Square lattice Ising model

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In statistical mechanics, the **two-dimensional square lattice Ising model** is a simple lattice model of interacting magnetic spins. The model is notable for having nontrivial interactions, yet having an analytical solution. The model was solved by Lars Onsager for the special case that the external magnetic field $H = 0$.[1] An analytical solution for the general case for $H \neq 0$ has yet to be found.

Defining the partition function [edit]

Consider a 2D Ising model on a square lattice Λ with *N* sites and periodic boundary conditions in both the horizontal and vertical directions, which effectively reduces the topology of the model to a torus. Generally, the horizontal coupling J and the vertical coupling J^* are not equal. With $\beta = \frac{1}{kT}$ and absolute temperature T and Boltzmann's constant k, the partition function

$$Z_N(K \equiv \beta J, L \equiv \beta J^*) = \sum_{\{\sigma\}} \exp\left(K \sum_{\langle ij \rangle_H} \sigma_i \sigma_j + L \sum_{\langle ij \rangle_V} \sigma_i \sigma_j \right).$$

Critical temperature [edit]

The critical temperature T_c can be obtained from the Kramers–Wannier duality relation. Denoting the free energy per site as $F(K, L)$, one has:

$$\beta F(K^*, L^*) = \beta F(K, L) + \frac{1}{2} \log\left[\sinh(2K)\sinh(2L)\right]$$

where

$$\sinh(2K^*)\sinh(2L) = 1$$
$$\sinh(2L^*)\sinh(2K) = 1$$

Assuming there is only one critical line in the (K,L) plane, the duality relation implies that this is given by:

$$\sinh(2K)\sinh(2L) = 1$$

For the isotropic case $J = J^*$, one finds the famous relation for the critical temperature T_c

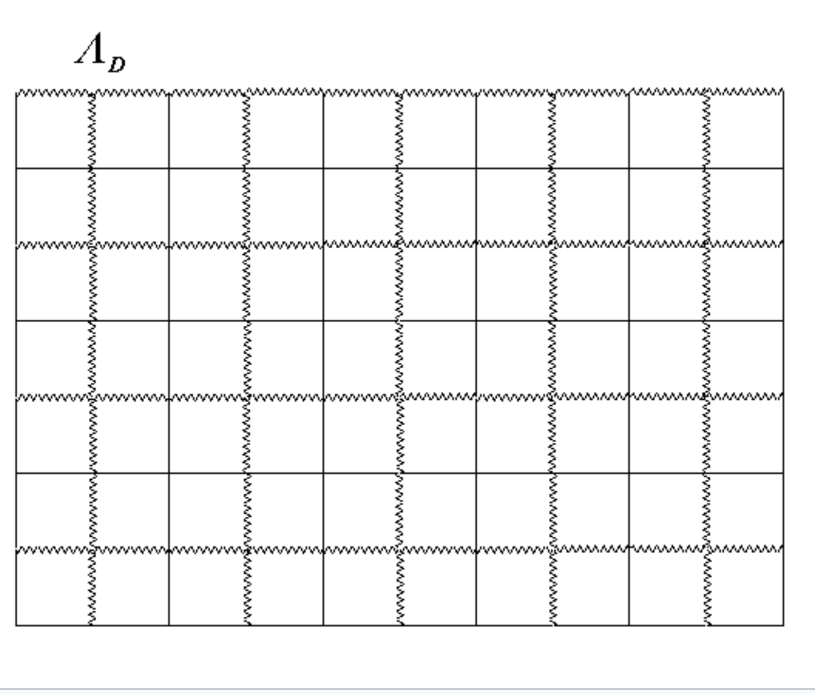
$$\frac{kT_c}{J} = \frac{2}{\ln(1+\sqrt{2})} \approx 2.26918531421$$

Dual lattice [edit]

Consider a configuration of spins $\{\sigma\}$ on the square lattice Λ. Let *r* and *s* denote the number of unlike neighbours in the vertical and horizontal directions respectively. Then the summand in Z_N corresponding to $\{\sigma\}$ is given by

$$e^{K(N-2s)+L(N-2r)}$$

Construct a dual lattice Λ_D as depicted in the diagram. For every configuration $\{\sigma\}$, a polygon is associated to the lattice by drawing a line on the edge of the dual lattice if the spins separated by the edge are unlike. Since by traversing a vertex of Λ the spins need to change an even number of times so that one arrives at the starting point with the same charge, every vertex of the dual lattice is connected to an even number of lines in the configuration, defining a polygon.

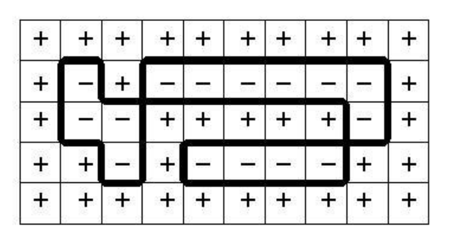


Dual lattice

This reduces the partition function to

$$Z_N(K, L) = 2e^{N(K+L)} \sum_{P \subset \Lambda_D} e^{-2Lr-2Ks}$$

summing over all polygons in the dual lattice, where *r* and *s* are the number of horizontal and vertical lines in the polygon, with the factor of 2 arising from the inversion of spin configuration.



Spin configuration on a dual lattice

Low-temperature expansion [edit]

At low temperatures, *K, L* approach infinity, so that as $T \to 0, \;\; e^{-K}, e^{-L} \to 0$, so that

$$Z_N(K, L) = 2e^{N(K+L)} \sum_{P \subset \Lambda_D} e^{-2Lr-2Ks}$$

defines a low temperature expansion of $Z_N(K, L)$.

High-temperature expansion [edit]

Since $\sigma\sigma' = \pm 1$ one has

$$e^{K\sigma\sigma'} = \cosh K + \sinh K(\sigma\sigma') = \cosh K(1 + \tanh K(\sigma\sigma')).$$

Therefore

$$Z_N(K, L) = (\cosh K \cosh L)^N \sum_{\{\sigma\}} \prod_{\langle ij \rangle_H} (1 + v\sigma_i\sigma_j) \prod_{\langle ij \rangle_V} (1 + w\sigma_i\sigma_j)$$

where $v = \tanh K$ and $w = \tanh L$. Since there are *N* horizontal and vertical edges, there are a total of 2^{2N} terms in the expansion. Every term corresponds to a configuration of lines of the lattice, by associating a line connecting *i* and *j* if the term $v\sigma_i\sigma_j$ (or $w\sigma_i\sigma_j$) is chosen in the product. Summing over the configurations, using

$$\sum_{\sigma_i = \pm 1} \sigma_i^n = \begin{cases} 0 & \text{for } n \text{ odd} \\ 2 & \text{for } n \text{ even} \end{cases}$$

shows that only configurations with an even number of lines at each vertex (polygons) will contribute to the partition function, giving

$$Z_N(K, L) = 2^N (\cosh K \cosh L)^N \sum_{P \subset \Lambda} v^r w^s$$

where the sum is over all polygons in the lattice. Since tanh *K*, tanh *L* $\to 0$ as $T \to \infty$, this gives the high temperature expansion of $Z_N(K, L)$.

The two expansions can be related using the Kramers–Wannier duality.

Exact solution [edit]

The free energy per site in the limit $N \to \infty$ is given as follows. Define the parameter k as

$$k = \frac{1}{\sinh(2K)\sinh(2L)}$$

The Helmholtz free energy per site F can be expressed as

$$-\beta F = \frac{\log(2)}{2} + \frac{1}{2\pi} \int_0^{\pi} \log\left[\cosh(2K)\cosh(2L) + \frac{1}{k}\sqrt{1 + k^2 - 2k\cos(2\theta)}\right] d\theta$$

For the isotropic case $J = J^*$, from the above expression one finds for the internal energy per site:

$$U = -J \coth(2\beta J) \left[1 + \frac{2}{\pi}(2\tanh^2(2\beta J) - 1) \int_0^{\pi/2} \frac{1}{\sqrt{1 - 4k(1+k)^{-2}\sin^2(\theta)}} d\theta \right]$$

and the spontaneous magnetization is, for $T < T_c$,

$$M = \left[1 - \sinh^{-4}(2\beta J)\right]^{1/8}$$

and $M = 0$ for $T \geq T_c$.

Notes [edit]

1. ^ Onsager, Lars (1944-02-01). "Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition". *Physical Review*. **65** (3–4): 117–149. doi:10.1103/PhysRev.65.117.

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