Recursive Polynomial Construction

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Theorem 0.1: Unique Interpolating Polynomial

Given: a polynomial, q, of degree $n-1$, and an (x_0, y_0) pair, the following constructs a unique interpolating polynomial. The interpolating polynomial will be of degree n,

$$p(x_0) = y_0 \tag{1}$$

$$p(x+h) = p(x) + q(x+h) \tag{2}$$

where,

$$p \in \mathbb{P}_n, q \in \mathbb{P}_{n-1}, h \neq 0$$

Lemma 0.1: Polynomial Translation

Define polynomials p and r as,

$$p(x) = \sum_{i=0}^{n} a_i x^i,\ r(x) = \sum_{i=0}^{n} b_i x^i,$$

$$b_i = \sum_{j=i}^{n} \binom{j}{i} (-h)^{j-i} a_j$$

Where,

$$p(x) = r(x-h)$$

The equation for p's translation, $p(x+h)$, in terms of x, and an offset, h, can be found by applying the binomial theorem,

$$p(x+h) = \sum_{i=0}^{n} x^i \sum_{j=i}^{n} \binom{j}{i} h^{j-i} a_j$$

Proof 0.1: Proof using binomial theorem

Expanding $p(x+h)$ using binomial theorem and reversing summation indices,

$$p(x+h) = \sum_{i=0}^{n} a_i (x+h)^i =$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{i} a_i x^j \binom{i}{j} h^{i-j} = \sum_{i=0}^{n} x^i \sum_{j=i}^{n} \binom{j}{i} h^{j-i} a_j$$

$\square$

This translation can be represented as a system using an $n+1 \times n+1$ matrix, D_h, for $p(x+h)$'s (or $r(x)$'s) representation,

$$\vec{p} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}, \vec{r} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$D_h = \begin{pmatrix} d_{00} & d_{01} & \dots & d_{0n} \\ 0 & d_{11} & \dots & d_{1n} \\ \vdots & \vdots & d_{ij} & \vdots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}, d_{ij} = \binom{j}{i} (-h)^{j-i}$$

Then the coefficients for the translated polynomial $p(x+h)$ or $r(x)$ can be found by solving the following system,

$$D_h\vec{p} = \vec{r} \tag{2.2}$$

System Representation of equation (2)

$$p(x+h) = p(x) + q(x+h) \tag{2}$$

Using vector representations of $p, q, and x$ and an $n+1 \times n+1$ matrix, D_h, for $p(x+h)$'s representation,

$$\vec{p} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}, \ \vec{q} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}, \ \vec{x}^T = \begin{pmatrix} x^0 \\ x^1 \\ \vdots \\ x^n \end{pmatrix}$$

$$D_h = \begin{pmatrix} d_{00} & d_{01} & \dots & d_{0n} \\ 0 & d_{11} & \dots & d_{1n} \\ \vdots & \vdots & d_{ij} & \vdots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}, d_{ij} = \binom{j}{i}(-h)^{j-i}$$

Then equation (2) can be represented,

$$\vec{x}D_h\vec{p} = \vec{x}\vec{p} + \vec{x}D_h\vec{q} \tag{2.2}$$

Proof 0.1: Existence Proof

Existance

Rearranging equation (2.2),

$$D_h^{-1}(D_h - I)\vec{p} = \vec{q}$$

The properties of d_{ij} make the diagonal elements of D_h all equal to one,

$$d_{ij} = \binom{j}{i}(-h)^{j-i} = 1 \, if \, j = i$$

When the identity matrix is subtracted from D_h, the matrix remains upper triangular but with zero elements along its diagonal, Let,

$$A = D_h - I$$

$$A = \begin{pmatrix} 0 & d_{01} & \dots & d_{0n} \\ 0 & 0 & \dots & d_{1n} \\ \vdots & \vdots & 0 & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

This leads to an inconsistent system for equation (2.2) if $\vec{q}$ has a non zero number as its final entry. Therefore if $p \in \mathbb{P}_n$ then $q \in \mathbb{P}_{n-1}$ for a solution to exist.

$$\begin{pmatrix} 0 & d_{01} & \dots & d_{0n} \\ 0 & 0 & \dots & d_{1n} \\ \vdots & \vdots & 0 & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ 0 \end{pmatrix} \tag{2.2}$$

Additionally, due to the other properties of d_{ij}, if $h \neq 0$, then every element of A above its main diagonal will $\neq 0$, leading to a pivot column in all other rows. Additionally, $q(x+h)$ can be written as another polynomial of the same degree so D_h^{-1} can temporarily be ignored.

We now have a system of n equations that can be used to solve for the n unknowns $a_1, a_2, \dots, a_n$.

$$\begin{pmatrix} d_{01} & \dots & d_{0n-1} \\ 0 & d_{12} \dots & d_{1n-1} \\ \vdots & & \vdots \\ 0 & \dots & d_{n-1n-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{2.3}$$

Therefore,

$$\forall q \in \mathbb{P}_{n-1}, \exists p \in \mathbb{P}_n \,|\, p(x+h) = p(x) + q(x+h) \tag{existance}$$

$\square$

Solving for p

The coefficients, $a_1, a_2, \ldots, a_n$ can be solved for using the system outlined in (2.3). With n unknowns and n equations. The unique solution that satisfies conditions (1) and (2) can by using equation (1) to solve for a_0.

$$p(x_0) = y_0 \tag{1}$$

This shows that a unique solution exists and can be solved for, for each system with $p \in \mathbb{P}_k$ where $k >= n$.

$$\forall q \in \mathbb{P}_{n-1}, \quad \exists p \in \mathbb{P}_n, \, where \ h \neq 0 \quad s.t.,$$

$$p(x_0) = y_0 \tag{1}$$

$$p(x+h) = p(x) + q(x+h) \tag{2}$$

Proof 0.1: Uniqueness Proof

Uniqueness

For any polynomial, $q \in \mathbb{P}_{n-1}$, there exists a unique polynomial, p, that satisfies

$$p(x_0) = y_0 \tag{1}$$

$$p(x+h) = p(x) + q(x+h), \, h \neq 0 \tag{2}$$

The unique polynomial, p, is of finite degree n.

proof

Above it was shown that a polynomial of finite degree n exists that satisfies equations (1) and (2),(taking n to be the smallest degree where the system is not inconsistent, $n = (deg(q) + 1)$.

An infinite set of points, S, that p must interpolate can be generated as follows,

$$\textit{Given}\,(x_0, y_0),$$
$$y_{i+1} = y_i + q(x_i),$$
$$x_{i+1} = x_i + h,$$
$$S = \{(x_i, y_i)\}_{i=0}^{\text{inf}}$$

This method gives an infinite set of points, S, all distinct. Assume another polynomial, r, of degree $m > n$ interpolates all the points in S. Then by subtracting p and r we get a third polynomial, s, of maximum degree m,

$$s = p - r, s \in \mathbb{P}_m$$

Because p and r both interpolate each point in the infinite set S, the polynomial, s will have an infinite amount of zeros.

$$s(x_i) = 0, \forall x_i \in S$$

Because m is finite, S must be the zero polynomial. Therefore,

$$p(x) = r(x)$$

$\square$