Let,

$$\sigma_\varrho(x) = \sum_{i=1}^{x} i^\varrho,\ for\,\varrho, x \in \mathbb{N}$$

be a function that sums the first x integers with the exponent ϱ.
A polynomial, p, interpolates $\sigma_\varrho, \forall n \in \mathbb{N}$ iff,

$$p(0) = 0, \tag{C_1}$$

$$p(x+1) = p(x) + (x+1)^\varrho, \tag{C_2}$$

1 Uniqueness

Bellow is the derivation for a method for finding an interpolating polynomial, $p_d \in \mathbb{P}_{d+1}$, of the dth order power sum, for $\forall d \in \mathbb{N}$, such that,

$$p_d(n) = \sum_{k=1}^{n} k^d, \quad \forall\, n \,\in\, \mathbb{N}$$

The following property holds for any dth order power sum,

$$\sum_{k=1}^{n+1} k^d = \sum_{k=1}^{n} k^d + (n+1)^d$$

By induction the following two properties are sufficient conditions for a polynomial, p, to interpolate a dth order power sum,

$$p_d(1) = \sum_{k=1}^{1} k^d, \tag{c1}$$

$$p_d(n+1) = p_d(n) + (n+1)^d,\ for\, all\, n \,\in\, \mathbb{N} \tag{c2}$$

$$p_d(n) = a_0 + a_1 n + \ldots + a_{d+1} n^{d+1}$$

Using the above constraints we can construct a system of equations for solving the $d+2$ coefficients of the polynomial interpolant $p_d \in \mathbb{P}_{d+1}$,

The condition $(c1)$ gives the first equation,

$$p_d(1) = \sum_{k=1}^{1} k^d = 1, \tag{c1}$$

$$p_d(1) = a_0 + a_1 + \ldots + a_{d+1} = 1,$$

$$\sum_{i=0}^{d+1} a_i = 1 \tag{1}$$

The second condition $(c2)$ leads to the remaining equations.

$$p_d(n+1) = p_d(n) + (n+1)^d, \; for \, all \, n \; \in \; \mathbb{N} \tag{c2}$$

$$(n+1)^d = \sum_{i=0}^{d} \binom{d}{i} n^i$$

$$p_d(n) = a_0 + a_1 n + \ldots + a_{d+1} n^{d+1} = \sum_{i=0}^{d+1} a_i n^i$$

$$p_d(n+1) = a_0 + a_1(n+1) + \ldots + a_{d+1}(n+1)^{d+1} = \sum_{i=0}^{d+1} a_i \sum_{j=0}^{i} \binom{i}{j} n^j$$

Grouping terms by powers of n,

$$p_d(n+1) = \sum_{i=0}^{d+1} a_i \sum_{j=0}^{i} \binom{i}{j} n^j = \sum_{i=0}^{d+1} n^i \sum_{j=i}^{d+1} \binom{j}{i} a_j$$

Rearranging condition $(c2)$,

$$p_d(n+1) - p_d(n) = (n+1)^d$$

Substituting the expanded representations,

$$\sum_{i=0}^{d+1} n^i \sum_{j=i}^{d+1} \binom{j}{i} a_j - \sum_{i=0}^{d+1} a_i n^i = \sum_{i=0}^{d} \binom{d}{i} n^i$$

This simplifies to,

$$\sum_{i=0}^{d} n^i \sum_{j=i+1}^{d+1} \binom{j}{i} a_j = \sum_{i=0}^{d} \binom{d}{i} n^i \tag{2}$$

Equation (2) leads to the following system,

$$\begin{pmatrix} \binom{1}{0} & \binom{2}{0} & \dots & \binom{d}{0} & \binom{d+1}{0} \\ 0 & \binom{2}{1} & \dots & \binom{d}{1} & \binom{d+1}{1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \binom{d+1}{d} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{d+1} \end{pmatrix} = \begin{pmatrix} \binom{d}{0} \\ \binom{d}{1} \\ \vdots \\ \binom{d}{d} \end{pmatrix} \tag{1}$$

Since A is upper triangular with non-zero diagnol entries, its columns are linearly independent, and so there will always be a solution.

Any system from a polynomial with degree less than d+1 will be inconsistent.

A system from a polynomial with degree d+1 will always have exactly one solution.

And...

using zeros of a polynomial and proof by contradiction by assuming a polynomial with greater than degree d+1 but finite also exists it is easy to show that the only poly interp is the one you solved for.