Lecture 23
Seiberg–Witten Theory II

Outline

This is the second of three lectures on the exact Seiberg-Witten solution of $\mathcal{N} = 2$ SUSY theory.

The second lecture:

- Elliptic Curves:

  The mathematical machinery for analyzing monodromy.

- Seiberg-Witten theory:

  Basics of $SO(3)$ gauge theory with $F = 1$ flavor $\Rightarrow \mathcal{N} = 2$ SUSY.

  Exact quantum duality transformation.

  Analysis of monodromies.

Reading: Terning 13.4-13.5.

Modular Symmetry

Setting: a low energy effective theory with a $U(1)$ gauge field.

Holomorphy: the holomorphic coupling τ of the effective low energy theory depends holomorphically on the physical parameters.

In some theories the holomorphic coupling τ is not a conventional function: as physical parameters vary, returning to their original values, τ fails to be single-valued.

Monodromy: the value that τ returns to is different, but physically equivalent $\Rightarrow$ τ transforms under an element of $SL(2,\mathbb{Z})$.

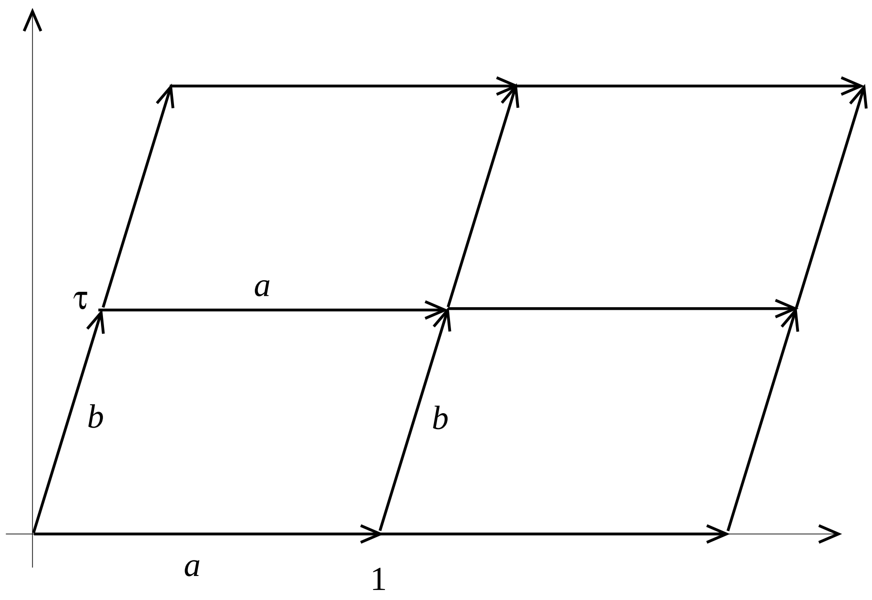
Terminology: τ is a section of an $SL(2,\mathbb{Z})$ bundle.

The mathematical structure underlying this situation is geometrical: $SL(2,\mathbb{Z})$ is the modular symmetry group of a torus:

Section of $SL(2,\mathbb{Z})$ $\leftrightarrow$ Modular parameter of a torus

Modular Parameter of a Torus

Construct a lattice of points in $\mathbb{C}$, using τ and 1 as basis vectors



Identify cells related by a lattice vector → torus with modular parameter τ.

Modular Transformation

Employ new basis vectors for the lattice: $\alpha\tau + \beta$ and $\gamma\tau + \delta$ where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$.

Invertibility: for $\alpha\delta - \beta\gamma = 1$, the transformation is invertible with another set of integers, so the new lattice is equivalent to the old one.

Geometrically, $\alpha\delta - \beta\gamma = 1$ ensures that the new parallelogram encloses one primitive cell, albeit a different one.

The two conditions $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ and $\alpha\delta - \beta\gamma = 1$ define the group $SL(2, \mathbb{Z})$.

The ration of the two basis vectors transform as

$$\tau \longrightarrow \tfrac{\alpha\tau+\beta}{\gamma\tau+\delta} \ .$$

Thus:

$SL(2, \mathbb{Z})$ of torus $\leftrightarrow$ $SL(2, \mathbb{Z})$ of the $U(1)$ gauge theory

Elliptic Curves

Mathematical challenge: the holomorphic coupling τ is a section of $SL(2, \mathbb{Z})$, it is not single valued.

Mathematical construction: two complex parameters $x, y \in \mathbb{C}$ related by one cubic (elliptic) equation

$$y^2 = x^3 + Ax^2 + Bx + C \ ,$$

form an elliptic curve.

Remark 1: a submanifold parametrized by a single parameter is a curve, although here the "curve" has one complex parameter, as befits a torus.

Remark 2: the equation respects complex structure because it is holomorphic $\Rightarrow$ it defines $SL(2, \mathbb{Z})$.

This representation of a torus is useful because A, B, C are single-valued functions of the moduli and parameters of the gauge theory.

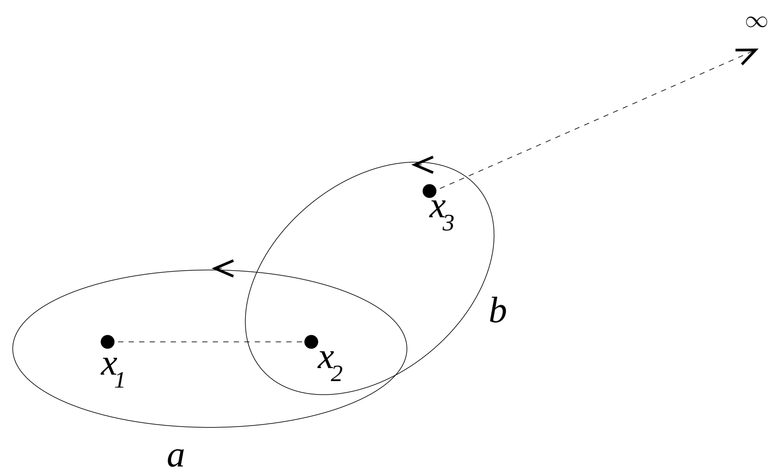
Elliptic Curve and the Torus

Factorize cubic polynomial

$$y^2 = x^3 + Ax^2 + Bx + C \equiv (x - x_1)(x - x_2)(x - x_3) \ .$$

Consider y a function of x: it is a square root $\Rightarrow$ the complex x plane is two sheets that meet along branch cuts.

The cubic has three zeroes: choose one branch cut between two of the zeroes, the other branch cut between the third zero and ∞.



Including the point at ∞ , the cut plane is topologically $\sim$ two spheres connected by two tubes.

Topologically, this in turn is a torus, with a and b a the cycles of torus.

Modular Parameter of the Torus

The periods of the torus are

$$\omega_1 = \int_a \frac{dx}{y} \ , \ \ \omega_2 = \int_b \frac{dx}{y} \ ,$$

where $\{a, b\}$ form a basis of one-cycles for the torus.

a, b cycles $\leftrightarrow$ two sides of the parallelogram.

The modular parameter is given by the ratio of the periods:

$$\tau(A, B, C) = \frac{\omega_2}{\omega_1} \ .$$

Important limits: two roots meet, or one of the roots goes to ∞.

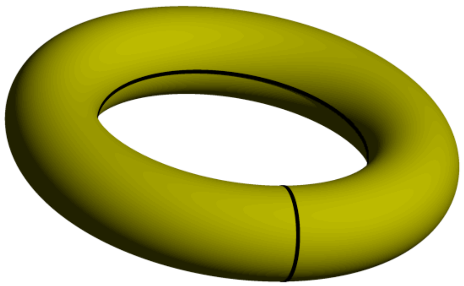
Mathematically: a cycle shrinks to zero, the branch cuts disappears, torus is singular.

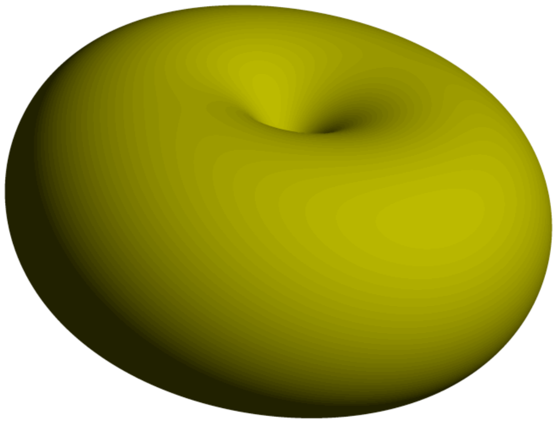
Physically: the holomorphic coupling τ is singular when two roots meet.

Singular Tori

Two roots are equal if the discriminant vanishes

$$\Delta = \Pi_{i<j}(x_i - x_j)^2 = 4A^3C - B^2A^2 - 18ABC + 4B^3 + 27C^2 = 0 \ .$$





The single-valued A, B, C are easier to determine than the multi-valued τ, but given A, B, and C we can calculate τ.

$SO(N)$ with $F = N - 2$

As we move in moduli space (the $z = \det M$ plane), singular points are encountered where new massless particles appear:

- $z = 0$ (magnetic excitations become light).
- $z = 16\Lambda_{N,N-2}^{2N-4}$ (dyonic excitations become light).

At these points the charged massless particles drive the dual photon coupling to zero so the dual holomorphic coupling is singular.

This determines the elliptic curve:

$$y^2 = x(x - 16\Lambda_{N,N-2}^{2N-4})(x - z)\ .$$

Consistency Checks

Consistency checks:

- A, B, and C must be holomorphic functions of the moduli and $\Lambda_{N,N-2}$, so that τ is holomorphic.

- In the weak coupling limit $\Lambda_{N,N-2} \to 0$, the curve becomes

  $$y^2 = x^2(x - z) \ .$$

  Two poles coincide (so the torus is singular) for all $z = \det M$: in an asymptotically free theory the gauge coupling runs to zero in the UV.

- The elliptic curve must be compatible with the global symmetries: $\det M$, $\Lambda^b_{N,N-2}$ both have R-charge, spurious $U(1)_A$ charge $(0, 2F)$. This is consistent with charge assignments $(0, 2F)$, $(0, 3F)$ for x, y.

Consistency Checks: Monodromies

Near a singularity $\epsilon = z - z_0 = \mathrm{detM} - \mathrm{detM}_0$ is small and two roots approach each other: $x_0 \pm a\epsilon^{n/2}$.

Notation: n is the order of the zero in the discriminant $\Delta \sim \epsilon^n$.

In the elliptic curve, shift x by x_0 and rescale x and y

$$\begin{aligned} y^2 &= (x - x_1)(x - x_0 - a\epsilon^{n/2})(x - x_0 + a\epsilon^{n/2}) \\ &\Rightarrow \quad y^2 = (x - \tilde{x})(x^2 - \epsilon^n) \ . \end{aligned}$$

Evaluation of period integrals:

$$\begin{aligned} \omega_1 &= \int_{-\epsilon^{n/2}}^{\epsilon^{n/2}} \frac{dx}{y} \approx \int_{-\epsilon^{n/2}}^{\epsilon^{n/2}} \frac{dx}{i\sqrt{\tilde{x}}\,\sqrt{x^2-\epsilon^n}} \approx -\frac{\pi}{\sqrt{\tilde{x}}} \ , \\ \omega_2 &= \int_{\epsilon^{n/2}}^{\tilde{x}} \frac{dx}{y} \approx \int_{\epsilon^{n/2}}^{\tilde{x}} \frac{dx}{\sqrt{(x-\tilde{x})(x^2-\epsilon^n)}} \approx \frac{i}{\sqrt{\tilde{x}}} \ln \epsilon^{n/2} \ , \end{aligned}$$

$$\Rightarrow \ \tau = \frac{\omega_2}{\omega_1} \approx \frac{1}{2\pi i} \ln \epsilon^n \ .$$

Conclusion: the monodromy at the singular point z_0 is T^n .

$SO(N)$ with $F = N - 2$

For $z = \det M \ll \Lambda_{N,N-2}^{2N-4}$ the discriminant of the elliptic curve $\Delta \sim z^2$ $\Rightarrow$ the monodromy $\mathcal{M}_0 \sim T^2$ (up to a duality transformation D, so the full monodronomy is $D^{-1}T^2D$).

For rank $M = r$, $z = \det M$ has an order $(F - r)$ zero on moduli space $\Rightarrow$ the monodromy in τ on moduli space, is $\mathcal{M}_0 \sim T^{2(F-r)}$.

Near $z = z_d$ where the dyon becomes massless, we similarly have discriminant $\Delta \sim (z - z_d)^2$, corresponding to $\mathcal{M}_{z_d} \sim T^2$, and the monodromy over M is also $\mathcal{M}_{z_d}$.

$SO(N)$ with $F = N - 2$

Monodromy at ∞ (careful!): for large z the roots are $\simeq (0, 16\Lambda_{N,N-2}^{4N-8}/z, z)$,
$\Rightarrow$ two sets of singular points approach each other simultaneously.

Rescale the coordinates so that only two roots approach each other

$$x \to x'(8\Lambda_{N,N-2}^{2N-4} - z), \;\; y \to y'(8\Lambda_{N,N-2}^{2N-4} - z)^{3/2} \;,$$

which gives the curve

$$y'^2 = x'^3 + x'^2 + \frac{16\Lambda_{N,N-2}^{4N-8}}{(8\Lambda_{N,N-2}^{2N-4} - z)^2} x' \;.$$

So near $z = \infty$, $\Delta \sim z^{-2} \Leftrightarrow \mathcal{M}_\infty \sim T^{-2}$, while the monodromy in the moduli space is $\mathcal{M}_\infty \sim T^{-2F}$.

Back in the original $x - y$ plane, the change of variables gives a factor $\sim 1/\sqrt{z}$ in $dx/y \Rightarrow$ an additional sign flip in τ.

Final result: $\mathcal{M}_\infty = -T^{-2}$.

$SO(N)$ with $F = N - 2$

Taking the monodromy around the magnetic singularity by definition involves the duality transformation S:

$$\mathcal{M}_0 = S^{-1} T^2 S \ .$$

Then the simplest solution of the consistency condition

$$\mathcal{M}_0 \mathcal{M}_{z_d} = \mathcal{M}_\infty \ ,$$

is the monodromy around the dyonic singularity

$$\mathcal{M}_{z_d} = (ST^{-1})^{-1} T^2 S T^{-1} \ .$$

Elliptic Curves in Mathematics

Elliptic curves appear in many contexts:

- The analysis of $U(1)$ theories with monopoles (present context).
- String geometry: notably F-theory, where the complexified coupling of type IIB string theory is an $SL(2,\mathbb{Z})$ section on a Calabi-Yau space.
- Number theory: elliptic curves are used for factoring large numbers, even in practical contexts like encryption in cell phones.
- The proof of Fermat's last theorem: the final step was the proof of the Taniyama-Shimura conjecture, relating elliptic curves over rational numbers to modular forms.

$\mathcal{N} = 2$: Seiberg–Witten Theory

Setting: $\mathcal{N} = 1$ SUSY $SO(3)$ gauge theory with $F = 1$ flavor.

The adjoint of $SO(3)$ is the vector so the theory has $\mathcal{N} = 2$ SUSY.

There is no superpotential, but a classical D-term potential:

$$V = \frac{1}{g^2} \mathrm{Tr} \left[\phi, \phi^\dagger \right]^2 \ ,$$

where ϕ is the scalar component of the adjoint chiral superfield.

Classical moduli space: the ϕ such that $\left[\phi, \phi^\dagger \right] = 0$,
$\Rightarrow$ we can take $\phi = \frac{1}{2} a \sigma^3$, up to gauge transformations.

Parameterize moduli space by the gauge invariant $u = \mathrm{Tr} \phi^2$,
$\Rightarrow$ classically we have $u = \frac{1}{2} a^2$.

At a generic point in moduli space the gauge group is broken $SO(3) \to U(1)$.

Symmetries

The $\mathcal{N}=2$ R-symmetry: $SU(2)_R \times U(1)_R$.

Due to $\mathcal{N}=2$ SUSY, the fermion superpartner of the matter field ϕ must have the unit $U(1)_R$ charge, just as the gaugino λ^a (it is just a convention which of these fermions is “the” gaugino).

Thus the R-charge of the scalar field ϕ is 2.

There are just two fermions with R-charge and they both have unit charge, so $U(1)_R$ is anomalous.

More precisely, the $U(1)_R$ is equivalent to $\theta \to \theta - \alpha \sum_{\mathbf{r}} 2T(\mathbf{r}) = \theta - 4\alpha$.

Transformation with $\alpha = \frac{2\pi}{4}k$, $k = 0, 1, 2, 3$ remains a symmetry, so instantons dynamically break $U(1)_R \to \mathbb{Z}_4$.

The VEV for u breaks $\mathbb{Z}_4 \to \mathbb{Z}_2$ which acts on u by taking $u \to -u$

$\mathcal{N} = 2$

$\mathcal{N} = 2$ SUSY $\Rightarrow$ the superpotential and the leading (up to two-derivative, or four fermion) terms from the Kähler function are related to a prepotential $P(A)$.

Note: In terms of $\mathcal{N} = 1$ SUSY, the $\mathcal{N} = 2$ supermultiplet contains a $\mathcal{N} = 1$ chiral supermultiplet A (with scalar component a).

The general low-energy effective $U(1)$ theory with $\mathcal{N} = 2$ SUSY (including kinetic terms depending on the Kähler potential, and also the gauge kinetic terms) can be written as

$$\mathcal{L} = \frac{1}{8\pi i} \int d^4\theta \frac{\partial P}{\partial A} \overline{A} + \frac{1}{16\pi i} \int d^2\theta \frac{\partial^2 P}{\partial A^2} W^\alpha W_\alpha + \text{h.c.}\ .$$

Note: an important role will be played by the effective gauge coupling (and its dependence on the scalar field):

$$\tau(A) = \frac{\partial^2 P}{\partial A^2}\ .$$

$\mathcal{N} = 2$: Duality

An important feature of the $\mathcal{N} = 2$ setting is that duality can be made very explicit.

In the electric description there are no magnetic sources in the Bianchi identity $\partial^\mu \tilde{F}_{\mu\nu} = 0$. The corresponding superspace Bianchi identity: $\mathrm{Im} D^\alpha W_\alpha = 0$.

Implement this condition in the path integral (off-shell: fluctuating fields!) through a Lagrange multiplier vector multiplet V_D with action:

$$\begin{aligned}
\tfrac{1}{4\pi}\mathrm{Im}\int d^4x d^4\theta V_D D^\alpha W_\alpha &= \tfrac{1}{4\pi}\mathrm{Re}\int d^4x d^4\theta i D^\alpha V_D W_\alpha \\
&= -\tfrac{1}{4\pi}\mathrm{Im}\int d^4x d^2\theta W_D^\alpha W_\alpha \ .
\end{aligned}$$

The logic:

- Fundamental theory: if the path integral integrates over V_D, this term implements the absence of magnetic sources.
- Dual theory: we may as well integrate first over the "fundamental" vector field V, and treat V_D as dynamical.

$\mathcal{N} = 2$: Dual Description

The path integral over the electric vector superfield V gives the dual $d^2\theta$ term:

$$\frac{1}{16\pi i} \int d^2\theta \left(-\frac{1}{\tau(A)} \right) W_D^\alpha W_{D\alpha} + \text{h.c.} \ .$$

The coupling of the dual theory is $\tau_D = -\frac{1}{\tau}$, as expected.

The complete duality map also needs to specify the dual matter field A_D (with scalar component a_D),

$$A_D \equiv \frac{\partial P}{\partial A} \ .$$

This is the identification that renders the matter kinetic term

$$\frac{1}{8\pi i} \int d^4\theta \, \frac{\partial P}{\partial A} \overline{A} + \text{h.c.} = \frac{1}{8\pi i} \int d^4\theta \, A_D \overline{A} + \text{h.c.}$$

invariant under the duality transformation $A_D \to A$, $A \to -A_D$.

The coupling constant of the low energy effective $U(1)$ theory is

$$\tau = \frac{\partial^2 P}{\partial A^2} = \frac{\partial a_D}{\partial a} \ .$$

The corresponding coupling constant of the magnetic dual theory is

$$\tau_D = -\frac{1}{\tau} = -\frac{\partial a}{\partial a_D} \ ,$$

as expected given the duality transformation $A_D \to A$, $A \to -A_D$.

Status:

- The $\mathcal{N} = 2$ Lagrangian is specified by the prepotential $P(A)$.
- The original (electric) theory can be duality transformed to magnetic variables directly in the path integral formalism.

$SL(2,\mathbb{Z})$

Electric-magnetic duality implements $S : \tau \to -\frac{1}{\tau}$.

Shift symmetry $T : \tau \to \tau + 4$ is also a symmetry of this theory.

So: a nontrivial subgroup of $SL(2,\mathbb{Z})$ acts on τ (It is denoted $\Gamma_0(4)$).

Duality also acts on moduli space. The explicit realization of the S transformation in the path integral related the coupling to the magnetic dual scalar field a_D as

$$\tau = \frac{\partial^2 P}{\partial A^2} = \frac{\partial a_D}{\partial a} \ .$$

Since T^n shifts τ by n we find the transformation properties

$$a_D \to a_D + n\, a \ , \ \ a \to a \ .$$

Thus the $SL(2,\mathbb{Z})$ generators S and T acts on the scalar fields (a_D, a) as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \ \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Weak Coupling

Perturbatively, the prepotential is completely determined by the anomaly (or equivalently the β function).

For the present case of $SU(2)$ gauge theory with adjoint matter

$$b = 3T(\mathbf{Ad}) - T(\mathbf{Ad}) = 2T(\mathbf{Ad}) = 2N = 4 \ ,$$

so the running holomorphic coupling (at scale $\mu = a$) is

$$\tau(\mu = a) = \tfrac{ib}{2\pi} \ln \tfrac{\mu}{\Lambda} = \tfrac{2i}{\pi} \ln \tfrac{a}{\Lambda} \ .$$

Allowing for nonperturbative k-instantons corrections (suppressed by $\Lambda^{bk} = \Lambda^{4k}$) and integrating $\tau = P''$ twice, we find the prepotential

$$P(A) = \tfrac{i}{2\pi} A^2 \ln \tfrac{A^2}{\Lambda^2} + A^2 \textstyle\sum_{k=1}^{\infty} p_k \left(\tfrac{\Lambda}{A}\right)^{4k} \ .$$

Weak Coupling

Weak coupling corresponds to large value of the VEV

$$u = \mathrm{Tr}\phi^2 \simeq \tfrac{1}{2}a^2 \ ,$$

$$\Rightarrow a = \sqrt{2u} \ , \ \ a_D = \tfrac{\partial P}{\partial a} = \tfrac{2ia}{\pi} \ln\left(\tfrac{a}{\Lambda}\right) + \tfrac{2ia}{\pi} \ .$$

(Reminder: $P(A) = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2} + \ldots$)

Traversing a loop around large $|u|$: $u \to e^{2\pi i} u$ we find

$$\begin{aligned} \ln a &\to \ln a + i\pi \ , \\ a &\to -a \ , \\ a_D &\to -a_D + 2a \ . \end{aligned}$$

The monodromy matrix acting on $(a_D, a)^T$ at ∞ is

$$\mathcal{M}_\infty = -T^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \ .$$

Singular Points

Reminder: to avoid $\mathrm{Im}\,\tau = 4\pi/g^2$ going negative somewhere, there must be at least two more singular points, with monodromies that do not commute with $\mathcal{M}_\infty$.

Simplest model: a state with electric charge $(n_m, n_e) = (0, 1)$ becomes massless near a singular point $u_j \Rightarrow a(u) \approx c_j(u - u_j)$ near u_j.

Near this point, the $U(1)$ gauge coupling flows to zero in the IR with β-function:

$$\tau(a(u)) \approx -\frac{i}{\pi} \ln \frac{a(u)}{\Lambda} \ .$$

Monodromy under $(u - u_j) \to e^{2\pi i}(u - u_j)$:

$$a_D(u) \to a_D(u) + 2a(u) \ , \ \ a(u) \to a(u) \ ,$$

$\Rightarrow$ monodromy matrix around this singular point:

$$\mathcal{M}_{u_j} = T^2 \ .$$

Singular Points

Another model: a dyon with charge (n_m, n_e) (n_m, n_e mutually prime) becomes massless at $u = u_k \Rightarrow$ find an $SL(2,\mathbb{Z})$ transformation D_{u_k} that converts this to charge $(0,1)$

$$\begin{aligned}
\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} &= D_{u_k} \begin{pmatrix} a_D \\ a \end{pmatrix} = \begin{pmatrix} \alpha a_D + \beta a \\ \gamma a_D + \delta a \end{pmatrix} , \\
\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= D_{u_k}^{-1} \begin{pmatrix} n_m \\ n_e \end{pmatrix} = \begin{pmatrix} \delta n_m - \gamma n_e \\ -\beta n_m + \alpha n_e \end{pmatrix} .
\end{aligned}$$

Result: monodromy in the original variables is

$$\begin{aligned}
\mathcal{M}_{u_k} = D_{u_k}^{-1} T^2 D_{u_k} &= \begin{pmatrix} 1 + 2\gamma\delta & 2\delta^2 \\ -2\gamma^2 & 1 - 2\gamma\delta \end{pmatrix} , \\
&= \begin{pmatrix} 1 + 2n_e n_m & 2n_e^2 \\ -2n_m^2 & 1 - 2n_e n_m \end{pmatrix} .
\end{aligned}$$

Two Singular Points

Simplest possibility: two singular points at finite u, related by the $\mathbb{Z}_2$ symmetry $u \to -u$.

Assumption: a massless monopole with charge $(1, 0)$ becomes massless at u_1 and some other excitation become massless at u_{-1}.

Compute first $\mathcal{M}_{u_1}$ (for a monopole) and then $\mathcal{M}_{u_{-1}}$ from

$$\mathcal{M}_{u_1}\mathcal{M}_{u_{-1}} = \mathcal{M}_\infty \ .$$

Result:

$$\mathcal{M}_{u_1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \ , \ \mathcal{M}_{u_{-1}} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \ ,$$

$\Rightarrow$ massless state at u_{-1} is a dyon with charge $(-1, 1)$ (or $(1, -1)$, this is physically equivalent through the $SL(2, \mathbb{Z})$ transformation $-I$).

Since $\mathcal{M}_\infty$ changes the electric charge by 2, we can obtain all the classical dyons with charge $(\pm 1, 2n+1)$ from phase redefinitions of u.

In summary, the monodromy around the monopole point:

$$\mathcal{M}_{u_1} = S^{-1}T^2S \ , \quad D = S \ .$$

And the monodromy around the dyon point:

$$\mathcal{M}_{u_{-1}} = (ST^{-1})^{-1}T^2ST^{-1} \ , \quad D = TS^{-1} \ .$$

These are same as the monodromies previously determined for $SO(N)$ with $N-2$ flavors.

The Seiberg-Witten Curve

The Seiberg-Witten curve is the elliptic curve with singularities in the u-plane at the correct three points:

$$y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u) \ .$$

In the next lecture we will analyze this curve.