Summary of Algorithms

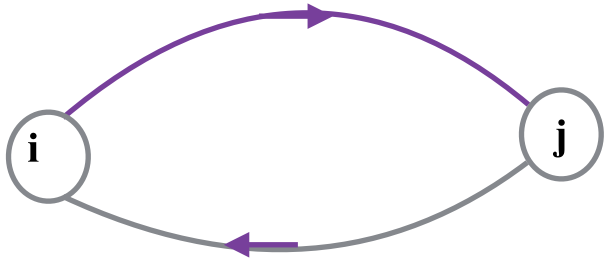
- Capacity Graphs — CLRS 26
  - Ford-Fulkerson's Max Flow
  - Theorem: Min cut = Max Flow
- Scheduling
  - Integer/fraction Knapsack — CLRS 16.2
  - Job Schedule — Ex. 16-2 (a)
  - Deadline Schedule — CLRS 16.5
- KMP String matching — CLRS 32
- Union/Find Algorithm — CLRS 21
  - Union by height/size
  - Path compression
    - $O(M \text{ Log}^*(N))$ explained
  - Binomial Queue: Union/Find

Capacity Graphs

- Capacity graph: link i -> j has capacity c(i,j)

- Max Flow Problem: Max flow into source node (s) out of terminal node (t)
  - constraints on flow i to j: f(i,j) = - f(j,i) $\sum_j f(i,j) = 0$

    capacity: 0 <= f(i,j) <= c(i,j) ;



If there is return arc -c(j,i) <= f(i,j) <= c(i,j)

Augmentation, Flow, Residual

- *Initialize residual graph G(N,A1), A1 = A.*

- *Initialize all f(i,j) = 0.* $0 \cdot r(i,j) \cdot c(i,j)$

  *a. Find path P from s to t (**augmenting path)***

  b. Find minimum capacity arc on path P. Denote capacity by C.

  c. Add C units of flow f(i,j) ! f(i,j) + C to each arc on path Modify residual graph (N,A1) accordingly.

  $-f(i,j) <= r(i,j) <= c(i,j) - f(i,j)$

  d. Repeat a-c until no path P from s to t in G(N,A1)

- *Properties:*

  a) Total flow increases and terminate when no augmentation possible: O(Max(C) |A|)

  b) With minimum hop path for augmentation $O(|N|\ |A|^2)$

Solution without backtracking

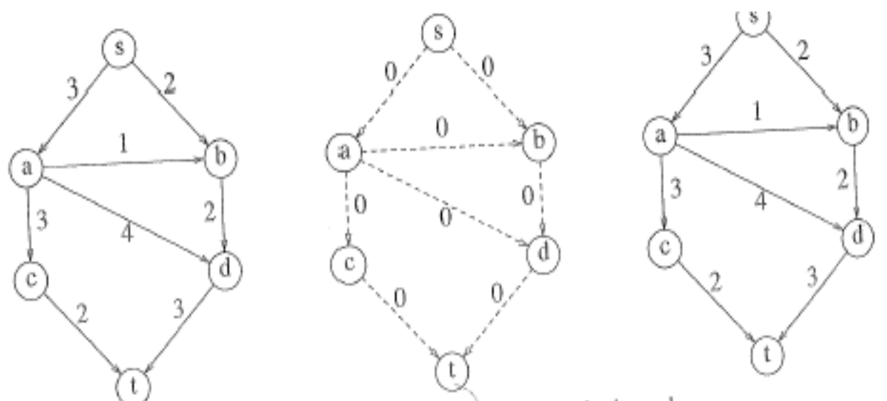


Figure 9.40 Initial stages of the graph, flow graph, and residual graph

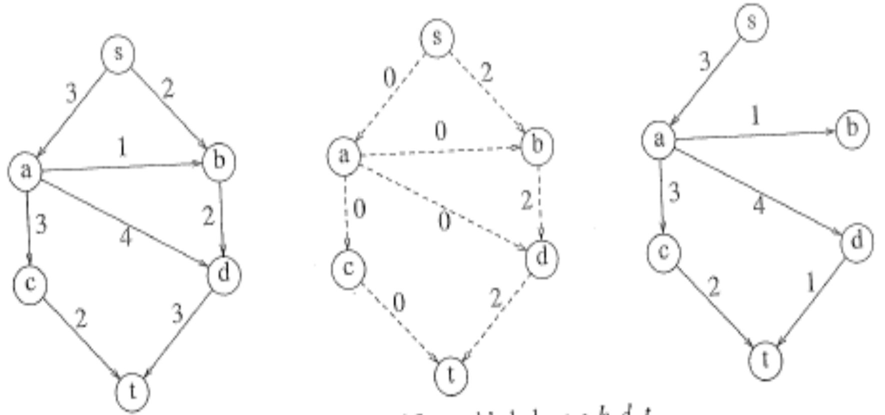


Figure 9.41 G, G_f, G_r after two units of flow added along s, b, d, t

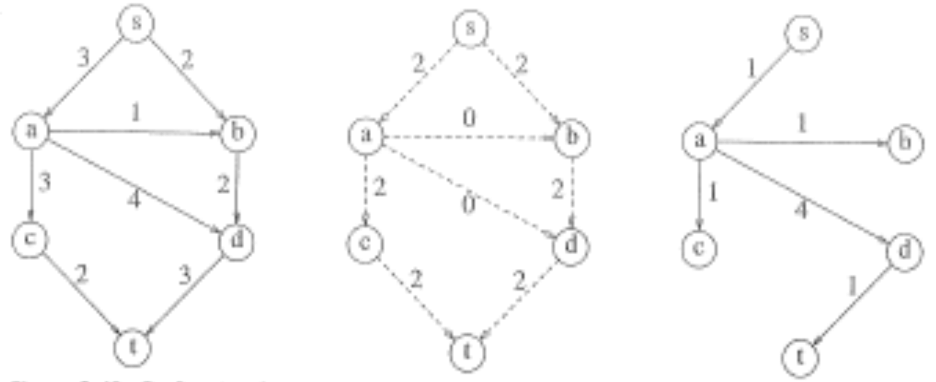


Figure 9.42 G, G_f, G_r after two units of flow added along s, a, c, t

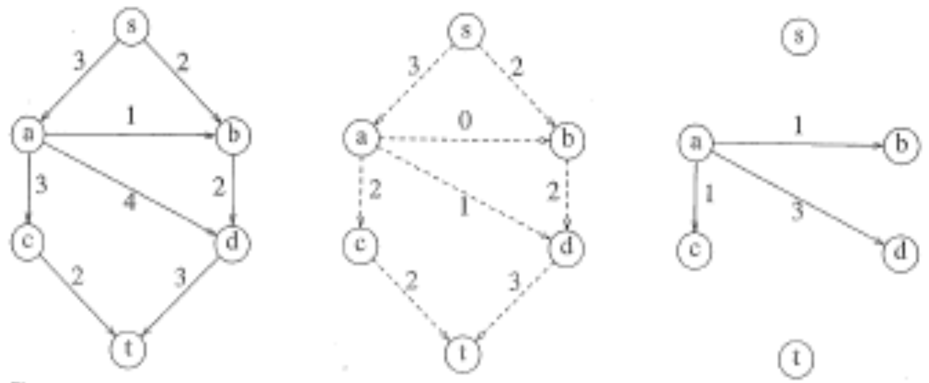


Figure 9.43 G, G_f, G_r after one unit of flow added along s, a, d, t—algorithm terminates

Solution with backtracking

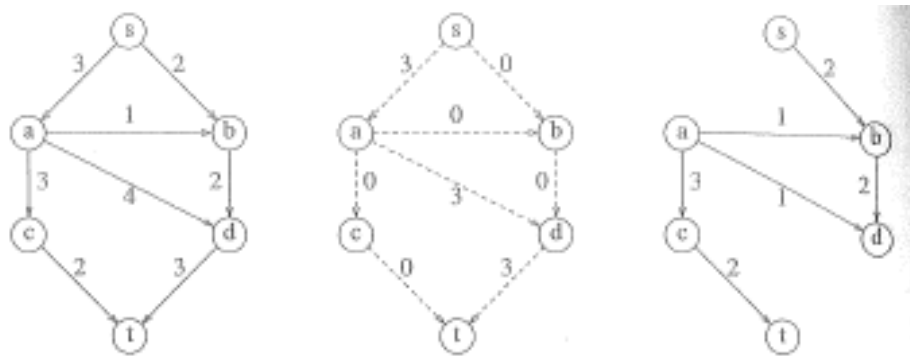


Figure 9.44 G, G_f, G_r if initial action is to add three units of flow along s, a, d, t—algorithm terminates with suboptimal solution

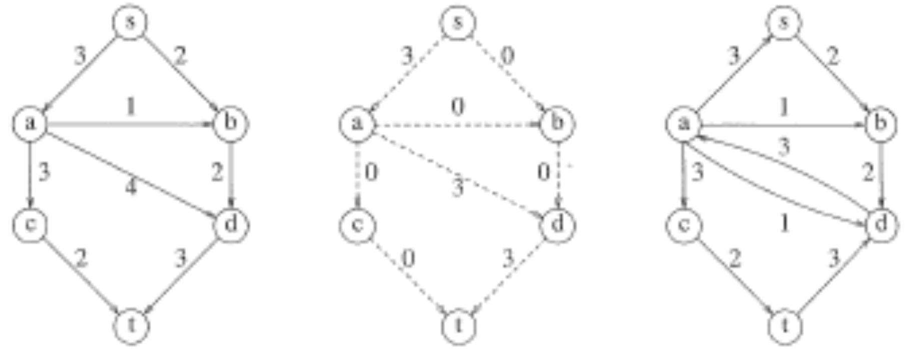


Figure 9.45 Graphs after three units of flow added along s, a, d, t using correct algorithm

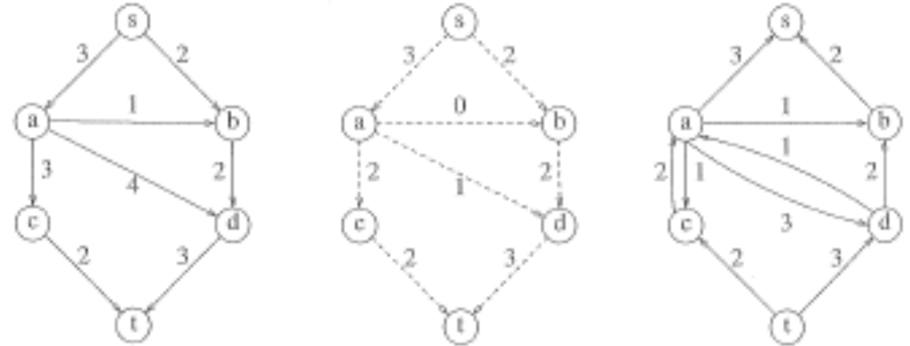


Figure 9.46 Graphs after two units of flow added along s, b, d, a, c, t using correct algorithm

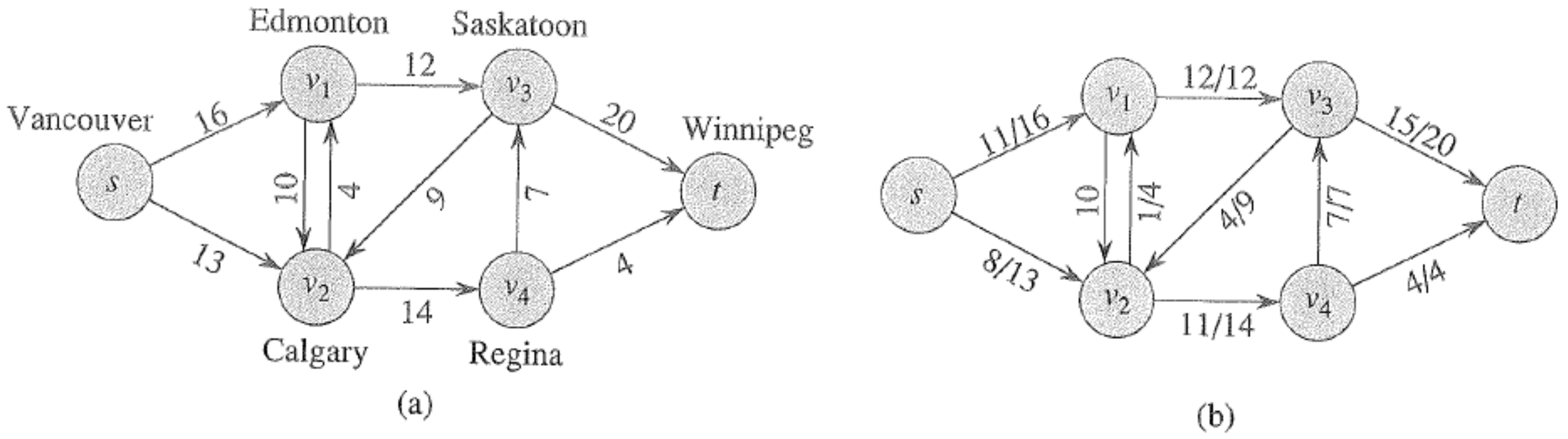


Figure 26.1 **(a)** A flow network $G = (V, E)$ for the Lucky Puck Company's trucking problem. The Vancouver factory is the source s, and the Winnipeg warehouse is the sink t. Pucks are shipped through intermediate cities, but only $c(u, v)$ crates per day can go from city u to city v. Each edge is labeled with its capacity. **(b)** A flow f in G with value $|f| = 19$. Only positive flows are shown. If $f(u, v) > 0$, edge (u, v) is labeled by $f(u, v)/c(u, v)$. (The slash notation is used merely to separate the flow and capacity; it does not indicate division.) If $f(u, v) \leq 0$, edge (u, v) is labeled only by its capacity.

Residual Graph

Flow/Capacity Graph

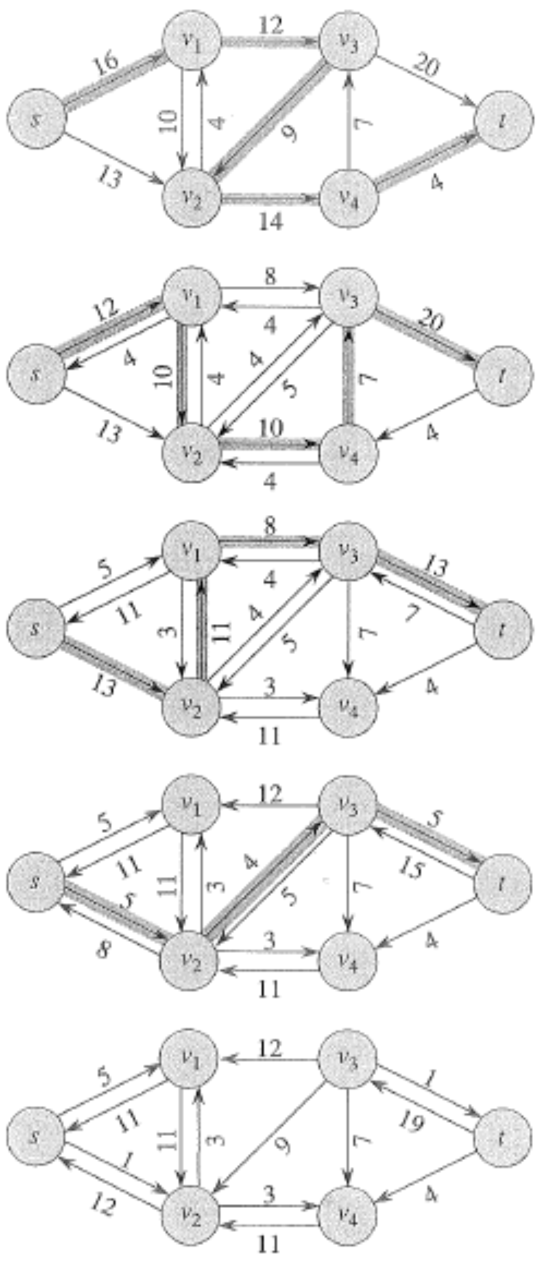
(a)

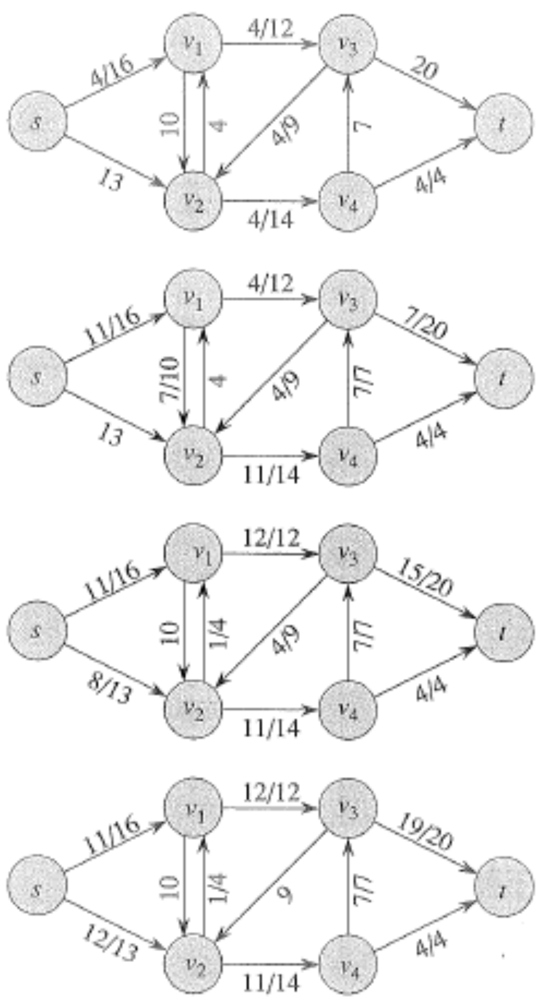
(b)

(c)

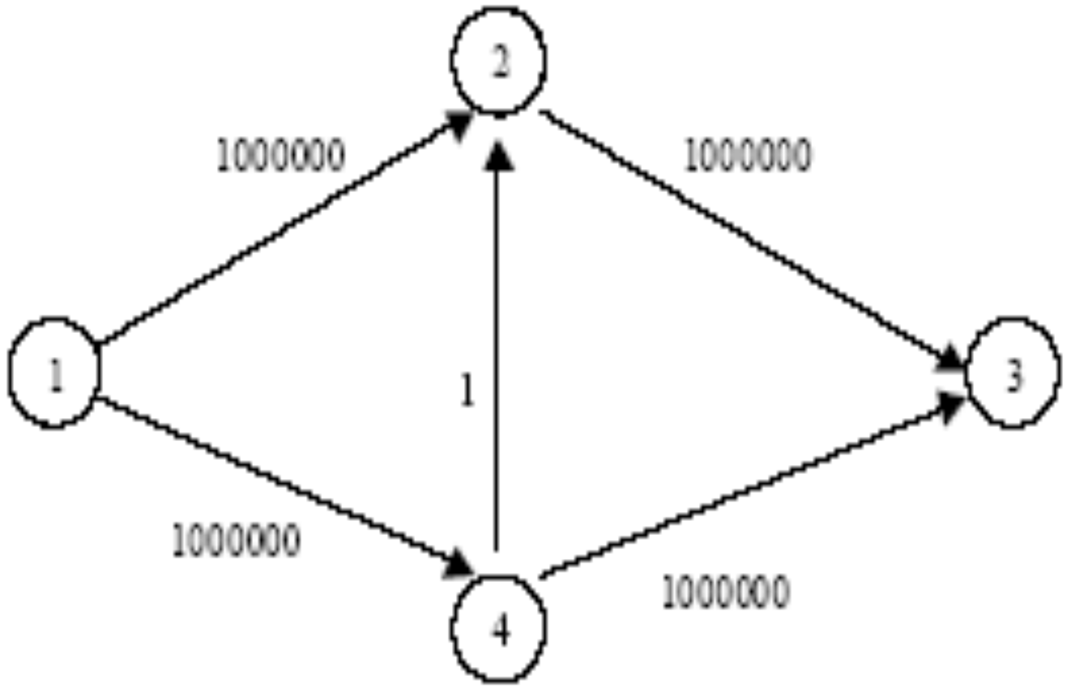
(d)

(e)





Classical Bad Case



Scheduling & Linear Programming

- *Knapsack*
- *Queuing*
- *Schedule with Deadlines*

Fractional Knapsack

- *Given set of task that take "time"* $t_1, t_2, t_3, \ldots, t_N$ *and*

  values $v_1, v_2, v_3, \ldots, v_N$

Allocate to maximize objective

$$S(x_i) = \Sigma_i x_i V_i$$

with "time" or "size" limited resource T:

$$\Sigma_i x\_i\ t_i <= T$$

Problem is to find assignment fractions

$$0 <= x_i <= 1 \quad \text{ok} \quad (\text{NP Hard if } xi = 0, 1)$$

(Put most valuable parts of N objects in to your sack)

Solution:

- Sort in increasing value per time v_i/t_i and take them in order as far as possible:
- All: $x_1 = 1, x_2 = 1, \ldots, x_{m-1} = 1$, Some of $0 < x_m < 1$, None: $x_{i+m} = 0, \ldots, x_N = 0$
- So that $t_1 + t_2 + \ldots t_{m-1} + x_m t_m = T$ exactly $0 \cdot x_m = (T - t_1 - t_2 - \ldots - t_{m-1})/t_m \cdot 1$

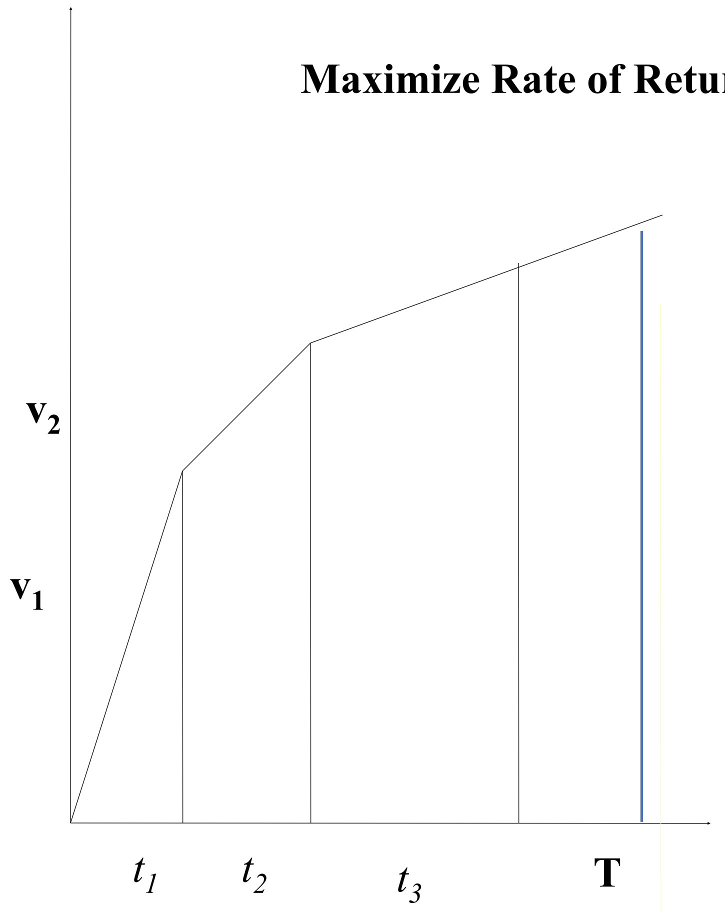
Integer Knapsack: x_i = 0 or 1 is VERY HARD to SOLVE!

Example: T = 50

Objects: v/t = \$60 / 10, \$100/20, \$120/30

Greedy \$60 + \$100 = \$160
Try it: Best integer is \$100 + \$120 = \$220
Best fractional: \$240

See CRLS Fig. 16.2



Queuing: Minimize "time in line"

- Jobs in queue take time t_i
- Minimize total waiting time:

$t_1 + (t_1 + t_2) + (t_1+t_2+t_3) + \ldots + (t_1 + \ldots + t_N)$

$W = N\, t_1 + (N-1)\, t_2 + \ldots + t_N$

- Solution: Min W by sorting t's in ascending order (shorter jobs first)!
- Recall sorting is by "swap theorem"

$$\text{Sort} \;\equiv\; \text{Max over permutation} \sum_i i a[i]$$

Scheduling with deadlines

- Task all take same time: $\Delta = 1 <$ T but they have different values and deadlines

- Sort in descending values (or penalties!):

  $v_1, v_2, \ldots, v_N$

  $d_1, d_2, \ldots, d_N$

- Feasible Solutions can always be in EFF!
- So this is the contraction:
  - Select them one at a time from ordered list of v's
  - Schedule them in order in "early first form" (EFF).
  - If schedule fails drop last one selected and continue to end of list.

  Alternative solution: maximum procrastination!

CRLS Fig 16.7

Max Procrastination solution: Sequence

Table 1

a	1	2	3	4	5	6	7
d	4	2	4	3	1	4	6
v	70	60	50	40	30	20	10
step 1				a1			
step 2		a2					
step 3			a3				
step 4	a4						
step 7						a7	

Select a1, a2, a3 and a4   Reject a5 and a6 Select a7

Knuth-Morris-Pratt string matching

- Text: T(1),T(2),...,T(N)
- Pattern: P(1), P(2),...,P(M)
- Match if
  - T(s+1) =P(1), T(s+2) = P(2),.., T(M) = P(M)
  - or T(s+1:s+M) = P(1:M)
- Trivial scan O(M (N-M+1))
- KMP algorithm O(N+M) by amortized analysis

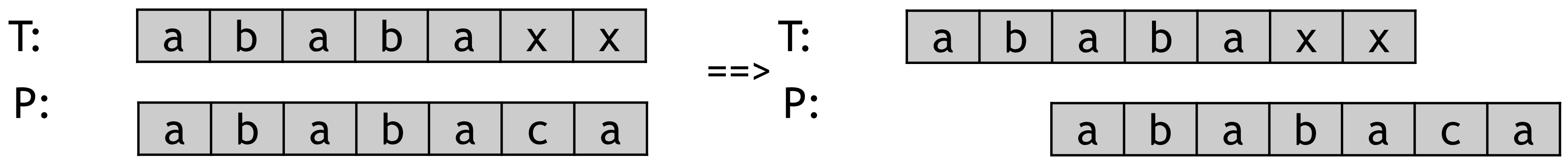
Prefix function:

Give a partial match of q letter what is the max match pi(q)?
or min shift s = q - pi(q)

- Given a partial match P(1:q) = T(s+1:s+q) what is the smallest shift that matches end of T(s+1:s+q)
- It is q – pi(q) where pi(q) prefix function for the max match of prefix to suffix of P(1:q)
- pi(q) = Max{k<q s.t. P(1:k) = P(1+q- k:q}

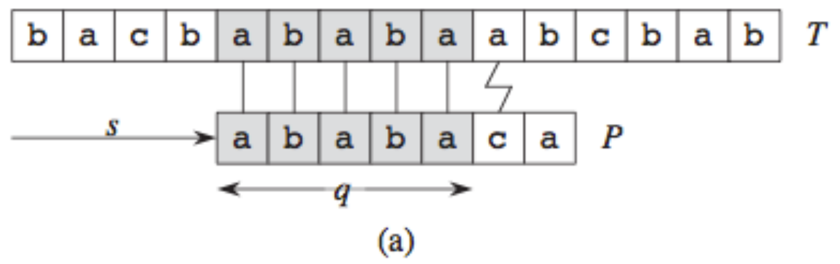
i.e. q matches become pi(q) ds = q – pi(q)

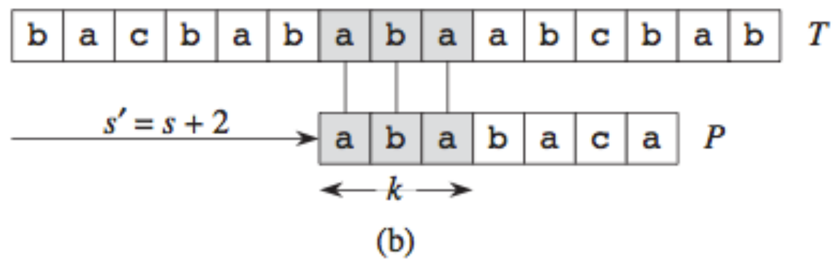
- Strategy pre-compute pi(q) and use it to advance the match



q = 5

pi(5) = 4





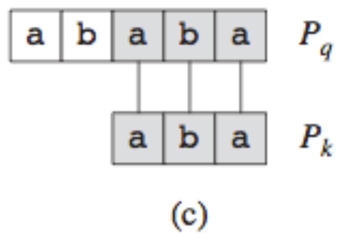
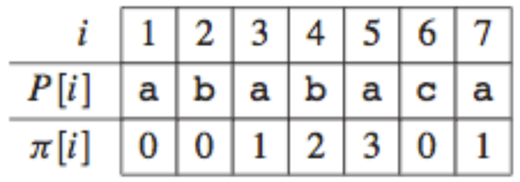
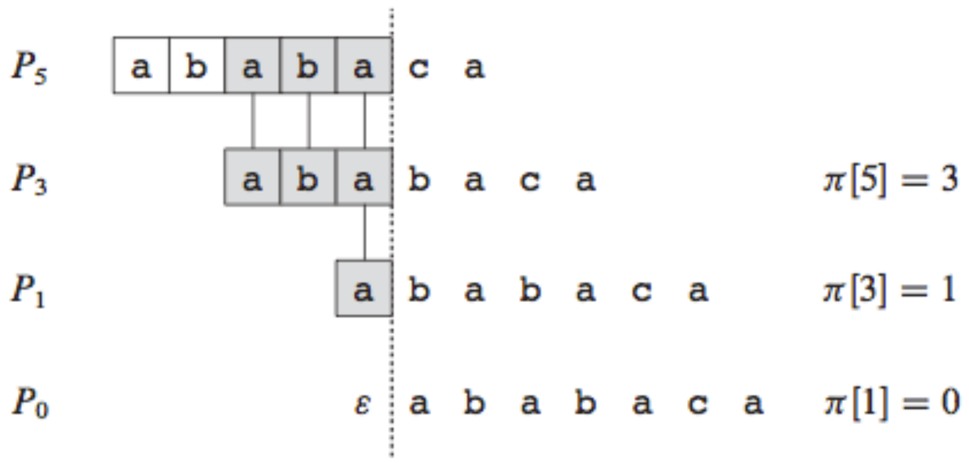


Figure 32.10 The prefix function π. **(a)** The pattern P = ababaca aligns with a text T so that the first $q = 5$ characters match. Matching characters, shown shaded, are connected by vertical lines. **(b)** Using only our knowledge of the 5 matched characters, we can deduce that a shift of $s + 1$ is invalid, but that a shift of $s' = s+2$ is consistent with everything we know about the text and therefore is potentially valid. **(c)** We can precompute useful information for such deductions by comparing the pattern with itself. Here, we see that the longest prefix of P that is also a proper suffix of P_5 is P_3. We represent this precomputed information in the array π, so that $\pi[5] = 3$. Given that q characters have matched successfully at shift s, the next potentially valid shift is at $s' = s + (q - \pi[q])$ as shown in part (b).



i	1	2	3	4	5	6	7
$P[i]$	a	b	a	b	a	c	a
$\pi[i]$	0	0	1	2	3	0	1

(a)



(b)

Computing pi(q)

```
■ Set  pi(1)  = 0;  k = 0
■ For q= 2,.., M
   {
      while k>0 and P(k+1) \= P(q) set k =pi(k) ;
      if P(k+1) == P(q)   k = k+1;
      pi(q) = k;
   }
```

KMP Algorithm

q matchs out **q matchs in**

- Compute pi(q) set q = 0
- For i = 1, N

```
{   // count c(i) to find new partial match
    // c(i) <=  q(i-1) – q(i) +1  or c(i) + q(i) - q(i) <=1
    while q>0 and P(q+1) \= T(i) set q =pi(q) ; // new q
     if P(q+1) = T(i) then q = q+1 ;
     if q = M {
          report pattern found T(i – M;i);
          q = pi(q);  // New match substring
                 }
 }
```

KMP Amortize analysis

- At While statement
- All sets are O(N) except at the while statement dq is called c(i) times
- c(i) + q(i) – q(i-1) <= 1 worst case for each i.
- sum_i [c(i) + q(i) – q(i-1)] <=N
- sum_i c(i) <= N + q(0) – q(N) <= N
- Therefore is O(N)
- Construction of pi(M) is O(M)
- So algorithm is O(N + M).

Relations: *Boolean valued Matrix* *R[a,b]*

- Set: S = {a,b,c,....}
- Relation (a,b) 2 S x S: a R b is True?
- Properties:
  - Reflexive: a R a is True
  - Anti-symmetric: a R b and b R a $\rightarrow$ a = b
  - Transitive: a R b and b R c $\rightarrow$ a R c
  - Total Ordering: a R b or b R a (inclusive or)
  - Self dual: a R b $\leftrightarrow$ b R a
  - Transpose: a R b $\leftrightarrow$ b R^T a
- RAT is partial ordering: e.g. descendants in a tree!

(e.g. <= is total ordering for int but g(N) = O(f(N)) is partial ordering!)

- Equivalence class is Reflexive, Transitive and Symmetric
  - Symmetric a R b if and only if b R a

Union/Find

- Equivalence class and Sets.
- O(1) Find
- O(1) Union
  - Union by Height/Size
  - Path Compressions
  - Log*(N) function
- Binomial Queue

Dynamic Equivalence

- Object a[i] can be numbered 0,...,N-1 (like nodes)
- Sequence of new equivalence a ~ b
- Two operations:
- FIND a in set S(i) ?
  - Must return T/F for find(a) ~find(b)
- UNION S(k) = S(i) U S(j)
  - Operation of find(a) not~ find(b)
- Want M finds and upto N unions: O(M+N)?
- Almost but actually impossible!

Determine if a ~ b

- O(1) answer is a ~`b if use array label a by of Set #
- Set up a 2-d are for SxS and look up T/F
- Alternatively “partition” S into equivalence classes (like connected component) is Ci for all a’s ~ b’s
- S = C1 U C2 U ...U Cn and C’s are disjoint: Ci ^ Ci = 0 (null set).

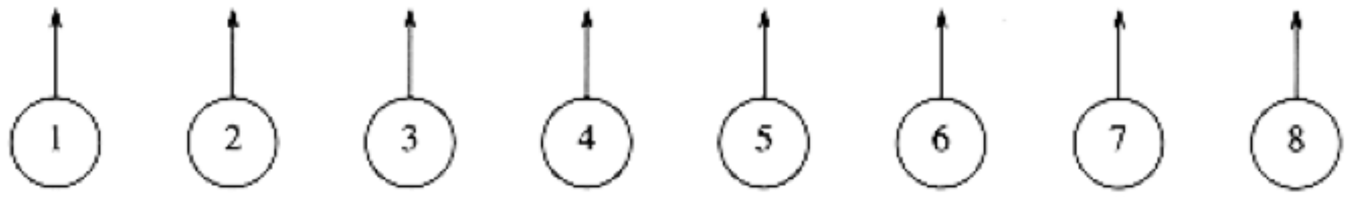


Figure 8.1 Eight elements, initially in different sets

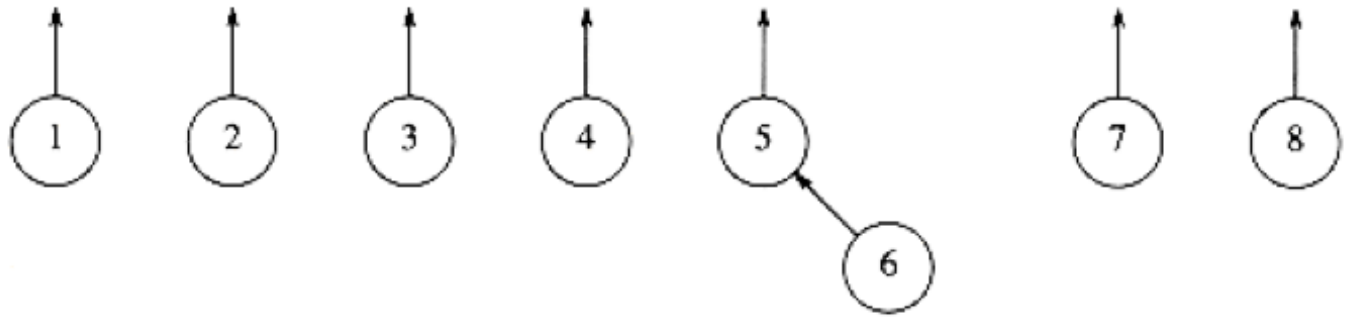


Figure 8.2 After union (5, 6)

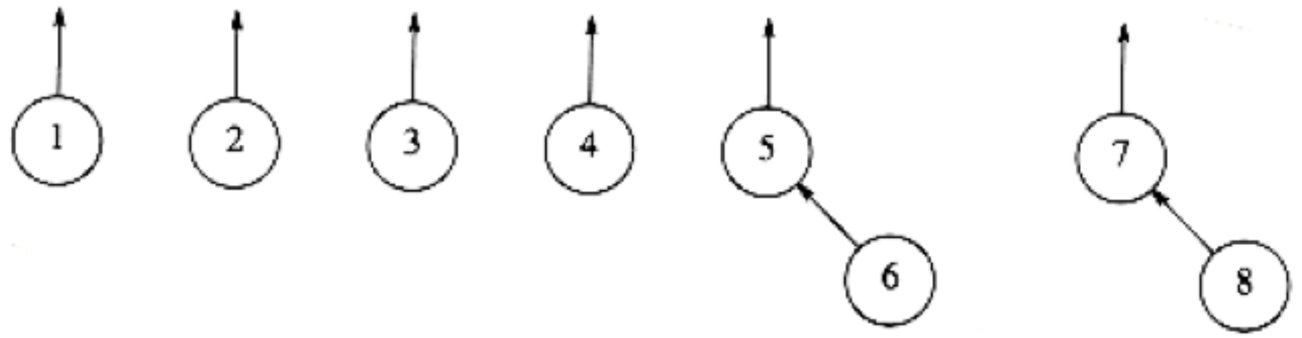


Figure 8.3 After union (7, 8)

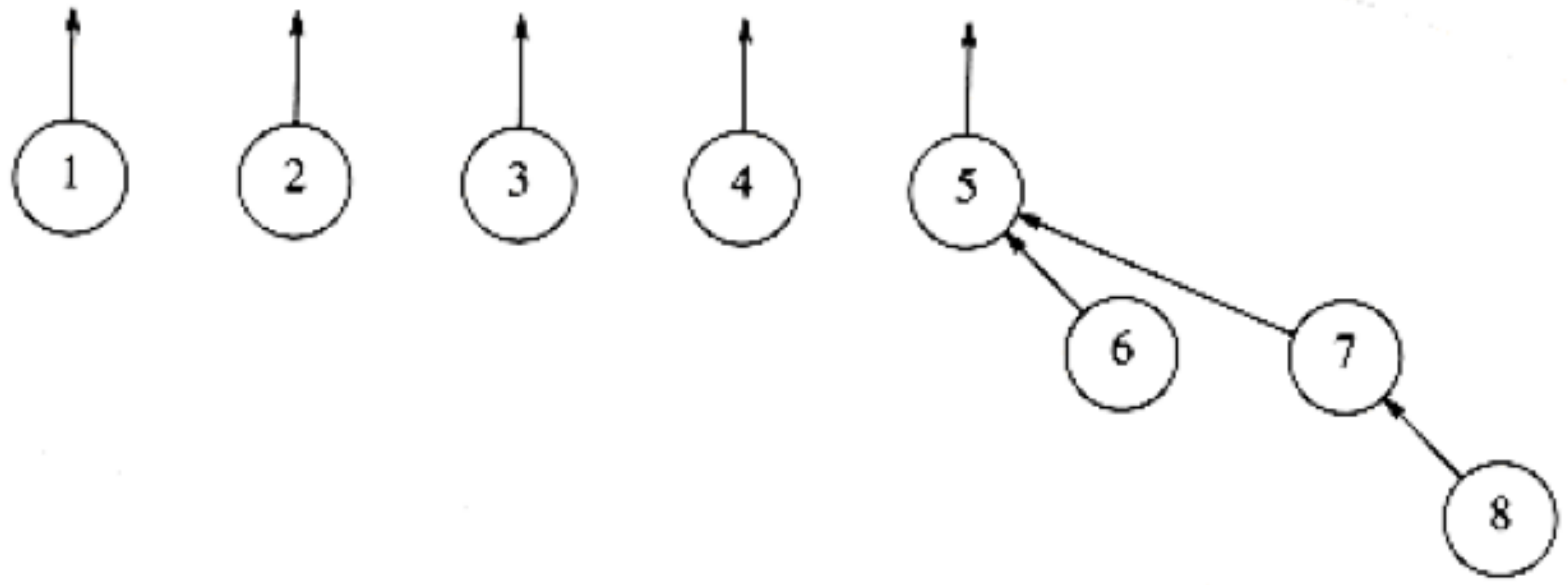


Figure 8.4 After union (5, 7)

0	0	0	0	0	5	5	7
1	2	3	4	5	6	7	8

Union (4,5)

Small tree links to large

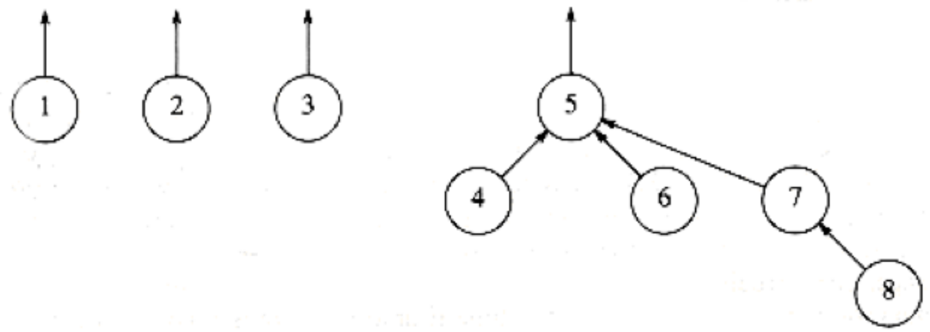


Figure 8.10 Result of union-by-size

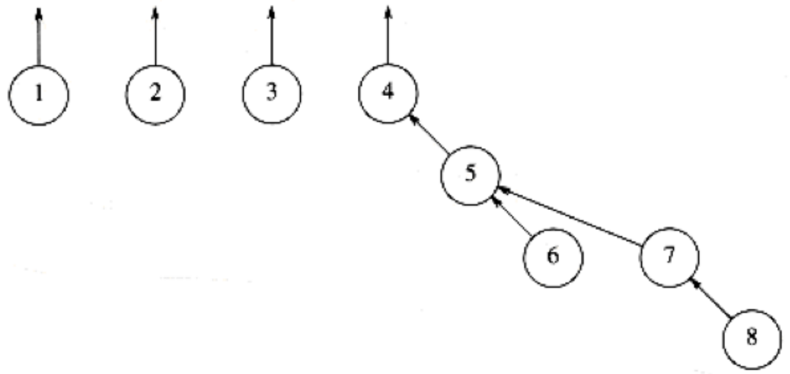


Figure 8.11 Result of an arbitrary union

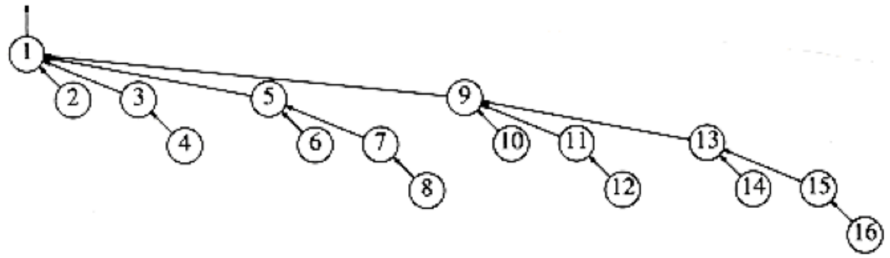
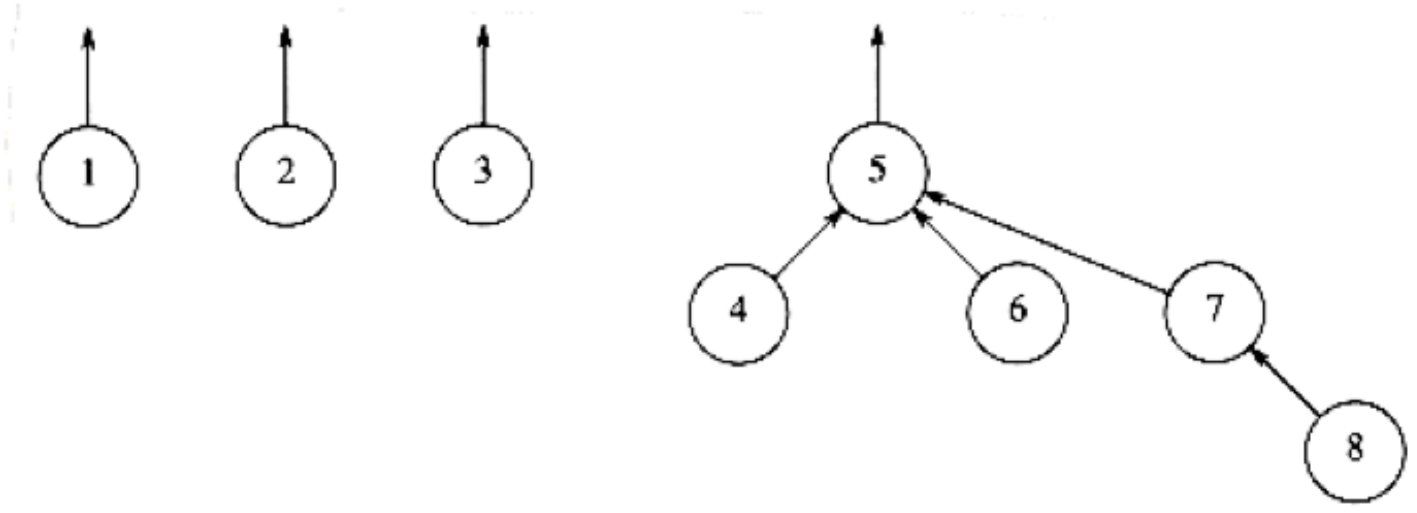


Figure 8.12 Worst-case tree for n = 16

Worst case of union-by-size depth is O(log(N))

The following figures show a tree and its implicit representation for both union-by-size and union-by-height. The code in Figure 8.13 implements union-by-height.



−1	−1	−1	5	−5	5	5	7
1	2	3	4	5	6	7	8

- size

0	0	0	5	−2	5	5	7
1	2	3	4	5	6	7	8

- height

Log*(N) inverse Ackermann function:

- *Ackerman Function:*
  - *$A(i) = 2^{A(i-1)}$; $A(0) = 1$*
  - *$A(1) = 2$, $A(2) = 4$, $A(3) = 16$, $A(4) = 2^{16} = 65536$, $A(5) = 2^{65636}$,*
  - *A(6) = VERY VERY VERY BIG!*
- *Inverse Ackerman:*
  - *i= Log*(N) = min number times you take $\log_2$ to get equal or smaller than below 1.*
  - *Worst case Union-by-Rank with path compression is O(M log^*(N) for M unions.*

Binomial Queue

- Combine: Priority Queue and Union/Find: log(N) Forest of trees:

  $H = n_0\, B_0 + n_1\, B_1 + n_2\, B_2 + \ldots + n_p\, B_p$

  with $n_i = 0,1$
- B_0 = root, $B_{k+1} = B_k + B_k$ attached to root of first. Since B_k has 2^k nodes this is just at binary bit representation $N = (n_p \ldots, n_0)$ for nodes of size $N \cdot 2^p$
- Build so that min key is in root of B_k.
- Adding H_1 to H_2 is binary arithmetic O(p = log(N))
- Always combing B_k + B_k \rightarrow B_{k+1} with min at root.

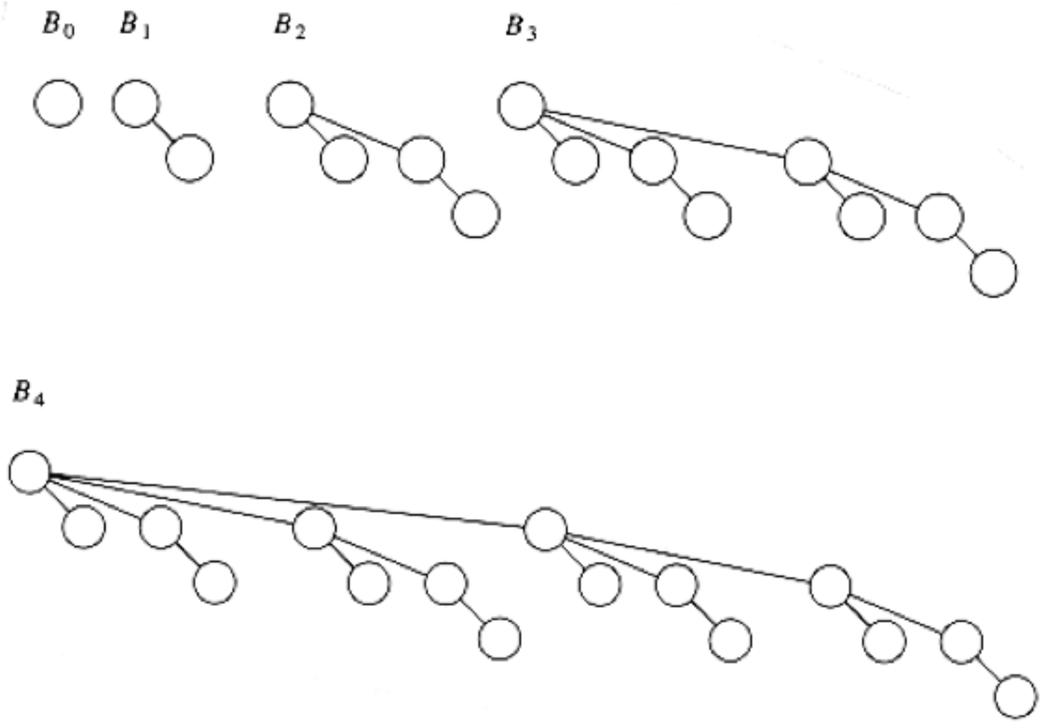
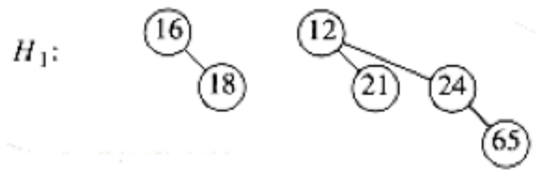


Figure 6.34 Binomial trees B_0, B_1, B_2, B_3, and B_4



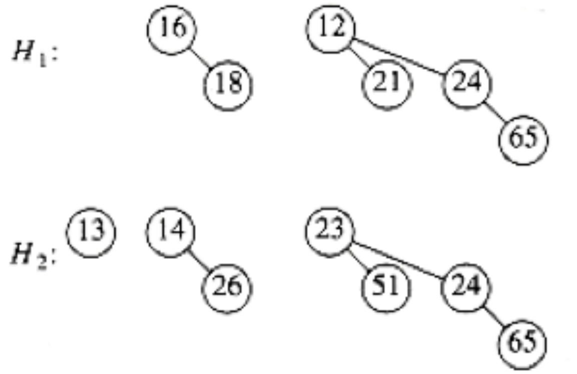


Figure 6.36 Two binomial queues H_1 and H_2

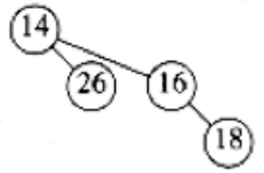
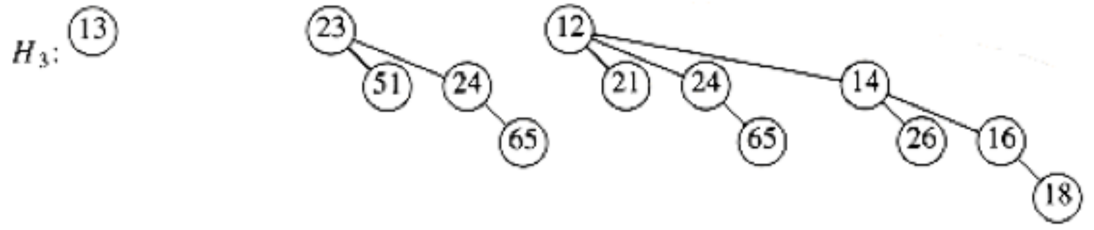


Figure 6.37 Merge of the two B_1 trees in H_1 and H_2


H_3:
13
23
51
24
65
12
21
24
65
14
26
16
18