Hw 4 Q1.

P is at most degree 3 because it is made up of cubic hermite interpolants.

Using Thm. on p. 344,

$$f(x) - p(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{i=0}^{n} (x-x_i)^2, \quad \xi \in [a,b]$$

p is of degree 3 $\Rightarrow$ $n=1$, so,

$$\max_{x\in[a,b]} |f(x)-p(x)| \leq \max_{\xi\in[a,b]} \frac{|f^{(4)}(\xi)|}{4!} \prod_{i=0}^{1} (\underbrace{x-x_i}_{h})^2$$

h is the distance between knots

and so h is also an upper bound on the term $(x-x_i)$ which disapears at the nodes.

$$\max_{x\in[a,b]} |f(x)-p(x)| \leq \max_{x\in[a,b]} \frac{|f^{(4)}(x)|\, h^2 \cdot h^2}{4!}$$

$$\max_{x\in[a,b]} |f(x)-p(x)| \leq \frac{h^4}{4!} \max_{x\in[a,b]} |f^{(4)}(x)|$$

$$C = \frac{1}{4!}$$ ~ constant independent of h and f.

Q2. $f \in C^2([a,b])$, $a = t_0 < t_1 < \dots < t_n = b$

s is the cubic spline interpolant of f at the knots $\{t_i\}_{i=0}^{n}$

s satisfies the boundary conditions $s'(a) = f'(a)$ and $s'(b) = f'(b)$

Prove,
$$\int_a^b [s''(x)]^2 dx \leq \int_a^b [f''(x)]^2 dx$$

let $g = f - s$ then,

$$f'' = g'' + s'' \Rightarrow (f'')^2 = (s'')^2 + (g'')^2 + 2s''g''$$

$$(*) \quad \int_a^b [f'']^2 dx = \int_a^b [s'']^2 dx + \int_a^b [g'']^2 dx + 2\int_a^b s''g'' dx$$

Because $\int_a^b [g'']^2 dx \geq 0$ if we show,

$$\boxed{2\int_a^b s''g'' dx \geq 0}$$ then the proof is complete

By Parts,

$$2\int_a^b s''g'' dx = 2s''g'\Big|_a^b - 2\int_a^b g's''' dx$$

let $A = 2s''g'\Big|_a^b$ and $B = -2\int_a^b g's''' dx$

<u>show</u> $A + B \geq 0$

$g' = f' - s'$

$A = 2s''g' \Big|_a^b = 2s''(f' - s') \Big|_a^b$

using the boundry conditions for S we get,

$$A = 2s''(b)(f'(b) - \underbrace{s'(b)}_{f'(b)}) - 2s''(a)(f'(a) - \underbrace{s'(a)}_{f'(a)})$$

$$A = 2s''(b) \cdot 0 - 2s''(a) \cdot 0 = 0$$

$$\boxed{A = 0}$$

$$g' = f' - s'$$

$$B = -2\int_a^b g' s''' \, dx = -2\int_a^b s'''(f' - s')\, dx = 2\int_a^b s''' s' \, dx - 2\int_a^b f' s''' \, dx$$

Apply parts:

$$B = 2\left[s''' s \Big|_a^b - \int_a^b s\, s^{(4)} \, dx \right] - 2\left[s''' f \Big|_a^b - \int_a^b f\, s^{(4)} \, dx \right]$$

Because S is a cubic spline $S^{(4)} = 0$

$$B = 2\left[s''' s \Big|_a^b \right] - 2\left[s''' f \Big|_a^b \right]$$

Because S interpolates f at a and b,

$$\cancel{B=}\ 2\left[s''' s \Big|_a^b \right] = 2\left[s''' f \Big|_a^b \right]$$

then,

$$\boxed{B = 0}$$

Since $A + B = 0$ then

$$2\int_a^b s'' g'' \, dx = A + B = 0$$

So $\quad 2\int_a^b s'' g'' \, dx \geq 0$ and the proof is complete.