Krammers-Wannier Duality in Lattice Systems

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I. INTRODUCTION

It was shown by R. Pierls [1] that in two dimensions or higher, the Ising model on a lattice necessarily has a region of low temperature with nonzero spontaneous magnetization. This implies the presence of a phase transition between such a low temperature ordered phase and a higher temperature disordered phase. A method for the calculation of this phase transition point was provided by H. A. Kramers and G. H. Wannier [2], using a *duality relation* between the high and low temperature expansions of the partition function of the system. The type of duality observed depends on the lattice structure. In this paper, the Kramers-Wannier duality in two dimensions will be studied, using the square, triangular, and hexagonal lattices as examples. Some extensions will also be briefly discussed.

II. SELF DUALITY IN THE SQUARE LATTICE

The strategy of expanding a partition function in a series expansion starts with an exactly solvable limit, and sum over perturbations of the solutions in these limits [3]. The simplest example of a two dimensional lattice is the square lattice. With regards to the duality relations, the high and low temperature partition function expansions obtained lead to an elegant geometrical interpretation, which, through *self duality*, allow one to calculate the phase transition point exactly. It is noted that all calculations in this paper are done in the absence of an external field. The N-particle partition function of the Ising model in the square lattice is given by

$$\mathcal{Z}_N = \sum_{\{\sigma\}} \exp\left(K \sum_{i,j} \sigma_i \sigma_j + L \sum_{i,k} \sigma_i \sigma_k \right)$$

The first and second sum run over horizontal and vertical links, respectively. The number of horizontal and vertical links are both taken to be equal to M. It is assumed that the coupling is different in the two directions (J and J') , while $K \equiv \beta J$ and $L \equiv \beta J'$.

A. High Temperature Expansion

In the high temperature expansion, the exactly solvable limit where the spins are independent is used. Using the fact that $\sigma_i = \pm 1$, one has the identity

$$\exp(x\sigma_i\sigma_j) = \cosh(x)\left(1 - \sigma_i\sigma_j \tanh(x)\right) \tag{1}$$

the N spin partition function can be rewritten as

$$\mathcal{Z}_N = (\cosh(K)\cosh(L))^M \sum_{\{\sigma\}} \prod_{i,j} (1 + v\sigma_i\sigma_j) \prod_{i,k} (1 + w\sigma_i\sigma_k) \tag{2}$$

where $v \equiv \tanh(K)$ and $w \equiv \tanh(L)$. In Equation (2), the parameters v and w are less than unity (except when their arguments are equal to infinity, which corresponds to zero temperature). In the high temperature region, i.e. $\beta \to 0$, these parameters are small. The product (of products) in Equation (2) contains 2^{2M} terms. The process of expansion is made easier by making the following graphical correspondence with terms in the expansion:

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1. With every factor of $v\sigma_i\sigma_j$, a line is drawn along the horizontal link between spins i and j.
2. With every factor of $w\sigma_i\sigma_k$, a line is drawn along the vertical link between spins i and k.
3. No line is associated with a pure factor of 1.

The expansion can now be made in terms of the lines on the graphs that are formed on the lattice using the rules ascribed above. The generic terms in the expansion have the form

$$v^r w^s \sigma_1^{n_1} \sigma_2^{n_2} ... \sigma_N^{n_N}$$

where r and s are the number of horizontal and vertical lines, respectively. n_i is the number of lines where the spin σ_i is an end-point. As the Ising spins each take values of ± 1, it can be seen that in an expansion of the above kind of terms, the only surviving contributions after summing over spins will be those where all the n_i's are even. This sum would result in $2^N v^r w^s$ for each such contribution. This gives us the following form for the partition function

$$\mathcal{Z}_N = 2^N (\cosh(K)\cosh(L))^M \sum_{\mathcal{P}} v^r w^s \tag{3}$$

where the sum is over graphs on the lattice with an even number of lines on each site. This corresponds to *closed polygons* on the lattice. It can be seen that the term outside the sum, 2^N, is indeed the partition function for the limit where all the spins are independent, and the sum represents (all) perturbations about this limit.

B. Low Temperature Expansion

In the low temperature limit, the exactly solvable limit is that where all the spins are aligned and perturbations about this are 'islands' [3] of different spin (also known as *domains*) (Fig. 2). The expansion of the partition function in this limit is made by creating a distinction between links where the joined spins are anti-parallel and parallel. Let r and s be the number of vertical and horizontal links between anti-parallel spins. The naming of these seems to be in opposition with the previous section, however, the reason for this will be made clear shortly. This configuration gives a contribution to the partition function that depends only on the number of *anti-parallel links*:

$$\exp\left(K(M-2s) + L(M-2r)\right)$$

A convenient tool in the following analysis is that of the *dual lattice*. For the purpose of this paper, the definition of the dual lattice will be taken as the lattice obtained by placing sites at the center of the original lattice and joining pairwise those relative to adjacent faces [4] (Fig. 1). Using this definition, the square lattice has a dual lattice which is also a square lattice. The model is hence said to have *self duality*. The spins, which were previously defined on the vertices of the original lattice, can now be equivalently defined on the center of the unit cells in the dual lattice. Using this correspondence, a new graphical correspondence can be made, using the following rules:

1. For every set of anti-parallel spins, a line is drawn on the dual lattice such that it crosses the spins.
2. No line is drawn for parallel spins.

Using the above scheme, it can be seen that there will be r horizontal lines and s vertical lines drawn on the dual lattice. Since spin flips correspond to different spins across faces in the original lattices, the number of lines we draw in the dual lattice must be *even*. These sets of lines that we draw correspond to closed polygons in the dual lattice, which in this low temperature expansion correspond to the various possible *magnetic domains*. The partition function becomes a sum over the closed polygons over the dual lattice

$$\mathcal{Z}_N = 2\sum_{\tilde{\mathcal{P}}} \exp\left(K(M-2s) + L(M-2r)\right) = 2\exp(M(K+L)) \sum_{\tilde{\mathcal{P}}} \exp\left(-2(Lr + Ks)\right) \tag{4}$$

The factor of 2 comes from the fact that the low temperature region has a broken $\mathbb{Z}_2$ symmetry, and for each configuration of polygons, there are two possible cases, which are related by flipping all the spins. In the low temperature region, $\beta \to \infty$, K and L are large and the dominant terms in the expansion are given by the terms which contain only small values of r and s. Once again, it can be seen that the term outside the sum is the partition function for a system of fully aligned spins.

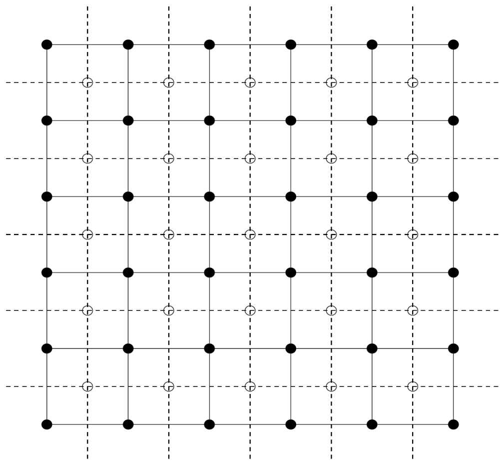

FIG. 1: The original lattice (solid dots) and it's dual (hollow dots). Self duality is evident [4]

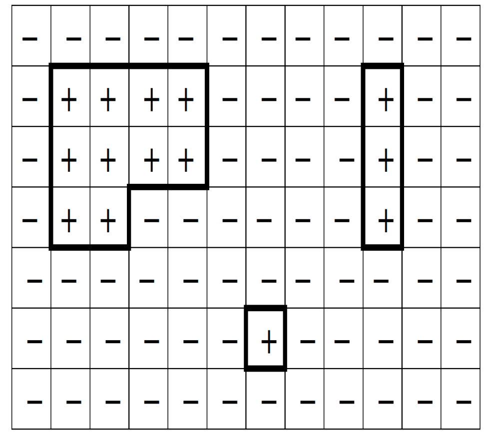

FIG. 2: Closed polygons on the dual lattice - domains [4]

C. Self Duality between the Expansions

It can be seen that both Equations (3) and (4) contain a sum over closed polygons over lattices. In the thermodynamic limit for a square lattice, where $N = M \to \infty$, with $N/M = 1$, the boundary of the sample, which is the only region where the difference between the lattice and its dual can be seen, becomes insignificant. Hence in this limit, the sums have the same geometric form. Due to this fact, there exists a *duality relation* between them, which allows for a mapping between the high temperature and low temperature regions of the model. If the couplings for the different temperature regions are made distinct ($\tilde{K}, \tilde{L}$ for the low temperature phase and K, L for the high temperature phase), this relation is

$$e^{-2\tilde{L}} \leftrightarrow \tanh K \quad , \quad e^{-2\tilde{K}} \leftrightarrow \tanh L$$

The above, written explicitly in terms of the function Φ gives

$$\Phi(e^{-2\tilde{L}}, e^{-2\tilde{K}}) = \Phi(v, w)$$
$$\implies \frac{\mathcal{Z}_N[\bar{K}, \bar{L}]}{2\exp(N(\bar{K} + \bar{L}))} = \frac{\mathcal{Z}_N[K, L]}{2^N \left(\cosh K \cosh L\right)^N}$$

Using the duality relation, an explicit relationship between the high and low temperature partition function can be obtained

$$\mathcal{Z}_N[\tilde{K}, \tilde{L}] = 2\left(\sinh 2K \sinh 2L\right)^{-N/2} \mathcal{Z}_N[K, L] \tag{5}$$

Peierls' argument states that the two regions describe distinct phases. Hence, there must be a critical point which the free energies derived from the two available partition functions must be equal. From Equation (5), using the usual relationship between the partition function and the free energy (per spin), one obtains

$$f(\tilde{K}, \tilde{L}) = f(K, L) + \frac{1}{2\beta} \ln\left[\sinh(2K)\sinh(2L)\right]$$

Due to the continuity of the free energy at the critical point (which is denoted using the subscript c), the second term must vanish at this point, i.e.

$$\sinh(2K_c)\sinh(2L_c) = 1$$

The above equation gives a set of fixed points, across which the system transitions from an ordered, low temperature phase, to a disordered, high temperature phase. In the *isotropic* case, i.e. where $K = L$ and $\bar{K} = \bar{L}$, the critical condition becomes particularly simple and the transition temperature can be found

$$\sinh(2K_c) = 1 \implies T_c^{sq} \approx 2.269J$$

The above agrees with Onsager's exact solution for the square lattice Ising model [5].

III. DUALITY IN THE HEXAGONAL AND TRIANGLE LATTICES

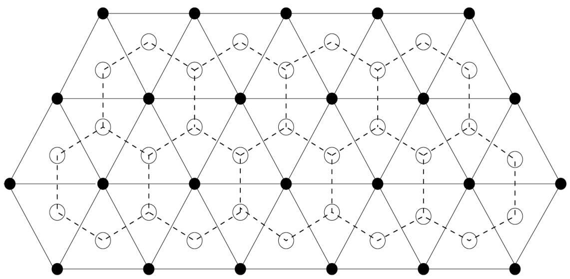

FIG. 3: Duality of the Hexagonal and Triangular Lattices [4]

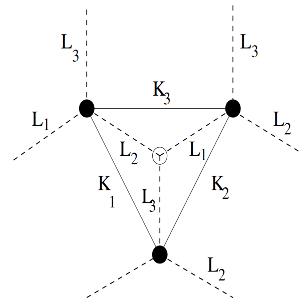

FIG. 4: Couplings in the lattices [4]

The duality of expansions in low and high temperature phases can be extended to other lattice systems. The Hexagonal and Triangular lattices are intimately linked since they are each others' dual lattices (Fig. 3). However, since they are not self dual, the process of relating their expansions is more involved. If the three types of links in the hexagonal and triangular lattice are assumed to have a different strengths, corresponding to K_i (triangular) and L_i (hexagonal), with $i = 1, 2, 3$ (Fig. 4), the partition functions (taking the number of links in each direction to be M) may be expressed as

$$\mathcal{Z}_N^H = \sum_{\{\sigma\}} \exp\left(\mathcal{L}_1 \sum_{l,i} \sigma_l \sigma_i + \mathcal{L}_2 \sum_{l,j} \sigma_l \sigma_j + \mathcal{L}_3 \sum_{l,k} \sigma_l \sigma_k\right)$$

$$\mathcal{Z}_N^T = \sum_{\{\sigma\}} \exp\left(\mathcal{K}_1 \sum_{l,i} \sigma_l \sigma_i + \mathcal{K}_2 \sum_{l,j} \sigma_l \sigma_j + \mathcal{K}_3 \sum_{l,k} \sigma_l \sigma_k\right)$$

where $\mathcal{L}_i = \beta L_i$ and $\mathcal{K}_i = \beta K_i$. The insight for this case that there exists a duality between the high temperature expansion of $\mathcal{Z}_N^T$ and the low temperature expansion of $\mathcal{Z}_N^H$. As in the case of the square lattice, the identity (1) is used to give

$$\mathcal{Z}_N^T = 2^N \left(\cosh \mathcal{K}_1 \cosh \mathcal{K}_2 \cosh \mathcal{K}_3\right)^M \sum_{\mathcal{P}} v_1^{r_1} v_2^{r_2} v_3^{r_3}$$

where $v_i = \tanh \mathcal{K}_i$, analogous to the case of the square lattice, the sum is over closed polygons on the triangular lattice. Subsequently, the low temperature expansion of the hexagonal lattice may be considered. As in the case for the square lattice, the low temperature expansion is conducted using the dual lattice. For a hexagonal lattice with N spins, the dual lattice is a triangular lattice with $2N$ spins. The partition function is thus written with the subscript $2N$,

$$\mathcal{Z}_{2N}^H = \exp(N(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3)) \sum_{\mathcal{P}} \exp\left(-2(\mathcal{L}_1 r_1 + \mathcal{L}_2 r_2 + \mathcal{L}_3 r_3)\right)$$

Again the sums are of the same geometric nature, and hence the duality relation may be written as (Using the $\mathcal{K}_i^*$'s for $\mathcal{Z}_N^T$ and $\mathcal{L}_i$'s for $\mathcal{Z}_{2N}^H$)

Error Reverse This

$$\tanh \mathcal{K}_i^* \leftrightarrow \exp(-2\mathcal{L}_i) \iff \sinh 2\mathcal{K}_i^* \leftrightarrow \frac{1}{\sinh 2\mathcal{L}_i}$$

Using the above, the partition functions have the following relationship

$$\mathcal{Z}_{2N}^H(\mathcal{K}^*) = \left(2 \sinh 2\mathcal{L}_1 \sinh 2\mathcal{L}_2 \sinh 2\mathcal{L}_3\right)^{N/2} \mathcal{Z}_N^T(\mathcal{L}) \tag{6}$$

A hurdling block now presents itself. Since the lattice and it's dual are not the same, i.e. there is no *self duality*, the argument that the free energies must be equal at the critical point cannot be employed to locate the critical point.

The critical point is identified with the help of the *Star-Triangle identity.* The identity will be presented here, without proof (the curious reader is referred to [4] for a succinct proof)

$$\sinh 2\mathcal{L}_i \sinh 2\mathcal{K}_i = \frac{(1-v_1^2)(1-v_2^2)(1-v_3^2)}{4[(1+v_1v_2v_3)(v_1+v_2v_3)(v_2+v_1v_3)(v_3+v_2v_1)]^{1/2}} \equiv h$$

The above identity relates the couplings of the hexagonal and triangular lattices, in any temperature region. Given that Equation (6) relates the hexagonal partition function at high temperature and the triangular partition function at low temperature, the Star-Triangle identity gives a relationship between the high and low temperature partition functions for a single type of lattice. For the triangular lattice, this relationship is

$$\mathcal{Z}_N^T(\mathcal{K}) = h^{N/2}\mathcal{Z}_N^T(\mathcal{K}^*)$$

Just Need to Check this numericaclly

Now the free energy argument may be applied, to obtain the critical condition as

$$h_c = 1 \iff \frac{(1-v_{c,1}^2)(1-v_{c,2}^2)(1-v_{c,3}^2)}{4[(1+v_{c,1}v_{c,2}v_{c,3})(v_{c,1}+v_{c,2}v_{c,3})(v_{c,2}+v_{c,1}v_{c,3})(v_{c,3}+v_{c,2}v_{c,1})]^{1/2}} = 1$$

In the isotropic limit, i.e. $v_i = v$, the above equation can be solved to give a single physical solution

tanh[K] = $$v_c = 2-\sqrt{3} \implies T_c^{\mathrm{tr}} = 3.416K$$

The same critical condition applies for the hexagonal lattice, but the v_i's get substituted from the duality condition, which gives the critical temperature of the hexagon as

$$T_c^{\mathrm{hex}} = 1.519L$$

Observing the isotropic cases for the three kinds of lattices considered, if one was to consider the same coupling strength, the critical temperature is directly proportional to the coordination number in the lattice. This is an indication of the fact that systems with a smaller number of spin-spin interactions will magnetize at a higher temperature.

IV. GENERALIZATIONS OF DUALITY

Kramers and Wannier's work on duality in two dimensions has far reaching generalizations, both in other lattice types, as well as other dimensionalities. In the three dimensional Ising model on a hypercubic lattice, one requires a partition function describing a *lattice gauge theory* to find the duality relation [3]. In general, any Abelian theory may be 'solved' using a duality transformation [6], however, as seen in the case of the triangular and hexagonal lattices, the dual of the theory may be completely different from the original.

V. CONCLUSION

The duality transformation is both conceptually simple, and in practice a relatively systematic method for identifying the critical points in a theory with a phase transition. In contrast to Onsager's original solution for the two dimensional Ising model on a square lattice, this method is simpler. While these duality relations in general do not reveal information about critical exponents, they do place constraints on the shapes of phase boundaries [3].

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