EC 504 – Fall 2023 – Homework 1 SOLUTION

Due Friday Sept 22, 2023 at 11:59 PM Boston Time

Start reading Chapters 1, 2, 3 and 4 in CLRS . They give a very readable introduction to Algorithms.. Also glance at Appendix A for math tricks which we wills use from time to time.

NOTE: THESE SOLUTIONS OFTEN HAVE FAR MORE DETAILS THAN RQUIRED. The purpose is to provide instrutive to sotumete in-class qustions and in-class dicustions.

1. (40 pts) In CRLS do Exercise 3.1-4 on page 53, Problem 3.2-7 on page 60, Problem 3-2 on page 61 , Problem 4.3-9 on page 88.

Missing Solution to Execise 3.1-4

Is 2^(n+1) = O(2^n) TRUE since 2^(n+1) = 2 * 2^n = constant * 2^n

is 2^(2 n) = O(2^n) FALSE since 2^(2 n) = 2^n * 2^n there is not constant multiply: note limit 2^(2 n)/2^n = 2^n —> infinity

Problem 1a : Problem 3.2-7

Prove by induction that the i^{TH} Fibonacci number satisfies the equality:

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$$

Recall:

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

Base case: We know that $F(0) = 0$ and $F(1) = 1$. We sill show that the claim holds for both $i = 0$ and $i = 1$:

$$F_0 = \left(\frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}\right)\Bigg|_{i=0} = \frac{1-1}{\sqrt{5}}$$

$$\therefore F_0 = 0 \quad \square$$

$$F_1 = \left(\frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}\right)\Bigg|_{i=1} = \frac{\phi - \hat{\phi}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right)$$

$$\therefore F_1 = 1 \quad \square$$

Induction step: Assume the claim holds for i and $i-1$ - our induction hypothesis is as follows:

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}, \quad F_{i-1} = \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}}$$

Show that if the claim holds for i and $i-1$, the claim must hold for $i+1$, namely:

$$F_{i+1} = \frac{\phi^{i+1} - \hat{\phi}^{i+1}}{\sqrt{5}}$$

By definition of the Fibonacci numbers:

$$F_{i+1} = F_i + F_{i-1}$$

Using the induction hypothesis:

$$F_{i+1} = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} + \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}}$$

$$F_{i+1} = \frac{1}{\sqrt{5}}\left(\phi^i + \phi^{i-1}\right) - \frac{1}{\sqrt{5}}\left(\hat{\phi}^i + \hat{\phi}^{i-1}\right)$$

Using $\phi = \frac{1+\sqrt{5}}{2}$:

$$\phi^i + \phi^{i-1} = \left(\frac{1+\sqrt{5}}{2}\right)^i + \left(\frac{1+\sqrt{5}}{2}\right)^{i-1}$$

$$\phi^i + \phi^{i-1} = \left(\frac{1+\sqrt{5}}{2}\right)^i \left[1 + \frac{2}{1+\sqrt{5}}\right]$$

$$\phi^i + \phi^{i-1} = \phi^i \left(\frac{1 + \sqrt{5} + 2}{1 + \sqrt{5}} \right) = \phi^i \left(\frac{3 + \sqrt{5}}{1 + \sqrt{5}} \right)$$

$$\phi^i + \phi^{i-1} = \phi^i \left(\frac{3 + \sqrt{5}}{1 + \sqrt{5}} \right) \left(\frac{1 - \sqrt{5}}{1 - \sqrt{5}} \right) = \phi^i \left(\frac{3 - 3\sqrt{5} + \sqrt{5} - 5}{1 - \sqrt{5} + \sqrt{5} - 5} \right) = \phi^i \left(\frac{-2 - 2\sqrt{5}}{-4} \right) = \phi^i \left(\frac{-2(1 + \sqrt{5})}{-4} \right)$$

$$\phi^i + \phi^{i-1} = \phi^i \times \left(\frac{1 + \sqrt{5}}{2} \right) = \phi^i \times \phi$$

$$\therefore \phi^i + \phi^{i-1} = \phi^{i+1}$$

Similarly, using $\hat{\phi} = \frac{1-\sqrt{5}}{2}$:

$$\hat{\phi}^i + \hat{\phi}^{i-1} = \left(\frac{1 - \sqrt{5}}{2} \right)^i + \left(\frac{1 - \sqrt{5}}{2} \right)^{i-1}$$

$$\hat{\phi}^i + \hat{\phi}^{i-1} = \left(\frac{1 - \sqrt{5}}{2} \right)^i \left[1 + \frac{2}{1 - \sqrt{5}} \right]$$

$$\hat{\phi}^i + \hat{\phi}^{i-1} = \hat{\phi}^i \left(\frac{1 - \sqrt{5} + 2}{1 - \sqrt{5}} \right) = \hat{\phi}^i \left(\frac{3 - \sqrt{5}}{1 - \sqrt{5}} \right)$$

$$\hat{\phi}^i + \hat{\phi}^{i-1} = \hat{\phi}^i \left(\frac{3 - \sqrt{5}}{1 - \sqrt{5}} \right) \left(\frac{1 + \sqrt{5}}{1 + \sqrt{5}} \right) = \hat{\phi}^i \left(\frac{3 + 3\sqrt{5} - \sqrt{5} - 5}{1 + \sqrt{5} - \sqrt{5} - 5} \right) = \hat{\phi}^i \left(\frac{-2 + 2\sqrt{5}}{-4} \right) = \hat{\phi}^i \left(\frac{-2(1 - \sqrt{5})}{-4} \right)$$

$$\hat{\phi}^i + \hat{\phi}^{i-1} = \hat{\phi}^i \times \left(\frac{1 - \sqrt{5}}{2} \right) = \hat{\phi}^i \times \hat{\phi}$$

$$\therefore \hat{\phi}^i + \hat{\phi}^{i-1} = \hat{\phi}^{i+1}$$

Putting it all together:

$$F_{i+1} = \frac{1}{\sqrt{5}} \left(\phi^i + \phi^{i-1} \right) - \frac{1}{\sqrt{5}} \left(\hat{\phi}^i + \hat{\phi}^{i-1} \right) = \frac{1}{\sqrt{5}} \phi^{i+1} - \frac{1}{\sqrt{5}} \hat{\phi}^{i+1}$$

$$\therefore F_{i+1} = \frac{\phi^{i+1} - \hat{\phi}^{i+1}}{\sqrt{5}} \quad \square$$

Therefore, by induction, we have shown that the i^{TH} Fibonacci number can be expressed as $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$.

Problem 1a : Problem 3-2

Indicate, for each pair of expressions (A, B) in the table below whether A is $\{O,\ o,\ \Omega,\ \omega,\ \Theta\}$ of B. Assume that $k \geq 1$, $\epsilon > 0$, and $c > 1$.

A	B	O	o	Ω	ω	Θ
$\lg^k n$	n^ϵ	✓	✓	×	×	×
n^k	c^n	✓	✓	×	×	×
$\sqrt{n}$	$n^{\sin n}$	×	×	×	×	×
2^n	$2^{n/2}$	×	×	✓	✓	×
$n^{\lg c}$	$c^{\lg n}$	✓	×	✓	×	✓
$\lg n!$	$\lg n^n$	✓	×	✓	×	✓

Recalling the graphical relationship between the various asymptotic notations:

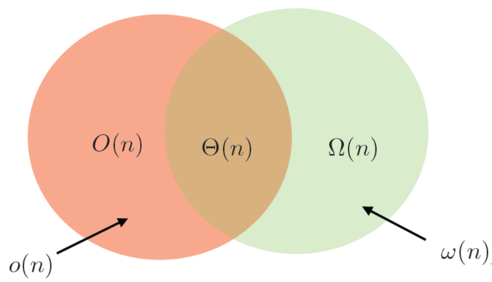

Figure 1: Relationship between asymptotic notations

We can write the formal definitions:

$$f(n) \in O\big((g(n))\big) \iff \exists c > 0,\ n_0\ s.t.\ f(n) \leq cg(n)\ \forall n \geq n_0$$

$$f(n) \in o\big((g(n))\big) \iff \forall c > 0,\ n_0\ s.t.\ f(n) < cg(n)\ \forall n \geq n_0$$

$$f(n) \in \Omega\big((g(n))\big) \iff \exists c > 0,\ n_0\ s.t.\ f(n) \geq cg(n)\ \forall n \geq n_0$$

$$f(n) \in \omega\big((g(n))\big) \iff \forall c > 0,\ n_0\ s.t.\ f(n) > cg(n)\ \forall n \geq n_0$$

We can write the limit definitions:

$$f(n) \in O\big((g(n))\big) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$f(n) \in o\big((g(n))\big) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

$$f(n) \in \Omega\big((g(n))\big) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

$$f(n) \in \omega\big((g(n))\big) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

$$f(n) \in \Theta\big(g(n)\big) \iff f(n) \in O\big(g(n)\big) \wedge f(n) \in \Omega\big(g(n)\big)$$

It is important to note that if $f(n) \in o\big(g(n)\big)$, then $f(n) \in O\big(g(n)\big)$; this follows from the limit definitions above. Similarly, if $f(n) \in \omega\big(g(n)\big)$, then $f(n) \in \Omega\big(g(n)\big)$.

$\lg^k n \ vs. \ n^{\epsilon}$:
Let $n = 2^m, \ m \geq 0$. Then we can express the limit as:

$$\lim_{n\to\infty} \frac{\lg^k n}{n^{\epsilon}} = \lim_{n\to\infty} \frac{(\lg n)^k}{n^{\epsilon}} = \lim_{m\to\infty} \frac{(\lg 2^m)^k}{(2^m)^{\epsilon}}$$

Then:

$$\lim_{m\to\infty} = \frac{(m \lg 2)^k}{2^{m\epsilon}} = \lim_{m\to\infty} \frac{m^k}{2^{m\epsilon}} = \lim_{m\to\infty} \frac{m^k}{(2^{\epsilon})^m}$$

Let $\alpha = 2^{\epsilon}$; since $\epsilon > 0$, $\alpha = 2^m > 1$:

$$\lim_{m\to\infty} \frac{m^k}{\alpha^m} = 0$$

Thus, $\lg^k n \in o\left(n^{\epsilon}\right)$; since $\lg^k n \in o\left(n^{\epsilon}\right)$, we also have that $\lg^k n \in O\left(n^{\epsilon}\right)$.

$n^k \ vs. \ c^n$:
Taking the limit:

$$\lim_{n\to\infty} \frac{n^k}{c^n}$$

Since $c > 1$, we note the same asymptotic behavior as the previous problem:

$$\lim_{n\to\infty} \frac{n^k}{c^n} = 0$$

Thus, $n^k \in o\left(c^n\right)$; since $n^k \in o\left(c^n\right)$, we also have that $n^k \in O\left(c^n\right)$.

$\sqrt{n} \ vs. n^{\sin n}$:
We cannot make any asymptotic comparison between the two functions. Recall that $\sqrt{n}$ is a strictly increasing function. Thus, if $\sqrt{n} \in O(g(n))$, $\exists c > 0$ such that $g(n)$ is also strictly increasing for $n \geq n_0$. The function $g(n) = n^{\sin n}$ cannot satisfy this requirement; Let $N = \{n : n \geq n_0, \ \sin n = -1\}$[1]. Since $\sin n$ is a periodic function, there are an infinite number of points in N. However, for all points in N:

$$n^{\sin n} = n^{-1} = \frac{1}{n} \ \forall n \in N$$

However, $\sqrt{n} \geq \frac{1}{n} \ \forall n \geq 1$; we can always find an infinite number of points where $n \geq n_0$ such that $f(n) = \sqrt{n} > g(n) \ \forall n \in N$, violating the bound. We cannot use the constant c to help reverse the inequality, as we would need c to change for an infinite number of points; thus we would never be able to find a constant c to always satisfy the inequality $f(n) \leq cg(n) \ \forall n \geq n_0$. Therefore, $\sqrt{n} \notin O\left(n^{\sin n}\right)$. Since $\sqrt{n} \notin O\left(n^{\sin n}\right)$, we can immediately conclude that $\sqrt{n} \notin o\left(n^{\sin n}\right)$ and $\sqrt{n} \notin \Theta\left(n^{\sin n}\right)$.

We can repeat the argument, except for a lower bound - the argument proceeds in the exact same way, and we can conclude that $\sqrt{n} \notin \Omega\left(n^{\sin n}\right)$. Since $\sqrt{n} \notin \Omega\left(n^{\sin n}\right)$, we can immediately conclude that $\sqrt{n} \notin \omega\left(n^{\sin n}\right)$.

$2^n \ vs. \ 2^{n/2}$
Computing the limit, we see:

$$\lim_{n\to\infty} \frac{2^n}{2^{n/2}} = \lim_{n\to\infty} \frac{2^n}{\left(2^{1/2}\right)^n} = \lim_{n\to\infty} \left(\frac{2}{\sqrt{2}}\right)^n \lim_{n\to\infty} \left(\sqrt{2}\right)^n = \infty$$

[1] I am implicitly extending n to the reals here - if n is an integer then the equality will not hold, but the general structure of the argument still holds. It's conceptually easier to argue from the reals.

Thus, $2^n \in \omega\left(2^{n/2}\right)$; since $2^n \in \omega\left(2^{n/2}\right)$, we also have that $2^n \in \Omega\left(2^{n/2}\right)$.

$n^{\lg c}$ *vs.* $c^{\lg n}$:
Let $\lg c = q$; then $c = 2^q$. Let $n = 2^x$. Then we can express $n^{\lg c}$ as:

$$n^{\lg c} = (2^x)^{\lg c} = \left(2^{\lg c}\right)^x = c^x$$

But since $n = 2^x$, we have $x = \lg n$. Thus:

$$n^{\lg c} = c^x = c^{\lg n}$$

$$\therefore n^{\lg c} = c^{\lg n}$$

Since $n^{\lg c} = c^{\lg n}$, we have $f(n) = g(n)$; thus, by the reflexivity property, $f(n) \in O\big(f(n)\big)$, $f(n) \in \Omega\big(f(n)\big)$ $f(n) \in \Theta\big(f(n)\big)$.
Thus, $n^{\lg c} \in O\left(c^{\lg n}\right)$, $n^{\lg c} \in \Omega\left(c^{\lg n}\right)$, and $n^{\lg c} \in \Theta\left(c^{\lg n}\right)$.

$\lg n!$ *vs.* $\lg n^n$:
Using Stirling's approximation, we can express $n!$ as:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

Thus $\lg n!$ can be expressed as:

$$\lg n! = \lg\left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right] = \lg\left[\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right] + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\lg n! = \lg\sqrt{2\pi n} + \lg\left(\frac{n}{e}\right)^n + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\lg n! = \lg\left(2\pi n\right)^{1/2} + n\lg\left(\frac{n}{e}\right) + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\lg n! = \frac{1}{2}\lg\left(2\pi n\right) + n\left(\lg n - \lg e\right) + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\lg n! = \frac{1}{2}\left(\lg 2 + \lg\pi + \lg n\right) + n\lg n - n\lg e + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\lg n! = \frac{1}{2} + \frac{\lg\pi}{2} + \frac{1}{2}\lg n + n\lg n - n\lg e + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

$$\therefore \lg n! = n\lg n + \frac{1}{2}\lg n - (\lg e)n + \left(\frac{1 + \lg\pi}{2}\right) + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]$$

We can easily express $\lg n^n$ as $n\lg n$. Taking the limit:

$$\lim_{n\to\infty}\frac{\lg n!}{\lg n^n} = \lim_{n\to\infty}\frac{n\lg n + \frac{1}{2}\lg n - (\lg e)n + \left(\frac{1+\lg\pi}{2}\right) + \lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]}{n\lg n}$$

$$\lim_{n\to\infty}\frac{\lg n!}{\lg n^n} = \lim_{n\to\infty}\frac{n\lg n}{n\lg n} + \frac{1/2\lg n}{n\lg n} - \frac{(\lg e)n}{n\lg n} + \frac{(1 + \lg\pi)/2}{n\lg n} + \frac{\lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]}{n\lg n}$$

$$\lim_{n\to\infty}\frac{\lg n!}{\lg n^n} = 1 + \frac{1}{2n} - \frac{\lg e}{\ln n} + \frac{(1 + \lg\pi)/2}{n\lg n} + \frac{\lg\left[\left(1 + \Theta\left(\frac{1}{n}\right)\right)\right]}{n\lg n}$$

$$\lim_{n\to\infty} \frac{\lg n!}{\lg n^n} = 1 + \lim_{n\to\infty} \frac{\lg\left[\left(1+\Theta\left(\frac{1}{n}\right)\right)\right]}{n \lg n}$$

By definition, for sufficiently large n, if $f(n) = \Theta\left(\frac{1}{n}\right)$, then $\exists c_2 > c_1 > 0,\ s.t.\ \frac{c_1}{n} \leq f(n) \leq \frac{c_2}{n}$. Thus:

$$\lg\left[1+\Theta\left(\frac{1}{n}\right)\right] \sim \lg\left(1+\frac{c}{n}\right) = \lg\left(\frac{n+c}{n}\right) = \lg(n+c) - \lg n$$

Using this result in the limit:

$$\lim_{n\to\infty} \frac{\lg n!}{\lg n^n} = 1 + \lim_{n\to\infty} \frac{\lg\left[\left(1+\Theta\left(\frac{1}{n}\right)\right)\right]}{n \lg n} = 1 + \lim_{n\to\infty} \frac{\lg(n+c) - \lg n}{n \lg n}$$

$$\therefore \lim_{n\to\infty} \frac{\lg n!}{\lg n^n} = 1$$

Thus, we can conclude that $\lg n! \in O(\lg n^n)$, $\lg n! \in \Omega(\lg n^n)$, and thus $\lg n! \in \Theta(\lg n^n)$.

Problem 1a : Problem 4.3-9

Solve the recurrence $T(n) = 3T(\sqrt{n}) + \log n$ by making a change of variables.

For simplicity, assume $\log n = \lg n$; let $n = 2^m$. Then:

$$T(n) = 3T(\sqrt{n}) + \lg n \Longleftrightarrow T\left(2^m\right) = 3T\left(\sqrt{2^m}\right) + \lg 2^m$$

$$T\left(2^m\right) = 3T\left(2^{m/2}\right) + m \lg 2$$

$$T\left(2^m\right) = 3T\left(2^{m/2}\right) + m$$

Let $\tilde{T}(m) = T\left(2^m\right)$; then we can write:

$$\tilde{T}\left(m\right) = 3\tilde{T}\left(\frac{m}{2}\right) + m$$

This recurrence relation fits the form of the master equation, with $a = 3$, $b = 2$, $f(m) = m$, and $k = 1$. Computing $\gamma = \frac{\ln a}{\ln b} = \log_2 3 > 1$, we see that $\gamma > k$; thus, we can immediately write the solution to the equation:

$$\tilde{T}(m) = \Theta\left(m^\gamma\right)$$

Recalling that $n = 2^m$, we can reverse the change of variables and find an expression for $T(n)$:

$$\tilde{T}(m) \in \Theta\left(m^\gamma\right)$$

$$\tilde{T}(\lg n) \in \Theta\left((\lg n)^\gamma\right)$$

Since $\tilde{T}(m) = T(2^m)$:

$$T\left(2^{\lg n}\right) \in \Theta\left((\lg n)^\gamma\right)$$

$$\therefore T(n) \in \Theta\left((\lg n)^\gamma\right)$$

Thus, $T(n) \in \Theta\left(\lg^\gamma n\right)$ with $\gamma = \log_2 3 > 1$.

Problem 1b

Place the following functions in order from asymptotically smallest to largest using $f(n) \in O(g(n))$ notation:

$$n^2 + 3n\log n + 5, \quad n^2 + n^{-2}, \quad n^{n^2} + n!, \quad n^{\frac{1}{n}}, \quad n^{n^2-1}, \quad \ln n, \quad \ln\ln n, \quad 3^{\ln n}, \quad 2^n$$

$$(1+n)^n, \quad n^{1+\cos n}, \quad \sum_{k=1}^{\log n} \frac{n^2}{2^k}, \quad 1, \quad n^2 + 3n + 5, \quad \log n!, \quad \sum_{k=1}^{n} \frac{1}{k}, \quad \prod_{k=1}^{n}\left(1 - \frac{1}{k^2}\right), \quad \left(1 - \frac{1}{n}\right)^n$$

Function $f(n) \in O(g(n))$ if $\exists c > 0$ such that:

$$f(n) \leq cg(n) \; \forall n \geq n_0$$

Equivalently, we can use the limit definition:

$$f(n) \in O(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$f_1(n) = n^2 + 3n\log n + 5$:

$$\lim_{n\to\infty} \frac{f_1(n)}{n^2} = \lim_{n\to\infty}\left(\frac{n^2 + 3n\log n + 5}{n^2}\right) = \lim_{n\to\infty}\left(1 + \frac{3\log n}{n} + \frac{5}{n^2}\right) = 1$$

$\therefore f_1(n) = n^2 + 3n\log n + 5 \in O(n^2)$

$f_2(n) = n^2 + n^{-2}$:

$$\lim_{n\to\infty} \frac{f_2(n)}{n^2} = \lim_{n\to\infty}\left(\frac{n^2 + n^{-2}}{n^2}\right) = \lim_{n\to\infty}\left(1 + \frac{1}{n^4}\right) = 1$$

$\therefore f_2(n) = n^2 + n^{-2} \in O(n^2)$

$f_3(n) = n^{n^2} + n!$:

$$\lim_{n\to\infty} \frac{f_3(n)}{n^{n^2}} = \lim_{n\to\infty} \frac{n^{n^2} + n!}{n^{n^2}} = \lim_{n\to\infty}\left(1 + \frac{n!}{n^{n^2}}\right) = 1 + \lim_{n\to\infty} \frac{n!}{n^{n^2}}$$

By definition:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

$$n! = \prod_{i=1}^{n} i < \prod_{i=1}^{n} n < n^n < n^{n^2}$$

Therefore:

$$\lim_{n\to\infty} \frac{n!}{n^{n^2}} = 0$$

$\therefore f_3(n) = n^{n^2} + n! \in O(n^{n^2})$

$f_4(n) = n^{\frac{1}{n}}$:

$$\lim_{n\to\infty} \frac{f_4(n)}{1} = \lim_{n\to\infty}\left(\frac{n^{\frac{1}{n}}}{1}\right) = \lim_{n\to\infty} n^{1/n}$$

Evaluating this limit, we arrive at the indeterminate form (∞^0); we can solve by transforming the limit as follows:

$$\lim_{n\to\infty} f(x)^{g(x)} = \exp \lim_{n\to\infty} \frac{\ln f(x)}{\left(\frac{1}{g(x)}\right)} = \exp\left(\lim_{n\to\infty} g(x) \ln f(x)\right)$$

Therefore, with $f(x) = n,\ g(x) = \frac{1}{n}$:

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \exp\left(\lim_{n\to\infty}\left(\frac{1}{n}\right)(\ln n)\right) = \exp\left(\lim_{n\to\infty}\left(\frac{\ln n}{n}\right)\right) = \exp 0 = 1$$

$\therefore\ f_4(n) = n^{\frac{1}{n}} \in O(1)$

$f_5(n) = n^{n^2-1}$:

$$\lim_{n\to\infty} \frac{f_5(n)}{n^{n^2}} = \lim_{n\to\infty} \frac{n^{n^2-1}}{n^{n^2}} = \lim_{n\to\infty}\left(\frac{1}{n^{n^2}} \times \frac{n^{n^2}}{n}\right) = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\therefore\ f_5(n) = n^{n^2-1} \in O(n^{n^2})$

$f_6(n) = \ln n$:

$$\lim_{n\to\infty} \frac{f_6(n)}{\ln n} = \frac{\ln n}{\ln n} = 1$$

$\therefore\ f_6(n) = \ln n \in O(\ln n)$

$f_7(n) = \ln\ln n$:

$$\lim_{n\to\infty} \frac{f_7(n)}{\ln\ln n} = \frac{\ln\ln n}{\ln\ln n} = 1$$

$\therefore\ f_7(n) = \ln\ln n \in O(\ln\ln n)$

$f_8(n) = 3^{\ln n}$:
Let $\ln n = m$, and let $e^x = 3,\ x = \ln 3$; then:

$$3^{\ln n} = (e^x)^m = e^{xm} = (e^m)^x$$

$$3^{\ln n} = \left(e^{\ln n}\right)^x = n^x = n^{\ln 3} = n^\gamma,\ \gamma = \ln 3$$

$\therefore\ f_8(n) = 3^{\ln n} \in O\left(n^\gamma\right),\ \gamma < 2$

$f_9(n) = 2^n$:

$$\lim_{n\to\infty} \frac{f_{9(n)}}{2^n} = \lim_{n\to\infty} \frac{2^n}{2^n} = 1$$

$\therefore\ f_9(n) = 2^n \in O(2^n)$

$f_{10}(n) = (1+n)^n$:

$$\lim_{n\to\infty} \frac{f_{10(n)}}{n^n} = \lim_{n\to\infty} \frac{(1+n)^n}{n^n}$$

Using the binomial theorem to expand $(1+n)^n$:

$$(1+n)^n = \sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} n^k = \sum_{k=0}^{n} \binom{n}{k} n^k = n^n + \sum_{k=0}^{n-1} \binom{n}{k} n^k = n^n + \sum_{k=0}^{n-1} c_k n^k$$

$$\lim_{n\to\infty} \frac{(1+n)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n^n + \sum_{k=0}^{n-1} c_k n^k}{n^n} \right) = \lim_{n\to\infty} \left(1 + \sum_{k=0}^{n-1} c_k \frac{n^k}{n^n} \right) = 1$$

$\therefore f_{10}(n) = (1+n)^n \in O(n^n)$

$f_{11}(n) = n^{1+\cos n}$:
Use the definition:

$$f(n) \in O(g(n)) \iff \exists c < 0 \; s.t. \; f(n) \leq cg(n) \; \forall n \geq n_0$$

By definition, $\cos n \in [-1, 1]$; thus, $(1+\cos n) \in [0, 2]$. Since $(1+\cos n) \leq 2 \; \forall n \geq 1$, we can bound $n^{1+\cos n}$ accordingly: $n^{1+\cos n} \leq n^2 \; \forall n \geq 1$.
$\therefore f_{11}(n) = n^{1+\cos n} \in O(n^2)$

$f_{12}(n) = \sum_{k=1}^{\log n} \frac{n^2}{2^k}$:
Let $b = \log n$; then the sum can be expressed as:

$$f_{12}(n) = \sum_{k=1}^{\log n} \frac{n^2}{2^k} = n^2 \sum_{k=1}^{b} \frac{1}{2^k} = n^2 \sum_{k=1}^{b} \left(\frac{1}{2}\right)^k = n^2 \sum_{k=1}^{b} \alpha^k$$

where $\alpha = 1/2$. Let $s_b = \sum_{k=1}^{b} \alpha^k$. Then:

$$s_b = \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{b-1} + \alpha^b$$

$$\alpha s_b = \alpha \left(\alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{b-1} + \alpha^b \right) = \alpha^2 + \alpha^3 + \ldots + \alpha^b + \alpha^{b+1}$$

$$\begin{aligned} s_b - \alpha s_b = {} & \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{b-1} + \alpha^b \\ & - \alpha^2 - \alpha^3 - \ldots - \alpha^{b-1} - \alpha^b - \alpha^{b+1} \end{aligned}$$

$$s_b - \alpha s_b = \alpha - \alpha^{b+1}$$

$$s_b(1-\alpha) = \alpha \left(1 - \alpha^b\right)$$

$$\therefore s_b = \frac{\alpha \left(1 - \alpha^b\right)}{1-\alpha}$$

Using $\alpha = 1/2$:

$$s_b = \frac{\alpha \left(1 - \alpha^b\right)}{1-\alpha} = \frac{(1/2)\left(1 - (1/2)^b\right)}{(1-1/2)} = \frac{(1/2)\left(1 - \frac{1}{2^b}\right)}{(1/2)}$$

$$\therefore s_b = 1 - 2^{-b}$$

Thus, we can write the function $f_{12}(n)$ as:

$$f_{12}(n) = \sum_{k=1}^{\log n} \frac{n^2}{2^k} = n^2 s_b = \left(1 - 2^{-b}\right) n^2$$

Recall $b = \log n$; for $n > 1$, we have that $\log n > 0$, and and thus $b > 0$. Therefore, for $n > 1$, $2^{-b} < 1$, and thus the whole prefactor $\left(1 - 2^{-b}\right) < 1$. Therefore, for $n > 1$, $\left(1 - 2^{-b}\right) n^2 < n^2$.
$\therefore f_{12}(n) = \sum_{k=1}^{\log n} \frac{n^2}{2^k} \in O(n^2)$

$f_{13}(n) = 1 \implies f_{13}(n) \in O(1)$

$f_{14}(n) = n^2 + 3n + 5$:

$$\lim_{n\to\infty} \frac{f_{14}(n)}{n^2} = \lim_{n\to\infty} \left(\frac{n^2+3n+5}{n^2}\right) = \lim_{n\to\infty} \left(1 + \frac{3}{n} + \frac{5}{n^2}\right) = 1$$

$\therefore\ f_{14}(n) = n^2 + 3n + 5 \in O(n^2)$

$f_{15}(n) = \log n!$:
We can express $\log n!$ as:

$$\log n! = \log\left(n \times n!\right) = \log n + \log\left((n-1)!\right)$$

Using similar logic, we can express $\log\left((n-1)!\right)$ as:

$$\log\left((n-1)!\right) = \log\left((n-1) \times (n-2)!\right) = \log\left(n-1\right) + \log\left((n-2)!\right)$$

Substituting above:

$$\log n! = \log n + \log\left(n-1\right) + \log\left((n-2)!\right)$$

We can carry out the process for each integer 1 to n, arriving at the sum:

$$\log n! = \sum_{m=1}^{n} \log m$$

Since $m \le n$, we can upper-bound the right sum:

$$\log n! = \sum_{m=1}^{n} \log m \le \sum_{m=1}^{n} \log n = \log n \times \sum_{m=1}^{n} 1$$

$$\therefore\ \log n! \le n \log n\ \forall n \ge 1$$

$\therefore\ f_{15}(n) = \log n! \in O(n \log n)$

$f_{16}(n) = \sum_{k=1}^{n} \frac{1}{k}$:
Using equation A.7 in the text:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$$

Thus, taking the limit:

$$\lim_{n\to\infty} \frac{f_{16(n)}}{\ln n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{1}{k}}{\ln n} = \lim_{n\to\infty} \frac{H_n}{\ln n} = \lim_{n\to\infty} \frac{\ln n + O(1)}{\ln n} = \lim_{n\to\infty} \left(1 + \frac{O(1)}{\ln n}\right) = 1$$

$\therefore\ f_{16}(n) = \sum_{k=1}^{n} \frac{1}{k} \in O(\ln n)$

$f_{17}(n) = \prod_{k=1}^{n} \left(1 - \frac{1}{k^2}\right)$:
Since the product starts at $k = 1$:

$$\left(1 - \frac{1}{k^2}\right)\Big|_{k=1} = 0$$

Therefore, $\forall n$:

$$\prod_{k=1}^{n} \left(1 - \frac{1}{k^2}\right) = 0 \le 1$$

$\therefore f_{17} = \prod_{k=1}^{n}\left(1 - \frac{1}{k^2}\right) = 0 < O(1)$

$f_{18}(n) = \left(1 - \frac{1}{n}\right)^n$:
For $n > 1$, we know that $\frac{1}{n} < 1$; thus for $n > 1$, $\left(1 - \frac{1}{n}\right) < 1$.
Therefore:

$$\left(1 - \frac{1}{n}\right)^n \leq 1^n \leq 1 \; \forall n > 1$$

$\therefore f_{18}(n) = \left(1 - \frac{1}{n}\right)^n \in O(1)$

Finally, we can sort the functions in ascending order:

$O(g(n))$	$f(n)$
$T(n) = 0$	$f_{17}(n) = \prod_{k=1}^{n}\left(1 - \frac{1}{k^2}\right)$
$O(1)$	$f_4(n) = n^{\frac{1}{n}}$ $f_{13}(n) = 1$ $f_{18}(n) = \left(1 - \frac{1}{n}\right)^n$
$O(\ln \ln n)$	$f_7(n) = \ln \ln n$
$O(\ln n)$	$f_6(n) = \ln n$ $f_{16}(n) = \sum_{k=1}^{n} \frac{1}{k}$
$O(n^\gamma),\ \gamma = \ln 3$	$f_8(n) = 3^{\ln n}$
$O(n \ln n)$	$f_{15}(n) = \log n!$
$O(n^2)$	$f_1(n) = n^2 + 3n \log n + 5$ $f_2(n) = n^2 + n^{-2}$ $f_{11}(n) = n^{1+\cos n}$ $f_{12}(n) = \sum_{k=1}^{\log n} \frac{n^2}{2^k}$ $f_{14}(n) = n^2 + 3n + 5$
$O(2^n)$	$f_9(n) = 2^n$
$O(n^n)$	$f_{10}(n) = (1+n)^n$
$O(n^{n^2})$	$f_3(n) = n^{n^2} + n!$ $f_5(n) = n^{n^2-1}$

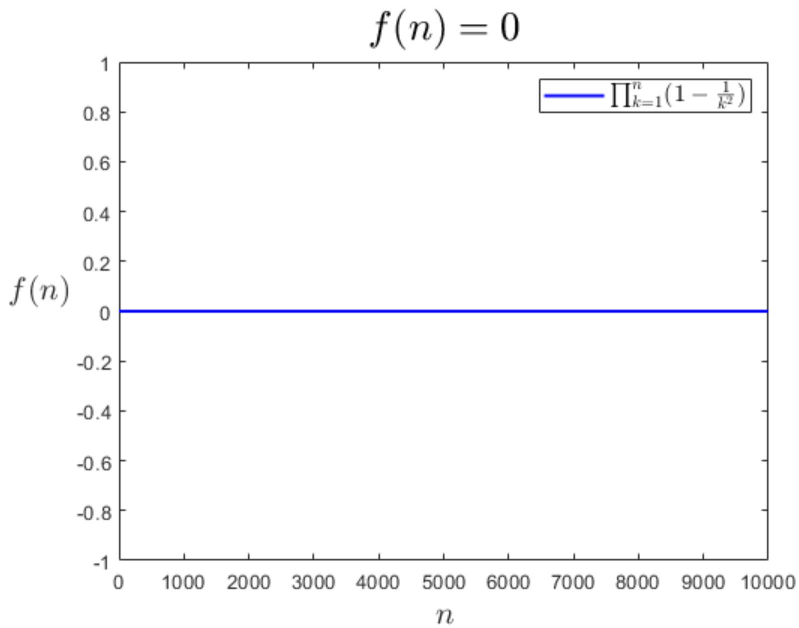

Figure 2: $f(n) = 0$

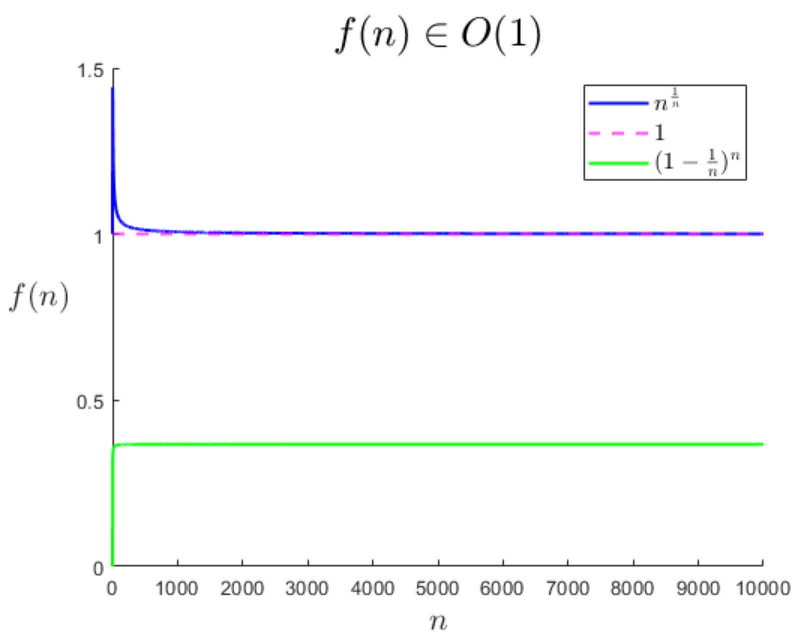

Figure 3: $f(n) \in O(1)$

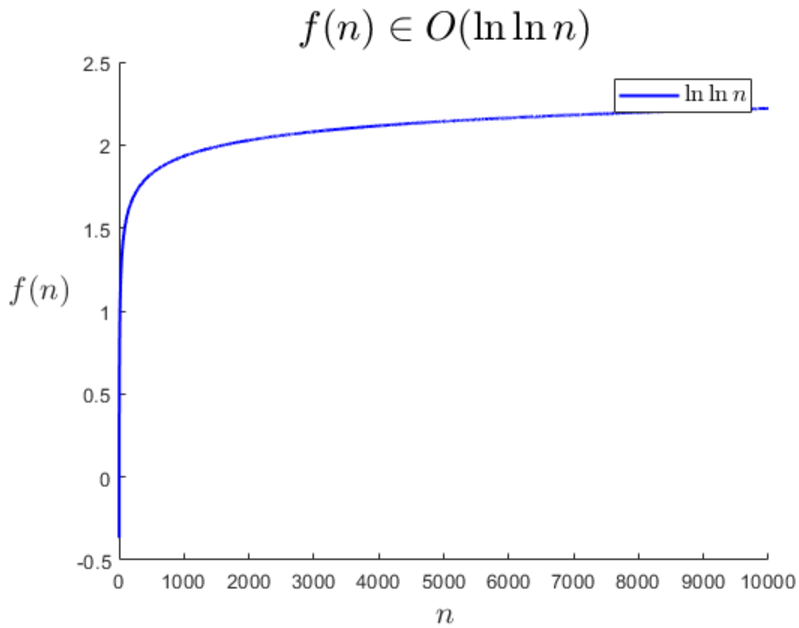

Figure 4: $f(n) \in O(\log \log n)$

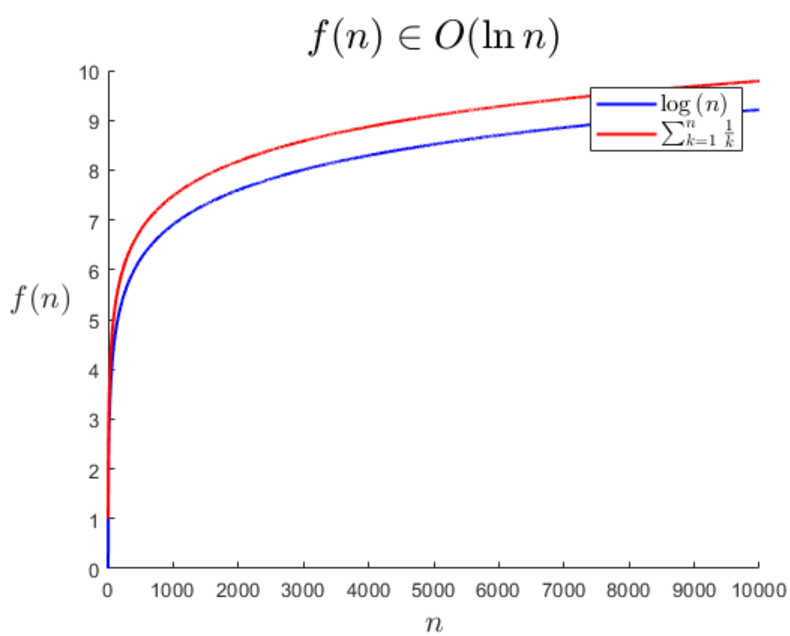

Figure 5: $f(n) \in O(\log n)$

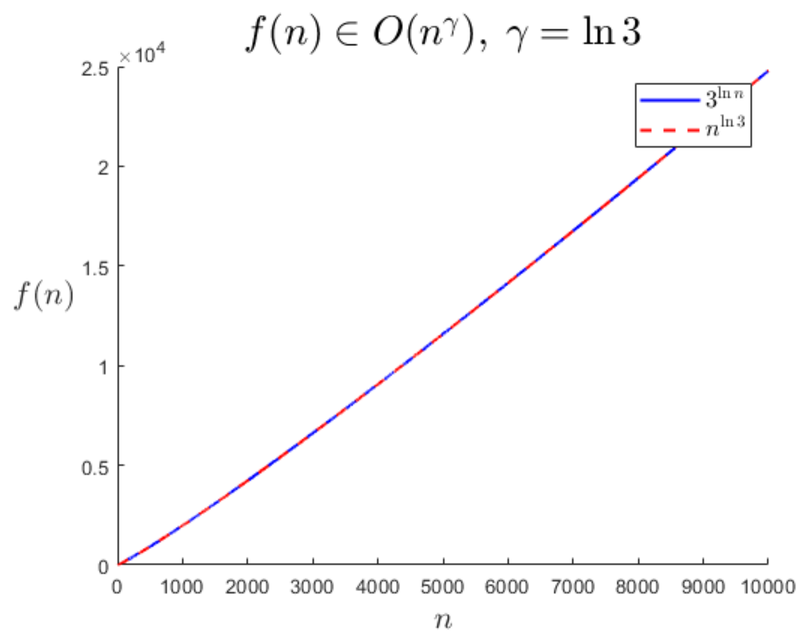

Figure 6: $f(n) \in O(n^{\gamma})$

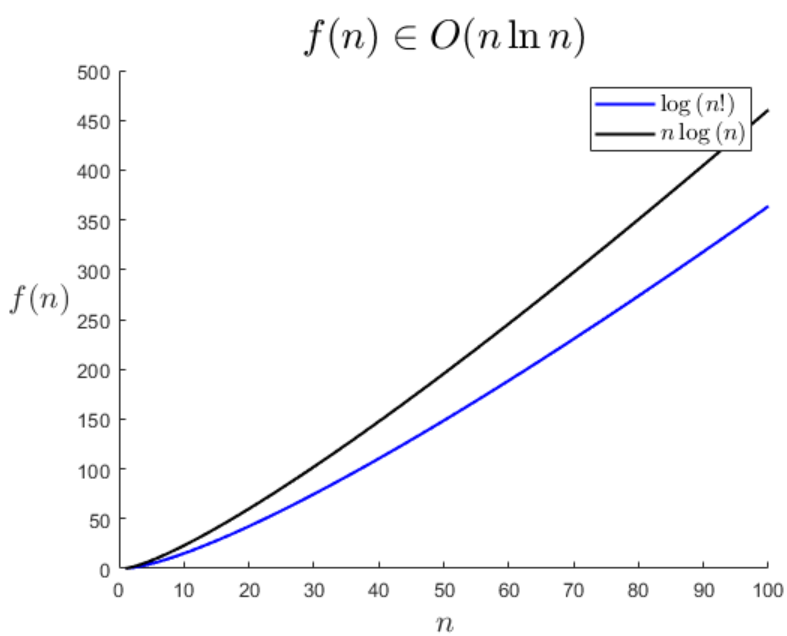

Figure 7: $f(n) \in O(n \log n)$

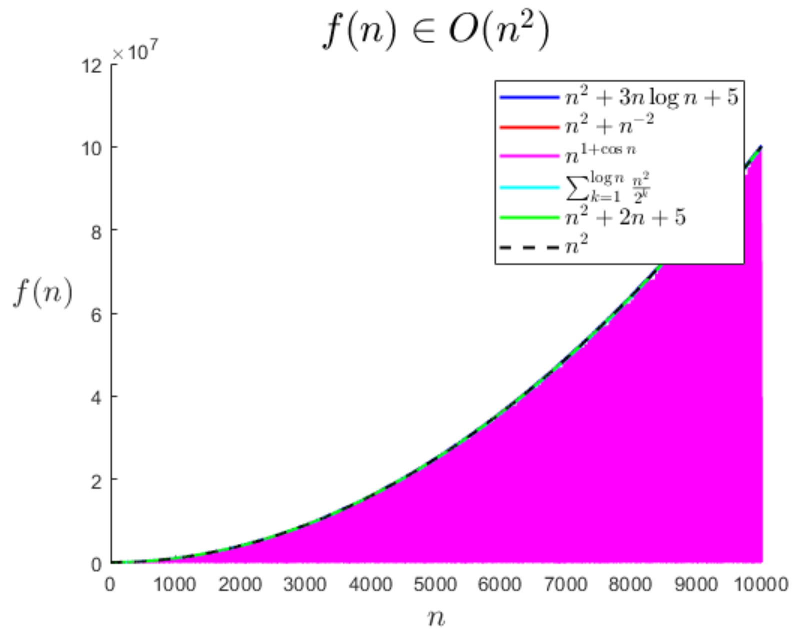

Figure 8: $f(n) \in O(n^2)$

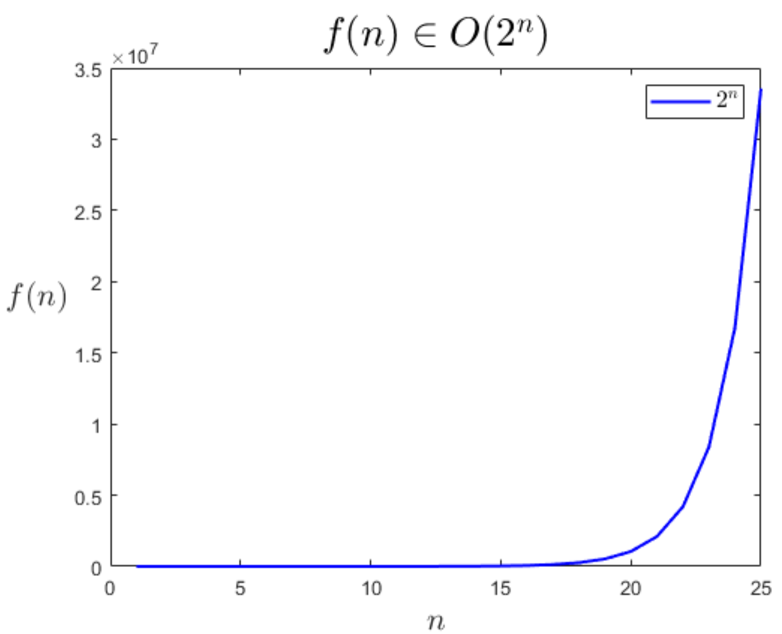

Figure 9: $f(n) \in O(2^n)$

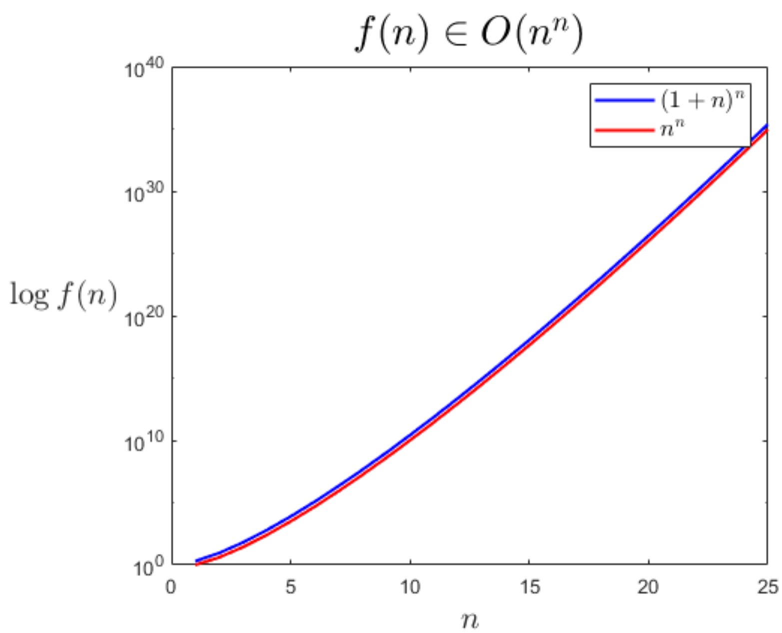

Figure 10: $f(n) \in O(n^n)$

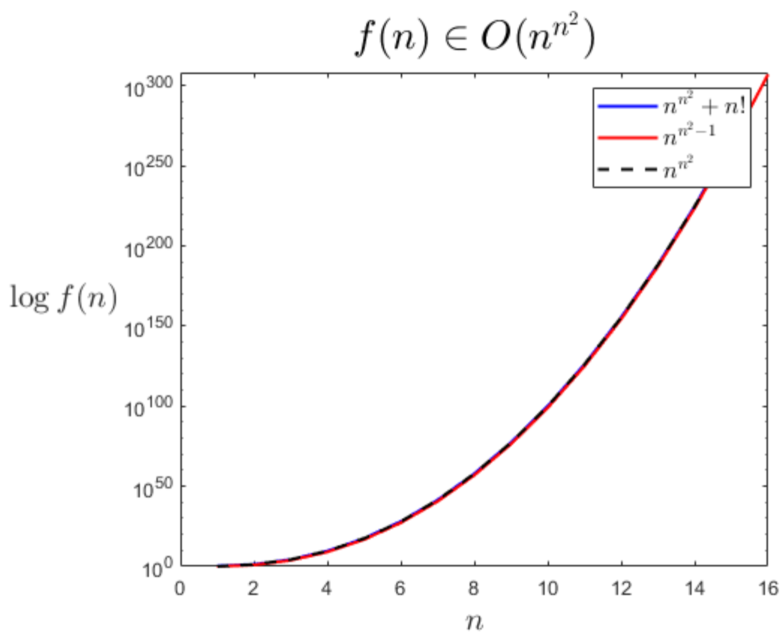

Figure 11: $f(n) \in O(n^{n^2})$

Problem 2

2A

Substitute:

$$T(n) = c_1 n + c_2 n \log_2 n$$

into:

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

to find the values of $c_1,\ c_2$ to determine the exact solution.

$$T(n) = 2T\left(\frac{n}{2}\right) + n \Longleftrightarrow c_1 n + c_2 n \log_2 n = 2\left[c_1\left(\frac{n}{2}\right) + c_2\left(\frac{n}{2}\right)\log_2\left(\frac{n}{2}\right)\right] + n$$

$$c_1 n + c_2 n \log_2 n = c_1 n + c_2 n\big(\log_2 n - \log_2 2\big) + n$$

$$c_1 n + c_2 n \log_2 n = c_1 n + c_2 n\big(\log_2 n - 1\big) + n$$

$$c_1 n + c_2 n \log_2 n = c_1 n + c_2 n \log_2 n - c_2 n + n$$

$$c_1 n + c_2 n \log_2 n = (c_1 - c_2 + 1)n + c_2 n \log_2 n$$

We require:

$$c_1 = c_1 - c_2 + 1$$

$$\therefore c_2 = 1$$

To find c_1, assume that $T(1) = t$. Then:

$$T(2) = 2T(1) + 2$$

$$c_1(2) + c_2(2)\log_2(2) = 2t + 2$$

$$2c_1 + 2(1)(1) = 2t + 2 \Longrightarrow 2c_1 + 2 = 2t + 2$$

$$\therefore c_1 = t = T(1)$$

Therefore, $T(n) = tn + n\log_2 n \quad \square$.

2B

Generalize this to the case for:

$$T(n) = aT\left(\frac{n}{b}\right) + n^k$$

with the trial solution:

$$T(n) = c_1 n^\gamma + c_2 n^k$$

using $a = b^\gamma$.

$$T(n) = aT\left(\frac{n}{b}\right) + n^k \iff c_1 n^\gamma + c_2 n^k = a\left[c_1\left(\frac{n}{b}\right)^\gamma + c_2\left(\frac{n}{b}\right)^k\right] + n^k$$

$$c_1 n^\gamma + c_2 n^k = \frac{ac_1 n^\gamma}{b^\gamma} + \frac{ac_2 n^k}{b^k} + n^k$$

Since $a = b^\gamma$:

$$c_1 n^\gamma + c_2 n^k = \frac{b^\gamma c_1 n^\gamma}{b^\gamma} + \frac{b^\gamma c_2 n^k}{b^k} + n^k$$

$$c_1 n^\gamma + c_2 n^k = c_1 n^\gamma + b^{\gamma-k} c_2 n^k + n^k$$

$$c_1 n^\gamma + c_2 n^k = c_1 n^\gamma + \left(b^{\gamma-k} c_2 + 1\right) n^k$$

$$c_2 = b^{\gamma-k} c_2 + 1$$

$$c_2 - b^{\gamma-k} c_2 = 1 \implies c_2\left(1 - b^{\gamma-k}\right) = 1$$

$$\therefore c_2 = \frac{1}{1 - b^{\gamma-k}}$$

Again assuming $T(1) = t$, and evaluating at $n = b$:

$$T(b) = aT\left(\frac{b}{b}\right) + b^k$$

$$T(b) = aT(1) + b^k$$

$$c_1 b^\gamma + c_2 b^k = at + b^k$$

$$c_1 b^\gamma = at + b^k - c_2 b^k$$

Using $c_2 = \frac{1}{1-b^{\gamma-k}}$:

$$c_1 b^\gamma = at + b^k - \left(\frac{1}{1 - b^{\gamma-k}}\right) b^k$$

$$c_1 = \frac{at}{b^\gamma} + \frac{b^k}{b^\gamma}\left(1 - \frac{1}{1 - b^{\gamma-k}}\right)$$

Since $a = b^\gamma$:

$$c_1 = t + \frac{b^k}{b^\gamma}\left(\frac{1 - b^{\gamma-k} - 1}{1 - b^{\gamma-k}}\right)$$

$$c_1 = t + \frac{b^k}{b^\gamma}\left(\frac{-b^{\gamma-k}}{1 - b^{\gamma-k}}\right)$$

$$c_1 = t + \frac{1}{b^\gamma}\left(\frac{-b^{\gamma}}{1 - b^{\gamma-k}}\right)$$

$$\therefore c_1 = t - \frac{1}{1 - b^{\gamma-k}}$$

Therefore, $T(n) = \left(t - \frac{1}{1-b^{\gamma-k}}\right) n^\gamma + \left(\frac{1}{1-b^{\gamma-k}}\right) n^k \quad \square$.

2C

However, if $\gamma = k$, then the above equation is undefined; thus, the guess $T(n) = c_1 n^\gamma + c_2 n^k$ is no longer valid and we must choose another form. If we guess:

$$T(n) = c_1 n^\gamma + c_2 n^\gamma \log_2 n$$

We can find constants c_1 and c_2 such that the recurrence $T(n) = aT\left(\frac{n}{b}\right) + n^k$ is satisfied for $\gamma = k$.

$$T(n) = aT\left(\frac{n}{b}\right) + n^k \Longleftrightarrow c_1 n^\gamma + c_2 n^\gamma \log_2 n = a\left[c_1\left(\frac{n}{b}\right)^\gamma + c_2\left(\frac{n}{b}\right)^\gamma \log_2\left(\frac{n}{b}\right)\right] + n^k$$

$$c_1 n^\gamma + c_2 n^\gamma \log_2 n = \frac{ac_1 n^\gamma}{b^\gamma} + \frac{ac_2 n^\gamma}{b^\gamma}\log_2\left(\frac{n}{b}\right) + n^k$$

Since $\gamma = k$ and $a = b^\gamma$:

$$c_1 n^\gamma + c_2 n^\gamma \log_2 n = c_1 n^\gamma + c_2 n^\gamma \log_2\left(\frac{n}{b}\right) + n^\gamma$$

$$c_1 n^\gamma + c_2 n^\gamma \log_2 n = c_1 n^\gamma + c_2 n^\gamma\left(\log_2 n - \log_2 b\right) + n^\gamma$$

$$c_1 n^\gamma + c_2 n^\gamma \log_2 n = c_1 n^\gamma + c_2 n^\gamma \log_2 n - \left(\log_2 b\right) c_2 n^\gamma + n^\gamma$$

$$c_1 n^\gamma + c_2 n^\gamma \log_2 n = c_2 n^\gamma \log_2 n + \left(c_1 + 1 - c_2 \log_2 b\right) n^\gamma$$

$$c_1 = c_1 + 1 - c_2 \log_2 b$$

$$\therefore c_2 = \frac{1}{\log_2 b}$$

Again assuming $T(1) = t$ and evaluating at $n = b$:

$$T(b) = aT\left(\frac{b}{b}\right) + b^\gamma$$

$$T(b) = aT(1) + b^\gamma$$

$$c_1 b^\gamma + c_2 b^\gamma \log_2 b = at + b^\gamma$$

$$c_1 b^\gamma = at + b^\gamma - c_2 b^\gamma \log_2 b$$

$$c_1 = \frac{at}{b^\gamma} + \frac{b^\gamma}{b^\gamma}\left(1 - c_2 \log_2 b\right)$$

Using $a = b^\gamma$ and $c_2 = \frac{1}{\log 2b}$:

$$c_1 = t + \left(1 - \frac{\log_2 b}{\log_2 b}\right)$$

$$\therefore c_1 = t$$

Therefore, $T(n) = tn^\gamma + \left(\frac{\log_2 n}{\log_2 b}\right) n^\gamma \quad \square$.

(30 pts) Place the following functions in order from asymptotically smallest to largest. As a convenience you may use $f(n) < O(g(n))$ to mean $f(n) \in O(g(n))$ and $f(n) = g(n)$ to mean $f(n) \in \Theta(g(n)$. Please use $=$ when you are sure that it is $\Theta(g(n))$.)

$$n^2 + 3n\log(n) + 5 \;,\; n^2 + n^{-2} \;,\; n^{n^2} + n! \;,\; n^{\frac{1}{n}} \;,\; n^{n^2-1} \;,\; \ln n \;,\; \ln(\ln n) \;,\; 3^{\ln n} \;,\; 2^n \;,$$

$$(1+n)^n \;,\; n^{1+\cos n} \;,\; \sum_{k=1}^{\log n} \frac{n^2}{2^k} \;,\; 1 \;,\; n^2 + 3n + 5 \;,\; log(n!) \;,\; \sum_{k=1}^{n} \frac{1}{k} \;,\; \prod_{k=1}^{n}(1 - \frac{1}{k^2}) \;,\; (1 - 1/n)^n$$

Giving the the algebra and explanation for the tricky cases can get some extra credit (even if you get it wrong!). Don't have to be perfect to get a good score.

Solution: SOME OF THESE ARE PRETTY SUBTLE. A GOOD RESULT IS GETTING THE ORDER SOME WHAT RIGHT. Here I give much more details not required since they are instructive. In some cases I even give the exact limit as $\to xx$.

$$1 \in O(1) \quad , \quad \prod_{k=1}^{n}(1 - \frac{1}{k^2}) \to 0 \in O(1) \quad , \quad n^{\frac{1}{n}} \in O(1) \quad , \quad (1 - 1/n)^n \to e \in O(1)$$

$$\ln(\ln n) \in O(\ln(\ln n)) \quad , \quad ln(n) \in O(log(n)) \quad , \quad \sum_{k=1}^{n} \frac{1}{k} = O(\ln n)$$

$$\ln(n!) \in O(n\log(n)) \quad , \quad 3^{\ln n} \to n^{\ln(3)} \in O(n^{\ln(3)}) \quad , \quad n^{1+\cos n} \in O(n^2)$$

$$n^2 + 3n + 5 \in O(n^2) \quad , \quad n^2 + 3n\ln n + 5 \in O(n^2) \quad , \quad n^2 + n^{-2} \in O(n^2)$$

$$2^n \in O(2^n) \quad , \quad \sum_{k=1}^{\log n} \frac{n^2}{2^k} \in ? \quad , \quad n! \in O(e^{-n} n^{n+1/2}) \quad , \quad (1+n)^n \in O(n^n)$$

$$n^{n^2-1} \in O(\frac{1}{n} n^{n^2}) \quad , \quad n^{n^2} + n! \in O(n^{n^2})$$

Here are few explicit sums:

1:

For instance,

$$\lim_{n\to\infty} \prod_{k=1}^{n}(1 - \frac{1}{k^2}) = 0 \in O(0)$$

because the first term in the product is zero. If the first term were not 0, then the product would converge to a constant, as

$$\ln \prod_{k=2}^{n}(1 - \frac{1}{k^2}) = \sum_{k=1}^{n} \ln(1 - \frac{1}{k^2}) \approx \sum_{k=1}^{n} -\frac{1}{k^2} = \frac{-\pi^2}{6}$$

2:

$\sum_{k=1}^{\ln n} \frac{n^2}{2^k} = n^2 \sum_{k=1}^{\ln n} (\frac{1}{2})^k = O(n^2)$. Just realizing that a geometric series is convergent $\sum_{k=0}^{N} x^k = O(1)$ or if you believe in slugging it through (not necessary!). $\sum_{k=0}^{N} x^k = \frac{1-x^{N+1}}{1-x}$ so $\sum_{k=1}^{N} x^k = \frac{1-x^{N+1}}{1-x} - 1 == \frac{1-(1/2^{\ln(n)+1}}{1/2} - 1 = 1 - (1/2)^{\ln(n)} \in O(1)$ using $x = 1/2$ and $N = \ln(n)$.

3:

In general a very useful trick it to take `ln-exp!`of the function ($f(n) = e^{\ln(f(n))}$) followed by the large n limit. For example:

$$(1 - 1/n)^n = e^{n\ln(1-1/n)} \simeq e^{n(1/n+1/2n^2+\cdots)} \to e^1 = 2.718281828459$$

(I found this going to WolframAlfpha: https://www.wolframalpha.com!) Also, recall that $e^{\ln n} = n$.

4:

Finally the function $n^{1+\cos n}$ is tricky so we accept any reasonable placements. It doesn't have a smooth monotonic limit at large n. It oscillates between $\Theta(1)$ and $\Theta(n^2)$ getting arbitrarily close both even at integer values. Therefore strictly speaking the best bound is $n^{1+\cos n} \in O(n^2)$ but $n^\alpha \in O(n^{1+\cos n})$ implies $\alpha \leq 0$ by the definition in CLRS of "Big Oh".

1. (40 pts)

   (a) Given the equation, $T(n) = 2T(n/2) + n$, guess a solution of the form:

   $$T(n) = c_1 n + c_2 \, nlog_2(n) \; .$$

   Find the coefficients c_1, c_2 to determine the exact solution assuming a value $T(1)$ at the bottom the recursion.

   (b) Generalize this to the case to the equation $T(n) = aT(n/b) + n^k$ and guess the solution of the form:

   $$T(n) = c_1 n^\gamma + c_2 n^k$$

   using $b^\gamma = a$ and assuming $\gamma \neq k$. First show if you drop the n^k using the **homogeneous** equation $T(n) = aT(n/b)$ the form $c_1 n^\gamma$ is a solution! (What is c_1?) Second drop $aT(n/b)$ and show $c_2 n^k$ is a solution (What is c_2?) With both terms (and $\gamma \neq k$) the full solution is just the sum of the to terms but only one or the other dominates!

   (c) What happens when the two solution collide (i.e have the same power, i.e $\gamma = \log(a)/\log(b) = k$.) Now show that the leading solution is as n goes to infinity is $T(n) = \Theta(n^k logn)$ [1]

[1] If you are ambitious you find exact solutions for part b and c above are of the form $T(n) = c_1 n^\gamma + givec_2 n^k$ and $T(n) = c_1 n^k + c_2 nlog(n) n^k$ explicitly determine the $c's$ for each case respectively. We did that in part for example. BUT note we already know the leading terms for the Master Equation without this extra effort! Neat trick!

Solution: You are give a a `guess` with unknown constants c_1, c_2. To see is a good guess see if you can determine the constants by substituting the `guess` in the RHS (right hand side) and the LHS (left hand side).

First case:

$$\begin{aligned} c_1 n + c_2\ n log_2(n) &= 2[\ c_1 n/2 + c_2\ (n/2) log_2(n/2)\ ] + n \\ &= c_1 n + c_2\ n(log_2(n) - log_2(2)) + n \\ &= c_1 n + c_2\ n log_2(n) - n c_2 + n \end{aligned}$$

Now to see if `RHS = LHS` $c_1 = c_2$ you need to match the n and the $nlog_2(n)$ term

$$c_1 = c_1 - c_2 + 1 \quad \text{and} \quad c_2 = c_2 \tag{1}$$

so it works $c_2 = 1$ and any c_1. To determine this you need to base provide a base case of the recursion (e.g. $T(1) = c_1$)).

Second case: With $\gamma \neq k$ the general solution works. Again LHS vs RHS

$$c_1 n^\gamma + c_2 n^k = a[c_1 (n/b)^\gamma + c_2 (n/b)^k] + n^k \tag{2}$$

Two conditions to match term:

$$c_1 = c_1 a/b^\gamma \quad \text{and} \quad c_2 = a/b^k c_2 + 1 \tag{3}$$

Again it works for any c_1 but we need $c_2 = 1/(1 - a/b^k)$. Trouble if $k = \gamma$ because now $c_2 = \infty$ (i.e. it fails!)

So need to start over with something larger by a log. Again try to match LHS and RHS

$$c_1 n^\gamma + c_2 n^\gamma log_2(n) = a[c_1 (n/b)^\gamma + c_2 (n/b)^\gamma \log_2(n/b)] + n^\gamma \tag{4}$$

Now the leading term c_2 works and the lower power is determined

$$c_2 = c_2 \quad \text{and} \quad c_1 = c_1 - c_2 log_2(b) \tag{5}$$

lower term is again determined relative to it. This is general the larger term matches as n goes to infinity and the smaller needs a base number of $T(1)$.