Polynomial Interpolants of Power Sums

Power Sum Definition

let,

$$\sum_{k=1}^{n} k^d = 1^d + 2^d + ... + n^d$$

define a "dth" order power sum of the first n integers.

Sufficient Conditions for a Polynomial Interpolant

The following property holds for any dth order power sum,

$$\sum_{k=1}^{n+1} k^d = \sum_{k=1}^{n} k^d + (n+1)^d$$

By induction the following two properties are sufficient conditions for a polynomial, p, to interpolate a dth order power sum,

$$p(1) = \sum_{k=1}^{1} k^d, \tag{c1}$$

$$p(n+1) = p(n) + (n+1)^d,\ for\, all\, n\ \in \mathbb{N} \tag{c2}$$

General Solution

The following constructs a unique polynomial of degree n,

$$p(x_0) = y_0 \tag{1}$$

$$p(x+h) = p(x) + q(x) \tag{2}$$

where,

$$p \in \mathbb{P}_n, q \in \mathbb{P}_{n-1}, h \neq 0$$

Expanded Equations

Define polynomials p and q as,

$$p(x) = \sum_{i=0}^{n} a_i x^i,\ \ q(x) = \sum_{i=0}^{n} b_i x^i,$$

The equation for p's translation, $p(x+h)$, in terms of x, and an offset, h, can be found by applying the binomial theorem,

$$p(x+h) = \sum_{i=0}^{n} x^i \sum_{j=i}^{n} \binom{j}{i} h^{j-i} a_j$$

System Representation

$$\vec{p} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}, \ \vec{q} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}, \ \vec{x}^T = \begin{pmatrix} x^0 \\ x^1 \\ \vdots \\ x^n \end{pmatrix}$$

And an $n+1 \times n+1$ matrix, D_h, for $p(x+h)$'s representation,

$$D_h = \begin{pmatrix} d_{00} & d_{01} & \dots & d_{0n} \\ 0 & d_{11} & \dots & d_{1n} \\ \vdots & \vdots & d_{ij} & \vdots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}, d_{ij} = \binom{j}{i} h^{j-i}$$

Then equation (2) can be represented,

$$\vec{x}D_h\vec{p} = \vec{x}\vec{p} + \vec{x}\vec{q} \tag{2.2}$$

Existance

Rearranging equation (2.2),

$$\vec{x}(D_h - I)\vec{p} = \vec{x}\vec{q}$$

The properties of d_{ij} make the diagonal elements of D_h all equal to one,

$$d_{ij} = \binom{j}{i} h^{j-i} = 1\, if\; j = i$$

When the identity matrix is subtracted from D_h, the matrix remains upper triangular but with zero elements along its diagonal,

Let,

$$A = D_h - I$$

$$A = \begin{pmatrix} 0 & d_{01} & \dots & d_{0n} \\ 0 & 0 & \dots & d_{1n} \\ \vdots & \vdots & 0 & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

This leads to an inconsistent system for equation (2.2) if $\vec{q}$ has a non zero number as its final entry. Therefore if $p \in \mathbb{P}_n$ then $q \in \mathbb{P}_{n-1}$ for a solution to exist.

$$\begin{pmatrix} 0 & d_{01} & \dots & d_{0n} \\ 0 & 0 & \dots & d_{1n} \\ \vdots & \vdots & 0 & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ 0 \end{pmatrix} \tag{2.2}$$

Additionally, due to the other properties of d_{ij}, if $h \neq 0$, then every element of A above its main diagonal will be > 0, leading to a pivot column in all other rows. Therefore,

$$\forall q \in \mathbb{P}_{n-1}, \exists p \in \mathbb{P}_n \,|\, p(x+h) = p(x) + q(x) \tag{existance}$$

Solving for p

With $n+1$ unknowns, and n relevant rows in the matrix A, equation (1) can be prepended to the first row of equation (2.2)'s augmented system representation, and the final row is dropped maintaining $n+1 \times n+1$ dimensions. The final augmented system can be written,

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & \dots & 1 & 1 & y_0 \\ \hline 0 & 1 & d_{02} & \dots & d_{0n-1} & d_{0n} & b_{0n} \\ 0 & 0 & 1 & \dots & d_{1n-1} & d_{1n} & b_{1n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & d_{n-1n} & b_{n-2n} \\ 0 & 0 & \dots & 0 & 0 & 1 & b_{n-1n} \end{array}\right), \, d_{ij} = \binom{j}{i} h^{j-i}$$

This system shows that a unique solution exists and can be solved for, for each system with $p \in \mathbb{P}_k$ where $k >= n$.

$$\forall q \in \mathbb{P}_{n-1}, \quad \exists p \in \mathbb{P}_n, \, where \; h \neq 0 \quad s.t.,$$

$$p(x_0) = y_0 \tag{1}$$

$$p(x+h) = p(x) + q(x) \tag{2}$$

Uniqueness

For any polynomial, $q \in \mathbb{P}_{n-1}$, there exists a unique polynomial, p, that satisfies

$$p(x_0) = y_0 \tag{1}$$

$$p(x+h) = p(x) + q(x), \, h \neq 0 \tag{2}$$

The unique polynomial, p, is of finite degree n.

proof

Above it was shown that a polynomial of finite degree n exists that satisfies equations (1) and (2),(taking n to be the smallest degree where the system is not inconsistent, $n = deg(q) + 1$).

An infinite set of points, S, that p must interpolate can be generated as follows,

$$Given\,(x_0, y_0),$$
$$y_{i+1} = y_i + q(x_i),$$
$$x_{i+1} = x_i + h,$$
$$S = \{(x_i, y_i)\}_{i=0}^{\text{inf}}$$

This method gives an infinite set of points, S, all distinct. Assume another polynomial, r, of degree $m > n$ interpolates all the points in S. Then by subtracting p and r we get a third polynomial, s, of maximum degree m,

$$s = p - r, s \in \mathbb{P}_m$$

Because p and r both interpolate each point in the infinite set S, the polynomial, s will have an infinite amount of zeros.

$$s(x_i) = 0, \forall x_i \in S$$

Because m is finite, S must be the zero polynomial. Therefore,

$$p(x) = r(x)$$

Using Binomial Theorem

Expanding $p(x + h)$ using binomial theorem,

$$p(x+h) = \sum_{i=0}^{n} a_i (x+h)^i =$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{i} a_i x^j \binom{i}{j} h^{i-j}$$

And pulling the j iterator to the outside in order to pull the powers of x outside the inner summation by reversing indices,

$$\sum_{i=0}^{n}\sum_{j=0}^{i} f(i,j) = \sum_{i=0}^{n}\sum_{j=i}^{n} f(j,i) \tag{3}$$

$$p(x+h) = \sum_{i=0}^{n} x^i \sum_{j=i}^{n} \binom{j}{i} h^{j-i} a_j$$

Using Taylor Expansion

Taylor expansion of $p \in \mathbb{P}_n$ around $x+h$,

$$p(x+h) = \sum_{i=0}^{n} \frac{h^i}{i!} p^{(i)}(x),$$

The i^{th} derivative of $p \in \mathbb{P}_n$,

$$p^{(i)}(x) = \sum_{i=0}^{n} \frac{j!}{(j-i)!} a_j x^{j-i}$$

Combining Taylors expansion around $x+h$ and the formula for the i^{th} derivative of $p \in \mathbb{P}_n$,

$$p(x+h) = \sum_{i=0}^{n} \frac{h^i}{i!} \sum_{i=0}^{n} \frac{j!}{(j-i)!} a_j x^{j-i}$$

In order to pull the x terms outside the inner summation, two summation rules can be applied: The first to remove the $-i$ from the x terms exponents. The second to move the j iterator to the outside.

$$\sum_{i}^{n} f(i) = \sum_{i\pm\Delta}^{n\pm\Delta} f(i \mp \Delta) \tag{1}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i} f(i,j) = \sum_{j=0}^{n}\sum_{j=0}^{n-j} f(i,j) \tag{2}$$

These are applied as follows,

$$p(x+h) = \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{h^i}{i!} \frac{j!}{(j-i)!} a_j x^{j-i} =$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{h^i}{i!} \frac{(j+i)!}{j!} a_{j+i} x^j =$$

$$= \sum_{i=0}^{n} x^j \sum_{i=0}^{n-j} \frac{h^i}{i!} \frac{(j+i)!}{j!} a_{j+i}$$

To clean up the formula the inverse of (1) is applied and a substitution for $\binom{i}{j}$,

$$= \sum_{i=0}^{n} x^j \sum_{i=j}^{n} h^{i-j} \frac{i!}{(i-j)!j!} a_i =$$

$$= \sum_{i=0}^{n} x^j \sum_{i=j}^{n} h^{i-j} \binom{i}{j} a_i,$$

$$\binom{i}{j} = \frac{i!}{(i-j)!j!}$$