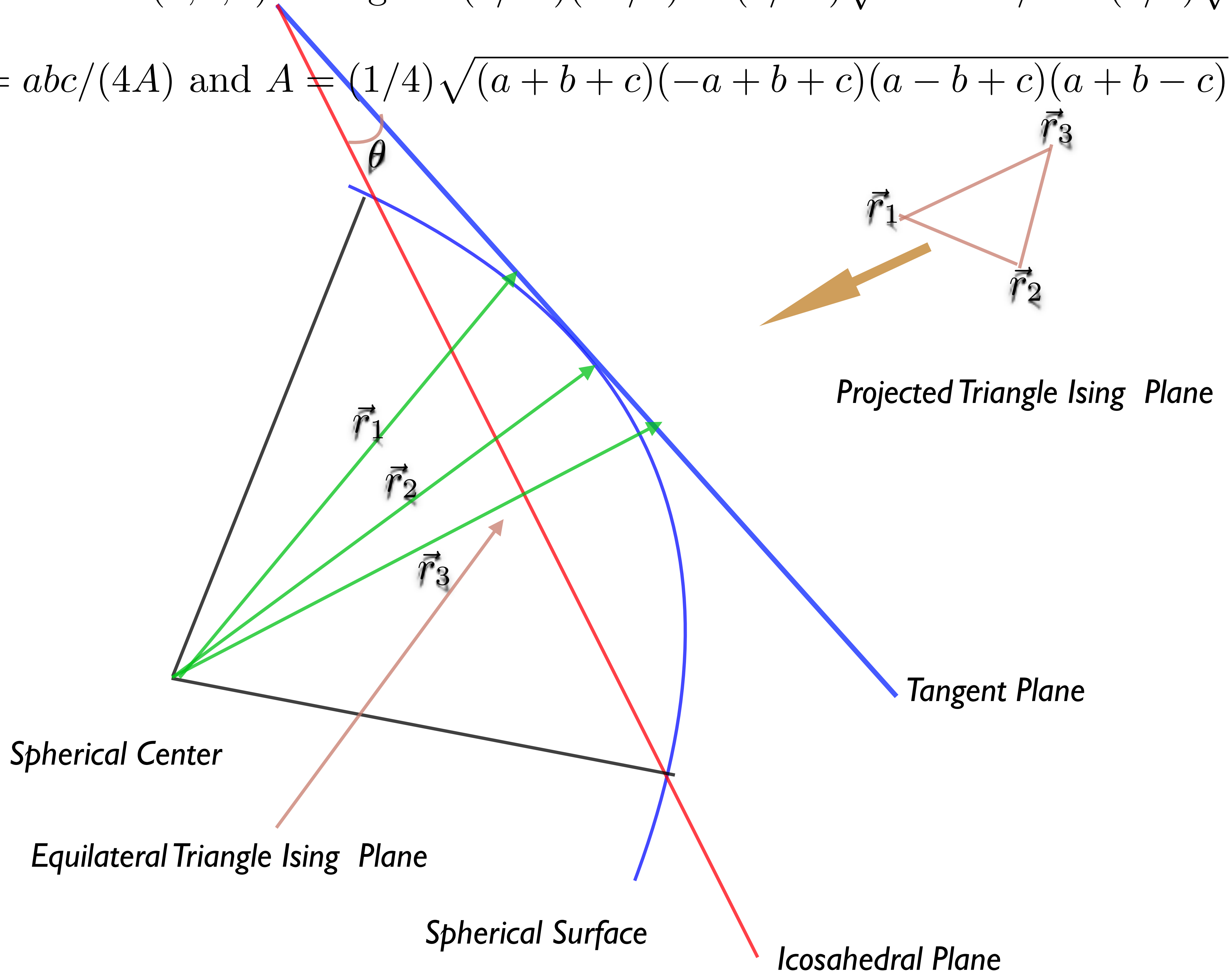
FEM Projective Geometry

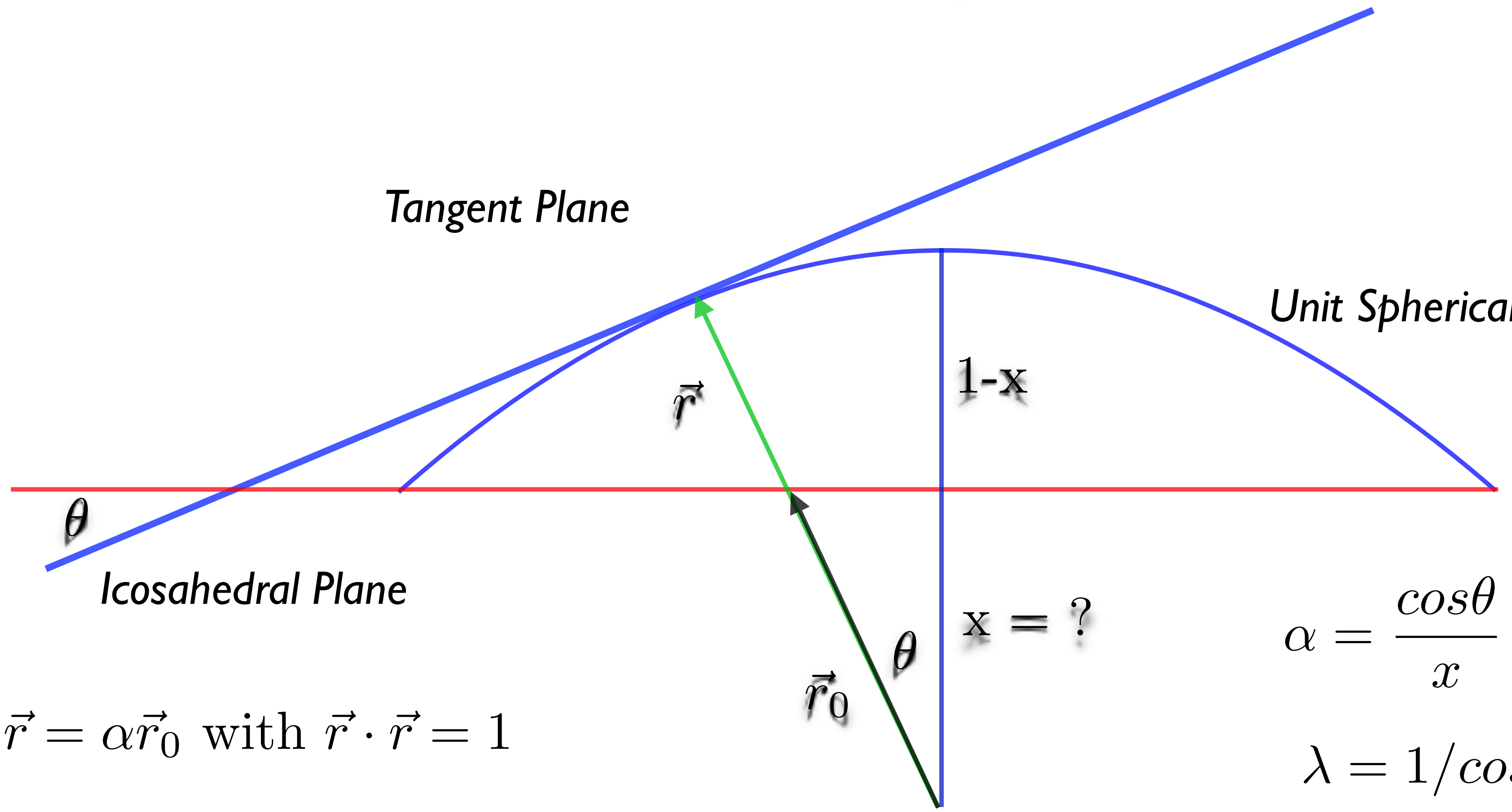
FE for $\triangle(a,b,c)$ on edge a: $(1/a^2)(ha/2) = (1/2a)\sqrt{R^2 - a^2/4} = (1/4)\sqrt{4R^2/a^2 - 1}$

$$R = abc/(4A) \text{ and } A = (1/4)\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$$



Projection Algebra

$\Delta r^{\perp} = \alpha \Delta r_0^{\perp}$ $\quad$ $\Delta r^{\|} = \alpha \lambda \Delta r_0^{\|}$



$\vec{r} = \alpha \vec{r}_0$ with $\vec{r} \cdot \vec{r} = 1$

$$\alpha = \frac{cos\theta}{x}$$

$$\lambda = 1/cos\theta$$

2D a',b,'c' vectors in Tangent Plane: $\vec{v} = (x, y)$ and $\hat{n} = (\cos\phi, \sin\phi)$

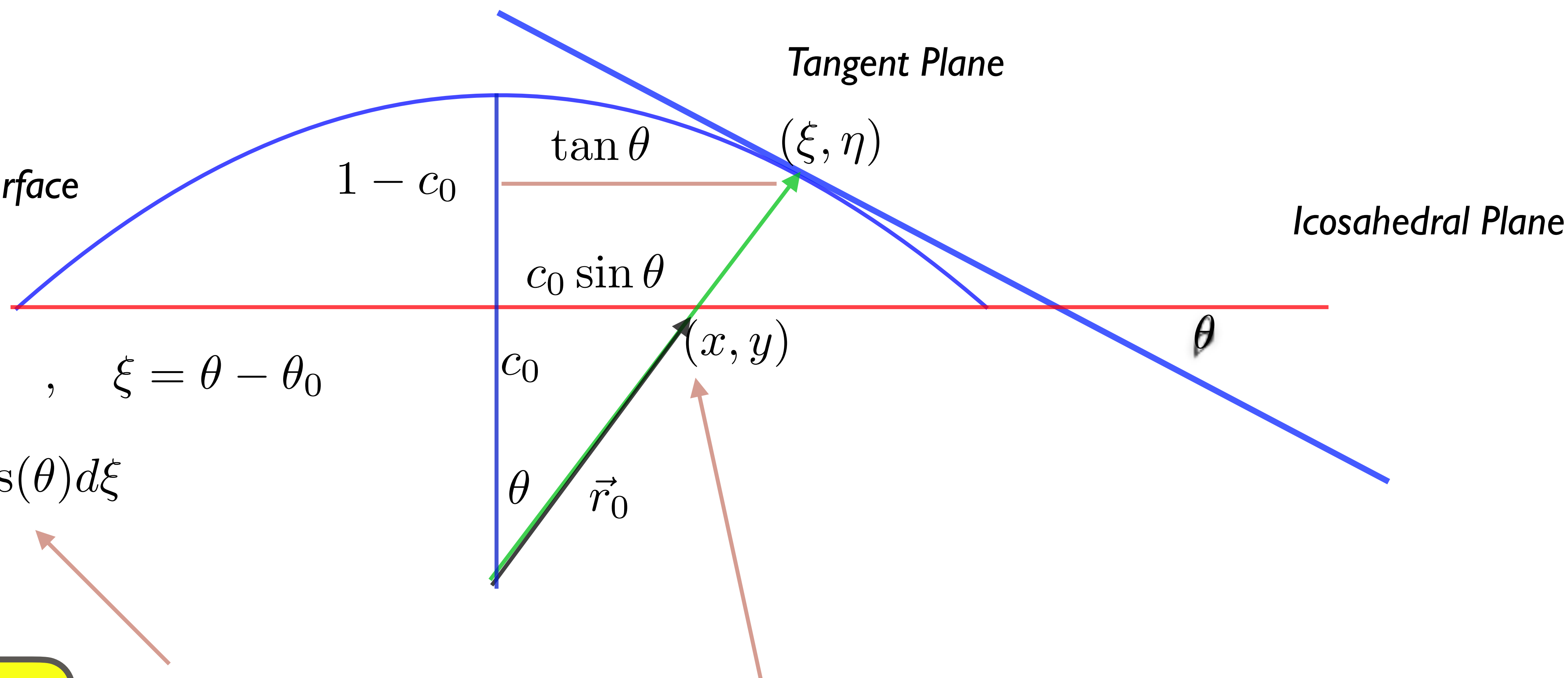
$$\vec{v}' = \alpha[\vec{v} + \lambda \hat{n}(\hat{n} \cdot \vec{v})] = \frac{1}{x}[\cos\theta \vec{v} + \hat{n}(\hat{n} \cdot \vec{v})]$$

$$x = R_{in}/R_{circ} = (1/6)(3 + \sqrt{5}) = 0.872677996249965$$

with $R_{in}/a = (\sqrt{3}/12)(3 + \sqrt{5})$, $R_{circ}/a = (1/4)\sqrt{10 + 2}$ https://en.wikipedia.org/wiki/Regular_icosahedron

One Again!

Consider a vector $\vec{v} = (v_x, v_y) \rightarrow \vec{v}' = (v_\xi, v_\eta)$



$x = c_0(\sin\theta - \sin\theta_0) \quad , \quad \xi = \theta - \theta_0$

$dx = c_0 d\sin\theta = c_0 \cos(\theta) d\xi$

$$\vec{v}_{\|} = (\hat{n} \cdot \vec{v})\hat{n}$$

$$\vec{v}_{\perp} = \vec{v} - (\hat{n} \cdot \vec{v})\hat{n}$$

TRICK: Place vertex at origin each co-ordinate system $(x, y) = (\xi, \eta) = (0, 0)$ and rotate: $\hat{n} = (1, 0)$

$$x_0 = R_{in}/R_{circ} = (1/6)(3 + \sqrt{5}) = 0.872677996249965$$

with $R_{in}/a = (\sqrt{3}/12)(3 + \sqrt{5})$, $R_{circ}/a = (1/4)\sqrt{10 + 2}$

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$$\vec{v}_{\|} = (\hat{n}\cdot\vec{v})\hat{n}$$

$$\vec{v}_{\perp} = \vec{v} - (\hat{n}\cdot\vec{v})\hat{n}$$

Continued

$$\vec{v}' = c_1\vec{v}_{\|} + c_2\vec{v}_{\perp} = c_1(\hat{n}\cdot\vec{v})\hat{n} + c_2(\vec{v} - (\hat{n}\cdot\vec{v})\hat{n})$$

Parallel

$$x = c_0(\sin\theta - \sin\theta_0) \quad , \quad \xi = \theta - \theta_0$$

$$dx = c_0 d\sin\theta = c_0\cos(\theta)d\xi$$

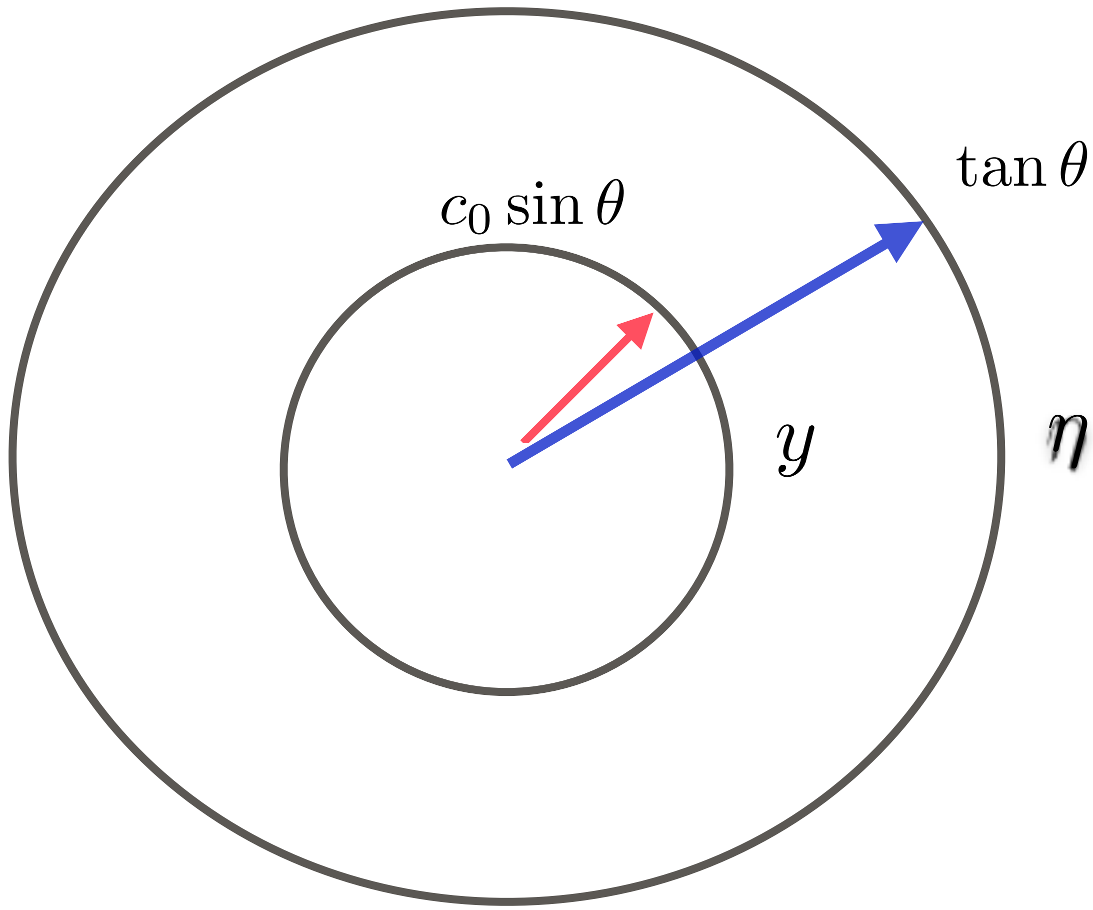
$$d\xi = \frac{1}{c_0\cos\theta}dx \qquad c_1 = \frac{c_0}{\cos\theta}$$

Perpendicular

$$dy = c_0\sin\theta d\phi \quad , \quad d\eta = \tan\theta d\phi$$

$$d\eta = \frac{\cos\theta}{c_0}dy \qquad c_2 = \frac{\cos\theta}{c_0}$$

(top view)



$$\vec{v}' = \frac{1}{c_0}\left[\frac{1}{\cos\theta}(\hat{n}\cdot\vec{v})\hat{n} + \cos\theta(\vec{v} - (\hat{n}\cdot\vec{v})\hat{n})\right]$$

$$\vec{v}' = \frac{1}{c_0\cos\theta}\left[\cos^2\theta\,\vec{v} + \sin^2\theta\,(\hat{n}\cdot\vec{v})\hat{n})\right]$$

where $\hat{n} = (\cos\phi, \sin\phi)$

$$\begin{bmatrix} v'_x \\ v'_y \end{bmatrix} = \frac{1}{c_0\cos\theta}\begin{bmatrix} \cos^2\theta + \sin^2\theta\cos^2\phi & \sin^2\theta\cos\phi\sin\phi \\ \sin^2\theta\cos\phi\sin\phi & \cos^2\theta + \sin^2\theta\sin^2\phi \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

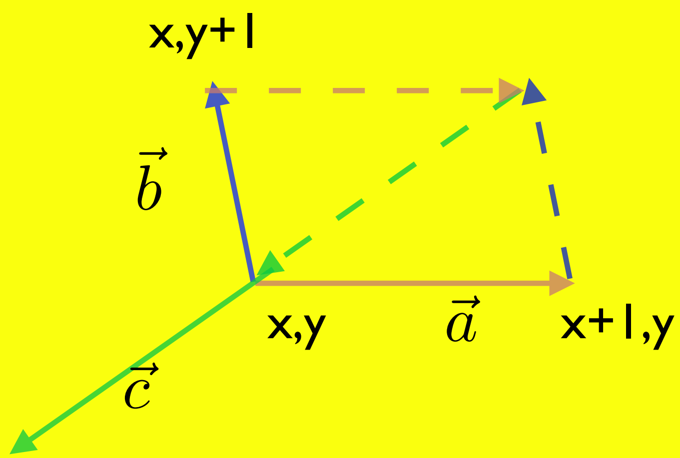
Apply this to the 6 fold vectors $(\pm\vec{a}, \pm\vec{b} \pm \vec{c})$ to get sheared lattice $(\pm\vec{a}', \pm\vec{b}', \pm\vec{c}')$ and then add new FEM weights

CODE: NOTE THE FEM ACTION ONLY DEPENDS ON THE ASPECT RATIO!

FE for $\triangle(a,b,c)$ on edge a: $(1/a^2)(ha/2) = (1/2a)\sqrt{R^2 - a^2/4} = (1/4)\sqrt{4R^2/a^2 - 1}$

$R = abc/(4A)$ and $A = (1/4)\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$

$\vec{a} + \vec{b} + \vec{c} = 0$



$$\vec{r}(x,y) = x\vec{a} + y\vec{b}$$

$$r^2 = \vec{r}\cdot\vec{r} = x^2a^2 + y^2b^2 + 2xy\vec{a}\cdot\vec{b}$$
$$= (x^2 - xy)a^2 + (y^2 - xy)b^2 + xyc^2$$

Triangular lattice: $a = b = c$

$A = (\sqrt{3}/4)a^2 \quad R = a^3/(\sqrt{3}a^2 = a/\sqrt{3}$

$w(i,j) = (1/2)\sqrt{4R^2/a^2 - 1} = 1/(2\sqrt{3})$

Square $a = b = c/\sqrt{2}$

$A = (1/4)\sqrt{c^2(4a^2 - c^2)} = (1/4)\sqrt{2a^2(4a^2 - 2a^2)} = a^2/2$

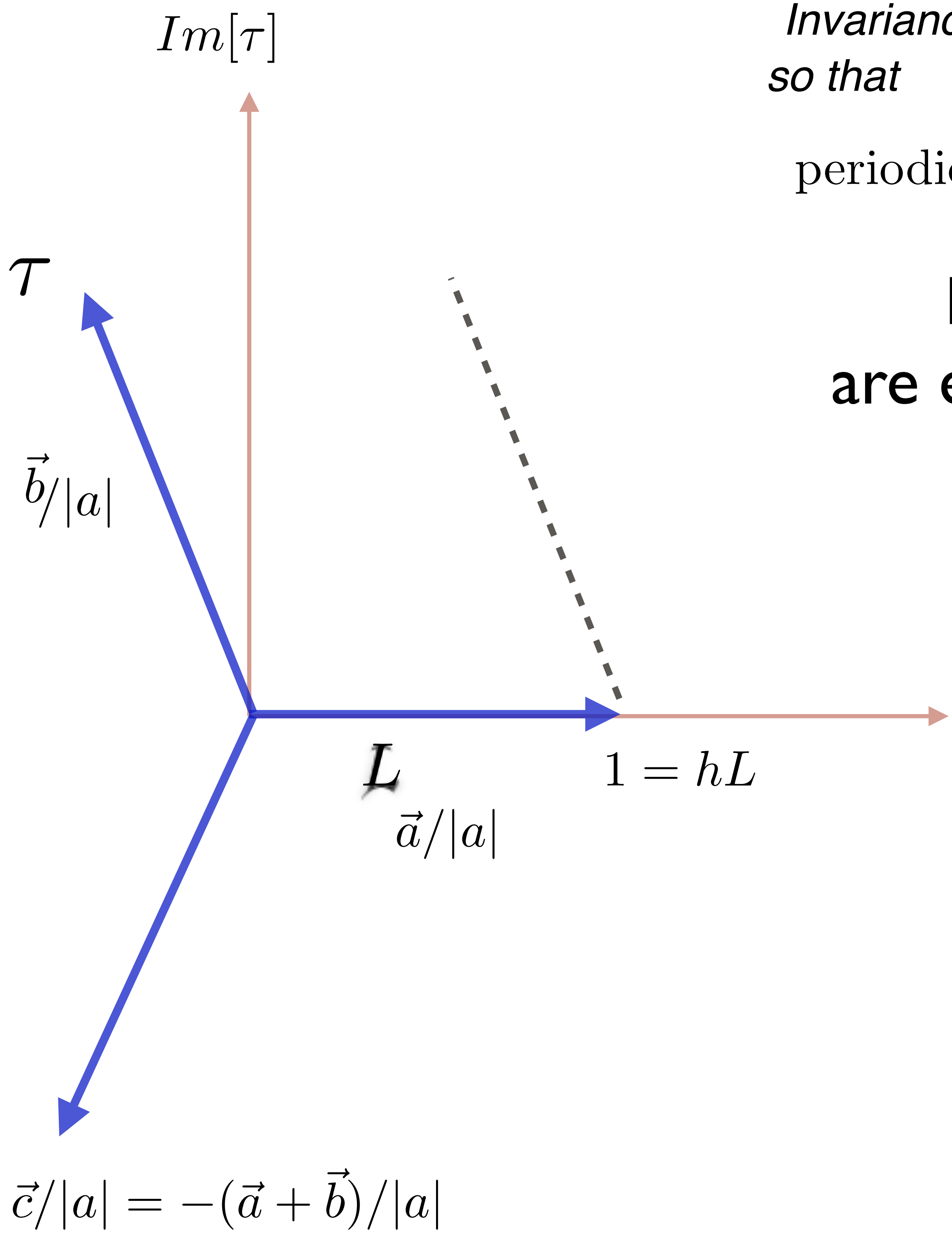
$R = abc/(4A) = \sqrt{2}a^3/(2a^2) = a/\sqrt{2}$

$w(i,j) = (1/2)\sqrt{4R^2/\ell^2 - 1} \to (1/2)\sqrt{2a^2/\ell^2 - 1} = 1/2, 1/2, 0$

so $\beta/(2\sqrt{3}) = \beta_\triangle = ln(3)/4$ and $\beta/2 = \beta_\square = \ln(1+\sqrt{2})/2$

$\beta_c = 2\sqrt{(3)}\log(3)/4 = 0.951426 \quad$ vs $\quad \beta_c = \log(1+\sqrt{2}) = 0.881374$

BINDER MODULAR PARAMETER



Invariance: Scaling and rotation to a = 1 + 0 i
so that $\tau = b$ *in complex co-ordinates*

periodicity implies $z \sim z + n + m\tau$

Different triangulation are exactly the modular group