TP SY09

May 6, 2018

1 Justification

$$d_M^2(x,y) = (x_y)^T M(x-y) \tag{1}$$

$$J(\{v_k, M_k\}_{k=1,\dots,K}) = \sum_{k=1}^{K}\sum_{i=1}^{n} z_{ik} d_{ik}^2 \tag{2}$$

$$v_k = \bar{x_k} = \frac{1}{n_k}\sum_{i=1}^{n} z_{ik} x_i \tag{3}$$

$$M_k^{-1} = (\rho_k det V_k)^{\frac{-1}{p}} V_k, avec V_k = \frac{1}{n_k}\sum_{i=1}^{n} z_{ik}(x_i - v_k)(x_i - v_k)^T \tag{4}$$

1.1 Question 10

$$\begin{aligned}
\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial v_k} &= \frac{\partial J(v_k, M_k)}{\partial v_k} - \frac{\partial(\sum_{k=1}^{K} \lambda_k(det(M_k) - \rho_k))}{\partial v_k} \\
&= \frac{\partial(\sum_{k=1}^{K}\sum_{i=1}^{n} z_{ik} d_{ik}^2)}{\partial v_k} - \sum_{k=1}^{K} \frac{\partial(\lambda_k(det(M_k) - \rho_k))}{\partial v_k} \\
&= \sum_{k=1}^{K}\sum_{i=1}^{n} \frac{\partial(z_{ik} d_{M_k}^2(x_i, v_k))}{\partial v_k} - 0 \\
&= \sum_{k=1}^{K}\sum_{i=1}^{n} \frac{\partial(z_{ik}(x_i - v_k)^T M_k (x_i - v_k))}{\partial v_k}
\end{aligned}$$

En fixant k, cela donne :

$$\begin{aligned}
&= \sum_{i=1}^{K} \frac{\partial(x_i - v_k)^T M_k (x_i - v_k)}{\partial v_k} \\
&= \sum_{i=1}^{K} \frac{\partial(x_i)^T M_k (x_i)}{\partial v_k} + \frac{\partial(-v_k)^T M_k(-v_k)}{\partial v_k} + \frac{\partial(x_i)^T M_k(-v_k)}{\partial v_k} + \frac{\partial(-v_k)^T M_k(x_i)}{\partial v_k} \\
&= \sum_{i=1}^{K} 0 + 2M_k v_k - 2M_k(x_i) \\
&= 2M_k \sum_{i=1}^{K} (v_k - (x_i))
\end{aligned}$$

1.2 Question 11

$$\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial M_k} = \sum_{k=1}^{K}\sum_{i=1}^{n} z_{ik}\frac{\partial((x_i - v_k)^T M_k (x_i - v_k))}{\partial M_k} - \sum_{k=1}^{K}\lambda_k \frac{\partial(det(M_k) - \rho_k)}{\partial M_k}$$

En utilisant (72) et (49) du matrix cook book:

$$= \sum_{k=1}^{K}\sum_{i=1}^{n}(z_{ik}(x_i - v_k)(x_i - v_k)^T) - \sum_{k=1}^{K}\lambda_k(det(M_k)(M_k^{-1})^T - \frac{\partial \rho_k}{\partial M_k})$$

En fixant encore une fois k:

$$= \sum_{i=1}^{n} x_i.v_k - \lambda_k(det(M_k)(M_k^{-1})^T - \frac{\partial \rho_k}{\partial M_k}$$

$$= \sum_{i=1}^{n} x_i.v_k - \lambda_k(det(M_k)(M_k^{-1})^T - 1$$

$$\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial \lambda_k} = \frac{\partial J(v_k, M_k)}{\partial \lambda_k} - \sum_{k=1}^{K}\frac{\partial(\lambda_k(det(M_k) - \rho_k))}{\partial \lambda_k}$$

$$= 0 - \sum_{k=1}^{K}(det(M_k) - \rho_k)$$