EC 504 – Spring 2023 – Homework 1

Due Wednesday Feb 8, 2023 at 8 PM Boston Time

NOTE: This HW is to be turned in to Gradescope as a PDF.

Start reading Chapters 1, 2, 3 and 4 in CLRS . They give a very readable introduction to Algorithms.. Also glance at Appendix A for math tricks which we will use from time to time.

1. (40 pts) In CRLS do Exercise 3.1-4 on page 53, Problem 3.2-7 on page 60, Problem 3-2 on page 61 , Problem 4.3-9 on page 88.

2. (30 pts) Place the following functions in order from asymptotically smallest to largest. As a convenience you may use $f(n) < O(g(n))$ to mean $f(n) \in O(g(n))$ and $f(n) = g(n)$ to mean $f(n) \in \Theta(g(n)$. Please use $=$ when you are sure that it is $\Theta(g(n))$.)

$$n^2 + 3n\log(n) + 5\ ,\ \ n^2 + n^{-2}\ ,\ n^{n^2} + n!\ ,\ n^{\frac{1}{n}}\ ,\ n^{n^2-1}\ ,\ \ln n\ ,\ \ln(\ln n)\ ,\ 3^{\ln n}\ ,\ 2^n\ ,$$

$$(1+n)^n\ ,\ n^{1+\cos n}\ ,\ \sum_{k=1}^{\log n}\frac{n^2}{2^k}\ ,\ 1\ ,\ n^2+3n+5\ ,\ log(n!)\ ,\ \sum_{k=1}^{n}\frac{1}{k}\ ,\ \prod_{k=1}^{n}(1-\frac{1}{k^2})\ ,\ (1-1/n)^n$$

Giving the the algebra and explanation for the tricky cases can get some extra credit (even if you get it wrong!). Don't have to be perfect to get a good score.

Solution: SOME OF THESE ARE PRETTY SUBTLE. A GOOD RESULT IS GETTING THE ORDER SOME WHAT RIGHT. Here I give much more details not required since they are instructive. In some cases I even give the exact limit as $\to xx$.

$$1 \in O(1)\quad ,\quad \prod_{k=1}^{n}(1-\frac{1}{k^2}) \to 0 \in O(1)\quad ,\quad n^{\frac{1}{n}} \in O(1)\quad ,\quad (1-1/n)^n \to e \in O(1)$$

$$\ln(\ln n) \in O(\ln(\ln n))\quad ,\quad ln(n) \in O(log(n))\quad ,\quad \sum_{k=1}^{n}\frac{1}{k} = O(\ln n)$$

$$\ln(n!) \in O(n\log(n))\quad ,\quad 3^{\ln n} \to n^{\ln(3)} \in O(n^{\ln(3)})\quad ,\quad n^{1+\cos n} \in O(n^2)$$

$$n^2+3n+5 \in O(n^2)\quad ,\quad n^2+3n\ln n+5 \in O(n^2)\quad ,\quad n^2+n^{-2} \in O(n^2)$$

$$2^n \in O(2^n)\quad ,\quad \sum_{k=1}^{\log n}\frac{n^2}{2^k} \in ?\quad ,\quad n! \in O(e^{-n}n^{n+1/2})\quad ,\quad (1+n)^n \in O(n^n)$$

$$n^{n^2-1} \in O(\frac{1}{n}n^{n^2})\quad ,\quad n^{n^2}+n! \in O(n^{n^2})$$

Here are few explicit sums:

1:

For instance,

$$\lim_{n\to\infty}\prod_{k=1}^{n}(1-\frac{1}{k^2}) = 0 \in O(0)$$

because the first term in the product is zero. If the first term were not 0, then the product would converge to a constant, as

$$\ln\prod_{k=2}^{n}(1-\frac{1}{k^2}) = \sum_{k=1}^{n}\ln(1-\frac{1}{k^2}) \approx \sum_{k=1}^{n}-\frac{1}{k^2} = \frac{-\pi^2}{6}$$

2:

$\sum_{k=1}^{\ln n}\frac{n^2}{2^k} = n^2\sum_{k=1}^{\ln n}(\frac{1}{2})^k = O(n^2)$. Just realizing that a geometric series is convergent $\sum_{k=0}^{N}x^k = O(1)$ or if you believe in slugging it through (not necessary!). $\sum_{k=0}^{N}x^k = \frac{1-x^{N+1}}{1-x}$ so $\sum_{k=1}^{N}x^k = \frac{1-x^{N+1}}{1-x} - 1 == \frac{1-(1/2^{\ln(n)+1}}{1/2} - 1 = 1-(1/2)^{\ln(n)} \in O(1)$ using $x = 1/2$ and $N = \ln(n)$.

3:

In general a very useful trick it to take `ln-exp!` of the function ($f(n) = e^{\ln(f(n))}$) followed by the large n limit. For example:

$$(1-1/n)^n = e^{n\ln(1-1/n)} \simeq e^{n(1/n+1/2n^2+\cdots)} \to e^1 = 2.718281828459$$

(I found this going to WolframAlfpha: https://www.wolframalpha.com!)
Also, recall that $e^{\ln n} = n$.

4:

Finally the function $n^{1+\cos n}$ is tricky so we accept any reasonable placements. It doesn't have a smooth monotonic limit at large n. It oscillates between $\Theta(1)$ and $\Theta(n^2)$ getting arbitrarily close both even at integer values. Therefore strictly speaking the best bound is $n^{1+\cos n} \in O(n^2)$ but $n^\alpha \in O(n^{1+\cos n})$ implies $\alpha \le 0$ by the definition in CLRS of "Big Oh".

1. (40 pts)

   (a) Given the equation, $T(n) = 2T(n/2) + n$, guess a solution of the form:

   $$T(n) = c_1 n + c_2\, n log_2(n)\ .$$

   Find the coefficients c_1, c_2 to determine the exact solution assuming a value $T(1)$ at the bottom the recursion.

(b) Generalize this to the case to the equation $T(n) = aT(n/b) + n^k$ and guess the solution of the form:

$$T(n) = c_1 n^\gamma + c_2 n^k$$

using $b^\gamma = a$ and assuming $\gamma \neq k$. First show if you drop the n^k using the **homogeneous** equation $T(n) = aT(n/b)$ the form $c_1 n^\gamma$ is a solution! (What is c_1?) Second drop $aT(n/b)$ and show $c_2 n^k$ is a solution (What is c_2?) With both terms (and $\gamma \neq k$) the full solution is just the sum of the to terms but only one or the other dominates!

(c) What happens when the two solution collide (i.e have the same power, i.e $\gamma = \log(a)/\log(b) = k$.) Now show that the leading solution is as n goes to infinity is $T(n) = \Theta(n^k logn)$ [1]

Solution: You are give a a `guess` with unknown constants c_1, c_2. To see is a good guess see if you can determine the constants by substituting the `guess` in the RHS (right hand side) and the LHS (left hand side).

First case:

$$\begin{aligned} c_1 n + c_2\ nlog_2(n) &= 2[\ c_1 n/2 + c_2\ (n/2) log_2(n/2)\ ] + n \\ &= c_1 n + c_2\ n(log_2(n) - log_2(2)) + n \\ &= c_1 n + c_2\ nlog_2(n) - nc_2 + n \end{aligned}$$

Now to see if `RHS = LHS` $c_1 = c_2$ you need to match the n and the $nlog_2(n)$ term

$$c_1 = c_1 - c_2 + 1 \quad \text{and} \quad c_2 = c_2 \tag{1}$$

so it works $c_2 = 1$ and any c_1. To determine this you need to base provide a base case of the recursion (e.g. $T(1) = c_1$)).

Second case: With $\gamma \neq k$ the general solution works. Again LHS vs RHS

$$c_1 n^\gamma + c_2 n^k = a[c_1 (n/b)^\gamma + c_2 (n/b)^k] + n^k \tag{2}$$

Two conditions to match term:

$$c_1 = c_1 a/b^\gamma \quad \text{and} \quad c_2 = a/b^k c_2 + 1 \tag{3}$$

Again it works for any c_1 but we need $c_2 = 1/(1 - a/b^k)$. Trouble if $k = \gamma$ because now $c_2 = \infty$ (i.e. it fails!)

So need to start over with something larger by a log. Again try to match LHS and RHS

$$c_1 n^\gamma + c_2 n^\gamma log_2(n) = a[c_1 (n/b)^\gamma + c_2 (n/b)^\gamma \log_2(n/b)] + n^\gamma \tag{4}$$

Now the leading term c_2 works and the lower power is determined

$$c_2 = c_2 \quad \text{and} \quad c_1 = c_1 - c_2 log_2(b) \tag{5}$$

lower term is again determined relative to it. This is general the larger term matches as n goes to infinity and the smaller needs a base number of $T(1)$.

[1] If you are ambitious you find exact solutions for part b and c above are of the form $T(n) = c_1 n^\gamma + givec_2 n^k$ and $T(n) = c_1 n^k + c_2 nlog(n) n^k$ explicitly determine the $c's$ for each case respectively. We did that in part for example. BUT note we already know the leading terms for the Master Equation without this extra effort! Neat trick!