EC 504 – Fall 2023 – Practrice Exam

This is sample of question. But ti is equally valuable to reveiw the Homework Exercise, slides on GitHUB and of course the disconsions in class.

1. (20 pts) Answer True or False to each of the questions below. Each question is worth 2 points. Answer true only if it is true as stated, with no additional assumptions. Give an explanation to earn partial credit even if the answer is wrong.

   (a) $2^{\log_{10} n} \in O(n)$

   Solution: True.
   $2^{\log_{10} n} = 10^{\log_{10}(2) \log_{10} n} = n^{\log_{10} 2} \in O(n)$.

   (b) $10^{\log_2(n)} \in O(n^3)$

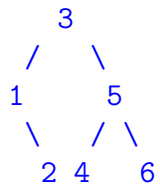
   Solution: False. $10^{\log_2 n} = 2^{\log_2(10) \log_2 n} = n^{\log_2 10} \in \omega(n^3)$.

   (c) If $f_i(n) \in O(n^2)$, then $\sum_{i=1}^{n} log_2(f_i(n)) \in O(n^2)$

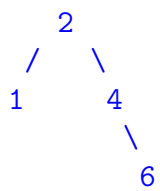
   Solution: False There are n terms, each of which is $O(n \log_2(n))$, so you can bound this by a constant times $n^2 \log_2(n)$, making it $O(n^2 \log_2(n))$.

   (d) In a binary search tree with no repeated keys, deleting the node with key x, followed by deleting the node with key y, will result in the same search tree as deleting the node with key y, then deleting the node with key x, under the assumption that you replace the deleted key with its predecessor if it exists, or with its successor otherwise.

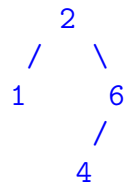
   Solution: True. You just have to think about it to convince yourself. However, if we leave ambiguous whether you can choose either a predecessor or a successor, then you can have counterexamples that make this false, simply by choosing a different replacement. Consider the tree

   and do the following: delete 3(replaced by its predecessor), then delete 5(replaced by its predecessor)to get:

Now, delete 5 (replaced by its successor), then delete 3 (replaced by its predecessor, to get:



(e) The second smallest element in a binary min-heap with all elements with distinct values will always be a child of the root.

Solution: True. The children of the root must be smaller than any of their children.

(f) Inserting numbers $1, \ldots, n$ into a binary min-heap in that order will take $\Theta(n)$ time.

Solution: True. Note that, since we are inserting them in increasing order, there will be no swaps, as inserting at the end means the parent will have a larger value.

(g) Consider a minimum spanning tree T in a connected, weighted undirected graph (V, E) where the weights have different values. Then, for every node i, the minimum spanning tree must contain the arc $\{i, j\}$ with the smallest weight connected to node i.

Solution: True. Otherwise, we can add that arc, and disconnect a current arc adjacent to node i, and get a smaller weight spanning tree.

(h) Any sorting algorithm on a set of 32 bit positive integers must have a run time complexity $f(N) \in \Omega(N \log(N))$.

Solution: False. The numbers must eventually repeat so you can do this in $O(N)$ by the histogram method

(i) There are instances of the integer knapsack problem which can be solved in polynomial time by a greedy algorithm.

Solution: Yes. Simplest one is when all of the tasks have unit capacity requirement.

(j) Consider a binary search tree with n keys. Finding whether a key value is already in the tree can be done in $O(\log(n))$.

Solution: False. The tree may not be balanced, so it is $O(n)$.

(k) The best union-find algorithm can perform a sequence random sequence of n unions and m finds in $T(n, m) \in O(n + m)$.

Solution: False. The Ackerman function is the best you can do which is slightly worse than this. Obscure fact!

(l) There are exactly 5 distinct binary search trees that include the numbers 1,2 anad 3.

Solution: True. There are 2 with 1 as root, 2 with 3 as root, but only 1 with 2 as root.

(m) The Johnson All to All distance algorithm for $G(N, A)$ introduces an augmented graph with an extra node and potential function to remove all the negative cycles so that Dijkstras one to all can be used.

Solution: False. It can not remove negative cycles. If there are no negative cycles it removes all negative bonds.

(n) The scheduling algorithm with deadlines and unit tasks when you us the *Maximun Procrastination* method is an example of amortized analysis.

Solution: False. This is typical greedy algorithm solutions method.

2. (10 pts) Consider a list of n elements with distinct values in the range $\{1, \ldots, n^2\}$. Identify which of the following sorting algorithms will produce a sorted list in worst case time $O(n \log n)$ (note I said $O()$, not $\Theta()$).

   (a) Counting (or Bin) sort

   (b) Radix sort with radix n

   (c) Merge sort

   (d) Insertion sort

   (e) Quicksort

   Solution: Note that counting sort will require counting n^2 buckets, so it is $\Theta(n^2)$. Radix sort will be $\Theta(n)$, with two rounds of radix sorting to cover the range n^2, so it is in $O(n \log n)$. Merge sort is $\Theta(n \log n)$, so it is also in $O(n \log n)$. Insertion sort and Quicksort are worst case $O(n^2)$

3. (15 pts) For each of these recursions, please give the tightest upper bound for the recursion. You can write your answer as $T(n) \in O(g(n))$ for your best choice of function $g(n)$.

   (a) $T(n) = 8T(n/2) + n^2$
   Solution: $\log_2(8) = 3$, so by the Master theorem, $T(n) \in \Theta(n^3) \in O(n^3)$.

   (b) $T(n) = 5T(n/2) + (n \log n)^2$
   Solution: $(n \log n)^2 \in O(n^{log_2(5)-\epsilon})$ for some $\epsilon > 0$, so $T(n) \in \Theta(n^{\log_2(5)})$.

   (c) $T(n) = 2T(n/2) + n \log n$
   Solution: This is similar to what we saw in one of the homework exercises. It does not satisfy the Master theorem, but by counting, we get

   $$T(n) \in \Theta(n(\log n)^2)$$

   (d) $T(n) = T(n/2) + n$
   Solution: This is a simple case of the Master theorem. the we get

   $$T(n) = \Theta(n)$$

(e) $T(n) = nT(n/2)^2$, with $T(1) = 1$. (HINT: Find the exact solution to the recursion relation for $S(n) = \log_2(T(n)$ and set $T(n) = 2^{S(n)}$.)

Solution: This is one of those very hard questions that are worth very little. The only real way to get at this is to look at the recursion values, for n a power of 2. Then,

$$T(2) = 2; T(4) = 16 = 4^2; T(8) = 2048, ...$$

This is certainly blowing up exponentially in n. Let's guess a solution of the form k^n: Then,

$$k^n? = n(k^{n/2})^2 = nk^n$$

and we see that k^n is too large a solution, because the recursion gives us a right hand value greater than k^n. To make it smaller, try $T(n) = \frac{k^n}{Cn}$. Then,

$$\frac{k^n}{Cn} = n(\frac{k^{n/2}}{Cn/2})^2 = \frac{k^n}{C^2 n/4}$$

and remarkably, we can get this to balance by picking $C = 4$. That still leaves a degree of freedom to choose k, which we match using the initial condition, as: $T(1) = k/4 = 1$, so $k = 4$. We verify this in the recursion:

$$T(2) = \frac{4^2}{4 \cdot 2} = 2; T(4) = \frac{4^4}{4 \cdot 4} = 16; T(8) = \frac{4^8}{4 * 8} = 2048$$

and thus, the problem is solved exactly: $T(n) \in \Theta(\frac{4^n}{n})$.

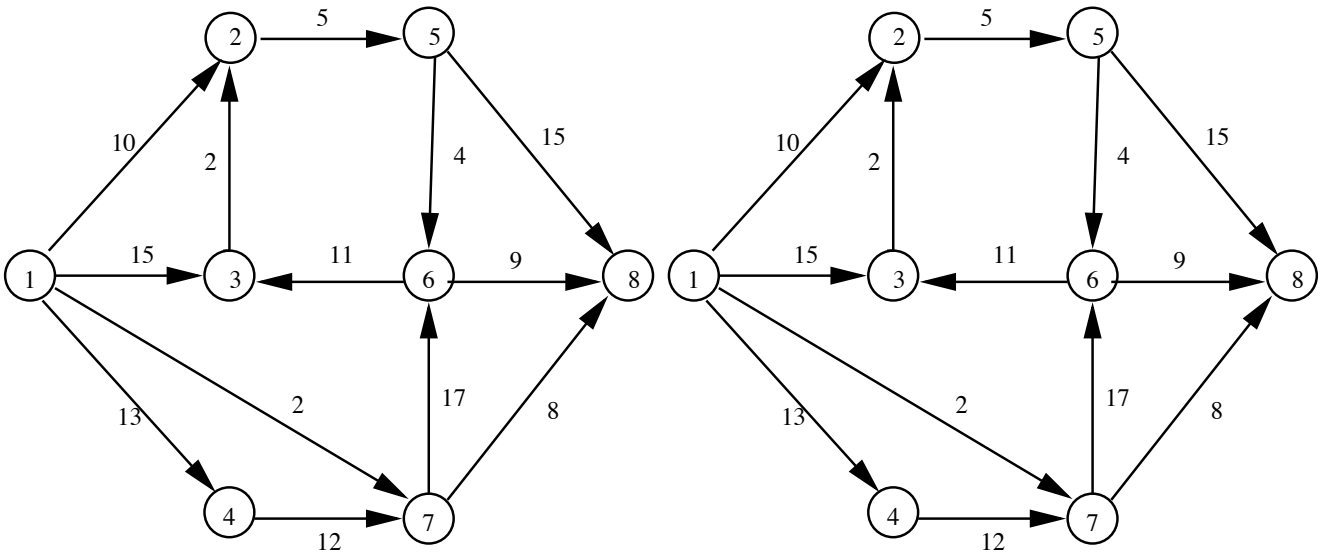
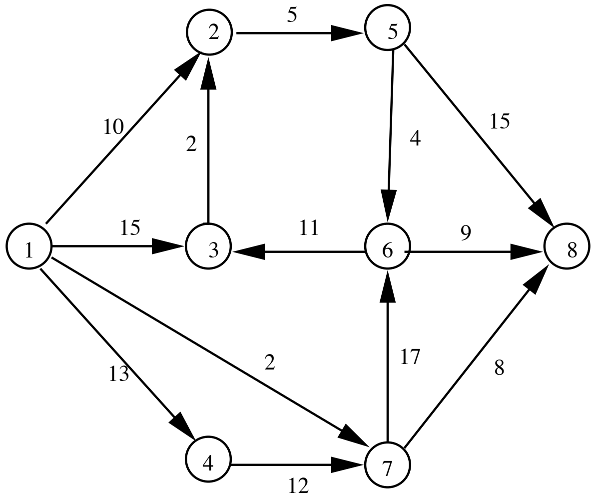


Figure 1:

4. (15 pts) Consider Fig. 1 as a directed, capacitated graph, where the numbers on an arc now indicate an arc's capacity to carry flow from node 1 to node 8. In the max-flow algorithm of Ford and Fulkerson, the key step is, once a path has been found, to augment the flow and construct the residual graph for the next iteration.

   (a) Suppose your program picks the first augmentation path to be $1 \to 7 \to 6 \to 3 \to 2 \to 5 \to 8$. Draw the augmented flow path on the left graph above and the residual graph above in the right side. What is the capacity of this path?

   (b) Now be smarter and beginning with a min hop $1 \to 2 \to 5 \to 8$ for the first path enumerate the additional paths that bring you to maximun flow. Draw **all** flow graphs and **all** the residual and after **each** augmentation. What is the maximum flow? Draw the minimum cut to show this right.



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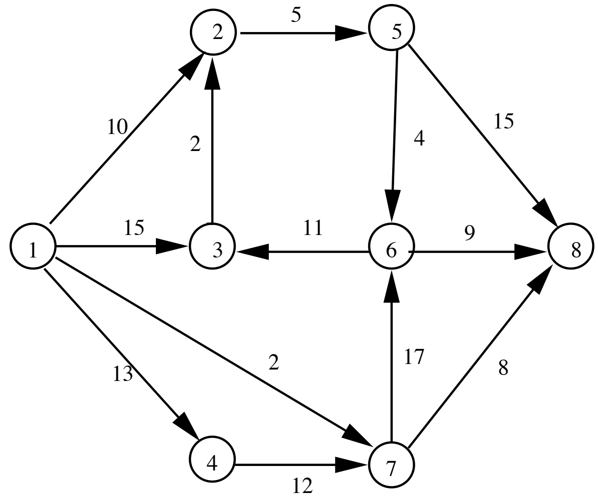

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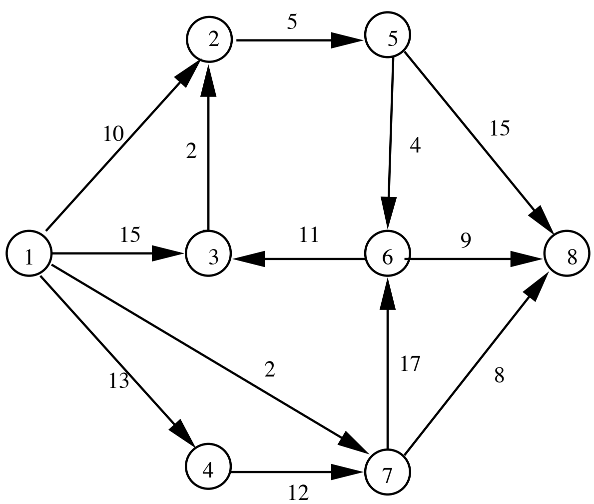

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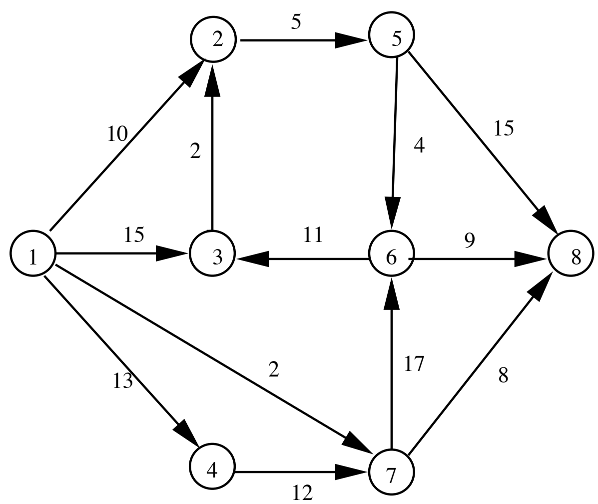

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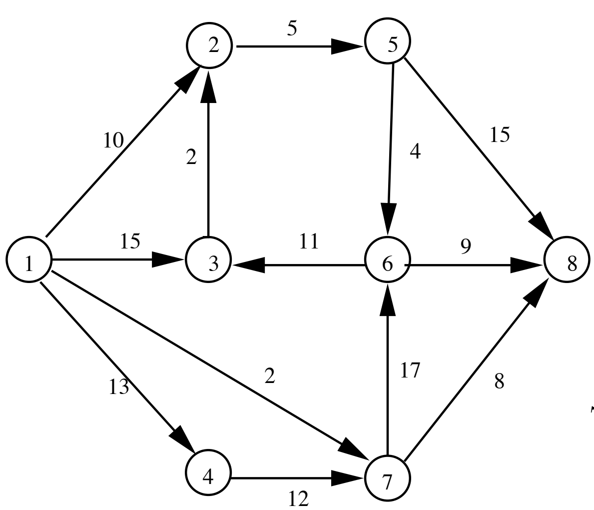

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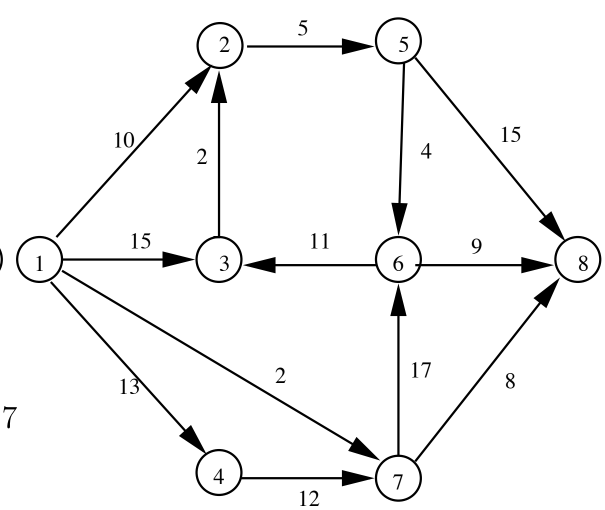

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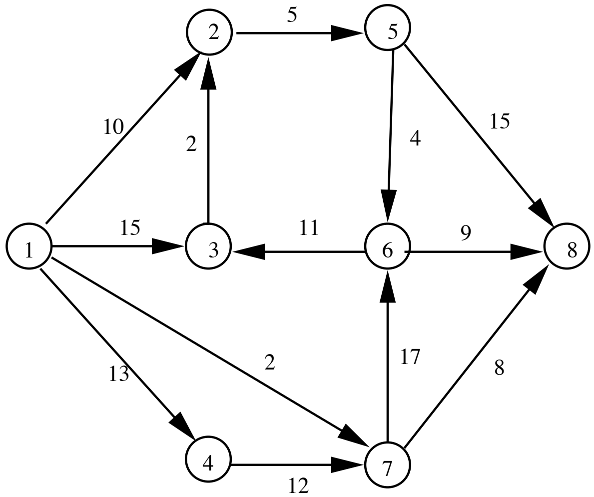
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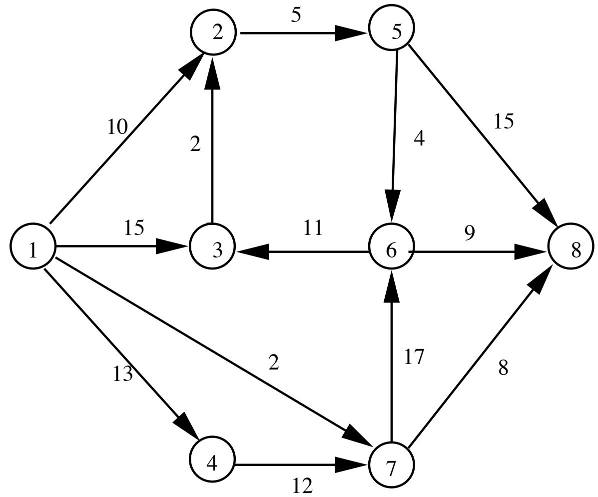
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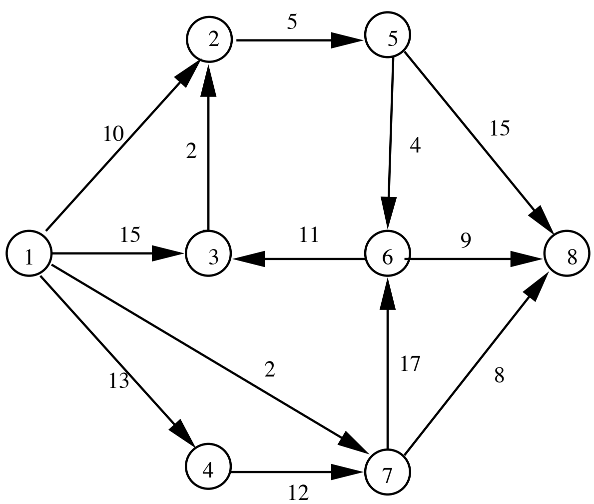
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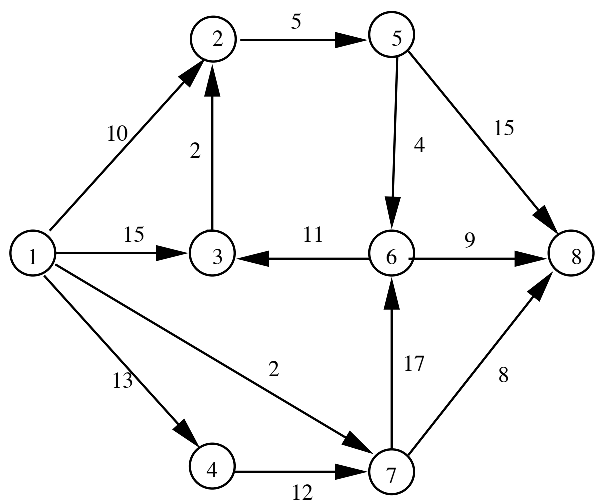
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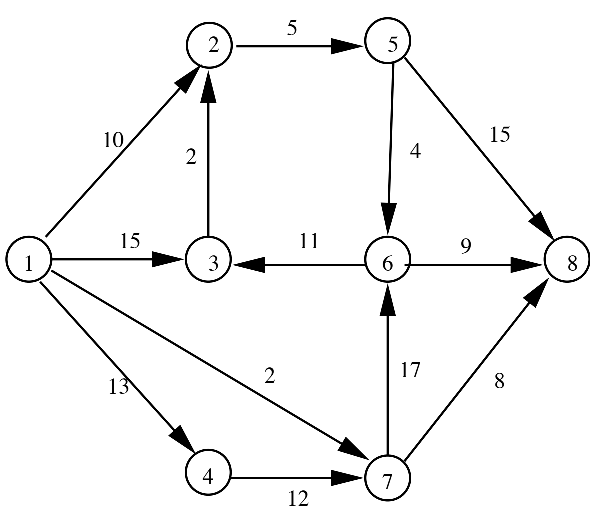
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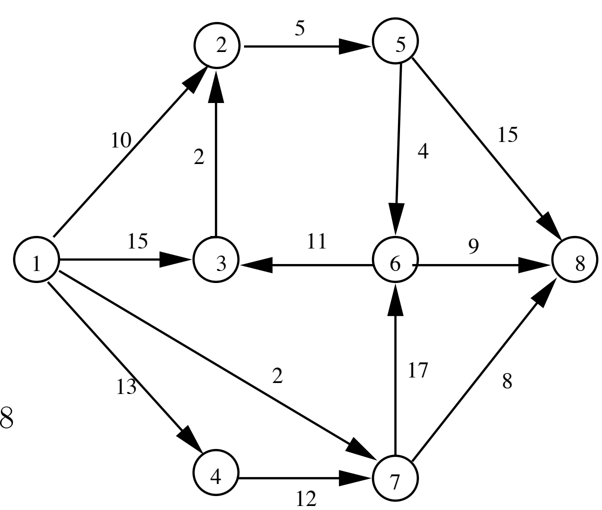
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5. (15 pts) You are interested in compression. Given a file with characters, you want to find the binary code which satisfies the prefix property (no conflicts) and which minimizes the number of bits required. As an example, consider an alphabet with 8 symbols, with relative frequency of appearance in a text file give below [1].

alphabet:	F	I	B	O	N	A	C	E
frequency:	4	20	6	14	8	25	15	37

(a) Determine the Huffman code by constructing a tree with **minimum external path length**. (Please use the "smaller weight to the left" convention.)

(b) With 0 bit to the left and 1 bit to right, identity the code for each letter and list the number of bits for each letter. (Note: as you descend the tree the bits are code for a letter goes right to left!) **You may want draw first on extra paper and then copy it below to get it this standare the form.**

(c) Compute the average number of bits per symbol in this code? Is it less than 3? (You can leave the answer as a fraction since getting the decimal value is difficult without a calculator.) Give an example of frequencies for these 8 symbols that would saturate 3 bits per letter?

[1] Fibonacci https://en.wikipedia.org/wiki/Fibonacci_sequence Ok I mispelled the name to make it 8 different letters!

6. (15 pts) Consider searching in the "text" $T(1:N)$ of length $N = 25$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
a	b	c	a	a	b	c	a	b	a	b	c	a	b	a	b	c	a	b	c	a	b	a	b	c

for the following string of length $M = 8$ as an array $P(1:M)$ (i.e. $P(i), i = 1, 2, \cdots M$)

b	a	b	c	a	b	c	a

using the KMP algorithm.

(a) Give the prefix function for above string: That is the value of $\pi(i)$ for $i = 1, \cdots 8$. [2] It is useful to copy the pattern in $P(1:M)$ to slide it against $P(1:M)$ After a failure at $q+1$ you can safely shift by $\pi(q)$ before starting to search again. Note overlapping finds are ok!)

(b) Find all the instances of this pattern in the text. That is give the start index in the text for the aligned pattern. (You can do this even if you have not got the right prefex function. Slide $P(1:M)$ against $T(1:N)$.)

(c) Specify all the non-trivial shift (i.e grater than 1 unit) that occurred in the scan using the KMP algorithm.

[2] **Recall that the prefex function looks at the first q values of $P(1:q)$ and ask how far you can shift to match to have largest prefi match the suffix in these q terms. $\pi(q) = MAX\{k < q$ such that$P(1:k) = P(1+q-k:q)\}$.**

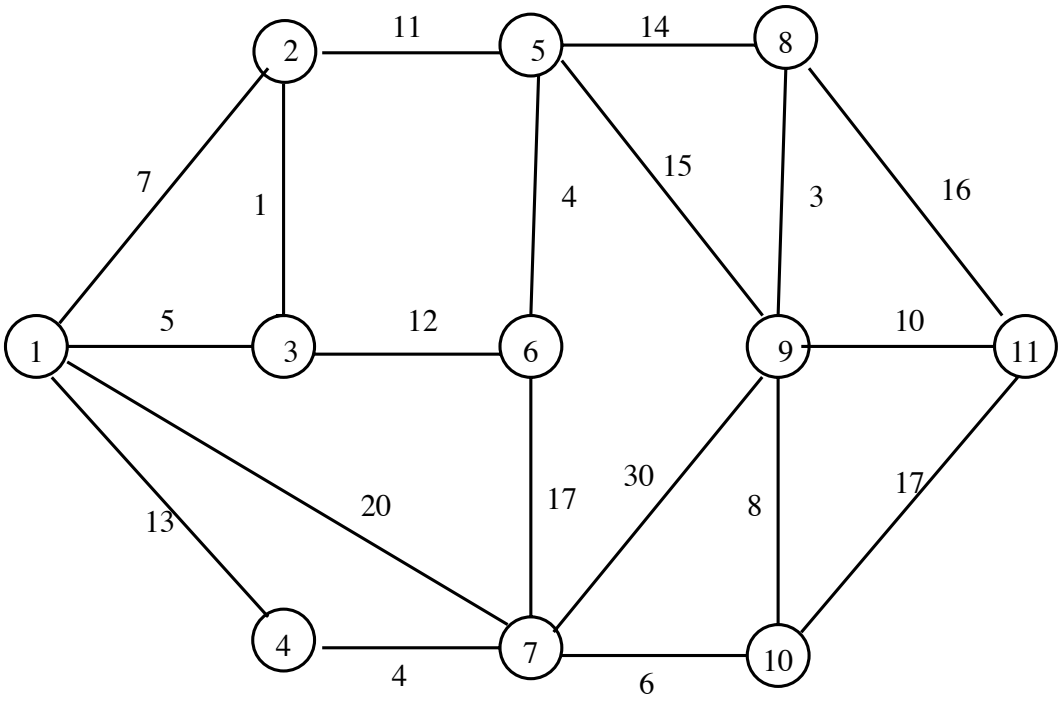


Figure 2:

7. (20 pts) Consider the undirected weighted graph in Figure 2. (There are copies of this graph at the end of the exam.) [3]

   (a) Use Dijkstra's algorithm to find the shortest paths from node 1 to all of the nodes. (Dijkstra's, like Prim's minimum spaning tree algorithm, adds one node at a time.)

   (i) Make a series of tables showing the values of the distance array $d(i)$ and the predecessor array $\pi(i)$ after each update

   (ii.) Draw the final tree on the figure with the final values of $d(i)$ and $p(i)$ at each node.

   (b) Repeat the exercise above except now use using Belllman-Ford this time (Bellman-Ford, like Kruskal's minimum spaning tree algorithm, adds one arc at a time.)

   (i.) Make a series of tables showing the values of the distance array $d(i)$ and the predecessor array $\pi(i)$ after each update.

   (ii.) Draw the final tree on the figure with the final values of $d(i)$ and $p(i)$ at each node.

[3] **Note in part a and b below you start with $d(1) = 0$, $d(i > 1) = \infty$ and $\pi(i) = -1$. The table at each step only needs to show the values of $d(i)$ and $\pi(i)$ that change!**

(c) Computed the minimum spanning tree considering all arcs to be bi-directional (in spite of the figure!) using Prim's algorithm starting form node 1, listing in order the total weight and predecessors as they are modified at each step.

(d) Are the final trees in all these cases the same? Explain

8. (15 pts) Quick questions:

   (a) How many distinct binary search trees are there that includes the numbers 1,2,3,and 4?

   Solution: 14. True There are 5 with 1 as root, 5 with 4 as root, but only 2 with either 2 or 3 as root.

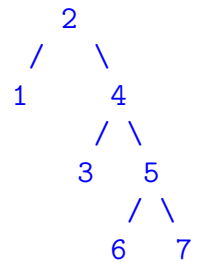
   Solution: 14. There are 2 with 1 as root, 2 with 3 as root, but only 1 with 2 as root.

   (b) What are the sizes of the binomial trees in a binomial heap that has 13 elements?

   Solution: There are 3 trees, as there are only 3 nonzero bits in the binary expansion of $13 = 8 + 4 + 1$. Hence, there is a tree of size 1, one of size 4 and one of size 8.

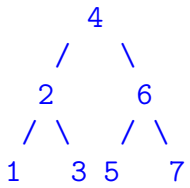
   (c) If you insert the numbers 1,2,3,4,5,6,7 into a red-black tree, what is the tree that results?

   Solution:



   (d) if you insert the numbers 1,2,3,4,5,6,7 into an AVL tree, what is the tree that results?

   Solution:



9. (15 pts) Assume that there is a set of tasks with deadlines and values listed below, each of which takes unit time to complete.

Task #:	1	2	3	4	5	6	7	8
Values:	2	4	3	8	3	6	5	3
Deadline:	1	8	2	6	4	1	3	5

(a) Explain clearly in four **short** statements the optimization algorithm that maximizes the value for task that can be completed by the deadlines. **(Unnecessarily long explanations will get LESS credit!)**

Solution: Do this deadline by `Maximun Procrastination` as described in class. You order then in decreasing value and schedule each at latest time to finish if possible. If not you drop the task and go to the next. That is perfect Gready algorithm.

(b) Use this algorithm to find the schedule for this set of tasks that maximizes the value for all tasks completed by the deadline.

Solution:

Take task in value order/deadlline:

4/6, 6/1, 7/3, 2/8, 3/2, 5/4. 8/5, 1/1

Hours :	1	2	3	4	5	6	7	8
Tasks:	6	3	7	5	8	4	--	2
Deadline:	1	2	3	4	5	6		8

This is really a bad choice of deadlines ! The first 7 are ok and you only drop the last one! **If I do one on the Final I will make more conflicts – like real life!!**

10. (15 pts) Consider the following scheduling problem: There are 10 tasks, each of which require a certain amount of processing time and have a value, as illustrated in the table below:

Task #:	1	2	3	4	5	6	7	8	9	10
Value:	7	3	4	6	3	4	8	5	7	2
Time:	3	9	3	6	2	6	8	4	5	7

(a) Suppose that there is a single machine with total capacity of 25 units of time, and that one gets partial credit for partially processing a task, so that processing a task of value V_i for time $x_i t_i$ when the requirements are t_i units provides value $x_i V_i$. **What is the optimal set of tasks to schedule, and the value achieved by this optimal schedule?**

Solution: This is just the fractional Knap sack problem!. Order the task in Maximun earning per unit time:

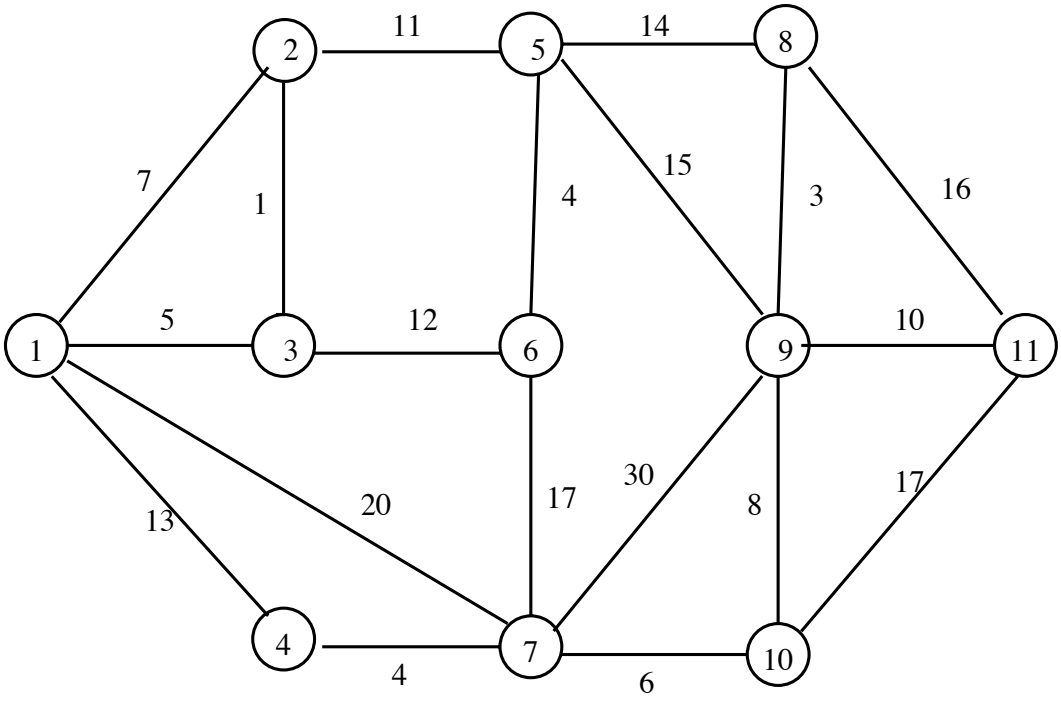
Task #:	1	2	3	4	5	6	7	8	9	10
Value/Time	7/3	1/3	4/3	1	3/2	2/3	1	5/4	7/5	2/7

Task #:	1	5	9	3	8	4	7	6	2	10
Value/Time	7/3	3/2	7/5	4/3	5/4	1	1	2/3	1/3	2/7
Time:	3	2	5	3	4	6	8	6	9	7

So cal do all of Tasks : $1, 5, 9, 3, 8, 4$ in 23 hours and one quarter if 7

(b) Suppose that there is a one machine with total capacity of 15 units of time on which you get no partial credit. Suppose there is another machine with 10 units on which you get partial credit. Modify the above result to do your best to schedule on the two machine. (You will get credit for any reasonable efficiency.)

Solution: No best solution required. But if I haven't make a mistake here is optimal solution: The first 4 above is 13 so swapping task 4 with 2 gives a perfect fill for 15 and then you can proceed to fill 10 as above and get a perfect answer!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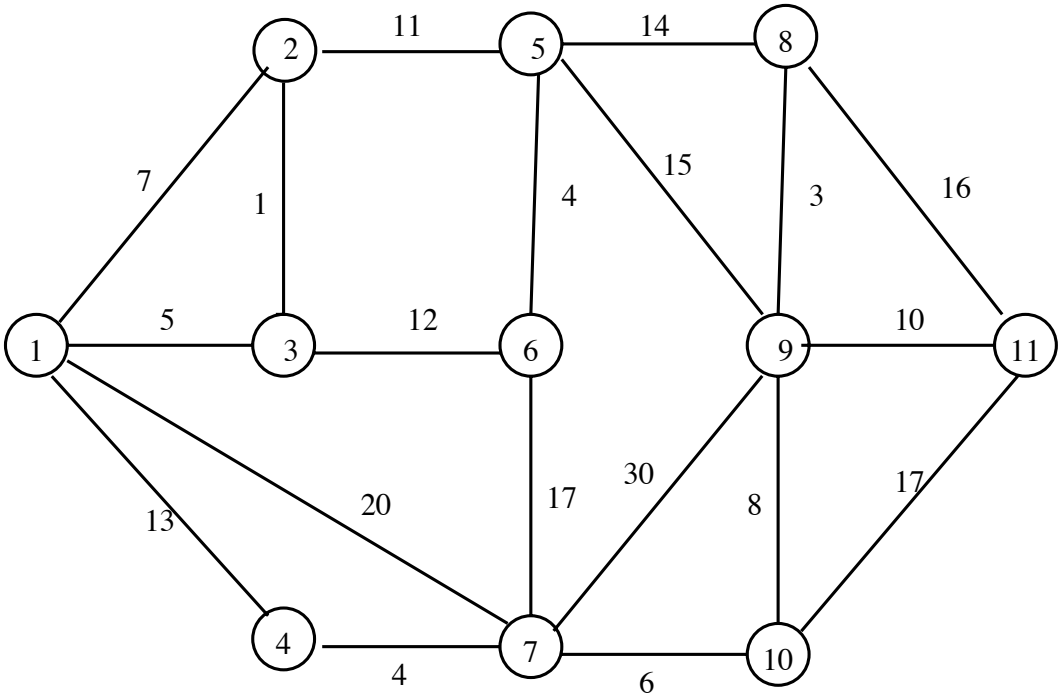
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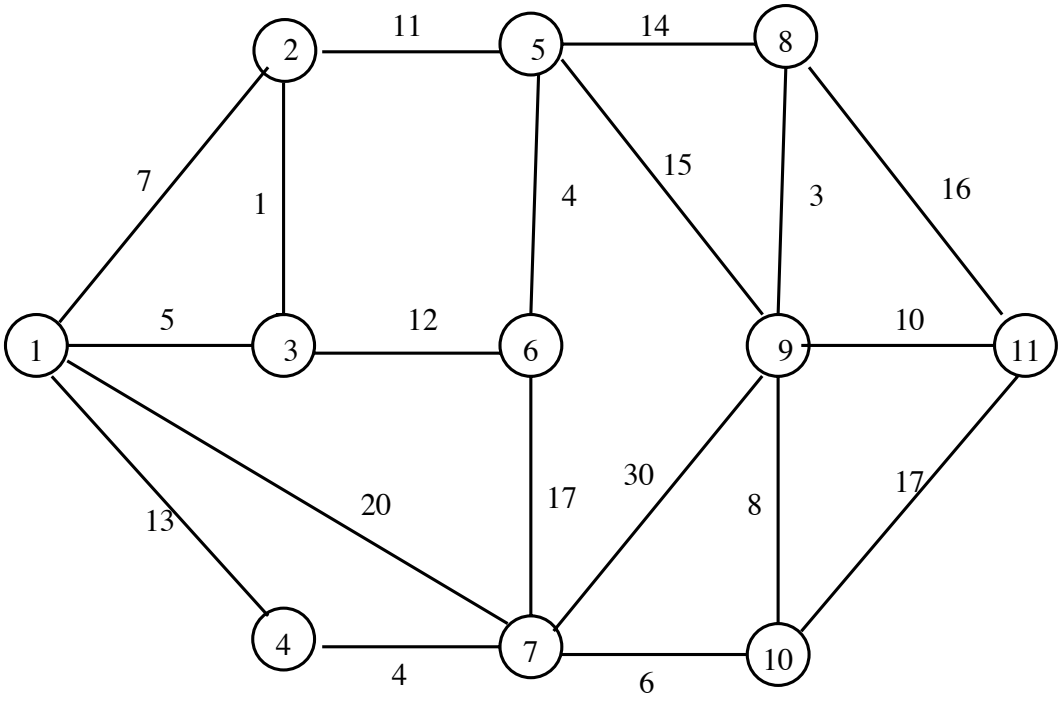
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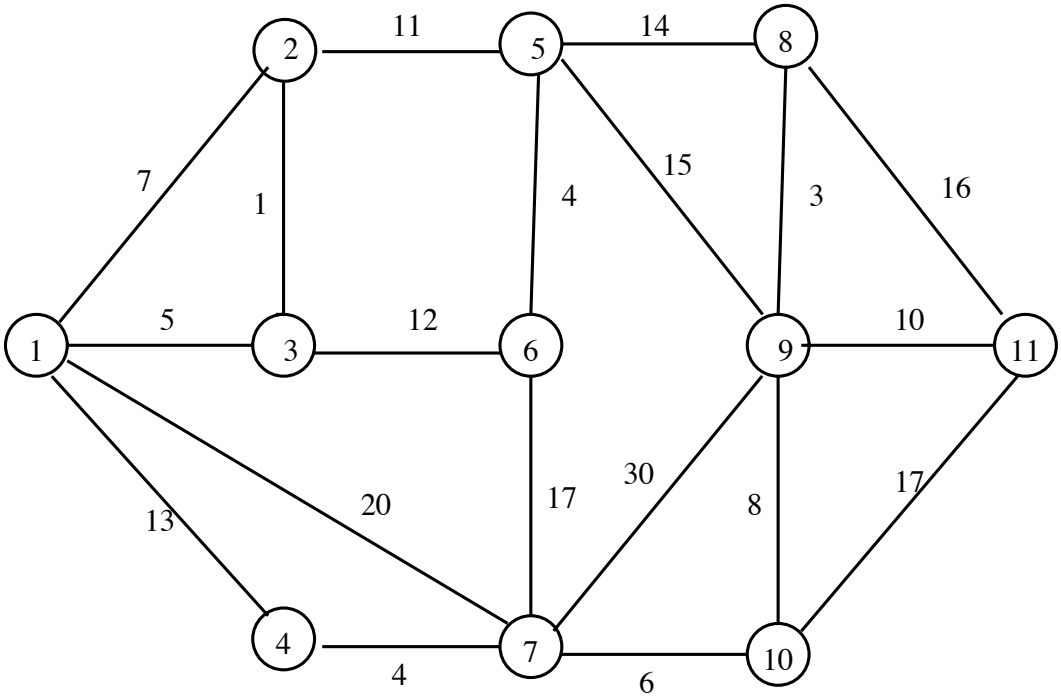
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