Polynomial Translation

Given an n^{th} degree polynomial, p, it's translation by a horizontal offset, $h \neq 0$, can be expressed as a new n^{th} degree polynomial q,

$$p(x+h) = q(x),$$

where,

$$p(x) = \sum_{i=0}^{n} a_i x^i,$$

$$q(x) = \sum_{i=0}^{n} b_i x^i,$$

$$b_i = \sum_{j=i}^{n} \binom{j}{i} (-h)^{j-i} a_j$$

This comes directly from the binomial theorem

Expanding $p(x+h)$ using binomial theorem and reversing summation indices,

$$p(x+h) = \sum_{i=0}^{n} a_i (x+h)^i = \sum_{i=0}^{n} \sum_{j=0}^{i} a_i x^j \binom{i}{j} h^{i-j} = \sum_{i=0}^{n} x^i \sum_{j=i}^{n} \binom{j}{i} h^{j-i} a_j$$

Represented as a system...

Using an $(n+1) \times (n+1)$ matrix, D_h, for q's representation,

$$\vec{p} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad \vec{q} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}, \quad D_h = \begin{pmatrix} d_{00} & d_{01} & \dots & d_{0n} \\ 0 & d_{11} & \dots & d_{1n} \\ \vdots & \vdots & d_{ij} & \vdots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}, \quad d_{ij} = \binom{j}{i} (-h)^{j-i}$$

defining $\binom{j}{i}$ as,

$$\binom{j}{i} = \begin{cases} \frac{j!}{(j-i)!i!}, & \text{if } j \geq i \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Then the coefficients for the translated polynomial $p(x+h)$ or $r(x)$ can be found by solving the following system (as demonstrated bellow in code),

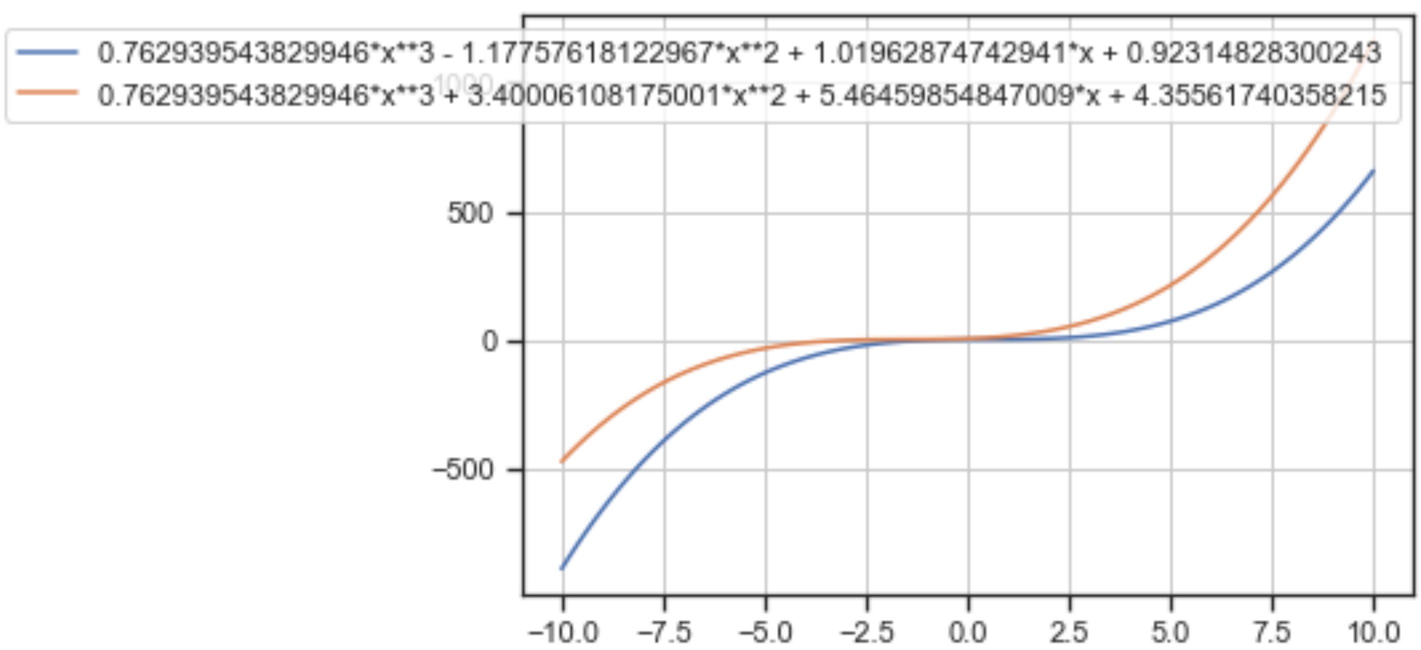
$$D_h \vec{p} = \vec{q} \tag{2}$$

In [140]:

```
import random

import numpy as np
from numpy.polynomial.polynomial import polyval
from numpy.linalg import matrix_power
from numpy.linalg import matrix_rank

from scipy.linalg import lu
from scipy.linalg import pascal

import matplotlib.pyplot as plt

import sympy
import sympy as sp
from sympy.abc import x

from IPython.display import display, Math, Latex

#
# Display Numpy Poly1d's
#

def plot_polys(polys, a=-10, b=10, steps=100):
    sp.init_printing()
    from sympy.abc import x
    X = np.linspace(a,b,steps)

    for p in polys:
        y = p(X)
        l = sp.Poly(p.c,x).as_expr()
        plt.plot(X,y, label=l)

    plt.grid()
    plt.legend()
    plt.show()
```

```
#
# create Horizontal Shift Matrix
#

def poly_shift_matrix(order, h):
    # h -- horizontal shift amount

    # n+1 coeficients to solve for -- n+1 x n+1 shift matrix
    n = order + 1

    # pascal matrix
    P = pascal(n, kind='upper')

    # create H
    H = np.eye(n)
    _h = 1
    for k in range(n):
        for i in range(n-k):
            H[i,i+k] = _h
        _h *= h

    # create shift matrix - flip for numpy convention
    S = np.flip(H * P)

    # return
    return S

#
# Horizontal Shift
#

def poly_horizontal_translation(p, delta_x):

    # r(x) = p(x + delta_x)
    S = poly_shift_matrix(p.order,delta_x)
    r = np.poly1d( S @ p )

    # return r(x) -- numpy polynomial
    return r

#
# Vertical Shift
#

def poly_vertical_translation(poly, delta_y):
    # r(x) = p(x) + delta_y
    r = poly + delta_y
    return r


#
# Translate Numpy Polynomial
#
```

```
#

def translate_polynomial(p, delta_x=0, delta_y=0):
    r = p

    if delta_x != 0:
        # r(x) = p(x + delta_x)
        r = poly_horizontal_translation(r, delta_x)

    if delta_y != 0:
        # r(x) = p(x) + delta_y
        r = poly_vertical_translation(r, delta_y)

    return r

#
# Random Testing
#

# SETTINGS
MIN_DEGREE = 2
MAX_DEGREE = 5
SHIFT_RANGE = 2 # MAX

# generate random base polynomial
n = random.randint(MIN_DEGREE,MAX_DEGREE)
p = np.poly1d( np.random.randn(n))

# translate base polynomial
x_shift, y_shift = random.randint(-SHIFT_RANGE,SHIFT_RANGE), random.randint(-SHIFT_I
q = translate_polynomial(p, delta_x=x_shift, delta_y=y_shift)

# plot results
plot_polys([p, q], a=-10, b=10)
```



Interpolation of recursivley defined sequences

One interseting use case of the translation Matrix, D_h, defined above involves finding an interpolating polynomial for a seqence, S, defined recursivley using a generating polynomial, g, of finite degree n and a known (x_0, y_0) pair as follows,

With (x_0, y_0) and h and $g(x)$ given, the recusrsive formula defines an infinite set S,

$$S = \{(x_i, y_i)\}_{i=-\infty}^{\infty}$$

$$\begin{cases} x_{i+1} & = x_i + h, \\ y_{i+1} & = y_i + g(x_{i+1}) \end{cases}$$

Theorem: There exists a unique finite degree polynomial interpolant, p, of the infinite set S generated by the recursive formula above. If the generator polynomial $g \in \mathbb{P}_n$ then the interpolant P is of degree $n + 1$.

$$p(x_i) = y_i, \quad for \quad \forall\, x_i, y_i \in S$$

If the following two conditions are met, then p interpolates S,

$$p(x_0) = y_0 \tag{condition 1}$$

$$p(x_i + h) = p(x) + g(x_i + h), \qquad \forall x_i, y_i \in S \tag{condition 2}$$

$$h \neq 0$$

Existance

$$p(x_i + h) = p(x_i) + g(x_i + h) \tag{2}$$

Using vector representations of p, g, and x and an $n+1 \times n+1$ matrix, D_h, for $p(x+h)$'s representation,

$$\vec{p} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}, \vec{g} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}, \vec{x}^{\,T} = \begin{pmatrix} x^0 \\ x^1 \\ \vdots \\ x^n \end{pmatrix}$$

$$D_h = \begin{pmatrix} d_{00} & d_{01} & \dots & d_{0n} \\ 0 & d_{11} & \dots & d_{1n} \\ \vdots & \vdots & d_{ij} & \vdots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}, d_{ij} = \binom{j}{i}(-h)^{j-i}$$

Then equation (2) can be represented,

$$\vec{x}D_h\vec{p} = \vec{x}\vec{p} + \vec{x}D_h\vec{g} \tag{2.2}$$

Rearranging equation (2.2),

$$D_h^{-1}(D_h - I)\vec{p} = \vec{g}$$

The properties of d_{ij} make the diagonal elements of D_h all equal to one,

$$d_{ij} = \binom{j}{i}(-h)^{j-i} = 1 \; if \; j = i$$

When the identity matrix is subtracted from D_h, the matrix remains upper triangular but with zero elements along its diagonal,

Let,

$$A = D_h - I$$

$$A = \begin{pmatrix} 0 & d_{01} & \dots & d_{0n} \\ 0 & 0 & \dots & d_{1n} \\ \vdots & \vdots & 0 & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

This leads to an inconsistent system for equation (2.2) if $\vec{g}$ has a non zero number as its final entry. Therefore if $p \in \mathbb{P}_n$ then $g \in \mathbb{P}_{n-1}$ for a solution to exist.

$$\begin{pmatrix} 0 & d_{01} & \dots & d_{0n} \\ 0 & 0 & \dots & d_{1n} \\ \vdots & \vdots & 0 & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ 0 \end{pmatrix} \tag{2.2}$$

Additionally, due to the other properties of d_{ij}, if $h \neq 0$, then every element of A above its main diagonal will $\neq 0$, leading to a pivot column in all other rows. Additionally, $g(x+h)$ can be written as another polynomial of the same degree so D_h^{-1} can temporarily be ignored.

We now have a system of n equations that can be used to solve for the n unknowns $a_1, a_2, \dots, a_n$.

$$\begin{pmatrix} d_{01} & & \dots & d_{0n-1} \\ 0 & d_{12} & \dots & d_{1n-1} \\ \vdots & & & \vdots \\ 0 & & \dots & d_{n-1n-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{2.3}$$

Therefore,

$$\forall g \in \mathbb{P}_{n-1}, \exists p \in \mathbb{P}_n \mid p(x+h) = p(x) + g(x+h) \tag{existance}$$

Uniqueness

For any generating polynomial, $g \in \mathbb{P}_n$, for the a sequence S, there exists a unique polynomial, p, that satisfies

$$p(x_0) = y_0 \qquad \text{(condition 1)}$$

$$p(x+h) = p(x) + g(x+h),\ h \neq 0 \qquad \text{(condition 2)}$$

The unique polynomial, p, is of finite degree $n + 1$.

Above it was shown that a polynomial of finite degree $n + 1$ exists that satisfies conditions (1) and (2),(taking $n + 1$ to be the smallest degree where the system is not inconsistent, $n + 1 = (deg(g) + 1)$.

An infinite set of points, S, that p must interpolate can be generated as follows,

$$\begin{aligned} &Given\,(x_0, y_0), \\ y_{i+1} &= y_i + g(x_i), \\ x_{i+1} &= x_i + h, \\ S &= \{(x_i, y_i)\}_{i=0}^{\inf} \end{aligned}$$

This method gives an infinite set of points, S, all distinct. Assume another polynomial, r, of degree $m \geq n$ interpolates all the points in S. Then by subtracting p and r we get a third polynomial, s, of maximum degree m,

$$s = p - r, s \in \mathbb{P}_m$$

Because p and r both interpolate each point in the infinite set S, the polynomial, s will have an infinite amount of zeros.

$$s(x_i) = 0, \forall x_i \in S$$

Because m is finite, S must be the zero polynomial. Therefore,

$$p(x) = r(x)$$

And p is unique and of degree $n + 1$.

solving for p

The coefficients, $a_1, a_2, \ldots, a_n$ can be solved for using the system outlined in (2.3). With n unknowns and n equations. The unique solution that satisfies conditions (1) and (2) can by using equation (1) to solve for a_0.

$$p(x_0) = y_0 \tag{1}$$

This shows that a unique solution exists and can be solved for, for each system with $p \in \mathbb{P}_k$ where $k >= n$.

$$\forall g \in \mathbb{P}_n, \quad \exists p \in \mathbb{P}_{n+1}, \; where \;\; h \neq 0 \quad s.t.,$$

$$p(x_0) = y_0 \tag{1}$$

$$p(x+h) = p(x) + g(x+h) \tag{2}$$

Connection to Power Sums

let,

$$\sum_{k=1}^{n} k^d = 1^d + 2^d + \ldots + n^d$$

define a "dth" order power sum of the first n integers.

The above method can be used for finding an interpolating polynomial, $p_d \in \mathbb{P}_{d+1}$, of the dth order power sum, for $\forall d \in \mathbb{N}$, such that,

$$p_d(n) = \sum_{k=1}^{n} k^d, \quad \forall n \in \mathbb{N}$$

The following property holds for any dth order power sum,

$$\sum_{k=1}^{n+1} k^d = \sum_{k=1}^{n} k^d + (n+1)^d$$

By induction the following two properties are sufficient conditions for a polynomial, p, to interpolate a dth order power sum,

$$p_d(1) = \sum_{k=1}^{1} k^d, \tag{c1}$$

$$p_d(n+1) = p_d(n) + (n+1)^d, \; for \, all \, n \in \mathbb{N} \tag{c2}$$

CODE

In [141]:

```
def power_sum( n, d=1 ):
    """ power_sum( n ) - power_sum( n-1 ) = n**d """

    if np.isscalar(n):
        # must be an integer
        n = int(n)
        s = 0
        for i in range(1,n+1):
            s += i**d
        return s

    return np.array([ power_sum( ni, d=d ) for ni in n ])
```

In [142]:

```
#
# create Horizontal Shift Matrix
#

def poly_shift_matrix(order, h):
    # h -- horizontal shift amount

    # n+1 coeficients to solve for -- n+1 x n+1 shift matrix
    n = order + 1

    # pascal matrix
    P = pascal(n, kind='upper')

    # create H
    H = np.eye(n)
    _h = 1
    for k in range(n):
        for i in range(n-k):
            H[i,i+k] = _h
        _h *= h

    # create shift matrix - flip for numpy convention
    S = np.flip(H * P)

    # return
    return S


# recursion: p(x +h) = p(x) + g(x+h), p(x0) = y0
def interpolate_recursion(g, x0, y0, h=1):
    # g is a numpy polynomial

    # get degree of p, n, from degree of g, n-1
```

```
    n = g.order + 1


    # get shift matrix
    S = poly_shift_matrix(n, h)

    # Find A = S^-1(S - I), system Ap = q -- solve for p after removing row and colu

    I = np.eye(n+1)
    A = np.linalg.inv(S) @ (S - I)

    # remove first row and last column
    A = A[1:,:-1]

    p_pad = np.zeros(shape=(n+1,))
    p_pad[:-1] = np.linalg.solve(A,g)

    p_raw = np.poly1d(p_pad)

    # solve system for final coefficient
    diff = p_raw(x0)

    if x0 != 0:
        p_raw += (y0 - diff)/x0

    else:
        p_raw += y0

    return np.poly1d(p_raw)


#
# settings
#

g_degree = random.randint(1,30)

x0 = random.randint(0,100)
y0 = power_sum(x0, d=g_degree)

g = np.zeros(shape=(g_degree+1,))
g[0] = 1
g = np.poly1d(g)

# find interpolants
p = interpolate_recursion(g, x0, y0 )

#
# display
#
import matplotlib.pyplot as plt
import pandas as pd
import seaborn
```

```python
N = 25

# predicted
x_values = np.linspace(0,N,N*2 + 1)
y_hat_values = p(x_values)

# true
int_x_values = np.array(list(range(1,N)))
y_values = power_sum(int_x_values, d=g_degree)

df = pd.DataFrame(list(zip(x_values, y_hat_values)), columns=['x', 'interp'])
df = df.set_index('x')

seaborn.set(style='ticks')
fig, ax = plt.subplots()
ax.grid(True)
ax = seaborn.lineplot(data=df, color="coral")
ax = seaborn.scatterplot(x=int_x_values, y=y_values, color="red")
plt.show()
```