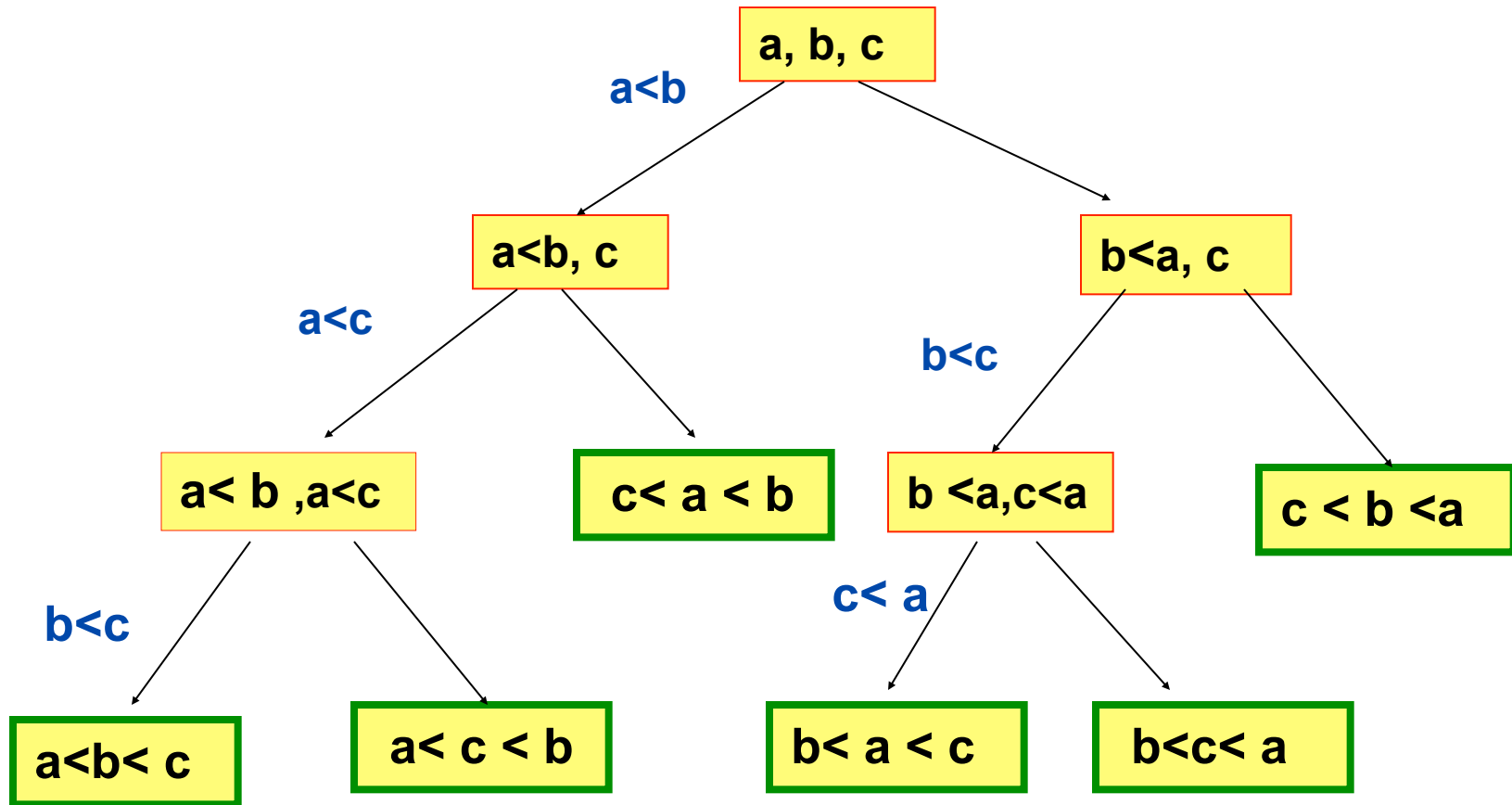


Searching and Sorting

- ❑ Searching
 - ❑ Linear $O(N)$
 - ❑ bisection $O(\log(N))$
 - ❑ dictionary $O(\log(\log(N)))$
- ❑ Sorting
 - ❑ Insertion, bubble, selection $O(N^2)$ CRLS: 2.1
 - ❑ Merge, Quick, Heap $O(N \log(N))$ CRLS: 2.2,
Proof: $\Omega(N^2)$ near neighbor exchange
- ❑ Proof: $\Omega(N \log(N))$ Comparison search
- ❑ Median (or k quick selection) Problem CLRS: 9
 - ❑ Bin (Count), Radix, Bucket $O(N)$ CLRS: 8
- ❑

Decision Tree

Proof of $\Omega(N \log(N))$



Binary decisions: $3! = 6$ possible outcomes. Longest path: $\log(3!)$

Searching: “Why Sort at All?”

■ int a[0], a[1], a[2], a[3], a[m], a[2], a[N-1]

Three Algorithms:

■ *Linear Search* →

O(N)

 (after Sorting)

■ *Bisection Search* →

O(log(N)).

■ *Dictionary Search* →

O(log[log[N]])

Bisection Search of Sorted List

■ int a[0], a[1], a[2], a[3], a[m],

a[N-2], a[N-1]

↑
i

↑
j

```
i = 0; j = N-1; m = N/2
while(b != a[m] && i != j){
    if(b > a[m]) i = m+1;
    if(b < a[m]) j = m-1;
    m = (j-i)/2 + i;
}
if(b == a[m]) "found it" else "not found"
```

Choose
mid point

$$T(N) = T(N/2) + c_0 \quad \rightarrow \quad T(N) \gg \text{Log}(N)$$

Dictionary: Sorting a nearly Uniform Sequence

■ `int a[0], a[1], a[2], a[3], ..., a[m], ..., a[2], a[N-1]`

↑
i

↑
j

Dictionary: Same code EXCEPT

estimate location of b

x = fractional distance ($0 < x < 1$)

$$x = (b - a[i]) / (a[j] - a[i]) ;$$

$$m = x(j - i) + i ;$$

$$N \rightarrow N^{\frac{1}{2}} \rightarrow N^{\frac{1}{4}} \rightarrow N^{\frac{1}{8}} \dots \rightarrow N^{\frac{1}{2^n}} = 1 \quad \text{or} \quad n = \log_2(\log_2(N))$$

$$T(N) \simeq T(N^{1/2}) + c_0 \quad \text{or with } N = 2^n, \quad T(n) = T(n/2) + c_0$$

$$\text{Master Equ.} \implies T(n) = O(\log(n)) = O(\log(\log_2(N)))$$

■ Extra Knowledge Helps: % Error » $1/N^{1/2}$

Insertion Sort --- Deck of Cards

- Insertion Sort(a[0:N-1]):
 for (i=1; i < n; i ++)
 for (j = i; (j>0) && (a[j]<a[j-1])); j--)
 swap a[j] and a[j-1] ;

Worst case $\Theta(N^2)$ number of “swaps” (i.e. time)

Outer loop trace for Insertion Sort

	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	
■	(Swaps)								
■	6	5	2	8	3	4	7	1	(1)
	5	← → 6							
■	5	6	2	8	3	4	7	1	
	(2)								
		2	← → 6						
	2	← → 5							
■	2	5	6	8	3	4	7	1	(0)
■	2	5	6	8	3	4	7	1	(3)
■	2	3	5	6	8	4	7	1	(3)
■	2	3	4	5	6	8	7	1	(1)
■	2	3	4	5	6	7	8	1	(7)

Bubble Sort --- Sweep R to L

- *Bubble Sort($a[0:N-1]$):*
 for $i=0$ to $n-1$
 for $j = n-1$ to $i + 1$
 if $a[j] < a[j-1]$ then
 swap $a[i]$ and $a[j]$

Worst case $\Theta(N^2)$ swaps (time)

Outer loop trace for Bubble Sort

	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	
■	(Swaps)								
■	6	5	2	8	3	4	7	1	(7)
■	1		6	5	2	8	4	7	(3)
■	1	2		6	5	3	8	4	(3)
■	1	2	3		6	5	4	8	(3)
■	1	2	3	4		6	5	7	(1)
■	1	2	3	4	5		6	7	(0)
■	1	2	3	4	5	6		7	(0)
■	1	2	3	4	5	6	7	8	(17)

◆ NOTE SAME # OF SWAPS? WHY?

Average Number of $N(N-1)/4$ swaps

- **Best Case:** *sorted order* → 0 swaps
- **Worst Case:** *reverse order* → $N(N-1)/2$ swaps
since $1 + 2 + \dots + N-1 = N(N-1)/2$
- **Average Case:** *Pair up each of the $N!$ permutations with its reverse order → Every pair must swap in one or the other: Thus average is half of all swaps → $(1/2) N(N-1)/2$ q.e.d.*

Selection Sort --- (Bubble only the index)

- Selection Sort($a[0:N-1]$):
 for $i=1$ to $n-2$
 { $min = i$
 for $j = n-1$ to $i + 1$
 if $a[j] < a[min]$ **then**
 $min = j$;
 swap $a[i]$ and $a[min]$;
 }

worst case $\Theta(N)$ swaps + $\Theta(N^2)$ comparisons

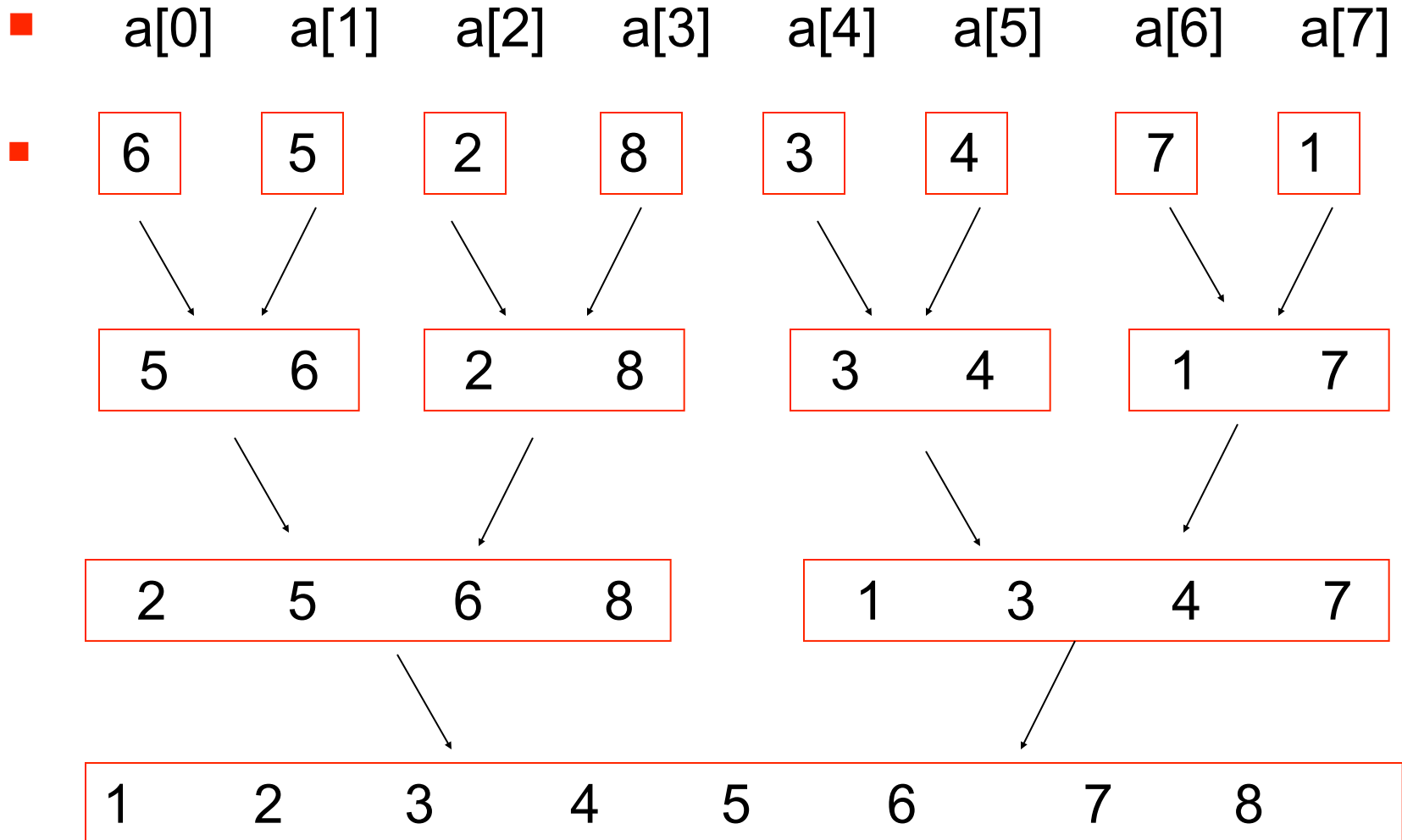
Outer loop trace for Selection Sort

	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]
■	6	5	2	8	3	4	7	1
(Swaps)								
(1)	1							6
■	1	5	2	8	3	4	7	6
(1)								
■		2			5			
■	1	2		5	8	3	4	7
(1)								
■	1	2	3		8	5	4	7
(1)								
■	1	2	3	4		5	8	7
(0)								
■	1	2	3	4	5		6	8

Merge Sort: *Worst Case* $\Theta(N \log(N))$

```
void mergesort(int a[ ], int l, int r)
    if (r > l) {
        m = (r+l)/2;
        mergesort(a, l, m);
        mergesort(a, m+1, r);
        for (i = l; i < m+1; i++) b[i] = a[i];
        for (j = m; j < r; j++)    b[r+m-j] = a[j+1];    // reverse
        for (k = l; k <= r; k++)
            a[k] = (b[i] < b[j]) ? b[i++] : b[j--]; }
```

Outer loop trace for Merge Sort



Lower Bound Theorem for Comparison Sort

Proof: Compute the maximum depth **D** of decision tree?

- Need $N!$ leaves to get all possible outcomes of a sorting routine.

$$1 \text{ (R)} \quad 2 \text{ (R)} \quad 4 \text{ (R)} \quad 8 \text{ (R)} \quad \bullet \bullet \bullet \text{ (R)} \quad 2^D$$

- Each level at most doubles:

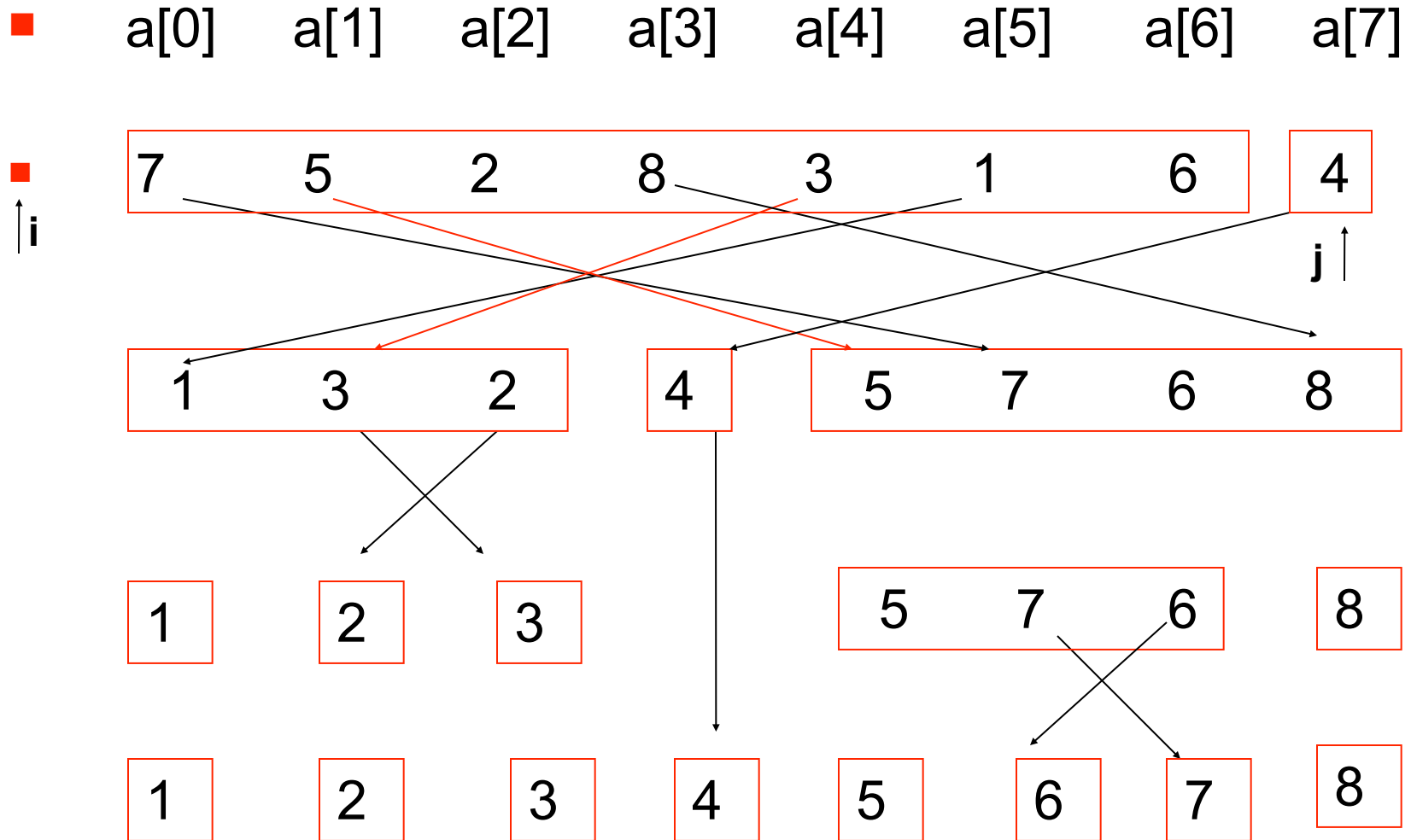
- Consequently for D levels: $N! \leq 2^D \Rightarrow D \geq \log_2(N!)$

$$\Rightarrow T(N) = \Omega(D) = \Omega(\log_2(N!)) = \Omega(N \log_2(N))$$

$$\textit{Information} = \log_2(N!) \cong N \log_2(N)$$

Number of bits to encode any (initial) state is information (- Entropy)¹⁹

Outer loop trace for Quick Sort (i moves before j)



Quick Sort: original Hoare partition

```
void quicksort(int a[ ], int l, int r)
{
    if (r > l){
        v = a[r]; i = l-1; j = r;
        for (;;){ while (a[++i] < v);      // move first i to right
                   while (a[--j] > v);    // then mover j left
                   if (i >= j) break ;
                   swap(&a[i], &a[j]); }
        swap(&a[i], &a[r]);                // move pivot in to center
        quicksort(a, l, i-1);
        quicksort(a, i+1, r);
    }
}
```

Quick Sort: *My Implementation*

```
void quicksort(int a[], int l, int r)
{
    int i, j, v;
    if (r > l)
    {
        v = a[r]; i = l-1; j = r;
        for (;;)
        {
            while (a[++i] < v && i >= j) ; // Move i first from left
            while (a[--j] >= v && i >= j) ; // Move j second from right
            if (i >= j) break;
            swap(&a[i], &a[j]); } // swap root to divide left and right sublist
        swap(&a[i], &a[r]);
        quicksort(a, l, i-1);    quicksort(a, i+1, r);
    }
}
```

Quick Sort: CLRS:7 Implementation

```
void quicksort(int a[ ], int l, int r)
{
    if (r > l){
        v = a[r]; i = l-1;
        for (int j= l ; j<r ; j++){
            if (a[j] <= v) {
                i +=1 ;           // move first i to right
                swap(&a[i], &a[j]) }
            swap(&a[i], &a[r]);    // move pivot in to center
        }
        quicksort(a, l, i-1);
        quicksort(a, i+1, r);
    }
}
```

Very cute: See CLRS page 172 Figure 7.1

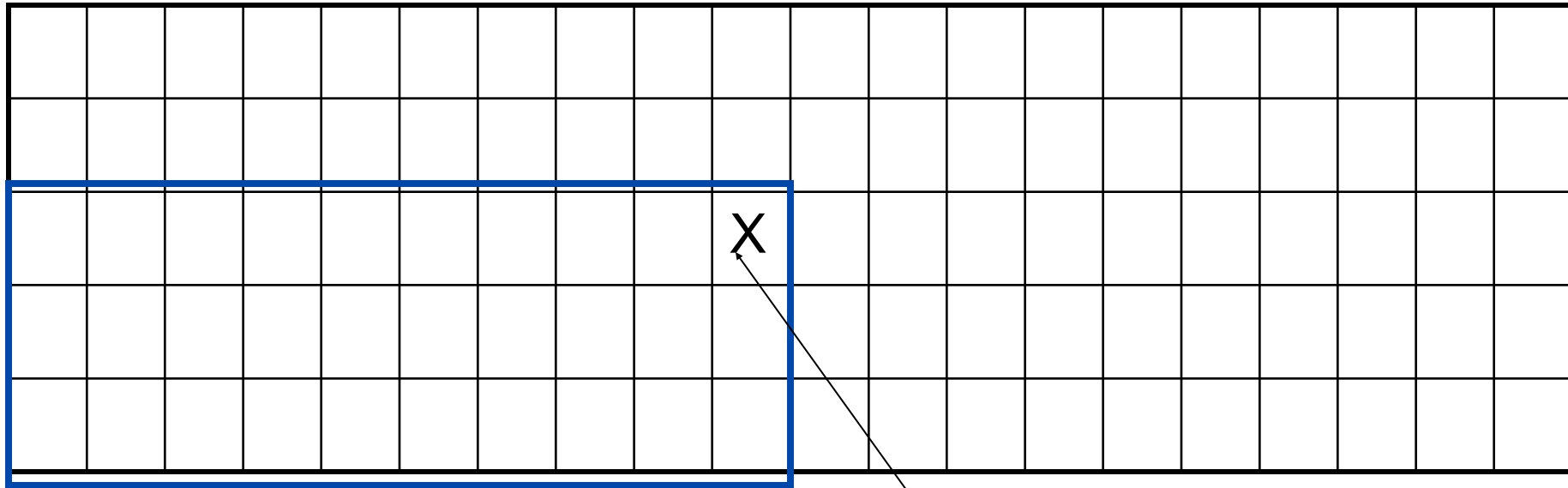
- Worst Case (choose smallest as pivot!):
 - ◆ $T(N) = T(N-1) + c N \rightarrow T(N) = O(N^2)$
- Best Case:(choose median as pivot)
 - ◆ $T(N) = 2 T(N/2) + cN \rightarrow T(N) = O(N \log(N))$
- Average Case:
 - ◆ $T(N) = 2[T(0) + T(1) + \dots + T(N-1)]/N + c N$
 - $T(N) = O(N \log (N))$
 - Using Calculus if you are lazy! ($x = N$)

$$xT(x) \simeq 2 \int_0^x dy T(y) + cx^2 \quad x \frac{dT(x)}{dx} = T(x) + 2cx \Rightarrow T(x) = 2cx \log(x) \quad 23$$

5 row of N/5 Columns

larger

larger



$\frac{3}{5} * \frac{1}{2} N$ Boxes smaller than X

Exact Medium of Middle Row

Can do better-- Worst Case $O(N \lg(N))$!

- *Approximate Media Selector for pivot v:*
 - ★ *Partition in 5 rows of $N/5$*
 - ★ *Sort each column*
 - ★ *Find (Exact) Medium of Middle list!*
- *Result there are $(3/5)(1/2)N$ elements smaller than pivot v*
- *K-th find is $O(N)$ --- Double recursion!*
 - ★ *Sort of $N/5$ col $O(N)$*
 - ★ *Find media of $T(N/5)$*
 - ★ *Find k-th in $T(7N/10)$ at worst*
- $T(N) < C_0 * (N/5) + T(N/5) + T(7N/10)$
 - ★ *Try solution: $T = C N$*
 - ★ $C(N - N/5 - 7N/10) = C N/10 < C_0 N/5$
 - ★ $C < 2 C_0$

$O(N)$: Bin, Radix & Bucket

- BIN Sort – make histogram (Counting sort CLRS 8.2):
 - ◆ N integers $0 < a[i] < M$ in the range $v = 0, \dots, M-1$.
 - ◆ Count number of occurrences in $a[i]$

```
for(v=0; v<M; v++) bin[ v ] =0;
for(i=0; i<N; i++) bin[a[i]] ++;
j=0;
for(v=0; v< M; v++) {
    for(i=0; i<bin[v]; i++)
        a[ j ] = v; j++;
}
```

→ $O(M + N)$ so if $M \gg N$ it is $O(N)$

Beating $N \log N$: Radix Sort (IBM Card Sorter!)

- Represent integers in $a[i]$ in base B : $n_0 + n_1 B + n_2 B^2 + \dots + n_p B^p$
- Sort into buckets by **low digits** first: n_0 , then n_1 , etc.

Queues: $B = 10$

Example: 64, 8, 216, 512, 27, 729, 0, 1, 343, 125

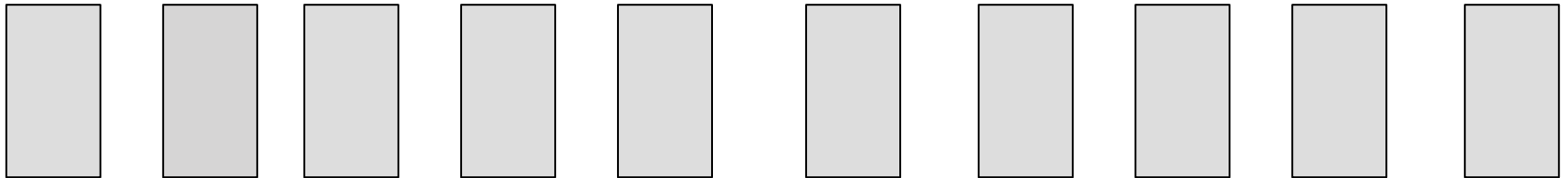
Bucket:	0	1	2	3	4	5	6	7	8	9
#1	0	1	512	343	64	125	216	27	8	729
	8 1 0	216 512	729 27 125		343		64			
#3	64 27 8 1 0	125	216	343		512		729		

$O(N P)$ where $B^P = N$ or $P = \log(N) / \log(B) = O(1)$

Bucket Sort

- Choose B Buckets as bins for **high digits** of $a[i]$
 - ♦ place N numbers in $a[i]$ in buckets
 - ♦ Sort average of N/B elements in each bucket.
 - ♦ CRLS uses insertion sort

Linked list:



Bucket: 0 1 2 3 4 5 6 7 8 9

$$\Rightarrow O(N + B(N/B) \log(N/B)) = O(N + N \log(N/B))$$
$$B = O(N)$$

Shell Sort:

$$O(N^\gamma) \quad 1 < \gamma < 2$$

Use insertion sort skip lists $a[i] < a[i+h]$ in descending order

$$1 = h_1 < h_2 < \dots < h_k < \dots < N$$

```
void shellsort(int a[], int N)
```

```
    int i, j, h, v;
```

```
    for (h = 1; h <= N/9; h = 3*h+1);
```

```
    for (; h > 0; h = h/3)
```

```
        for (i = h; i < N; i++)
```

```
            { v = a[i];
```

```
              for (j = i; (j >= h) && (a[j-h] > v)
```

```
                  { j -= h;
```

```
                    a[j] = a[j-h];
```

```
                  }
```

```
            a[j] = v;
```

```
        }
```

//Kunth 1969

// Find Largest h

// Descending skip distance

// Insertion sort

Properties of Shell Sort

- *Shell's sequence:*
 - ♦ $h = 1, 2, 4, 8, \dots, 2^N \rightarrow$ Worst Case: $O(N^2)$
- *Hibbards sequence:*
 - ♦ $1, 3, 7, 15, \dots, 2^k - 1 \rightarrow$ Average Case: $O(N^{5/4})$,
 \rightarrow Worst: $O(N^{3/2})$
- *Theorem:*

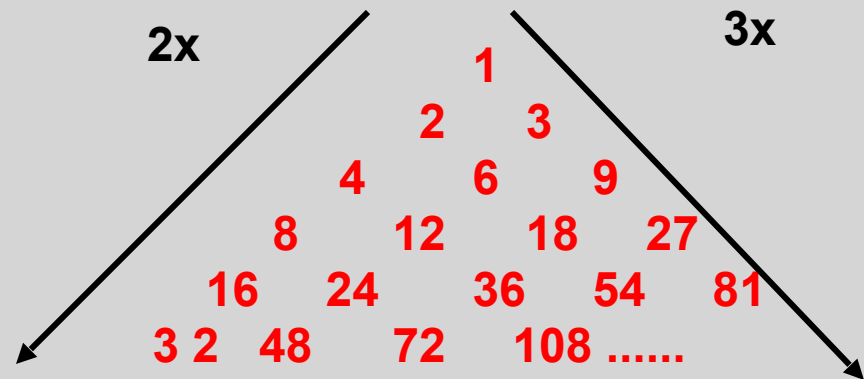
A “ $h = p$ ” sorted list remains p sorted after a “ $h = q$ ” sort!
(Proof is hard -- for me anyway)

Cute increment : $T(N) = \Omega(N \log^2(N))$ for Shell Sort!

- Each sort finds at most one *adjacent* element, $a[(i-1)h]$, *out order!*
 - Each pass $O(N)$

..... $a[(i-5)h]$, $a[(i-4)h]$, $a[(i-3)h]$, $a[(i-2)h]$, $a[(i-1)h]$, $a[ih]$
All other are $(2n + 3m)h$ away! (e.g $5 = 2 + 3$).
- The number of increments h 's smaller than $O(N)$ is the area:
 $O(\log(N) \log(N))$. q.e.d.

h-triangle: $2x/3x$ for left/right child →



$$x = \log_2(N) \text{ and } y = \log_3(N) \rightarrow \text{Area} = x * y/2 = O(\log(N)*\log(N))$$

Median Finding: Quick Select

- Median is the element $a[m]$ so that half is less/equal
- Generalize to finding k -th smallest in set S
- **Quick(S, k):** $|S|$ = size of S
 - ◆ If $|S| = 1$, the $k = 1$ in S
 - ◆ Pick pivot $v \in S$ & Partition $S - \{v\}$ into S_L & S_H
 - ◆ If $k < |S_L| + 1$ then k -th $\in S_L$: **Quick(S_L, k)**
 - ◆ If $k = |S_L| + 1$ k -th is v : exit
 - ◆ If $k > |S_L| + 1$ then k -th $\in S_R$: **Quick($S_R, k - |S_L| - 1$)**

Now: $T(N) = O(N)$ is average performance

$$T(N) = [T(0) + T(1) + \dots + T(N-1)]/N + cN$$

$$xT(x) \simeq \int_0^x dy T(y) + cx^2 \quad x \frac{dT(x)}{dx} = 2cx \Rightarrow T(x) = 2cx_{31}$$