater concern than model misspecification for hedging errors.

Finally, three additional effects are introduced to examine the impact on hedging errors: (1) a negative market price of risk, (2) a negative jump, and (3) less frequent rebalancing. The impact of a negative volatility risk premium should produce positive hedging errors for both portfolios, but the impact should be magnified for $\Pi^{\delta}_{30,1}$ given that the discretizion error now encompasses two terms. The effect of jumps should produce a more negative distribution, while less frequent rebalancing should increase the variability for both portfolios; however, each effect should be magnified for $\Pi^{\delta}_{30,1}$. Table 1 and Figure 3 demonstrate the results.

Examining each rebalancing simulation, the initial BS simulation, shown in Panel A of Figure 3, reveals positive and significant error for $\Pi^{\delta}_{30,1}$ but insignificant error for $\Pi^{\delta\gamma}_{30,1}$. The effect of the discretization can make up close to 80% of the dollar-hedged error in the case, as calculated from Equation (33) in Appendix A.

TABLE 1 Simulated dollar-hedging errors.

	One-day rebalance		Five-day rebalance		10-day rebalance	
	$\Pi^{\delta}_{30,1}$	$\Pi^{\delta\gamma}_{30,1}$	$\Pi^{\delta}_{30,1}$	$\Pi^{\delta\gamma}_{30,1}$	$\Pi^{\delta}_{30,1}$	$\Pi^{\delta\gamma}_{30,1}$
BS	\$0.05	\$0.00	\$0.05	\$0.00	\$0.04	\$0.00
	(\$0.33)	(\$0.20)	(\$0.70)	(\$0.37)	(\$0.97)	(\$0.47)
SV	\$0.53	\$0.08	\$0.52	\$0.07	\$0.53	\$0.07
	(\$0.37)	(\$0.23)	(\$0.72)	(\$0.33)	(\$0.97)	(\$0.48)
SVJ	\$0.41	\$0.07	\$0.42	\$0.06	\$0.42	\$0.06
	(\$0.64)	(\$0.26)	(\$0.88)	(\$0.37)	(\$1.08)	(\$0.50)

The table reports the simulated dollar-hedging errors of 40,000 price paths for a 30-day option for both a delta-hedged ($\Pi^{\delta}_{30,1}$) and a delta–gamma hedged portfolio ($\Pi^{\delta\gamma}_{30,1}$). Each portfolio is tested using three data-generating processes: (1) Black–Scholes (BS) using stock price and strike price equal to 50, a risk-free rate equal to 3%, a volatility equal to 30%, and a market price of risk equal to 0.41; (2) stochastic volatility (SV) using the BS inputs and a speed of mean-reversion equal to 7, a level of mean-reversion equal to 0.09, volatility of variance process equal to 0.3, a correlation of −0.53, and market price of volatility risk, λ_σ, equal to −2; and (3) stochastic volatility with jumps (SVJ) using the SV inputs and jump arrival rate of 1, a mean jump size of −10% and a variance of jump size of 0.025. The arrival rate is drawn from a Poisson distribution. Each portfolio is examined under the three-parametric distributions using three rebalancing frequencies; one-day, five-day and 10-day. For each portfolio, model and rebalancing period, the mean dollar-hedging error is reported as well as the dollar standard deviation, given in parentheses.

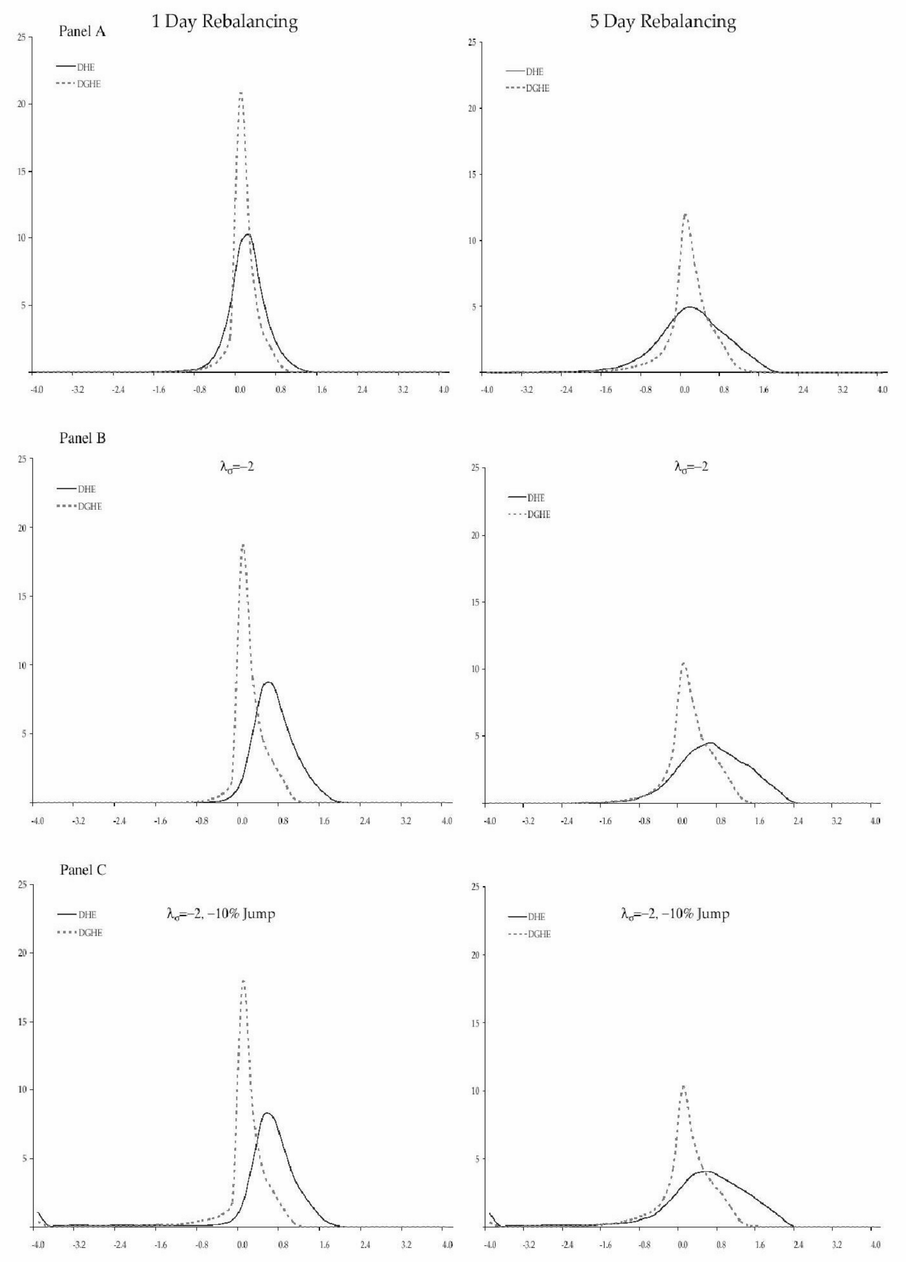
So while the distribution appears centered around zero, the effect of the discretization does result in positive, yet small, economic returns. The economic impact of these errors is small since they make up less than 4% of the option value. However, with the inclusion of stochastic volatility (SV) and a negative volatility risk premium, $\lambda_\sigma = -2$, shown in Panel B of Figure 3, result in a significant increase in positive dollar errors. The mean dollar-hedge error is \$0.52 and \$0.07 for $\Pi^{\delta}_{30,1}$ and $\Pi^{\delta\gamma}_{30,1}$, respectively. This finding is consistent with the theoretical prediction presented earlier. The effect of a negative volatility risk premium has resulted in a discretization error that is now close to 35% of the option value and economically significant for $\Pi^{\delta}_{30,1}$. Consequently, the ramifications of a large discretization error can lead to incorrect conclusions for the volatility risk premium using a delta-hedged portfolio.

The inclusion of a jump, drawn from the distribution $N \sim (-0.1, 0.025)$ and jump arrival rate of 1, has the effect of making the distribution of the errors more variable and negatively skewed. The variability in dollar-hedge error is \$0.64 and \$0.26 for $\Pi^{\delta}_{30,1}$ and $\Pi^{\delta\gamma}_{30,1}$, respectively. The effect of the jump is magnified for $\Pi^{\delta}_{30,1}$, where the dollar errors are almost double the variability using one-day rebalancing. The skewness can be seen in Panel C of Figure 3, where the tails are kinked up in the last panel. The inclusion of the jump can lead to outcomes where the hedged errors can exceed the call value. Thus, if prices are best described using a stochastic volatility with jumps (SVJ) process, then estimating the volatility risk premium using a delta-hedged portfolio will result in not only higher measurement error due to discretization but also lower power in the estimation due to higher variability as compared to the delta–gamma portfolio.

The effect of the fewer rebalancing periods results in greater variability in both portfolios but has less impact on the delta–gamma portfolio. This hold regardless of whether there is a significant market price of volatility risk and/or a jump included in the specification. This is interesting as it may have economic ramifications on hedging behavior. Using the SVJ data-generating process, the variability in 10-day rebalance for the delta–gamma hedged portfolio is less than the one-day variability for the delta-hedged portfolio. The high variability in the delta-hedged portfolio is not surprising, given the Robins and Schachter (1994) finding that it is not the minimum-variance hedge. Since daily rebalancing is costly, it appears economically viable to implement a delta–gamma hedge to protect profits with fewer rebalancing periods. However, as to estimating the volatility risk premium, the concern with costs of rebalancing is not necessary since it is an econometric exercise and not a practical application.

Figure 3 shows the distribution of simulated ATM dollar-hedging errors of 40,000 price paths for a 30-day option for both a delta-hedged ($\Pi^{\delta}_{30,1}$) and a delta–gamma hedged portfolio ($\Pi^{\delta\gamma}_{30,1}$). Each portfolio is tested using three data-generating processes: (1) Black–Scholes (BS) using stock price and strike price equal to 50, a risk-free rate equal to 3%, a volatility equal to 30%, and a market price of risk equal to 0.41; (2) stochastic volatility (SV) using the BS inputs and a speed of mean-reversion equal to 7, a level of mean-reversion equal to 0.09, volatility of variance process equal to 0.3, a correlation of −0.53, and market price

FIGURE 3 Total hedging errors.



of volatility risk, λ_σ, equal to −2; and (3) stochastic volatility with jumps (SVJ) using the SV inputs and jump arrival rate of 1, a mean jump size of −10%, and a variance of jump size of 0.025. The arrival rate is drawn from a Poisson distribution. Each portfolio is examined under the three-parametric distributions using two rebalancing frequencies: one-day and five-day. DHE is the delta-hedged error defined as the discrete form of Equation (1) from day 30 to day one. DGHE is the delta–gamma hedged error defined as the discrete form of Equation (2) from day 30 to day one.

While Bertsimas, Kogan, and Lo (2000) demonstrate how the realized delta-hedged error may have mean of zero, the discretization error does have a positive significant impact on the delta-hedging errors. Additionally, the variability of the distribution is significantly higher than the delta–gamma hedged errors and increases with the inclusion of jumps or as the length of time increases between rebalancing. These findings highlight the potential problems of ignoring tracking error while confirming the stability of the "distortion" parameter. However, the result is sensitive to different strike prices used to form the gamma hedge, as the more away-from-the-money the second option is, the higher the delta–gamma hedging error becomes.[6] However, any option used to control for gamma that is within 10% of the first option's strike price produces hedging errors that are one-half to a one-third of the variability of the errors associated with traditional delta-hedged portfolios.

4 DATA

To demonstrate the difference between delta- and delta–gamma hedged gains, and the impact of volatility, option prices from the S&P100 index and natural gas markets are examined. The daily S&P100 index options were acquired through Optionmetrics from January 1996 through December 2002. Optionmetrics reports the best closing bid and offer price for the options, so all option prices used in the sample are the mid-point of these two observations. For the natural gas options, daily closing prices were collected from NYMEX from January 2000 through December 2004. In particular, I report and focus on call options only, although put options can be used in a similar fashion to acquire similar results.

In constructing the sample, several filters were applied to reduce recording and measurement errors for the hedged portfolios. First, any option that violated an American arbitrage bound ($C_t > S_t e^{-d\tau}$, $C_t < S_t e^{-d\tau} - e^{-r\tau}K$) was excluded, where d is the dividend yield. Second, for a given strike price K, any option that had a missing observation from day t, the day the option was bought, to expiration, $t + \tau$, was removed. Third, any option that had a price less than \$0.25 or had a maturity less than seven days was deleted. Specific to the natural gas options, only maturities of 30 days or less were examined to avoid potential problems due to lack of trading volume.

In forming the delta–gamma hedged portfolios, additional constraints were implemented. Any option, C_{it}, sold at strike price K_i on date t was required to have a corresponding option, C_{jt}, at strike price K_j that could also be used to create the gamma position. The option C_{jt} had also to conform to the same delta-hedged requirements as C_{it}, but the absolute percentage difference in the strike

[6] Further tests on delta–gamma hedging errors used options that were 20% in- and out-of-the money. The results suggest that the more the option is either in- or out-of-the money, the less effective it becomes at reducing hedging errors. However, with higher levels of volatility, deeper ITM and OTM options can be used to eliminate the tracking error problem.

prices between the two options could not be greater than 10% for the S&P options. This requirement was relaxed to 25% for the natural gas contracts, but only options with strike/spot ratios between (0.9–1.1) were examined. Additionally, C_{jt} had to be greater than $(\Gamma_i/\Gamma_j)C_{jt}$ to ensure a short position in the option contract. Finally any Γ_i or Γ_j that was less than 0.001 was eliminated.

The total number of S&P delta-hedged portfolios fashioned was 127,597; for the delta–gamma hedged portfolio there are 100,546. This includes all portfolios over all moneyness and maturity ranges that survived the filtering process. For the natural gas sample, there are 5,366 and 1,866 delta-hedged and delta–gamma hedged portfolios respectively. There are significantly fewer portfolios for natural gas because of tighter restrictions on the filtering process.

The price of the S&P100 is price-adjusted for dividends, and the risk-free rate is the three-month treasury yield from the Federal Reserve. To estimate day-t volatility, $\tilde{\sigma}_t$, three measure are adopted, a GARCH(1,1) model, and two measures of sample standard deviation. The GARCH(1,1) is estimated for both the S&P100 and natural gas futures using the entire sample:

$$\mu_{t-1} = \bar{\mu} + \epsilon_t \tag{20}$$

$$\sigma_t^2 = \alpha + \beta\mu_{t-1}^2 + \gamma\sigma_{t-1}^2 \tag{21}$$

$$\epsilon_t = \sigma_t \nu_t \tag{22}$$

where $\mu_t = \ln(S_t/S_{t-1})$, σ_t is the conditional volatility, and $\nu \sim N(0,1)$. From this estimate, a daily time series of volatility, σ^h, is constructed allowing for estimation of Δ_t, Γ_t, $Vega_t$ and Ξ_t. The additional measures of volatility are sample standard deviations constructed with and without the mean return over the period:

$$\tilde{\sigma}_t^h = \sqrt{\frac{252}{\tau}\sum_{n=t-\tau}^{t}\left(R_n - \overline{R_{t,t+\tau}}\right)^2} \tag{23}$$

$$\tilde{\sigma}_t^{2h} = \sqrt{\frac{252}{\tau}\sum_{n=t-\tau}^{t} R_n^2} \tag{24}$$

where $\overline{R_{t,t+\tau}}$ is the average return over the given period. The advantage in using Equations (23)–(24) is that it produces volatility estimates with serially uncorrelated errors for non-overlapping periods. For each daily $\tilde{\sigma}_t$, 30 days of prior returns are used in constructing the estimate.[7]

[7] Additional robustness checks used shorter, 15-day, and longer, 60-day, samples to calculate the prior volatility estimate. In each case the effect of the time horizon had negligible impact on the results.

5 ESTIMATION OF DELTA- AND DELTA–GAMMA HEDGED PORTFOLIOS

The first step in determining the extent of the volatility premium is to calculate the discrete values of $\Pi^{\delta}_{t,t+\tau}$, and $\Pi^{\delta\gamma}_{t,t+\tau}$ as

$$\Pi^{\delta}_{t,t+\tau} = C_t - \max(S_{t+\tau} - K, 0) + \sum_{t=0}^{t+\tau-1} \Delta_t(S_{t+1} - S_t) + \sum_{t=0}^{t+\tau-1} r_t(C_t - \Delta_t S_t) \tag{25}$$

$$\begin{aligned}\Pi^{\delta\gamma}_{t,t+\tau} = {} & C_{it} - \max(S_{i,t+\tau} - K_i, 0) + \sum_{t=0}^{t+\tau-1} \Delta_{it}(S_{i,t+1} - S_t) + \sum_{t=0}^{t+\tau-1} r_t(C_{it} - \Delta_{it} S_t) \\ & + \sum_{t=0}^{t+\tau-1} \frac{\Gamma_{it}}{\Gamma_{jt}}(C_{j,t+1} - C_{jt}) - \sum_{t=0}^{t+\tau-1} \frac{\Gamma_{it}}{\Gamma_{jt}} \Delta_{jt}(S_{j,t+1} - S_t) - \sum_{t=0}^{t+\tau-1} \frac{\Gamma_{it}}{\Gamma_{jt}} r(C_{jt} - \Delta_{jt} S_t)\end{aligned} \tag{26}$$

where S_t is the time-t price of the S&P100 index. For the natural gas contracts, S_t is replaced by F_t, and the Black–Scholes hedge ratios, Δ and Γ, are amended by subtracting the risk-free rate, r, from Equation (19) as given in Black (1976). As shown in Section 3, the problems resulting from model misspecification are small and are confirmed in robustness checks in Bakshi and Kapadia (2003b).[8] Consequently, each time a hedge ratio is calculated, $\Delta_t = N[d_1(F_t, t)]$ and $\Gamma_t = n[d_1]/(S\sigma\sqrt{T})$, where $n[\cdot]$, $N[\cdot]$, and d_1 are defined as before.

In particular, the delta and delta–gamma portfolio gains are calculated in the following manner. At time t, call C_{it} is sold and hedge ratio Δ_{it} is calculated with Δ_{it} shares of S_t bought to hedge the exposure. In borrowing/lending $r_t(C_{it} - \Delta_t S_t)$, the delta hedge is created for day t such that $\Pi^{\delta}_t = 0$. For the gamma hedge, Γ_t is first calculated for C_i and C_j, and then Γ_{it}/Γ_{jt} shares of C_j are bought. The creation of the gamma position eliminates the initial delta hedge, requiring the selling of $(\Gamma_{it}/\Gamma_{jt})\Delta_{jt}$ shares of S_t and borrowing $(\Gamma_{it}/\Gamma_{jt}) \times r(C_{jt} - \Delta_{jt} S_t)$. This forms the delta–gamma hedge at time t such that $\Pi^{\delta\gamma}_t = 0$. The positions are then reformed on day $t+1$ with the new hedge ratios Δ_{t+1} and Γ_{t+1}. This process is completed until expiration and the daily gains are then summed to arrive at the value for $\Pi^{\delta}_{t,t+\tau}$ and $\Pi^{\delta\gamma}_{t,t+\tau}$.

5.1 S&P100

To estimate the gains from the delta- and delta–gamma hedged portfolios, each option is evaluated using Equations (25) and (26) respectively. The formation of delta-hedged gains is similar to that of Bakshi and Kapadia (2003a, b) except that in this case the primary option is sold vs. bought. While this does not change the

[8] They demonstrate that model misspecification is not a problem through an analysis of potential over- and under-hedging. Additionally noted, since the hedge ratio takes into account time-varying GARCH volatility, the model is less misspecified.

qualitative results, it does present a more realistic position of most option traders, as on average they tend to take the short side of the contract.

Table 2 presents the results for the dollar delta-hedged gains for S&P100 call options for various maturities and moneyness. Additionally reported are the scaled dollar gains by call price, Π_t^{δ}/C_{it}, and index level, Π_t^{δ}/S_t. Across all maturity and moneyness combinations, the delta-hedged strategy makes money and is significant, except when the option is deep OTM with a maturity of less than 14 days. The average return makes about 0.27% of the index and 18.9% of the call value. For the ATM options, the average return is 0.34% and 10.6% for the scaled gains by index and call price respectively. These results reflect greater delta-hedged gains than the findings of Bakshi and Kapadia (2003b). In particular, their findings showed an average gain of 3.31% of the call price for ATM S&P500 options with maturity between 15 and 30 days from January 1991 to December 1995.[9] As shown in Table 2, for the same moneyness and maturity category, but for the January 1996 to December 2002 period, the scaled gain is 8.78%. This increase can be attributed to the level of volatility in the market. The average 30-day realized volatility from 1991 to 1995 is 10.33%; for the period 1996–2002 the average volatility is 19.83%. Since delta-hedged gains are proportional to volatility, a difference of 5.47% seems plausible.

Also of note and consistent with the preceding findings, the dollar delta-hedged gains tend to increase with maturity and decrease with away-from-the-money options. This can be directly associated with the Vega of the option, as an option's Vega is highest at-the-money and increases with maturity. However, the deep ITM option also demonstrates positive gains, unlike the findings of Bakshi and Kapadia (2003b). This is part due to the increased liquidity for these options in the current sample, which reduced recording and price updating errors associated with their 1991–1995 sample.

While these results are highly significant and have large economic impact, they cannot be fully associated with a negative market price of volatility risk. Since there is significant tracking error when the gamma of the option is not controlled for, the gains to delta position can be attributed to either potential explanation. Consequently, it is relevant to associate the volatility premium with the findings for the delta–gamma position. The results for delta–gamma hedged gains can be found in Table 3.

The delta–gamma position is calculated as

$$\Pi_t^{\delta\gamma} = \frac{1}{n}\sum\nolimits_{j=1}^{n}\Pi_{jt}^{\delta\gamma}$$

where j represents all strike prices, K_j, that are greater than K_i. In averaging across all potential gamma hedges for a given option, the potential distortion value from the cross-partial derivative is dependent only on the strike/spot ratio

[9] Their reported gain is negative because their position buys the call and then sells the stock to form the delta-hedged.

TABLE 2 Delta-hedged gains for all S&P100 call options.

		Π^{δ}				Π^{δ}/S				Π^{δ}/C			
K/S	***N***	**<14**	**14–30**	**31–60**	**>61**	**<14**	**14–30**	**31–60**	**>61**	**<14**	**14–30**	**31–60**	**>61**
<0.875	20,076	0.76	0.88	1.60	1.44	0.12	0.15	0.26	0.22	0.66	0.86	1.43	1.16
		(0.04)	(0.05)	(0.04)	(0.08)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
0.875–0.925	16,485	0.878	1.26	1.69	1.70	0.14	0.21	0.28	0.30	1.42	2.02	2.48	2.21
		(0.08)	(0.08)	(0.07)	(0.11)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
0.925–0.975	21,506	0.69	1.19	1.87	2.34	0.12	0.19	0.30	0.40	2.21	3.28	4.40	4.45
		(0.10)	(0.08)	(0.07)	(0.11)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
0.975–1.025	24,086	0.71	1.15	2.08	3.42	0.12	0.17	0.33	0.55	12.10	8.78	10.52	11.49
		(0.09)	(0.06)	(0.07)	(0.09)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)
1.025–1.075	21,334	0.15	0.54	1.43	3.84	0.02	0.08	0.23	0.60	47.73	43.45	20.41	23.32
		(0.07)	(0.05)	(0.07)	(0.08)	(0.00)	(0.00)	(0.00)	(0.00)	(0.04)	(0.03)	(0.01)	(0.01)
1.075–1.125	9,994	−0.09	0.14	0.90	3.21	−0.01	0.02	0.17	0.51	−7.02	46.95	39.93	36.14
		(0.17)	(0.04)	(0.08)	(0.09)	(0.00)	(0.00)	(0.00)	(0.00)	(0.40)	(0.08)	(0.04)	(0.02)
>1.125	14,116	−0.06	0.02	0.59	1.53	0.03	0.00	0.13	0.27	−66.38	−15.43	73.95	87.97
		(0.40)	(0.04)	(0.04)	(0.05)	(0.00)	(0.00)	(0.00)	(0.00)	(1.72)	(0.09)	(0.07)	(0.04)

The table reports the returns to the dollar delta-hedged gains, Π^{δ}, dollar delta-hedged gains scaled by index price, Π^{δ}/S_t, and dollar delta-hedged gains scaled by call price, Π^{δ}/C_t. Each portfolio is rebalanced daily to create a time-t delta-neutral portfolio calculated from Equation (25).

$$\Pi^{\delta}_{t,t+\tau} = C_t - \max(S_{t+\tau} - K, 0) + \sum_{t=0}^{T-1} \Delta_t (S_{t+1} - S_t) + \sum_{t=0}^{T-1} r_t (C_t - \Delta_t S_t)$$

The Δ_t is updated daily and is calculated as the Black–Scholes hedge ratio using σ^h volatility. The return for each moneyness/maturity category represents the average return from the sample January 1996 to December 2002. The standard errors are shown in parenthesis. Returns are given in percentages for Π^{δ}/S_t and Π^{δ}/C_t.

TABLE 3 Delta–gamma hedged gains for all S&P100 call options.

		$\Pi^{\delta\gamma}$				$\Pi^{\delta\gamma}/\Phi$				$\Pi^{\delta\gamma}/X$			
K/S	***N***	**<14**	**14–30**	**31–60**	**>61**	**<14**	**14–30**	**31–60**	**>61**	**<14**	**14–30**	**31–60**	**>61**
<0.875	9,345	0.982	−0.052	0.108	2.098	0.25	−0.01	−0.06	0.46	1.59	0.02	−0.32	2.21
		(0.21)	(0.21)	(0.14)	(0.20)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
0.875–0.925	14,251	0.557	1.333	1.239	0.715	0.14	0.27	0.20	0.17	1.40	2.64	1.82	1.58
		(0.18)	(0.12)	(0.10)	(0.15)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
0.925–0.975	21,288	0.587	0.909	0.969	0.438	0.14	0.22	0.22	0.15	2.45	3.75	3.71	2.67
		(0.09)	(0.07)	(0.06)	(0.11)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
0.975–1.025	23,893	0.532	0.938	1.087	0.754	0.12	0.20	0.24	0.20	21.38	17.81	14.04	8.68
		(0.07)	(0.04)	(0.05)	(0.07)	(0.00)	(0.00)	(0.00)	(0.00)	(0.03)	(0.01)	(0.01)	(0.01)
1.025–1.075	18,573	−0.053	0.301	0.678	1.340	0.00	0.06	0.14	0.28	22.53	51.16	41.38	28.44
		(0.10)	(0.04)	(0.05)	(0.06)	(0.00)	(0.00)	(0.00)	(0.00)	(0.17)	(0.04)	(0.02)	(0.02)
1.075–1.125	7,026	0.190	0.152	0.347	1.143	0.03	0.03	0.08	0.23	76.18	36.58	53.14	54.98
		(0.16)	(0.05)	(0.07)	(0.06)	(0.00)	(0.00)	(0.00)	(0.00)	(0.54)	(0.10)	(0.08)	(0.04)
>1.125	6,170	0.886	0.417	0.724	0.975	0.22	0.09	0.16	0.21	414.02	134.65	123.40	111.92
		(0.36)	(0.10)	(0.06)	(0.03)	(0.00)	(0.00)	(0.00)	(0.00)	(1.55)	(0.38)	(0.15)	(0.05)

The table reports the returns to the dollar delta–gamma hedged gains, $\Pi^{\delta\gamma}$, dollar delta-hedged gains scaled by Φ, $\Pi^{\delta\gamma}/\Phi_t$, and dollar delta-hedged gains scaled by X, $\Pi^{\delta\gamma}/X_t$. Φ_t is equal to $S_t - (\Gamma_{it}/\Gamma_{jt})\Delta_{jt}S_t$; X_t is equal to $C_{it} - (\Gamma_{it}/\Gamma_{jt})C_{jt}$. Each portfolio is rebalanced daily to create a time-t delta–gamma neutral portfolio calculated from Equation (26).

$$\Pi^{\delta\gamma}_{t,t+\tau} = C_{it} - \max(S_{i,t+\tau} - K_i, 0) + \sum_{t=0}^{T-1}\Delta_{it}(S_{i,t+1} - S_t) + \sum_{t=0}^{T-1} r_t(C_{it} - \Delta_{it}S_t) + \sum_{t=0}^{T-1}\frac{\Gamma_{it}}{\Gamma_{jt}}(C_{j,t+1} - C_{jt}) - \sum_{t=0}^{T-1}\frac{\Gamma_{it}}{\Gamma_{jt}}\Delta_{jt}(S_{j,t+1} - S_t) - \sum_{t=0}^{T-1}\frac{\Gamma_{it}}{\Gamma_{jt}} r(C_{jt} - \Delta_{jt}S_t)$$

The Δ_t and Γ_t are updated daily and are calculated as the Black–Scholes hedge ratios using σ^h volatility. The return for each moneyness/maturity category represents the average return from the sample January 1996 to December 2002. The standard errors are shown in parentheses. Returns are given in percentages for $\Pi^{\delta\gamma}/\Phi_t$ and $\Pi^{\delta\gamma}/X_t$.

of the original option's value. Given what is observed in Figure 1, the dollar delta–gamma hedged gains should be highest for close-to and ATM options.[10]

Since the delta–gamma portfolio holds a long and short position in both the calls and the underlying simultaneously, the normalization must account for the difference. The difference in the underlying index positions is set equal to

$$\Phi_t = S_t - \frac{1}{n}\sum_{j=1}^{n}\frac{\Gamma_{it}}{\Gamma_{jt}}\Delta_{jt}S_t$$

For the difference in call positions,

$$X_t = C_{it} - \frac{1}{n}\sum_{j=1}^{n}\frac{\Gamma_{it}}{\Gamma_{jt}}C_{jt}$$

The averaging accounts for the different options used to gamma-hedge the same C_i.

The mean return for all delta–gamma portfolios normalized by Φ and X is 0.19% and 22.07% respectively. For the ATM options, the return is 0.21% and 13.8%. In each moneyness/maturity category, the gains are positive and statistically significant.[11] The results are consistent with the notion of a negative volatility premium found in Coval and Shumway (2001), Pan (2002) and Doran and Ronn (2006). While the results are surprisingly similar to those of the delta-hedged gains, general inferences can be made on the delta–gamma portfolio about the market price of volatility risk because there is no discretization error. Note that the ATM dollar errors are higher for the delta-hedged portfolio and have higher standard errors than the delta–gamma hedged errors. For the 14–30 day maturity sample, the ATM dollar delta-hedged errors are $1.15 while the dollar delta–gamma hedged errors are $0.938 with lower standard error, consistent with the simulation evidence presented earlier.

5.2 Natural gas contracts

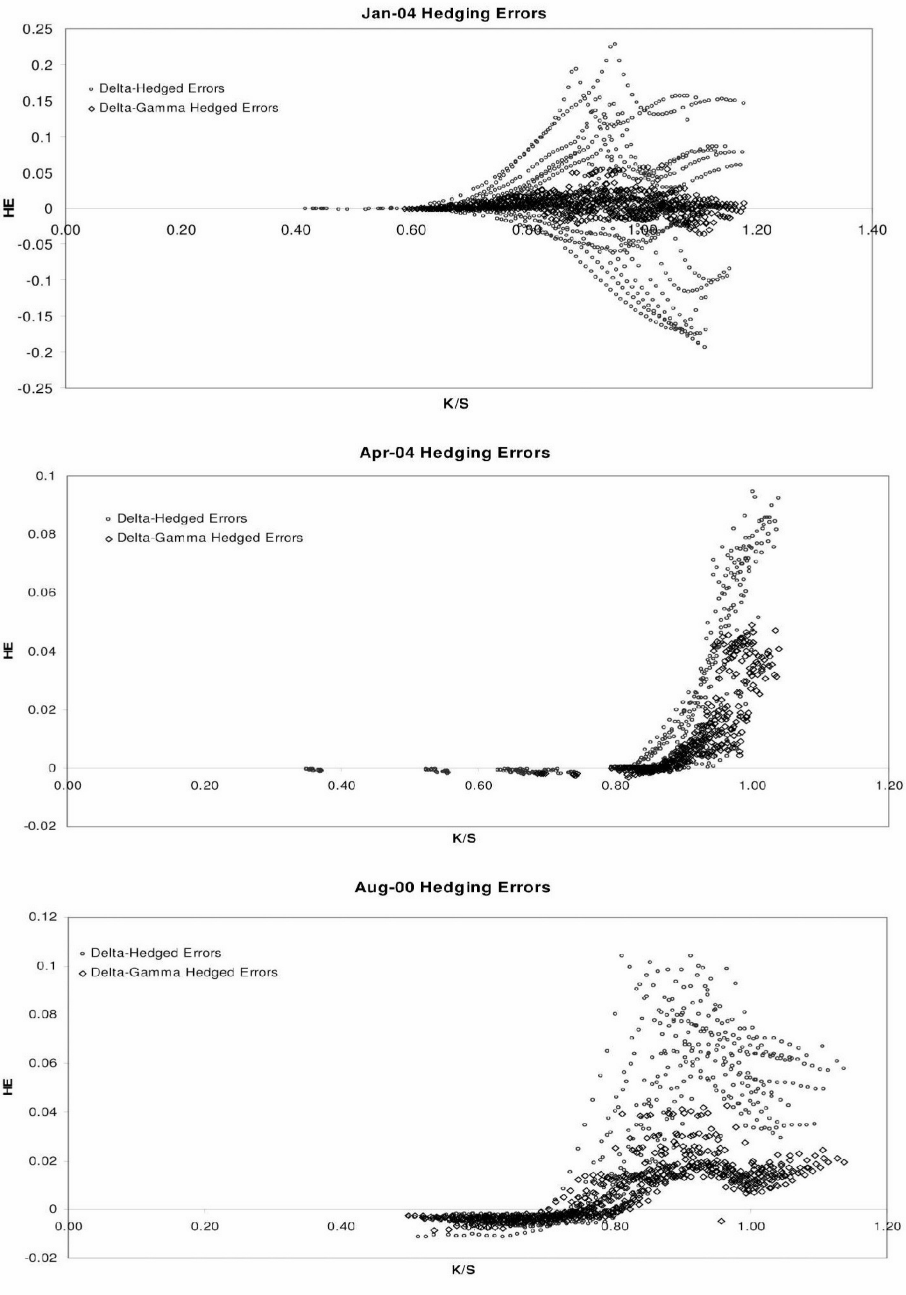
Doran and Ronn (2006) find a negative market price of volatility risk for natural gas, which implies that gains to the Π_t^{δ} and $\Pi_t^{\delta\gamma}$ portfolios should also be positive. However, unique to this market, there is significant variation in the month-to-month demand and price volatility for natural gas, which requires within-month and seasons estimation.[12] It is within this market, where close-to-maturity volatility can approach and exceed 100%, that the true impact of ignoring gamma can be assessed. This is specifically demonstrated in Figure 4, highlighting the variations in delta- and delta–gamma portfolios over moneyness for different monthly

[10] Ξ is maximized for close-to-the-money options.

[11] The value in parentheses is the standard error, reported as the standard deviation divided by the square root of observations.

[12] Doran and Ronn (2006) find that the market price of volatility risk is different across seasons for energy commodities.

FIGURE 4 Monthly hedging errors.



The figure shows the actual realization of the delta-hedged, $\Pi^{\delta}_{t,\,t+\tau}$, and delta–gamma hedged, $\Pi^{\delta\gamma}_{t,\,t+\tau}$, errors/gains for the months of January 2004, April 2004 and August 2000. Delta-hedged gains/errors are calculated from Equation (25) from day t to $t+\tau$. Delta–gamma hedged gains/errors are calculated from Equation (26) from day t to $t+\tau$. The maturity of a given realization ranges from 30 days to 14 days.

contracts. Since the volatility levels are almost double that experienced in the equity markets, higher values of the dollar delta–gamma gains will be associated with higher volatility premiums due to smaller values of the distortion. The results for the within-month and season estimation of Π_t^{δ} and $\Pi_t^{\delta\gamma}$ for expiration between 14–30 days can be found in Tables 4–6.

It is apparent from the findings that ignoring tracking error can result in large variation in the delta-hedged gains. For example, interpreting the findings for January would suggest that the delta-hedged portfolio returns on average −4.4% of X_t. This would be in direct contradiction of the existence of a negative market price of volatility risk. However, when gamma is controlled for, not only does the

TABLE 4 Delta-hedged gains for ATM natural gas options.

Variable	Observations	Π_t^{δ}	Π_t^{δ}/F_t	Π_t^{δ}/C_t	% > 0
January	613	−0.023	−0.28	−4.40	49.9
		(0.01)	(0.08)	(1.09)	
February	559	0.043	0.78	10.00	79.1
		(0.00)	(0.08)	(0.67)	
March	640	0.045	0.88	11.08	73.9
		(0.00)	(0.08)	(1.11)	
April	479	0.013	0.28	3.78	63.3
		(0.00)	(0.06)	(0.94)	
May	364	0.031	0.58	8.61	74.7
		(0.00)	(0.04)	(0.61)	
June	340	0.031	0.71	8.86	67.9
		(0.00)	(0.07)	(0.88)	
July	489	−0.043	−0.83	−10.45	17.8
		(0.00)	(0.04)	(0.53)	
August	310	0.036	0.76	9.60	85.5
		(0.00)	(0.06)	(0.68)	
September	319	−0.001	−0.12	−0.62	49.2
		(0.00)	(0.05)	(0.55)	
October	339	0.048	1.05	13.67	98.2
		(0.00)	(0.04)	(0.60)	
November	421	−0.018	−0.33	−4.73	31.4
		(0.00)	(0.04)	(0.58)	
December	493	0.038	0.92	11.20	80.3
		(0.00)	(0.05)	(0.69)	

The table reports the returns to the dollar delta-hedged gains, Π^{δ}, dollar delta-hedged gains scaled by futures price, Π^{δ}/F_t, and dollar delta-hedged gains scaled by call price, Π^{δ}/C_t. Each portfolio is rebalanced daily to create a time-t delta-neutral portfolio calculated from Equation (25) with the futures price F_t substituted in for S_t. The Δ_t is updated daily and is calculated as the Black hedge ratio using σ^h volatility. The return for each month represents the average return of each yearly contract used in the sample from January 2000 to December 2004. For natural gas, ATM is defined as the ratio of strike/price between 0.9 and 1.1. % > 0 is the percentage of observations greater than zero. The standard errors are shown in parentheses. Returns are given in percentages for Π^{δ}/S_t and Π^{δ}/C_t.

portfolio have positive return but the dollar gain as a percentage of X is nearly 21%. Grouping the months into seasons the same effect is observed, and it is especially strong for the winter months. It is in the winter months that the highest price volatility is observed for natural gas, and as a result, if gamma is ignored, that it can lead to an erroneous interpretation of the delta-hedged gain.[13] Note

TABLE 5 Delta–gamma hedged gains for ATM natural gas options.

Variable	Observations	Π_t^{δ}	Π_t^{δ}/Λ_t	Π_t^{δ}/X_t	% > 0
January	250	0.056	1.45	20.74	94.8
		(0.00)	(0.06)	(1.01)	
February	275	0.038	0.95	15.00	73.8
		(0.00)	(0.06)	(1.04)	
March	301	0.069	2.28	28.32	84.4
		(0.00)	(0.14)	(2.03)	
April	215	0.089	2.58	34.23	98.6
		(0.00)	(0.12)	(1.58)	
May	99	0.023	0.79	9.70	61.6
		(0.00)	(0.19)	(2.19)	
June	91	0.014	0.63	6.59	50.8
		(0.00)	(0.17)	(1.69)	
July	153	−0.048	−1.55	−18.67	12.4
		(0.00)	(0.15)	(1.70)	
August	77	0.064	2.57	28.51	100.0
		(0.00)	(0.14)	(1.59)	
September	58	−0.006	−0.30	−2.71	60.3
		(0.00)	(0.22)	(2.05)	
October	93	0.021	0.79	8.88	90.3
		(0.00)	(0.16)	(1.90)	
November	132	0.013	0.41	5.16	82.6
		(0.01)	(0.22)	(2.44)	
December	136	0.006	0.34	3.25	49.3
		(0.00)	(0.16)	(1.72)	

The table reports the returns to the dollar delta–gamma hedged gains, $\Pi^{\delta\gamma}$, dollar delta-hedged gains scaled by Λ, $\Pi^{\delta\gamma}/\Phi_t$, and dollar delta-hedged gains scaled by X, $\Pi^{\delta\gamma}/X_t$. Λ_t is equal to $F_t - (\Gamma_{it}/\Gamma_{jt})\Delta_{jt}F_t$; X_t is equal to $C_{it} - (\Gamma_{it}/\Gamma_{jt})C_{jt}$. Each portfolio is rebalanced daily to create a time-t delta–gamma neutral portfolio calculated from Equation (26) with the futures price F_t substituted in for S_t. The Δ_t and Γ_t are updated daily and are calculated as the Black hedge ratios using σ^h volatility. The return for each month represents the average return of each yearly contract used in the sample from January 2000 to April 2005. For natural gas, ATM is defined as the ratio of strike/price between 0.9 and 1.1. % > 0 is the percentage of observations greater than zero. The standard errors are shown in parentheses. Returns are given in percentages for $\Pi^{\delta\gamma}/\Phi_t$ and $\Pi^{\delta\gamma}/X_t$.

[13] For example, for the January 2001 contract, the futures price of natural gas on November 27, 2000, was \$6.473 per contract. At expiration of the call, the price was \$9.805, representing a price appreciation of 51.4% over a 30-day period.

TABLE 6 Delta- and delta–gamma hedged gains for ATM natural gas options across seasons.

	PANEL A					PANEL B				
Variable	**Observations**	Π_t^{δ}	Π_t^{δ}/F_t	Π_t^{δ}/C_t	**% > 0**	**Observations**	$\Pi_t^{\delta\gamma}$	$\Pi_t^{\delta\gamma}/\Lambda_t$	$\Pi_t^{\delta\gamma}/X_t$	**% > 0**
Winter	1,812	0.021	0.46	5.51	67.4	826	0.055	1.58	21.59	84.0
		(0.003)	(0.05)	(0.60)			(0.002)	(0.06)	(0.89)	
Spring	1,183	0.024	0.50	6.73	68.1	405	0.056	1.70	22.02	74.3
		(0.002)	(0.03)	(0.50)			(0.003)	(0.10)	(1.24)	
Summer	1,118	−0.009	−0.19	−2.08	45.5	288	−0.009	−0.20	−2.84	45.5
		(0.002)	(0.03)	(0.42)			(0.004)	(0.14)	(1.59)	
Fall	1,253	0.022	0.53	6.52	68.7	361	0.012	0.48	5.40	72.0
		(0.001)	(0.03)	(0.43)			(0.003)	(0.11)	(1.21)	
All	5,366	0.016	0.35	4.43	63.3	1,880	0.037	1.12	14.83	73.7
		(0.001)	(0.02)	(0.27)			(0.000)	(0.04)	(0.62)	

Panel A reports the average delta-hedged returns across seasons for Π^{δ}, Π^{δ}/F_t and Π^{δ}/C_t. Panel B reports the average delta–gamma hedged returns for $\Pi^{\delta\gamma}$, $\Pi^{\delta\gamma}/\Lambda_t$ and Π^{δ}/X_t. The winter season includes months January through March; spring includes April through June; summer includes July through September; and fall includes October through December. The delta-hedged portfolio is calculated from Equation (25); the delta–gamma hedged portfolio is calculated from Equation (26). The standard errors are shown in parentheses. Returns are given in percentages for Π^{δ}/S_t, Π^{δ}/C_t, $\Pi^{\delta\gamma}/\Phi_t$ and $\Pi^{\delta\gamma}/X_t$.

also that the percentage of realized gains that are greater than zero increases for all seasons when controlling for gamma, with an increase of more than 10% in the percent gain of $\Pi_t^{\delta\gamma}/X_t$ over Π_t^{δ}/C_t for all observations.

The average ATM natural gas delta–gamma gains for $\Pi_t^{\delta\gamma}/\Lambda_t$ and $\Pi_t^{\delta\gamma}/X_t$ are 1.12% and 14.83% respectively.[14] Comparing across markets, the ATM findings show a significantly higher gain in $\Pi_t^{\delta\gamma}/\Lambda_t$ as compared to the equity counterpart $\Pi_t^{\delta\gamma}/\Phi_t$. The implications of this distinction should be a function of both the degree of volatility present in each market and the actual magnitude of the volatility premium. This will be assessed shortly.

The results for July are initially worrisome since they suggest a positive and significant market price of volatility risk. However, a negative market risk premium with a negative volatility risk premium will result in a negative delta–gamma hedged gain. In examining the mean returns for the July contracts, all four years demonstrated negative and significant premiums. In particular, the 2001 futures price lost over 10% of its value in the remaining 22 business days of the contract's maturity. Consequently, the results for July lend further evidence in support of negative market price of volatility risk.

To assess the economic impact of the results it is necessary to gauge the market depth and volume traded for this commodity. As compared to the volume for all other commodities on NYMEX, natural gas futures and options make up close to 20% and 40% of the total actively traded contracts respectively.[15] For example, the total option volume traded for the 2002 year was 10,966,023 contracts, making up 46% of the total market.[16] With a mean dollar delta–gamma gain of \$0.037 for the ATM contracts, and the right to buy one futures contract at a contract size of \$10,000/mmbtu, the yearly dollar impact is roughly \$4 billion.

5.3 Inference on the volatility premium

To assess the impact of λ_σ as a percent of the call price, a regression was run on the delta- and delta–gamma hedged gains on the estimated GARCH volatility series. From Proposition 2 of Bakshi and Kapadia (2003b), the scaled delta-hedged gains for ATM options can be related to the market price of volatility risk.[17] This arises because an ATM option's Vega has close to zero dependence on volatility, allowing for estimation of volatility premium with a simple linear structure. However, this relationship holds only for the scaled delta-hedged gains, Π_t^{δ}/S_t, and not for the scaled delta–gamma hedged gains. or $\Pi_t^{\delta\gamma}/\Phi_t$ it is necessary to account for the value of the distortion created by holding two positions in the underlying and two options prior to estimation. Consequently, the expected

[14] Λ_t is set equal to $\Lambda_t = F_t - 1/n \sum_{j=1}^{n} (\Gamma_{it}/\Gamma_{jt})\Delta_{jt}F_t$.
[15] The only energy commodity traded that exceeds the volume of natural gas is light, sweet crude oil.
[16] Data provided by the Chicago Mercantile Exchange and NYMEX. The volume traded in 2001 was 5,974,240 and in 2003 was 8,742,277.
[17] Refer to Equations (19) and (22) of their paper.

delta–gamma hedged gains are equal to

$$\mathrm{E}\left[\Pi_t^{\delta\gamma}\right] = S_t \lambda_\sigma \sigma_t^h \left[\Xi_{it} - \frac{1}{n} \sum_{j=1}^{n} \frac{\Gamma_{it}}{\Gamma_{jt}} \Delta_{jt} \Xi_{jt} \right] \tag{27}$$

This requires that the scaled delta–gamma hedged gains, $\Pi_t^{\delta\gamma}/\Phi_t$, be amended to account for Ξ dependence on volatility as such:

$$\Phi_t^* = S_t \left[\Xi_{it} - \frac{1}{n} \sum_{j=1}^{n} \frac{\Gamma_{it}}{\Gamma_{jt}} \Delta_{jt} \Xi_{jt} \right] \tag{28}$$

where Ξ is defined as in Section 2.2, and j represents all strike prices greater than K_i. By accounting for Ξ in this manner, the following relationship arises:

$$\frac{\mathrm{E}\left[\Pi_t^{\delta\gamma}\right]}{\Phi_t^*} = \lambda_\sigma \sigma_t^h \tag{29}$$

which preserves the same linear regression framework as in Bakshi and Kapadia (2003b), allowing for identical estimation of volatility premiums using the delta–gamma portfolios.

5.3.1 Estimation results

In maintaining the linear structure, I now estimate the volatility risk premium using both the delta- and delta–gamma hedged gains with the following regression framework:

$$\frac{\Pi_t^\delta}{F_t} = \alpha + \beta \sigma_t^h + \epsilon_t \tag{30}$$

$$\frac{\Pi_t^{\delta\gamma}}{F_t} = \alpha + \beta \sigma_t^h + \epsilon_t \tag{31}$$

If the market price of volatility risk is negative, β should have a positive sign and α should be insignificant. Specific to the natural gas contracts, a fixed effects regression is run controlling for monthly effects.[18] The results of the regressions can be found in Tables 7 and 8. For the S&P100 options, β is not significant for the specification (30) but is positive and significant for specification (31). The effect of ignoring gamma results in interpreting an insignificant volatility risk premium when, in fact, the premium is significant. This is consistent with the findings in the simulation presented earlier that delta-hedged errors are two to

[18] For the natural gas contracts, the dependent variables are Π_t^δ/F_t and $\Pi_t^{\delta\gamma}/\Lambda_t^*$.

three times as variable as the delta–gamma hedged errors. In comparison to previous authors' findings, this sample period has higher levels of volatility and, consequently, higher variations in the gains to delta-hedged portfolios.[19]

TABLE 7 Regression of delta- and delta–gamma hedged gains for ATM index options.

	Π^{δ}/S			$\Pi^{\delta\gamma}/\Phi^{*}$		
	Full sample	**96–98**	**99–02**	**Full sample**	**96–98**	**99–02**
σ^h	0.006	0.004	0.008	0.012	0.012	0.012
	(0.91)	(0.55)	(0.92)	(1.97)*	(1.96)*	(1.95)*
$\alpha\ (\times 10^{-2})$	0.083	0.092	0.077	−0.013	0.042	−0.05
	(0.76)	(0.48)	(0.71)	(0.13)	(0.22)	(0.43)
Observations	438	262	176	436	174	262
R^2	0.11	0.13	0.02	0.06	0.02	0.08

The table reports the results of the regression of Π^{δ}/S and $\Pi^{\delta\gamma}/\Phi^{*}$ on GARCH volatility given in Equations (30) and (31) – ie, $\Pi_t^{\delta}/S_t = \alpha + \beta\sigma_t^h + \epsilon_t$ and $\Pi_t^{\delta\gamma}/\Phi_t^{*} = \alpha + \beta\sigma_t^h + \epsilon_t$, respectively. Φ^{*} is given as the stock price multiplied by the distortion value given in Equation (28). The sample consists of options that are ATM, defined between a strike/price ratio of 0.975 to 1.025, and maturity close to 30 days. The full sample used options from 1996 to 2002, with two sub-period estimations from 1996 through 1998 and 1999 through 2002.
Robust t-statistics are shown in parentheses. " * " indicates significant at 5%.

TABLE 8 Regression of delta- and delta–gamma hedged gains for ATM natural gas options.

	Π^{δ}/F	$\Pi^{\delta\gamma}/\Lambda^{*}$	$\overline{\Pi^{\delta}/F}$	$\overline{\Pi^{\delta\gamma}/\Lambda^{*}}$
σ^h	0.005	0.031	0.004	0.023
	(1.43)	(3.28)**	(0.86)	(3.52)**
$\alpha\ (\times 10^{-2})$	0.057	−0.91	0.12	−0.39
	(1.08)	(1.30)	(0.34)	(0.60)
Observations	5,366	1,880	51	51
R^2	0.15	0.34	0.13	0.27

The table reports the results of the regression of Π^{δ}/F and $\Pi^{\delta\gamma}/\Lambda^{*}$ on GARCH volatility given in Equations (30) and (31) – ie, $\Pi_t^{\delta}/F_t = \alpha + \beta\sigma_t^h + \epsilon_t$ and $\Pi_t^{\delta\gamma}/\Lambda_t^{*} = \alpha + \beta\sigma_t^h + \epsilon_t$, respectively. Λ^{*} is given as the futures price multiplied by the distortion value given in Equation (28). $\overline{\Pi^{\delta}/F}$ and $\overline{\Pi^{\delta\gamma}/\Lambda^{*}}$ represent the within-month average returns for the entire sample from January 2000 to April 2004. As a result, Equations (30) and (31) are amended to allow for monthly fixed effects. The sample consists of options that are ATM, defined between a strike/price ratio of 0.9 to 1.1, and maturity close to 30 days.
Robust t-statistics are shown in parentheses. " ** " indicates significant at 1%.

[19] Robustness test were done using Ξ_t as the independent variable controlling for levels of volatility and moneyness. A significant positive coefficient was found for Ξ_t, which is consistent with a negative volatility premium.

This result is reinforced in the findings from the natural gas markets. In addition to the fixed effects regression, the individual portfolio returns are averaged within months, $\overline{\Pi_t^{\delta\gamma}/\Lambda_t^*} = {}^1\!/\!_n \sum_{k=1}^{n} \Pi_{kt}^{\delta\gamma}/\Lambda_t^*$, to create a sample of 51 average monthly returns. This eliminates any within-month specific component incorporated in the data. The regression produces statistically insignificant results for the delta-hedged portfolio and significant results for the delta–gamma portfolio. Of equal importance is the magnitude of the difference in the estimates. While the coefficient in the delta-hedged regression is positive, it is close to an order of magnitude lower than the delta–gamma counterpart. This suggests that the true volatility risk premium is higher than has previously been found. This finding is consistent in both markets.

To assess the economic impact of the coefficient estimate, three levels of volatility were examined to infer the volatility premium. This was done for two additional reasons: first, to demonstrate the degree to which volatility differs between the natural gas and equity markets, and second, to highlight how using a delta-hedged portfolio will result in incorrect inference on the volatility premium.

As reported in Table 9, on the three specific dates used for each market the volatility premium was inferred from the underlying price of the asset as a percentage of the call price. For example, on April 26, 2001, the futures price of natural gas was $4.94 with a volatility of 38.8%. The call price for the closest ATM option was $0.291. Using the coefficient estimate of 0.031 translates into a volatility premium of 1.2% of the futures price, or $.059 of the ATM call. As a percentage of the call price, the volatility premium is 20.9%. By comparison, implementing the same methodology but using the delta portfolio coefficient implies a volatility premium of 3.29%. This suggests that sellers of options are

TABLE 9 Volatility premium estimates.

Date	σ^h	Price	Call	$\lambda_\sigma^{\delta\gamma}$	λ_σ^{δ}
S&P100					
June 18, 1998	16.0	537.1	14.00	7.37	3.69
May 23, 2002	24.7	547.6	12.25	13.26	6.63
March 23, 2000	29.1	831.7	21.00	13.84	6.92
Natural gas contracts					
April 26, 2001	38.8	4.94	0.291	20.4	3.29
October 29, 2001	68.3	3.34	0.290	24.4	3.93
February 25, 2003	120.1	6.58	0.645	37.9	6.12

The table reports the effect of the market price of volatility risk, $\lambda_\sigma^{\delta\gamma}$ and λ_σ^{δ}, as a percentage of the call price. Each risk measure is estimated from the fitted values from Equations (30) and (31). The periods selected demonstrate low, medium and high periods of GARCH volatility. The table reports the date of the observation, the volatility, the price of the underlying and the price of the call closest to ATM on that date.

making more money for bearing volatility risk than previously found. This is especially true in the natural gas markets, where volatility can be triple that found in the S&P.

6 CONCLUSION

The results of this study have enhanced previous findings in three important ways. First, by demonstrating the need to control for an option's gamma, more precise inference can be made on the volatility premium without the restriction of a parametric model. By reducing discretization and model misspecification error, a non-zero volatility premium is directly linked to the gains associated with a delta–gamma hedged portfolio. Second, for empirical tests of volatility premium, an option's Ξ, and not Vega, must be controlled for to allow a linear regression framework. Finally, the volatility premium found is higher than previously reported for S&P options, further enhancing the notion of why option traders prefer to be short.

In addition, identical methodology was implemented for the natural gas markets. This was done specifically to examine a market where the volatility close to expiration of the option/future can exceed 100%. It is within this market that the effects of ignoring an option's gamma can especially be observed. As found, inference on the volatility premium by forming delta-neutral portfolios lends very little insight into the size of the volatility premium. However, when a delta–gamma neutral portfolio is formed, the volatility risk premium can make up close to 40% of the call price in highly volatile periods. This result is enhanced by controlling for months of contract, eliminating seasonal variations in demand and volatility.

These results presented here corroborate and extend the findings of Doran and Ronn (2006), Bakshi and Kapadia (2003b), Buraschi and Jackwerth (2001) and others. By demonstrating positive net gains to the delta–gamma portfolio, I have further demonstrated why the market price of volatility risk is negative. Of interest, though, is why this is the case in natural gas markets. In equity markets, holding options acts as a hedge against positive volatility shocks. For energy markets, prices tend to increase in conjunction with volatility increases. As such, positive correlation between price and volatility innovations, and a negative volatility premium, imply a negative market price of risk. This conjecture, which has yet to be demonstrated, is worth exploring and could help enhance our understanding of energy markets.

It must be noted that the formation of such a portfolio will incur significant transaction costs in not only the formation of the gamma position but also the need to rebalance all three positions. The effect of commissions and wide bid–ask spreads may make the implementation of this strategy cost-ineffective. However, the simulation evidence shows that with longer rebalancing periods a more effective hedge is implemented by controlling for gamma than rebalancing daily by just using a delta-hedged position. Thus, there appears to be a practical benefit to controlling for the option's gamma beyond the econometric exercise of estimating volatility risk premium. This leads to the question of what is the optimal time to rebalance given actual transaction costs and alternative hedging strategies. Solving this could help improve the hedging and risk management practices of firms and investors.

APPENDIX A: PROOF OF PROPOSITION 1

The proof of Proposition 1 is as follows. Initially, let

$$\Phi_q = -\int_t^{t+1} \frac{\partial C_q}{\partial S}\,\mathrm{d}S_u - \int_t^{t+1} r\left(C_{qt} - \frac{\partial C_q}{\partial S} S_u\right)\mathrm{d}u$$

$$\Omega_q = \int_t^{t+1} \Delta_{qu}\,\mathrm{d}S_u + \int_t^{t+1} r\left(C_{qt} - \Delta_{qu} S_u\right)\mathrm{d}u$$

and

$$\hat{\Gamma} = \left(\frac{\partial^2 C_i/\partial S^2}{\partial^2 C_j/\partial S^2}\right)$$

Then, taking expectations of Equation (7) over the interval $[t, t+1]$ given filtration Ψ_t results in

$$\begin{aligned}
\mathrm{E}^P\left[\Pi_{t,t+1}^{\delta\gamma} \,\middle|\, \Psi_t\right] &= \mathrm{E}^P\left[(\Phi_i + \Omega_i) - \hat{\Gamma}(\Phi_j + \Omega_j) \,|\, \Psi_t\right] \\
&= -(\mu - r)\int_t^{t+1} \mathrm{E}^{P^*}\left[\frac{\partial C_i}{\partial S} - \Delta_{iu} \,|\, \Psi_t\right] \mathrm{E}^P\left[S_t \,|\, \Psi_t\right]\mathrm{d}u \\
&\quad + (\mu - r)\int_t^{t+1} \mathrm{E}^{P^*}\left[\hat{\Gamma} \,|\, \Psi_t\right] \mathrm{E}^{P^*}\left[\frac{\partial C_j}{\partial S} - \Delta_{ju} \,|\, \Psi_t\right] \mathrm{E}^P\left[S_t \,|\, \Psi_t\right]\mathrm{d}u
\end{aligned} \tag{32}$$

where E^{P^*} is the probability measure defined by the Radon–Nikodym derivative $\mathrm{d}P^*/\mathrm{d}P = S_t/\mathrm{E}^P[S_t]$. The discretization error arises due to the fixed hedge ratio, Δ_i, and evaluating the partial derivative within the expectation. Branger and Schlag (2004) demonstrate that over the interval $[t, t+1]$ the partial derivative changes by

$$\frac{\partial C_t}{\partial S} - \frac{\partial C_{t+1}}{\partial S} = \int_t^{t+1} \frac{\partial^2 C_u}{\partial S^2}\left(\mathrm{d}S_u - (r + \sigma^2) S_u \,\mathrm{d}u\right) \tag{33}$$

Since this hedging strategy incorporates two options, this additional term will appear twice. Thus it is necessary to eliminate this term to demonstrate that $\mathrm{E}[\Pi^{\delta\gamma}_{t,t+\tau} = 0]$. Noting that the drift of the stock under E^{P^*} is $\mu + \sigma^2$:

$$
\begin{aligned}
&\mathrm{E}^P\left[\Pi^{\delta\gamma}_{t,t+1} \mid \Psi_t\right] \\
&= -(\mu - r)\left(\frac{\partial C_i}{\partial S} - \Delta_{iu}\right)\int_t^{t+1} \mathrm{E}^P[S_t \mid \Psi_t]\,\mathrm{d}u + (\mu - r)^2 \int_t^{t+1} \mathrm{E}^*\left[\frac{\partial^2 C_i}{\partial S^2}\middle|\Psi_t\right]\mathrm{E}^P[S_t \mid \Psi_t]\,\mathrm{d}u \\
&\quad -(\mu - r)\left(\frac{\partial C_j}{\partial S} - \Delta_{ju}\right)\int_t^{t+1} \mathrm{E}^{P^*}\left[\frac{\partial^2 C_i/\partial S^2}{\partial^2 C_j/\partial S^2}\middle|\Psi_t\right]\mathrm{E}^P\left[S_t \middle| \Psi_t\right]\mathrm{d}u \\
&\quad +(\mu - r)^2 \int_t^{t+1} \mathrm{E}^{P^*}\left[\frac{\partial^2 C_i/\partial S^2}{\partial^2 C_j/\partial S^2}\middle|\Psi_t\right]\mathrm{E}^{P^*}\left[\frac{\partial^2 C_j}{\partial S^2}\middle|\Psi_t\right]\mathrm{E}^P[S_t \mid \Psi_t]\,\mathrm{d}u
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&\mathrm{E}^P\left[\Pi^{\delta\gamma}_{t,t+1} \mid \Psi_t\right] \\
&= -(\mu - r)\left(\frac{\partial C_i}{\partial S} - \Delta_{iu}\right)\int_t^{t+1} \mathrm{E}^P[S_t \mid \Psi_t]\,\mathrm{d}u \\
&\quad +(\mu - r)\left(\frac{\partial C_j}{\partial S} - \Delta_{ju}\right)\int_t^{t+1} \mathrm{E}^{P^*}\left[\frac{\partial^2 C_i/\partial S^2}{\partial^2 C_j/\partial S^2}\middle|\Psi_t\right]\mathrm{E}^P\left[S_t \middle| \Psi_t\right]\mathrm{d}u
\end{aligned} \tag{35}
$$

Since there is no concern about statically hedging gamma, the second partial derivative with respect to the underlying within the expectation can directly combine with the first partial derivative, reducing Equation (34) to the expression in (35). Thus, both second partials that arise due to discretization can be eliminated through division and subtraction. As a result, the expected delta–gamma hedged portfolio $\mathrm{E}[\Pi^{\delta\gamma}_{t,t+\tau} = 0]$ if $\partial C_i/\partial S = \Delta_i$ and $\partial C_j/\partial S = \Delta_j$.

APPENDIX B: PROOF OF PROPOSITION 2

The proof of Proposition 2 is as follows. From Equation (16) of the paper,

$$
\begin{aligned}
\Pi^{\delta\gamma}_{t,t+\tau} = &-\int_t^{t+\tau} \frac{\partial C_i}{\partial S}\mathrm{d}S_u - \int_t^{t+\tau} r\left(C_{it} - \frac{\partial C_i}{\partial S}\right)\mathrm{d}u \\
&+\int_t^{t+\tau} \Delta_{iu}\,\mathrm{d}S_u + \int_t^{t+\tau} r\left(C_{it} - \Delta_{iu}S_u\right)\mathrm{d}u
\end{aligned}
$$

$$+\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}\frac{\partial C_j}{\partial S}\,\mathrm{d}S_u+\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}r\left(C_{jt}-\frac{\partial C_j}{\partial S}S_u\right)\mathrm{d}u$$

$$-\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}\Delta_{ju}\,\mathrm{d}S_u-\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}r\left(C_{ju}-\Delta_{ju}S_u\right)\mathrm{d}u$$

$$-\int_t^{t+\tau}\lambda_\sigma\xi\sigma^2\frac{\partial C_i}{\partial\sigma}\,\mathrm{d}u-\int_t^{t+\tau}\xi\frac{\partial C_i}{\partial\sigma}\,\mathrm{d}Z_u^{\sigma^2}$$

$$+\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}\lambda_\sigma\xi\sigma^2\frac{\partial C_j}{\partial\sigma}\,\mathrm{d}u+\int_t^{t+\tau}\frac{\Gamma_{iu}}{\Gamma_{ju}}\xi\frac{\partial C_j}{\partial\sigma}\,\mathrm{d}Z_u^{\sigma^2}$$

which, incorporated with Equation (32) and generating differences in hedges ratio from $[t,t+1]$, results in

$$\begin{aligned}
&\mathrm{E}^P\left[\Pi^{\delta\gamma}_{t,t+1}\,\middle|\,\Psi_t\right]\\
&=-(\mu-r)\left(\frac{\partial C_i}{\partial S}-\Delta_{iu}\right)\int_t^{t+1}\mathrm{E}^P[S_t\,|\,\Psi_t]\,\mathrm{d}u\\
&+(\mu-r)^2\int_t^{t+1}\mathrm{E}^{P^*}\left[\frac{\partial^2 C_i}{\partial S^2}\,\middle|\,\Psi_t\right]\mathrm{E}^P[S_t\,|\,\Psi_t]\,\mathrm{d}u\\
&-\lambda_\sigma\int_t^{t+1}\mathrm{E}^{P^*}\left[\frac{\partial C_i}{\partial\sigma_i}\xi\sigma_i^2\,\middle|\,\Psi_t\right]\mathrm{d}u\\
&-\lambda_\sigma(\mu-r)\int_t^{t+1}\mathrm{E}^{P^*}\left[\frac{\partial^2 C_i}{\partial S\partial\sigma_i}\xi\sigma_i^2\,\middle|\,\Psi_t\right]\mathrm{E}^P[S_t\,|\,\Psi_t]\,\mathrm{d}u\\
&+(\mu-r)\left(\frac{\partial C_j}{\partial S}-\Delta_{ju}\right)\int_t^{t+1}\mathrm{E}^{P^*}\left[\frac{\partial^2 C_i/\partial S^2}{\partial^2 C_j/\partial S^2}\,\middle|\,\Psi_t\right]\mathrm{E}^P[S_t\,|\,\Psi_t]\,\mathrm{d}u\\
&-(\mu-r)^2\int_t^{t+1}\mathrm{E}^{P^*}\left[\frac{\partial^2 C_i/\partial S^2}{\partial^2 C_j/\partial S^2}\,\middle|\,\Psi_t\right]\mathrm{E}^{P^*}\left[\frac{\partial^2 C_j}{\partial S^2}\,\middle|\,\Psi_t\right]\mathrm{E}^P[S_t\,|\,\Psi_t]\,\mathrm{d}u\\
&+\lambda_\sigma\int_t^{t+1}\mathrm{E}^{P^*}\left[\frac{\partial^2 C_i/\partial S^2}{\partial^2 C_j/\partial S^2}\,\middle|\,\Psi_t\right]\mathrm{E}^{P^*}\left[\frac{\partial C_j}{\partial\sigma_j}\xi\sigma_j^2\,\middle|\,\Psi_t\right]\mathrm{d}u\\
&+\lambda_\sigma(\mu-r)\int_t^{t+1}\mathrm{E}^{P^*}\left[\frac{\partial^2 C_i/\partial S^2}{\partial^2 C_j/\partial S^2}\,\middle|\,\Psi_t\right]\mathrm{E}^{P^*}\left[\frac{\partial^2 C_j}{\partial S\partial\sigma_j}\xi\sigma_j^2\,\middle|\,\Psi_t\right]\mathrm{E}^P[S_t\,|\,\Psi_t]\,\mathrm{d}u \qquad (36)
\end{aligned}$$

where the drift of the variance process is equal to $\kappa(\theta - \sigma^2) + \xi\sigma\rho$. This expression can be further reduced such that term is dependent on only λ_σ or Δ.

$$
\begin{aligned}
&\mathrm{E}^P\left[\Pi^{\delta\gamma}_{t,t+1} \,|\, \Psi_t\right] \\
&= -(\mu - r)\left(\frac{\partial C_i}{\partial S} - \Delta_{iu}\right)\int_t^{t+1} \mathrm{E}^P[S_t \,|\, \Psi_t]\,\mathrm{d}u \\
&-\lambda_\sigma \int_t^{t+1} \mathrm{E}^{P^*}\left[\frac{\partial C_i}{\partial \sigma_i}\xi\sigma_i^2 \,\middle|\, \Psi_t\right]\mathrm{d}u \\
&-\lambda_\sigma(\mu - r)\int_t^{t+1} \mathrm{E}^{P^*}\left[\frac{\partial^2 C_i}{\partial S \partial \sigma_i}\xi\sigma_i^2 \,\middle|\, \Psi_t\right]\mathrm{E}^P[S_t \,|\, \Psi_t]\,\mathrm{d}u \\
&+(\mu - r)\left(\frac{\partial C_j}{\partial S} - \Delta_{ju}\right)\int_t^{t+1} \mathrm{E}^{P^*}\left[\frac{\partial^2 C_i/\partial S^2}{\partial^2 C_j/\partial S^2} \,\middle|\, \Psi_t\right]\mathrm{E}^P[S_t \,|\, \Psi_t]\,\mathrm{d}u \\
&+\lambda_\sigma \int_t^{t+1} \mathrm{E}^{P^*}\left[\frac{\partial^2 C_i/\partial S^2}{\partial^2 C_j/\partial S^2} \,\middle|\, \Psi_t\right]\mathrm{E}^{P^*}\left[\frac{\partial C_j}{\partial \sigma_j}\xi\sigma_j^2 \,\middle|\, \Psi_t\right]\mathrm{d}u \\
&+\lambda_\sigma(\mu - r)\int_t^{t+1} \mathrm{E}^{P^*}\left[\frac{\partial^2 C_i/\partial S^2}{\partial^2 C_j/\partial S^2} \,\middle|\, \Psi_t\right]\mathrm{E}^{P^*}\left[\frac{\partial^2 C_j}{\partial S \partial \sigma_j}\xi\sigma_j^2 \,\middle|\, \Psi_t\right]\mathrm{E}^P[S_t \,|\, \Psi_t]\,\mathrm{d}u
\end{aligned}
\tag{37}
$$

If $\lambda_\sigma = 0$, $\partial C_i/\partial S = \Delta_i$ and $\partial C_j/\partial S = \Delta_j$, then $\Pi^{\delta\gamma}_{t,t+1} = 0$.

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