EC504 Homework 1

Problem 1a : Exercise 3.1-4

Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$

Function $f(n) = O(g(n))$ if:

$$f(n) = O(g(n)) \iff \exists c > 0, n_0 \; s.t. \; f(n) \leq cg(n) \; \forall n \geq n_0$$

Equivalently, we can use the limit definition:

$$f(n) = O(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

Using the first definition, we see the first claim holds:

$$f(n) \leq cg(n) \implies 2^{n+1} \leq c\,(2^n)$$

Taking $c = 2$:

$$2^{n+1} \leq 2 \times (2^n) = 2^{n+1}$$

$$2^{n+1} \leq 2^{n+1} \; \forall n \geq 1$$

We see that for $n \geq n_0 = 1$, the above inequality holds. Thus, $2^{n+1} = O(2^n)$.
Equivalently, we arrive at the same result using the limit definition:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{2^n}{2 \times 2^n} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

Thus, $2^{n+1} = O(2^n)$.
However, the second claim does not hold - using the first definition, assume we pick $c = 2^x,\; x \in \mathbb{R}$. Then:

$$2^{2n} \leq c\,(2^n) = 2^x \times 2^n = 2^{x+n}$$

$$2^{2n} \leq 2^{x+n}$$

This inequality only holds if $2n \leq x + n$, or if $x \geq n$. But this arrives at a contraction; if $n > x$, then the inequality will no longer hold, meaning that choice of c was invalid. We can repeat this argument - for any selected c, it is always possible to find a point n_v where the inequality is violated; thus $2^{2n} \neq O(2^n)$.
Equivalently, we arrive at the same result using the limit definition:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{2^{2n}}{2^n} = \lim_{n\to\infty} \frac{\left(2^2\right)^n}{2^n} = \lim_{n\to\infty} \frac{4^n}{2^n} = \lim_{n\to\infty} 2^n = \infty$$

Thus, $2^{2n} \neq O(2^n)$.

Problem 1a : Problem 3.2-7

Prove by induction that the i^{TH} Fibonacci number satisfies the equality:

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$$

Recall:

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

Base case: We know that $F(0) = 0$ and $F(1) = 1$. We sill show that the claim holds for both $i = 0$ and $i = 1$:

$$F_0 = \left(\frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}\right)\bigg|_{i=0} = \frac{1-1}{\sqrt{5}}$$

$$\therefore F_0 = 0 \quad \square$$

$$F_1 = \left(\frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}\right)\bigg|_{i=1} = \frac{\phi - \hat{\phi}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right)$$

$$\therefore F_1 = 1 \quad \square$$

Induction step: Assume the claim holds for i and $i-1$ - our induction hypothesis is as follows:

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}, \quad F_{i-1} = \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}}$$

Show that if the claim holds for i and $i-1$, the claim must hold for $i+1$, namely:

$$F_{i+1} = \frac{\phi^{i+1} - \hat{\phi}^{i+1}}{\sqrt{5}}$$

By definition of the Fibonacci numbers:

$$F_{i+1} = F_i + F_{i-1}$$

Using the induction hypothesis:

$$F_{i+1} = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} + \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}}$$

$$F_{i+1} = \frac{1}{\sqrt{5}}\left(\phi^i + \phi^{i-1}\right) - \frac{1}{\sqrt{5}}\left(\hat{\phi}^i + \hat{\phi}^{i-1}\right)$$

Using $\phi = \frac{1+\sqrt{5}}{2}$:

$$\phi^i + \phi^{i-1} = \left(\frac{1+\sqrt{5}}{2}\right)^i + \left(\frac{1+\sqrt{5}}{2}\right)^{i-1}$$

$$\phi^i + \phi^{i-1} = \left(\frac{1+\sqrt{5}}{2}\right)^i \left[1 + \frac{2}{1+\sqrt{5}}\right]$$

$$\phi^i + \phi^{i-1} = \phi^i \left(\frac{1+\sqrt{5}+2}{1+\sqrt{5}}\right) = \phi^i \left(\frac{3+\sqrt{5}}{1+\sqrt{5}}\right)$$

$$\phi^i + \phi^{i-1} = \phi^i \left(\frac{3+\sqrt{5}}{1+\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{1-\sqrt{5}}\right) = \phi^i \left(\frac{3-3\sqrt{5}+\sqrt{5}-5}{1-\sqrt{5}+\sqrt{5}-5}\right) = \phi^i \left(\frac{-2-2\sqrt{5}}{-4}\right) = \phi^i \left(\frac{-2(1+\sqrt{5})}{-4}\right)$$

$$\phi^i + \phi^{i-1} = \phi^i \times \left(\frac{1+\sqrt{5}}{2}\right) = \phi^i \times \phi$$

$$\therefore \phi^i + \phi^{i-1} = \phi^{i+1}$$

Similarly, using $\hat{\phi} = \frac{1-\sqrt{5}}{2}$:

$$\hat{\phi}^i + \hat{\phi}^{i-1} = \left(\frac{1-\sqrt{5}}{2}\right)^i + \left(\frac{1-\sqrt{5}}{2}\right)^{i-1}$$

$$\hat{\phi}^i + \hat{\phi}^{i-1} = \left(\frac{1-\sqrt{5}}{2}\right)^i \left[1 + \frac{2}{1-\sqrt{5}}\right]$$

$$\hat{\phi}^i + \hat{\phi}^{i-1} = \hat{\phi}^i \left(\frac{1-\sqrt{5}+2}{1-\sqrt{5}}\right) = \hat{\phi}^i \left(\frac{3-\sqrt{5}}{1-\sqrt{5}}\right)$$

$$\hat{\phi}^i + \hat{\phi}^{i-1} = \hat{\phi}^i \left(\frac{3-\sqrt{5}}{1-\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{1+\sqrt{5}}\right) = \hat{\phi}^i \left(\frac{3+3\sqrt{5}-\sqrt{5}-5}{1+\sqrt{5}-\sqrt{5}-5}\right) = \hat{\phi}^i \left(\frac{-2+2\sqrt{5}}{-4}\right) = \hat{\phi}^i \left(\frac{-2(1-\sqrt{5})}{-4}\right)$$

$$\hat{\phi}^i + \hat{\phi}^{i-1} = \hat{\phi}^i \times \left(\frac{1-\sqrt{5}}{2}\right) = \hat{\phi}^i \times \hat{\phi}$$

$$\therefore \hat{\phi}^i + \hat{\phi}^{i-1} = \hat{\phi}^{i+1}$$

Putting it all together:

$$F_{i+1} = \frac{1}{\sqrt{5}}\left(\phi^i + \phi^{i-1}\right) - \frac{1}{\sqrt{5}}\left(\hat{\phi}^i + \hat{\phi}^{i-1}\right) = \frac{1}{\sqrt{5}}\phi^{i+1} - \frac{1}{\sqrt{5}}\hat{\phi}^{i+1}$$

$$\therefore F_{i+1} = \frac{\phi^{i+1} - \hat{\phi}^{i+1}}{\sqrt{5}} \quad \square$$

Therefore, by induction, we have shown that the i^{TH} Fibonacci number can be expressed as $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$.

Problem 1a : Problem 3-2

Indicate, for each pair of expressions (A, B) in the table below whether A is $\{O,\ o,\ \Omega,\ \omega,\ \Theta\}$ of B. Assume that $k \geq 1$, $\epsilon > 0$, and $c > 1$.

A	B	O	o	Ω	ω	Θ
$\lg^k n$	n^ϵ	✓	✓	×	×	×
n^k	c^n	✓	✓	×	×	×
$\sqrt{n}$	$n^{\sin n}$	×	×	×	×	×
2^n	$2^{n/2}$	×	×	✓	✓	×
$n^{\lg c}$	$c^{\lg n}$	✓	×	✓	×	✓
$\lg n!$	$\lg n^n$	✓	×	✓	×	✓

Recalling the graphical relationship between the various asymptotic notations:

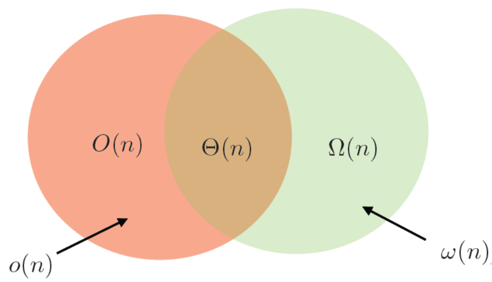

Figure 1: Relationship between asymptotic notations

We can write the formal definitions:

$$f(n) \in O\big((g(n))\big) \iff \exists c > 0,\ n_0\ s.t.\ f(n) \leq cg(n)\ \forall n \geq n_0$$

$$f(n) \in o\big((g(n))\big) \iff \forall c > 0,\ n_0\ s.t.\ f(n) < cg(n)\ \forall n \geq n_0$$

$$f(n) \in \Omega\big((g(n))\big) \iff \exists c > 0,\ n_0\ s.t.\ f(n) \geq cg(n)\ \forall n \geq n_0$$

$$f(n) \in \omega\big((g(n))\big) \iff \forall c > 0,\ n_0\ s.t.\ f(n) > cg(n)\ \forall n \geq n_0$$

We can write the limit definitions:

$$f(n) \in O\big((g(n))\big) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$f(n) \in o\big((g(n))\big) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$f(n) \in \Omega\big((g(n))\big) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

$$f(n) \in \omega\big((g(n))\big) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

$$f(n) \in \Theta\big(g(n)\big) \iff f(n) \in O\big(g(n)\big) \wedge f(n) \in \Omega\big(g(n)\big)$$

It is important to note that if $f(n) \in o\big(g(n)\big)$, then $f(n) \in O\big(g(n)\big)$; this follows from the limit definitions above. Similarly, if $f(n) \in \omega\big(g(n)\big)$, then $f(n) \in \Omega\big(g(n)\big)$.

$\lg^k n \ vs. \ n^\epsilon$:

Let $n = 2^m, \ m \geq 0$. Then we can express the limit as:

$$\lim_{n\to\infty} \frac{\lg^k n}{n^\epsilon} = \lim_{n\to\infty} \frac{(\lg n)^k}{n^\epsilon} = \lim_{m\to\infty} \frac{(\lg 2^m)^k}{(2^m)^\epsilon}$$

Then:

$$\lim_{m\to\infty} = \frac{(m \lg 2)^k}{2^{m\epsilon}} = \lim_{m\to\infty} \frac{m^k}{2^{m\epsilon}} = \lim_{m\to\infty} \frac{m^k}{(2^\epsilon)^m}$$

Let $\alpha = 2^\epsilon$; since $\epsilon > 0$, $\alpha = 2^m > 1$:

$$\lim_{m\to\infty} \frac{m^k}{\alpha^m} = 0$$

Thus, $\lg^k n \in o\left(n^\epsilon\right)$; since $\lg^k n \in o\left(n^\epsilon\right)$, we also have that $\lg^k n \in O\left(n^\epsilon\right)$.

$n^k \ vs. \ c^n$:

Taking the limit:

$$\lim_{n\to\infty} \frac{n^k}{c^n}$$

Since $c > 1$, we note the same asymptotic behavior as the previous problem:

$$\lim_{n\to\infty} \frac{n^k}{c^n} = 0$$

Thus, $n^k \in o\left(c^n\right)$; since $n^k \in o\left(c^n\right)$, we also have that $n^k \in O\left(c^n\right)$.

$\sqrt{n} \ vs. n^{\sin n}$:

We cannot make any asymptotic comparison between the two functions. Recall that $\sqrt{n}$ is a strictly increasing function. Thus, if $\sqrt{n} \in O\big(g(n)\big)$, $\exists c > 0$ such that $g(n)$ is also strictly increasing for $n \geq n_0$. The function $g(n) = n^{\sin n}$ cannot satisfy this requirement; Let $N = \{n : n \geq n_0, \ \sin n = -1\}$[1]. Since $\sin n$ is a periodic function, there are an infinite number of points in N. However, for all points in N:

$$n^{\sin n} = n^{-1} = \frac{1}{n} \ \forall n \in N$$

However, $\sqrt{n} \geq \frac{1}{n} \ \forall n \geq 1$; we can always find an infinite number of points where $n \geq n_0$ such that $f(n) = \sqrt{n} > g(n) \ \forall n \in N$, violating the bound. We cannot use the constant c to help reverse the inequality, as we would need c to change for an infinite number of points; thus we would never be able to find a constant c to always satisfy the inequality $f(n) \leq cg(n) \ \forall n \geq n_0$. Therefore, $\sqrt{n} \notin O\left(n^{\sin n}\right)$. Since $\sqrt{n} \notin O\left(n^{\sin n}\right)$, we can immediately conclude that $\sqrt{n} \notin o\left(n^{\sin n}\right)$ and $\sqrt{n} \notin \Theta\left(n^{\sin n}\right)$.

We can repeat the argument, except for a lower bound - the argument proceeds in the exact same way, and we can conclude that $\sqrt{n} \notin \Omega\left(n^{\sin n}\right)$. Since $\sqrt{n} \notin \Omega\left(n^{\sin n}\right)$, we can immediately conclude that $\sqrt{n} \notin \omega\left(n^{\sin n}\right)$.

$2^n \ vs. \ 2^{n/2}$

Computing the limit, we see:

$$\lim_{n\to\infty} \frac{2^n}{2^{n/2}} = \lim_{n\to\infty} \frac{2^n}{\left(2^{1/2}\right)^n} = \lim_{n\to\infty} \left(\frac{2}{\sqrt{2}}\right)^n \lim_{n\to\infty} \left(\sqrt{2}\right)^n = \infty$$

[1] I am implicitly extending n to the reals here - if n is an integer then the equality will not hold, but the general structure of the argument still holds. It's conceptually easier to argue from the reals.

Thus, $2^n \in \omega\left(2^{n/2}\right)$; since $2^n \in \omega\left(2^{n/2}\right)$, we also have that $2^n \in \Omega\left(2^{n/2}\right)$.

$n^{\lg c}$ $vs.$ $c^{\lg n}$:

Let $\lg c = q$; then $c = 2^q$. Let $n = 2^x$. Then we can express $n^{\lg c}$ as:

$$n^{\lg c} = (2^x)^{\lg c} = \left(2^{\lg c}\right)^x = c^x$$

But since $n = 2^x$, we have $x = \lg n$. Thus:

$$n^{\lg c} = c^x = c^{\lg n}$$

$$\therefore n^{\lg c} = c^{\lg n}$$

Since $n^{\lg c} = c^{\lg n}$, we have $f(n) = g(n)$; thus, by the reflexivity property, $f(n) \in O\big(f(n)\big)$, $f(n) \in \Omega\big(f(n)\big)$ $f(n) \in \Theta\big(f(n)\big)$.

Thus, $n^{\lg c} \in O\left(c^{\lg n}\right)$, $n^{\lg c} \in \Omega\left(c^{\lg n}\right)$, and $n^{\lg c} \in \Theta\left(c^{\lg n}\right)$.

$\lg n!$ $vs.$ $\lg n^n$:

Using Stirling's approximation, we can express $n!$ as:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

Thus $\lg n!$ can be expressed as:

$$\lg n! = \lg\left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right] = \lg\left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right] + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\lg n! = \lg\sqrt{2\pi n} + \lg\left(\frac{n}{e}\right)^n + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\lg n! = \lg\left(2\pi n\right)^{1/2} + n\lg\left(\frac{n}{e}\right) + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\lg n! = \frac{1}{2}\lg\left(2\pi n\right) + n\left(\lg n - \lg e\right) + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\lg n! = \frac{1}{2}\left(\lg 2 + \lg\pi + \lg n\right) + n\lg n - n\lg e + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\lg n! = \frac{1}{2} + \frac{\lg\pi}{2} + \frac{1}{2}\lg n + n\lg n - n\lg e + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\therefore \lg n! = n\lg n + \frac{1}{2}\lg n - (\lg e)n + \left(\frac{1 + \lg\pi}{2}\right) + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

We can easily express $\lg n^n$ as $n\lg n$. Taking the limit:

$$\lim_{n\to\infty}\frac{\lg n!}{\lg n^n} = \lim_{n\to\infty}\frac{n\lg n + \frac{1}{2}\lg n - (\lg e)n + \left(\frac{1+\lg\pi}{2}\right) + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]}{n\lg n}$$

$$\lim_{n\to\infty}\frac{\lg n!}{\lg n^n} = \lim_{n\to\infty}\frac{n\lg n}{n\lg n} + \frac{1/2\lg n}{n\lg n} - \frac{(\lg e)n}{n\lg n} + \frac{(1 + \lg\pi)/2}{n\lg n} + \frac{\lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]}{n\lg n}$$

$$\lim_{n\to\infty}\frac{\lg n!}{\lg n^n} = 1 + \frac{1}{2n} - \frac{\lg e}{\ln n} + \frac{(1 + \lg\pi)/2}{n\lg n} + \frac{\lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]}{n\lg n}$$

$$\lim_{n\to\infty} \frac{\lg n!}{\lg n^n} = 1 + \lim_{n\to\infty} \frac{\lg\left[\left(1+\Theta\left(\frac{1}{n}\right)\right)\right]}{n \lg n}$$

By definition, for sufficiently large n, if $f(n) = \Theta\left(\frac{1}{n}\right)$, then $\exists c_2 > c_1 > 0,\ s.t.\ \frac{c_1}{n} \le f(n) \le \frac{c_2}{n}$. Thus:

$$\lg\left[1+\Theta\left(\frac{1}{n}\right)\right] \sim \lg\left(1+\frac{c}{n}\right) = \lg\left(\frac{n+c}{n}\right) = \lg(n+c) - \lg n$$

Using this result in the limit:

$$\lim_{n\to\infty} \frac{\lg n!}{\lg n^n} = 1 + \lim_{n\to\infty} \frac{\lg\left[\left(1+\Theta\left(\frac{1}{n}\right)\right)\right]}{n \lg n} = 1 + \lim_{n\to\infty} \frac{\lg(n+c) - \lg n}{n \lg n}$$

$$\therefore \lim_{n\to\infty} \frac{\lg n!}{\lg n^n} = 1$$

Thus, we can conclude that $\lg n! \in O(\lg n^n)$, $\lg n! \in \Omega(\lg n^n)$, and thus $\lg n! \in \Theta(\lg n^n)$.

Problem 1a : Problem 4.3-9

Solve the recurrence $T(n) = 3T(\sqrt{n}) + \log n$ by making a change of variables.

For simplicity, assume $\log n = \lg n$; let $n = 2^m$. Then:

$$T(n) = 3T(\sqrt{n}) + \lg n \Longleftrightarrow T\left(2^m\right) = 3T\left(\sqrt{2^m}\right) + \lg 2^m$$

$$T\left(2^m\right) = 3T\left(2^{m/2}\right) + m \lg 2$$

$$T\left(2^m\right) = 3T\left(2^{m/2}\right) + m$$

Let $\tilde{T}(m) = T\left(2^m\right)$; then we can write:

$$\tilde{T}\left(m\right) = 3\tilde{T}\left(\frac{m}{2}\right) + m$$

This recurrence relation fits the form of the master equation, with $a = 3$, $b = 2$, $f(m) = m$, and $k = 1$. Computing $\gamma = \frac{\ln a}{\ln b} = \log_2 3 > 1$, we see that $\gamma > k$; thus, we can immediately write the solution to the equation:

$$\tilde{T}(m) = \Theta\left(m^\gamma\right)$$

Recalling that $n = 2^m$, we can reverse the change of variables and find an expression for $T(n)$:

$$\tilde{T}(m) \in \Theta\left(m^\gamma\right)$$

$$\tilde{T}(\lg n) \in \Theta\left((\lg n)^\gamma\right)$$

Since $\tilde{T}(m) = T(2^m)$:

$$T\left(2^{\lg n}\right) \in \Theta\left((\lg n)^\gamma\right)$$

$$\therefore T(n) \in \Theta\left((\lg n)^\gamma\right)$$

Thus, $T(n) \in \Theta\left(\lg^\gamma n\right)$ with $\gamma = \log_2 3 > 1$.

Problem 1b

Place the following functions in order from asymptotically smallest to largest using $f(n) \in O(g(n))$ notation:

$$n^2 + 3n\log n + 5, \quad n^2 + n^{-2}, \quad n^{n^2} + n!, \quad n^{\frac{1}{n}}, \quad n^{n^2-1}, \quad \ln n, \quad \ln\ln n, \quad 3^{\ln n}, \quad 2^n$$

$$(1+n)^n, \quad n^{1+\cos n}, \quad \sum_{k=1}^{\log n} \frac{n^2}{2^k}, \quad 1, \quad n^2 + 3n + 5, \quad \log n!, \quad \sum_{k=1}^{n} \frac{1}{k}, \quad \prod_{k=1}^{n}\left(1 - \frac{1}{k^2}\right), \quad \left(1 - \frac{1}{n}\right)^n$$

Function $f(n) \in O(g(n))$ if $\exists c > 0$ such that:

$$f(n) \leq cg(n) \; \forall n \geq n_0$$

Equivalently, we can use the limit definition:

$$f(n) \in O(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$f_1(n) = n^2 + 3n\log n + 5$:

$$\lim_{n\to\infty} \frac{f_1(n)}{n^2} = \lim_{n\to\infty}\left(\frac{n^2 + 3n\log n + 5}{n^2}\right) = \lim_{n\to\infty}\left(1 + \frac{3\log n}{n} + \frac{5}{n^2}\right) = 1$$

$\therefore f_1(n) = n^2 + 3n\log n + 5 \in O(n^2)$

$f_2(n) = n^2 + n^{-2}$:

$$\lim_{n\to\infty} \frac{f_2(n)}{n^2} = \lim_{n\to\infty}\left(\frac{n^2 + n^{-2}}{n^2}\right) = \lim_{n\to\infty}\left(1 + \frac{1}{n^4}\right) = 1$$

$\therefore f_2(n) = n^2 + n^{-2} \in O(n^2)$

$f_3(n) = n^{n^2} + n!$:

$$\lim_{n\to\infty} \frac{f_3(n)}{n^{n^2}} = \lim_{n\to\infty} \frac{n^{n^2} + n!}{n^{n^2}} = \lim_{n\to\infty}\left(1 + \frac{n!}{n^{n^2}}\right) = 1 + \lim_{n\to\infty} \frac{n!}{n^{n^2}}$$

By definition:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

$$n! = \prod_{i=1}^{n} i < \prod_{i=1}^{n} n < n^n < n^{n^2}$$

Therefore:

$$\lim_{n\to\infty} \frac{n!}{n^{n^2}} = 0$$

$\therefore f_3(n) = n^{n^2} + n! \in O(n^{n^2})$

$f_4(n) = n^{\frac{1}{n}}$:

$$\lim_{n\to\infty} \frac{f_4(n)}{1} = \lim_{n\to\infty}\left(\frac{n^{\frac{1}{n}}}{1}\right) = \lim_{n\to\infty} n^{1/n}$$

Evaluating this limit, we arrive at the indeterminate form (∞^0); we can solve by transforming the limit as follows:

$$\lim_{n\to\infty} f(x)^{g(x)} = \exp \lim_{n\to\infty} \frac{\ln f(x)}{\left(\frac{1}{g(x)}\right)} = \exp\left(\lim_{n\to\infty} g(x)\ln f(x)\right)$$

Therefore, with $f(x) = n,\ g(x) = \frac{1}{n}$:

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \exp\left(\lim_{n\to\infty}\left(\frac{1}{n}\right)(\ln n)\right) = \exp\left(\lim_{n\to\infty}\left(\frac{\ln n}{n}\right)\right) = \exp 0 = 1$$

$\therefore\ f_4(n) = n^{\frac{1}{n}} \in O(1)$

$f_5(n) = n^{n^2-1}$:

$$\lim_{n\to\infty} \frac{f_5(n)}{n^{n^2}} = \lim_{n\to\infty} \frac{n^{n^2-1}}{n^{n^2}} = \lim_{n\to\infty}\left(\frac{1}{n^{n^2}} \times \frac{n^{n^2}}{n}\right) = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\therefore\ f_5(n) = n^{n^2-1} \in O(n^{n^2})$

$f_6(n) = \ln n$:

$$\lim_{n\to\infty} \frac{f_6(n)}{\ln n} = \frac{\ln n}{\ln n} = 1$$

$\therefore\ f_6(n) = \ln n \in O(\ln n)$

$f_7(n) = \ln\ln n$:

$$\lim_{n\to\infty} \frac{f_7(n)}{\ln\ln n} = \frac{\ln\ln n}{\ln\ln n} = 1$$

$\therefore\ f_7(n) = \ln\ln n \in O(\ln\ln n)$

$f_8(n) = 3^{\ln n}$:
Let $\ln n = m$, and let $e^x = 3,\ x = \ln 3$; then:

$$3^{\ln n} = (e^x)^m = e^{xm} = (e^m)^x$$

$$3^{\ln n} = \left(e^{\ln n}\right)^x = n^x = n^{\ln 3} = n^\gamma,\ \gamma = \ln 3$$

$\therefore\ f_8(n) = 3^{\ln n} \in O\left(n^\gamma\right),\ \gamma < 2$

$f_9(n) = 2^n$:

$$\lim_{n\to\infty} \frac{f_{9(n)}}{2^n} = \lim_{n\to\infty} \frac{2^n}{2^n} = 1$$

$\therefore\ f_9(n) = 2^n \in O(2^n)$

$f_{10}(n) = (1+n)^n$:

$$\lim_{n\to\infty} \frac{f_{10(n)}}{n^n} = \lim_{n\to\infty} \frac{(1+n)^n}{n^n}$$

Using the binomial theorem to expand $(1+n)^n$:

$$(1+n)^n = \sum_{k=0}^{n} \binom{n}{k}(1)^{n-k}n^k = \sum_{k=0}^{n}\binom{n}{k}n^k = n^n + \sum_{k=0}^{n-1}\binom{n}{k}n^k = n^n + \sum_{k=0}^{n-1} c_k n^k$$

$$\lim_{n\to\infty} \frac{(1+n)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n^n + \sum_{k=0}^{n-1} c_k n^k}{n^n} \right) = \lim_{n\to\infty} \left(1 + \sum_{k=0}^{n-1} c_k \frac{n^k}{n^n} \right) = 1$$

$\therefore f_{10}(n) = (1+n)^n \in O(n^n)$

$f_{11}(n) = n^{1+\cos n}$:
Use the definition:

$$f(n) \in O(g(n)) \iff \exists c < 0 \; s.t. \; f(n) \leq cg(n) \; \forall n \geq n_0$$

By definition, $\cos n \in [-1, 1]$; thus, $(1 + \cos n) \in [0, 2]$. Since $(1 + \cos n) \leq 2 \; \forall n \geq 1$, we can bound $n^{1+\cos n}$ accordingly: $n^{1+\cos n} \leq n^2 \; \forall n \geq 1$.
$\therefore f_{11}(n) = n^{1+\cos n} \in O(n^2)$

$f_{12}(n) = \sum_{k=1}^{\log n} \frac{n^2}{2^k}$:
Let $b = \log n$; then the sum can be expressed as:

$$f_{12}(n) = \sum_{k=1}^{\log n} \frac{n^2}{2^k} = n^2 \sum_{k=1}^{b} \frac{1}{2^k} = n^2 \sum_{k=1}^{b} \left(\frac{1}{2}\right)^k = n^2 \sum_{k=1}^{b} \alpha^k$$

where $\alpha = 1/2$. Let $s_b = \sum_{k=1}^{b} \alpha^k$. Then:

$$s_b = \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{b-1} + \alpha^b$$

$$\alpha s_b = \alpha \left(\alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{b-1} + \alpha^b \right) = \alpha^2 + \alpha^3 + \ldots + \alpha^b + \alpha^{b+1}$$

$$\begin{aligned} s_b - \alpha s_b = \alpha &+ \alpha^2 + \alpha^3 + \ldots + \alpha^{b-1} + \alpha^b \\ &- \alpha^2 - \alpha^3 - \ldots - \alpha^{b-1} - \alpha^b - \alpha^{b+1} \end{aligned}$$

$$s_b - \alpha s_b = \alpha - \alpha^{b+1}$$

$$s_b(1 - \alpha) = \alpha \left(1 - \alpha^b \right)$$

$$\therefore s_b = \frac{\alpha \left(1 - \alpha^b\right)}{1 - \alpha}$$

Using $\alpha = 1/2$:

$$s_b = \frac{\alpha \left(1 - \alpha^b\right)}{1 - \alpha} = \frac{(1/2)\left(1 - (1/2)^b\right)}{(1 - 1/2)} = \frac{(1/2)\left(1 - \frac{1}{2^b}\right)}{(1/2)}$$

$$\therefore s_b = 1 - 2^{-b}$$

Thus, we can write the function $f_{12}(n)$ as:

$$f_{12}(n) = \sum_{k=1}^{\log n} \frac{n^2}{2^k} = n^2 s_b = \left(1 - 2^{-b}\right) n^2$$

Recall $b = \log n$; for $n > 1$, we have that $\log n > 0$, and and thus $b > 0$. Therefore, for $n > 1$, $2^{-b} < 1$, and thus the whole prefactor $\left(1 - 2^{-b}\right) < 1$. Therefore, for $n > 1$, $\left(1 - 2^{-b}\right) n^2 < n^2$.
$\therefore f_{12}(n) = \sum_{k=1}^{\log n} \frac{n^2}{2^k} \in O(n^2)$

$f_{13}(n) = 1 \implies f_{13}(n) \in O(1)$

$f_{14}(n) = n^2 + 3n + 5$:

$$\lim_{n\to\infty} \frac{f_{14}(n)}{n^2} = \lim_{n\to\infty} \left(\frac{n^2 + 3n + 5}{n^2}\right) = \lim_{n\to\infty} \left(1 + \frac{3}{n} + \frac{5}{n^2}\right) = 1$$

$\therefore f_{14}(n) = n^2 + 3n + 5 \in O(n^2)$

$f_{15}(n) = \log n!$:
We can express $\log n!$ as:

$$\log n! = \log (n \times n!) = \log n + \log ((n-1)!)$$

Using similar logic, we can express $\log ((n-1)!)$ as:

$$\log ((n-1)!) = \log ((n-1) \times (n-2)!) = \log (n-1) + \log ((n-2)!)$$

Substituting above:

$$\log n! = \log n + \log (n-1) + \log ((n-2)!)$$

We can carry out the process for each integer 1 to n, arriving at the sum:

$$\log n! = \sum_{m=1}^{n} \log m$$

Since $m \leq n$, we can upper-bound the right sum:

$$\log n! = \sum_{m=1}^{n} \log m \leq \sum_{m=1}^{n} \log n = \log n \times \sum_{m=1}^{n} 1$$

$$\therefore \log n! \leq n \log n \; \forall n \geq 1$$

$\therefore f_{15}(n) = \log n! \in O(n \log n)$

$f_{16}(n) = \sum_{k=1}^{n} \frac{1}{k}$:
Using equation A.7 in the text:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$$

Thus, taking the limit:

$$\lim_{n\to\infty} \frac{f_{16(n)}}{\ln n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{1}{k}}{\ln n} = \lim_{n\to\infty} \frac{H_n}{\ln n} = \lim_{n\to\infty} \frac{\ln n + O(1)}{\ln n} = \lim_{n\to\infty} \left(1 + \frac{O(1)}{\ln n}\right) = 1$$

$\therefore f_{16}(n) = \sum_{k=1}^{n} \frac{1}{k} \in O(\ln n)$

$f_{17}(n) = \prod_{k=1}^{n} \left(1 - \frac{1}{k^2}\right)$:
Since the product starts at $k = 1$:

$$\left(1 - \frac{1}{k^2}\right)\Big|_{k=1} = 0$$

Therefore, $\forall n$:

$$\prod_{k=1}^{n} \left(1 - \frac{1}{k^2}\right) = 0 \leq 1$$

$\therefore f_{17} = \prod_{k=1}^{n}\left(1 - \frac{1}{k^2}\right) = 0 < O(1)$

$f_{18}(n) = \left(1 - \frac{1}{n}\right)^n$:
For $n > 1$, we know that $\frac{1}{n} < 1$; thus for $n > 1$, $\left(1 - \frac{1}{n}\right) < 1$.
Therefore:

$$\left(1 - \frac{1}{n}\right)^n \le 1^n \le 1 \; \forall n > 1$$

$\therefore f_{18}(n) = \left(1 - \frac{1}{n}\right)^n \in O(1)$

Finally, we can sort the functions in ascending order:

$O(g(n))$	$f(n)$
$T(n) = 0$	$f_{17}(n) = \prod_{k=1}^{n}\left(1 - \frac{1}{k^2}\right)$
$O(1)$	$f_4(n) = n^{\frac{1}{n}}$ $f_{13}(n) = 1$ $f_{18}(n) = \left(1 - \frac{1}{n}\right)^n$
$O(\ln\ln n)$	$f_7(n) = \ln\ln n$
$O(\ln n)$	$f_6(n) = \ln n$ $f_{16}(n) = \sum_{k=1}^{n} \frac{1}{k}$
$O(n^\gamma),\ \gamma = \ln 3$	$f_8(n) = 3^{\ln n}$
$O(n \ln n)$	$f_{15}(n) = \log n!$
$O(n^2)$	$f_1(n) = n^2 + 3n\log n + 5$ $f_2(n) = n^2 + n^{-2}$ $f_{11}(n) = n^{1+\cos n}$ $f_{12}(n) = \sum_{k=1}^{\log n} \frac{n^2}{2^k}$ $f_{14}(n) = n^2 + 3n + 5$
$O(2^n)$	$f_9(n) = 2^n$
$O(n^n)$	$f_{10}(n) = (1+n)^n$
$O(n^{n^2})$	$f_3(n) = n^{n^2} + n!$ $f_5(n) = n^{n^2-1}$

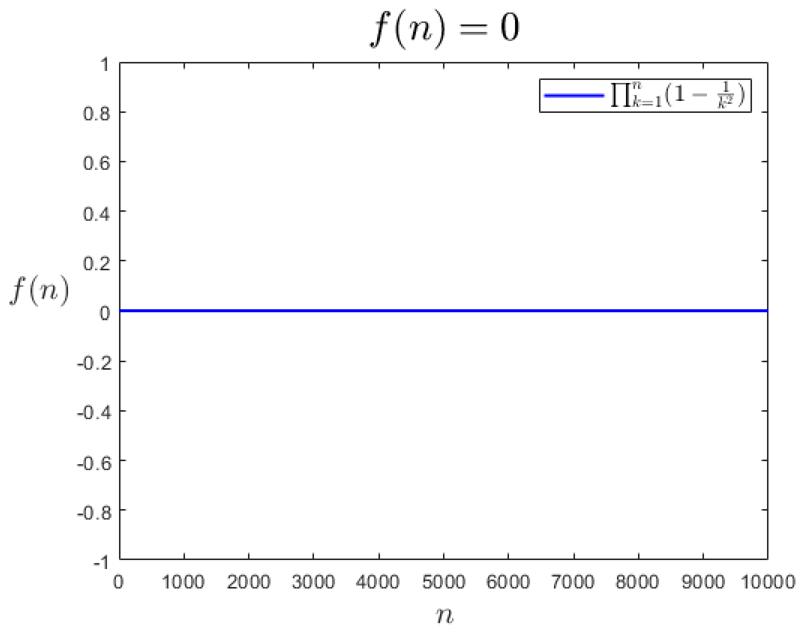

Figure 2: $f(n) = 0$

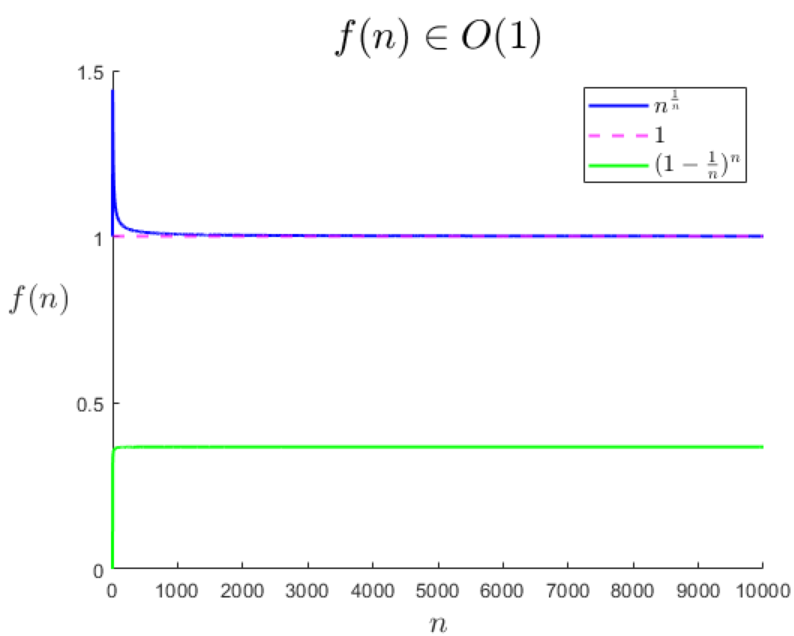

Figure 3: $f(n) \in O(1)$

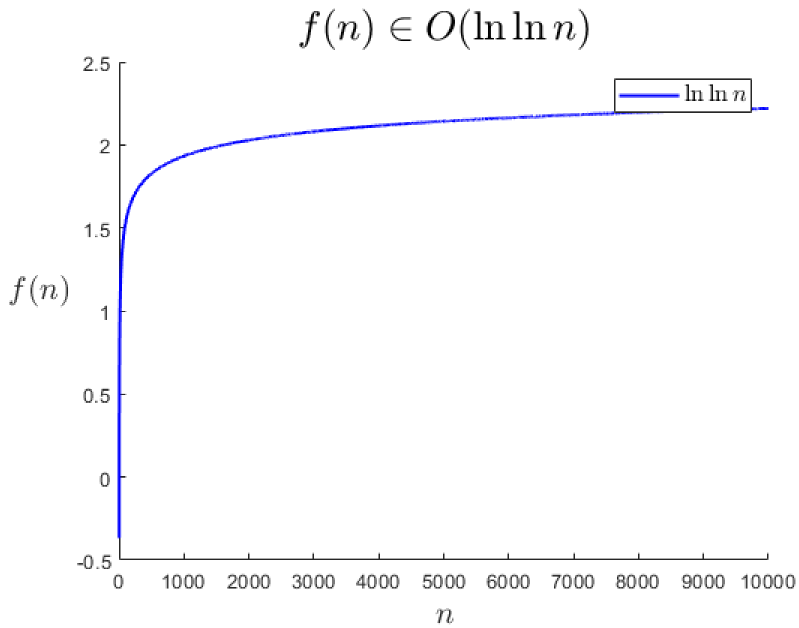

Figure 4: $f(n) \in O(\log \log n)$

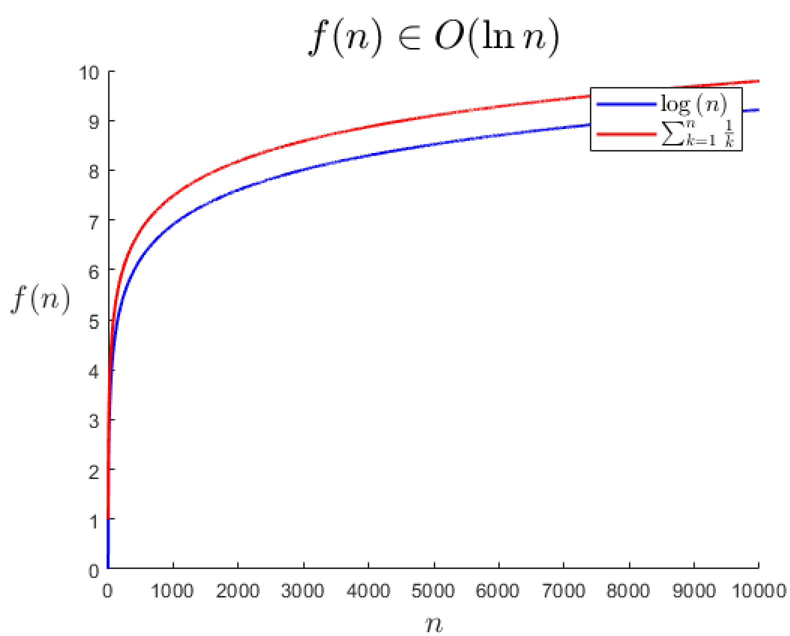

Figure 5: $f(n) \in O(\log n)$

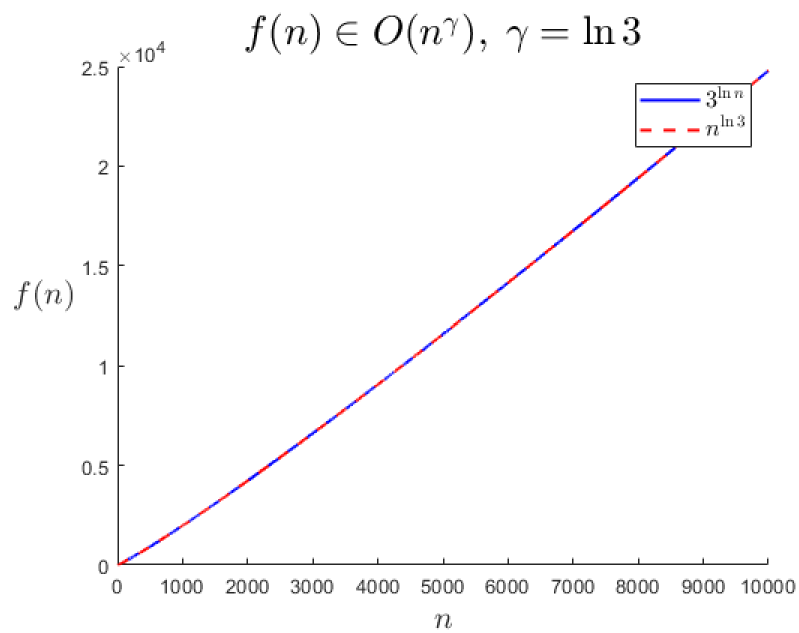

Figure 6: $f(n) \in O(n^{\gamma})$

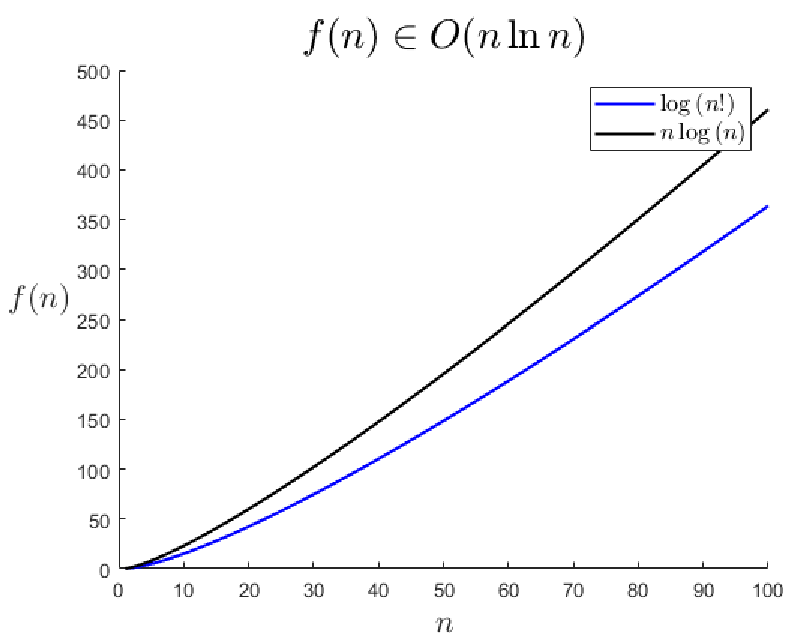

Figure 7: $f(n) \in O(n \log n)$

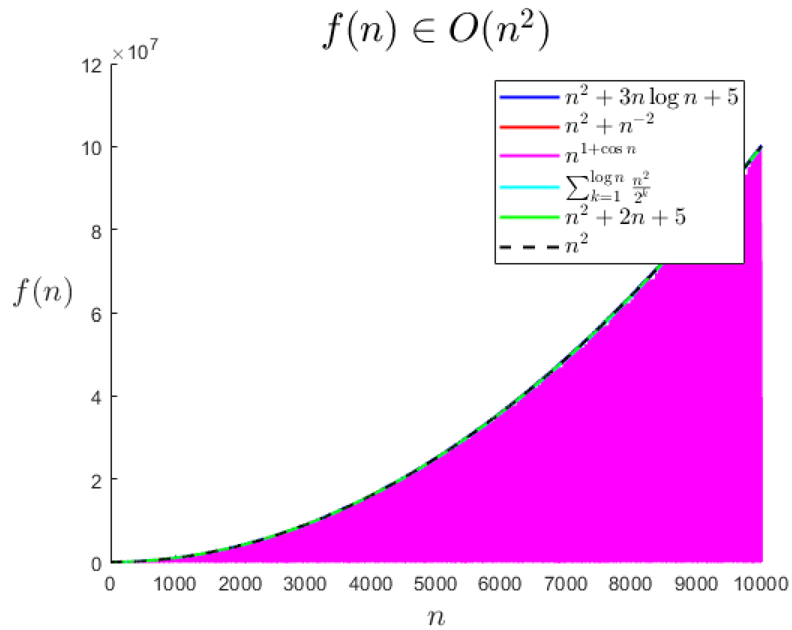

Figure 8: $f(n) \in O(n^2)$

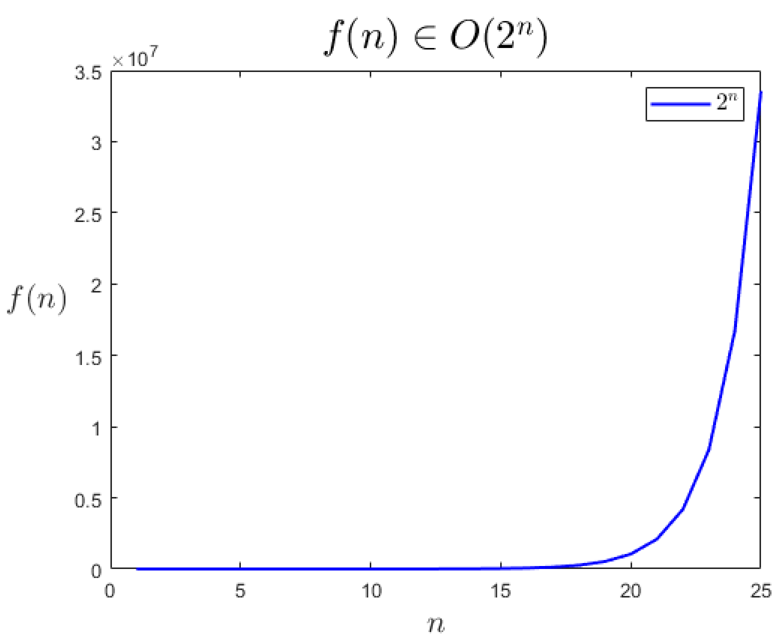

Figure 9: $f(n) \in O(2^n)$

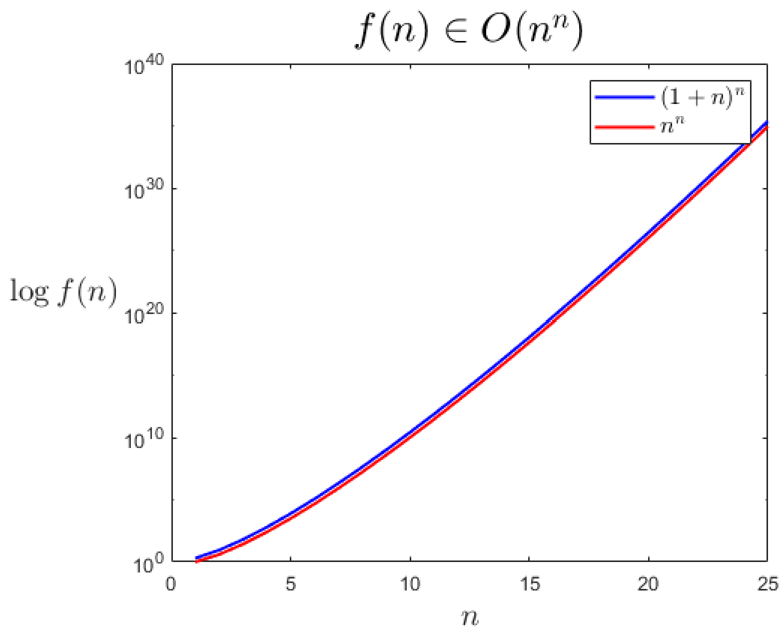

Figure 10: $f(n) \in O(n^n)$

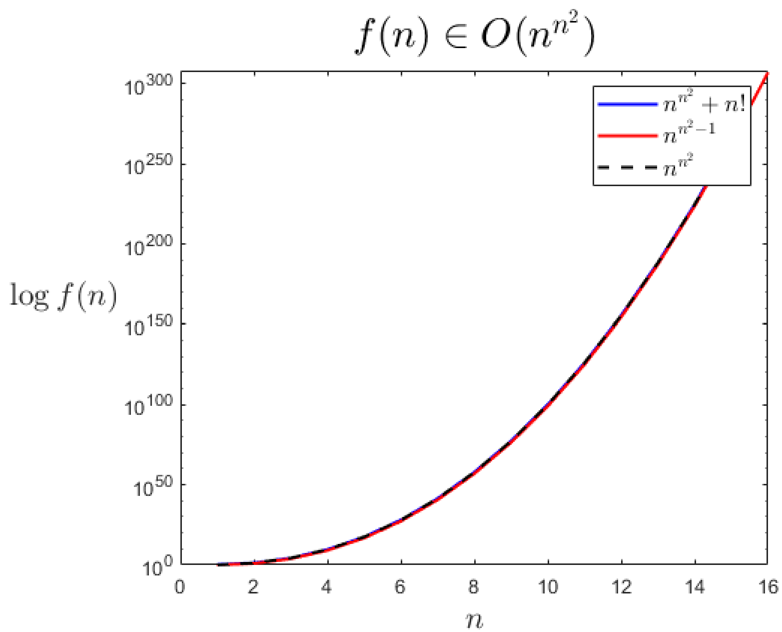

Figure 11: $f(n) \in O(n^{n^2})$

Problem 2

2A

Substitute:

$$T(n) = c_1 n + c_2 n \log_2 n$$

into:

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

to find the values of $c_1,\ c_2$ to determine the exact solution.

$$T(n) = 2T\left(\frac{n}{2}\right) + n \Longleftrightarrow c_1 n + c_2 n \log_2 n = 2\left[c_1\left(\frac{n}{2}\right) + c_2\left(\frac{n}{2}\right)\log_2\left(\frac{n}{2}\right)\right] + n$$

$$c_1 n + c_2 n \log_2 n = c_1 n + c_2 n\big(\log_2 n - \log_2 2\big) + n$$

$$c_1 n + c_2 n \log_2 n = c_1 n + c_2 n\big(\log_2 n - 1\big) + n$$

$$c_1 n + c_2 n \log_2 n = c_1 n + c_2 n \log_2 n - c_2 n + n$$

$$c_1 n + c_2 n \log_2 n = (c_1 - c_2 + 1)n + c_2 n \log_2 n$$

We require:

$$c_1 = c_1 - c_2 + 1$$

$$\therefore c_2 = 1$$

To find c_1, assume that $T(1) = t$. Then:

$$T(2) = 2T(1) + 2$$

$$c_1(2) + c_2(2)\log_2(2) = 2t + 2$$

$$2c_1 + 2(1)(1) = 2t + 2 \Longrightarrow 2c_1 + 2 = 2t + 2$$

$$\therefore c_1 = t = T(1)$$

Therefore, $T(n) = tn + n \log_2 n \quad \square$.

2B

Generalize this to the case for:

$$T(n) = aT\left(\frac{n}{b}\right) + n^k$$

with the trial solution:

$$T(n) = c_1 n^\gamma + c_2 n^k$$

using $a = b^\gamma$.

$$T(n) = aT\left(\frac{n}{b}\right) + n^k \Longleftrightarrow c_1 n^\gamma + c_2 n^k = a\left[c_1\left(\frac{n}{b}\right)^\gamma + c_2\left(\frac{n}{b}\right)^k\right] + n^k$$

$$c_1 n^\gamma + c_2 n^k = \frac{ac_1 n^\gamma}{b^\gamma} + \frac{ac_2 n^k}{b^k} + n^k$$

Since $a = b^\gamma$:

$$c_1 n^\gamma + c_2 n^k = \frac{b^\gamma c_1 n^\gamma}{b^\gamma} + \frac{b^\gamma c_2 n^k}{b^k} + n^k$$

$$c_1 n^\gamma + c_2 n^k = c_1 n^\gamma + b^{\gamma-k} c_2 n^k + n^k$$

$$c_1 n^\gamma + c_2 n^k = c_1 n^\gamma + \left(b^{\gamma-k} c_2 + 1\right) n^k$$

$$c_2 = b^{\gamma-k} c_2 + 1$$

$$c_2 - b^{\gamma-k} c_2 = 1 \Longrightarrow c_2\left(1 - b^{\gamma-k}\right) = 1$$

$$\therefore c_2 = \frac{1}{1 - b^{\gamma-k}}$$

Again assuming $T(1) = t$, and evaluating at $n = b$:

$$T(b) = aT\left(\frac{b}{b}\right) + b^k$$

$$T(b) = aT(1) + b^k$$

$$c_1 b^\gamma + c_2 b^k = at + b^k$$

$$c_1 b^\gamma = at + b^k - c_2 b^k$$

Using $c_2 = \frac{1}{1-b^{\gamma-k}}$:

$$c_1 b^\gamma = at + b^k - \left(\frac{1}{1 - b^{\gamma-k}}\right) b^k$$

$$c_1 = \frac{at}{b^\gamma} + \frac{b^k}{b^\gamma}\left(1 - \frac{1}{1 - b^{\gamma-k}}\right)$$

Since $a = b^\gamma$:

$$c_1 = t + \frac{b^k}{b^\gamma}\left(\frac{1 - b^{\gamma-k} - 1}{1 - b^{\gamma-k}}\right)$$

$$c_1 = t + \frac{b^k}{b^\gamma}\left(\frac{-b^{\gamma-k}}{1 - b^{\gamma-k}}\right)$$

$$c_1 = t + \frac{1}{b^\gamma}\left(\frac{-b^{\gamma}}{1 - b^{\gamma-k}}\right)$$

$$\therefore c_1 = t - \frac{1}{1 - b^{\gamma-k}}$$

Therefore, $T(n) = \left(t - \frac{1}{1-b^{\gamma-k}}\right) n^\gamma + \left(\frac{1}{1-b^{\gamma-k}}\right) n^k \quad \square.$

2C

However, if $\gamma = k$, then the above equation is undefined; thus, the guess $T(n) = c_1 n^\gamma + c_2 n^k$ is no longer valid and we must choose another form. If we guess:

$$T(n) = c_1 n^\gamma + c_2 n^\gamma \log_2 n$$

We can find constants c_1 and c_2 such that the recurrence $T(n) = aT\left(\frac{n}{b}\right) + n^k$ is satisfied for $\gamma = k$.

$$T(n) = aT\left(\frac{n}{b}\right) + n^k \iff c_1 n^\gamma + c_2 n^\gamma \log_2 n = a\left[c_1\left(\frac{n}{b}\right)^\gamma + c_2\left(\frac{n}{b}\right)^\gamma \log_2\left(\frac{n}{b}\right)\right] + n^k$$

$$c_1 n^\gamma + c_2 n^\gamma \log_2 n = \frac{ac_1 n^\gamma}{b^\gamma} + \frac{ac_2 n^\gamma}{b^\gamma}\log_2\left(\frac{n}{b}\right) + n^k$$

Since $\gamma = k$ and $a = b^\gamma$:

$$c_1 n^\gamma + c_2 n^\gamma \log_2 n = c_1 n^\gamma + c_2 n^\gamma \log_2\left(\frac{n}{b}\right) + n^\gamma$$

$$c_1 n^\gamma + c_2 n^\gamma \log_2 n = c_1 n^\gamma + c_2 n^\gamma \left(\log_2 n - \log_2 b\right) + n^\gamma$$

$$c_1 n^\gamma + c_2 n^\gamma \log_2 n = c_1 n^\gamma + c_2 n^\gamma \log_2 n - \left(\log_2 b\right) c_2 n^\gamma + n^\gamma$$

$$c_1 n^\gamma + c_2 n^\gamma \log_2 n = c_2 n^\gamma \log_2 n + \left(c_1 + 1 - c_2 \log_2 b\right) n^\gamma$$

$$c_1 = c_1 + 1 - c_2 \log_2 b$$

$$\therefore c_2 = \frac{1}{\log_2 b}$$

Again assuming $T(1) = t$ and evaluating at $n = b$:

$$T(b) = aT\left(\frac{b}{b}\right) + b^\gamma$$

$$T(b) = aT(1) + b^\gamma$$

$$c_1 b^\gamma + c_2 b^\gamma \log_2 b = at + b^\gamma$$

$$c_1 b^\gamma = at + b^\gamma - c_2 b^\gamma \log_2 b$$

$$c_1 = \frac{at}{b^\gamma} + \frac{b^\gamma}{b^\gamma}\left(1 - c_2 \log_2 b\right)$$

Using $a = b^\gamma$ and $c_2 = \frac{1}{\log 2b}$:

$$c_1 = t + \left(1 - \frac{\log_2 b}{\log_2 b}\right)$$

$$\therefore c_1 = t$$

Therefore, $T(n) = tn^\gamma + \left(\frac{\log_2 n}{\log_2 b}\right) n^\gamma \quad \square$.