Jonathan M Brown
Updated 2021-12-29

Let's chat about…Logistic Regression!

(NBA 2014-15 data from Basketball-Reference.com)

I. What is "logistic regression" and why do we need it?

Logistic regression is just like regression with a number of differences:

	Linear Regression	Logistic Regression
Nature of response variable	Quantitative	Binary: 0, 1
What the model predicts	Predicts values for the dependent variable.	Predicts the logarithm of the odds, or "log-odds" of something occurring (of the response variable = 1).
How the model is found (i.e, "estimation method')	OLS: Ordinary Least Squares. Minimizing the error, or more specifically, the sum of the squared residuals.	ML: Maximum Likelihood. Maximizes the probability of the data being true, given the parameters.
Measure of "goodness of fit"	R^2	Residual deviance
What the regression line looks like	Before any data transformations: a straight line, or a linear "surface" in multiple regression. After data transformations: a variety of curved lines or curved "surfaces".	An S-curve

OK, that's a bunch of stuff, can we simply what's happening here?

Yes, let's start with the basics and then give an example.

We use logistic regression whenever we have a response variable that takes on ONLY two values, generally, "0" or "1", which can stand for a number of things. We tend to think of "0" as a "failure" or "something not happening", and we tend to think of "1" as a "success" or "something happening".

Do you have some examples of this?

Examples of "0", "1" response variables:

- Admission into college:
  - 0 = for not admitted
  - 1 = for admitted
- Medical drug response:
  - 0 = for not recovered from illness
  - 1 = recovered from illness
- Employment status:
  - 0 = for unemployed
  - 1 = employed

That seems simple enough, why can't we run a simple linear regression on this?

Well, we can, but we run into a problem as you'll see.

To show this, let's use an example with the NBA data. Let's say that we want to predict whether or not an NBA team will make it into the playoffs. This is our **PLAYOFF** variable and it takes on "0" for non-playoff teams and "1" for playoff teams.

Let's say that we want to try to explain **PLAYOFF** using, average points per game **PTS.G**. That is, we think a team's average points per game help us predict whether or not a team makes the playoffs.

So, we want to make a model like this:

$$PLAYOFF = \beta_0 + \beta_1 PTS.G + \varepsilon$$

When you say "predict" whether or not a team will make the playoffs, what does that mean?

Good question!

Remember what our simple linear regression model predicts? It predicts the AVERAGE VALUE of the response variable at a given value of the explanatory variable!

The same is true here: we want to predict the "average" value for the PLAYOFF variable.

i. Average values of "0, 1" variables.

What does "average" mean when the variable is "0, 1"?

Even better question.

The short answer is that for a "0, 1" variable, the "average" value is the probability of a "success"!

Let's see some examples.

Let's pretend I only have 10 NBA teams, and let's pretend that none of them made the playoffs. What would that data look like?

	Team	Playoff
1	A	0
2	B	0
3	C	0
4	D	0
5	E	0
6	F	0
7	G	0
8	H	0
9	I	0
10	J	0

OK, so what's the average value for **Playoff**?

$$Playoff\ Average = \frac{0+0+0+0+0+0+0+0+0+0}{10} = 0$$

When no teams makes the playoffs, the Playoff average = 0, and we say that the probability of team's making the playoff is 0%! The proportion of teams making the playoffs = 0.

OK, that makes sense, what if ALL teams made the playoffs?

What would all teams making the playoffs look like in the data?

	Team	Playoff
1	A	1
2	B	1
3	C	1
4	D	1
5	E	1
6	F	1
7	G	1
8	H	1
9	I	1
10	J	1

Now, what's what average value for **Playoff**?

$$Playoff\ Average = \frac{1+1+1+1+1+1+1+1+1+1}{10} = \frac{10}{10} = 1$$

When ALL teams makes the playoffs, the Playoff average = 1, and we say that the probability of team's making the playoff is 100%! The proportion of teams making the playoffs = 1.

OK, but what about when some teams make the playoffs, but some don't?

Let's say that only FOUR teams made the playoffs. What would that data look like?

	Team	Playoff
1	A	0
2	B	1
3	C	0
4	D	0
5	E	1
6	F	1
7	G	0
8	H	0
9	I	0
10	J	1

Again, what's the average value for **Playoff**?

$$Playoff\ Average = \frac{0+1+0+0+1+1+0+0+0+1}{10} = \frac{4}{10} = 0.4$$

Aha! The average value of 0.4 means that the probability of a team making the playoffs is 40%. The proportion of teams making the playoffs = 0.4.

That's it?

Yes: **The average value for a "0,1" variable is the probability of having a "1" or of achieving "success".**

But wait, regression predicts average values of the response variable, sooo?

Exactly! **With a "0, 1" response variable, linear regression predicts the probability of success, however a "success" is defined.**

So, with our actual NBA data, let's find the mean of the **PLAYOFF** variable:

```
> mean(nba15$PLAYOFF)
[1] 0.5333333
```

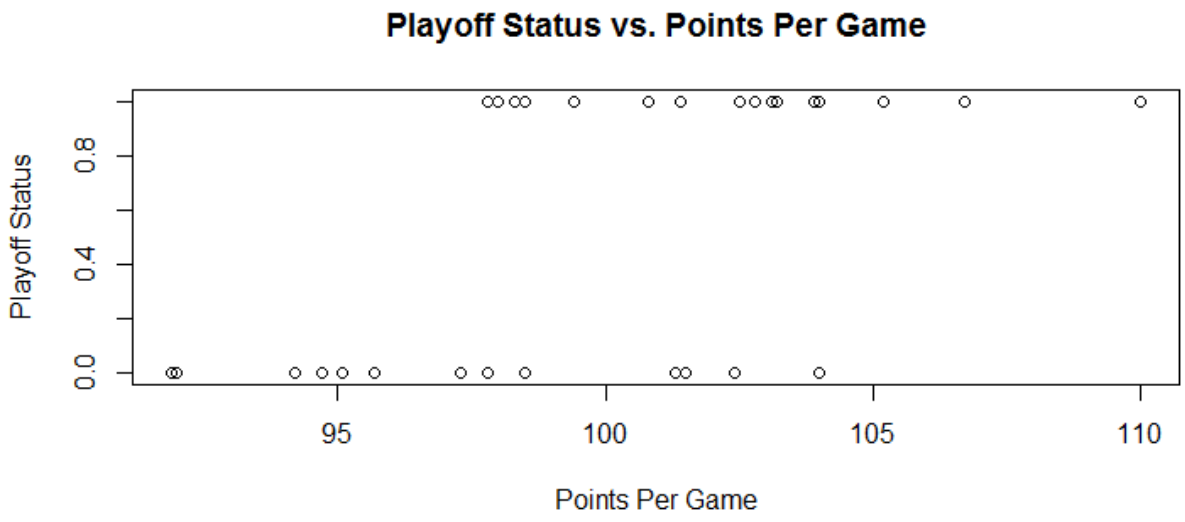
Thus, we see that since the proportion of teams making the playoffs = 0.53, there is 53% of making the playoffs!

But as we well know, this probability may depend on a number of factors! Just as housing price potentially depended on square feet, age, and other factors, the probability of making the playoffs may depend on other factors.

Let's try to model it.

ii. Plotting "0,1" data

Our first step in regression has always been to make a scatterplot of response vs. explanatory variable:



Huh, is that how it's supposed to look?

Yes! Since we ONLY have 0s and 1s, we will only see two rows of data.

That doesn't look linear, isn't that a problem?

In general, yes. And that's where logistic regression comes in, but let's keep going anyway.

iii. A simple linear regression with a "0, 1" variable

Let's just run the model above and see what happens.

```
> summary(lm(PLAYOFF ~ 1 + PTS.G, data = nba15))
```

```
Call:
lm(formula = PLAYOFF ~ 1 + PTS.G, data = nba15)

Residuals:
     Min       1Q   Median       3Q      Max
-0.79880 -0.32563  0.01443  0.29617  0.61439

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -6.13216    1.85762  -3.301  0.00263 **
PTS.G        0.06664    0.01856   3.591  0.00124 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.4273 on 28 degrees of freedom
Multiple R-squared:  0.3154,   Adjusted R-squared:  0.2909
F-statistic:  12.9 on 1 and 28 DF,  p-value: 0.001242
```

This gives us the following model:

$$\widehat{PLAYOFF} = -6.13 + 0.067 * PTS.G$$

Since we're predicting probabilities, another symbol is sometimes used:

$$\hat{\pi}_{Playoff} = -6.13 + 0.067 * PTS.G$$

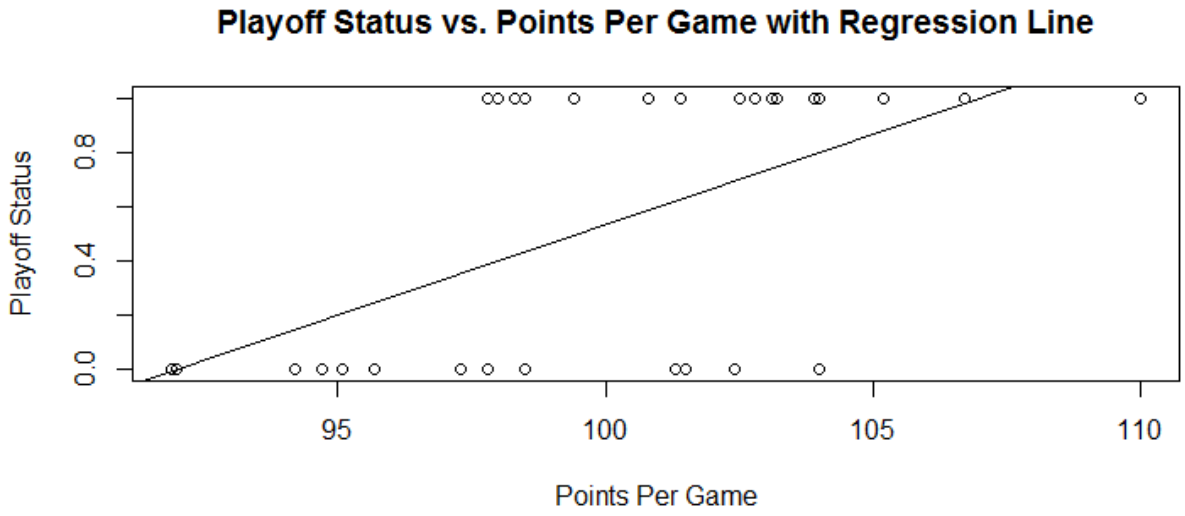
Where $\hat{\pi}$ is just the predicted probability of making it into the playoffs.

How do we interpret our coefficients?

This model is called the **linear probability model**, because it uses a straight line to predict probabilities. Our coefficient interpretations reflect this:

- $\widehat{\beta_0} = -6.13$ For a team that scores 0 points per game, the probability of making the playoffs is -613%. *Huh???* Yes. But the intercept often doesn't make sense.
- $\widehat{\beta_1} = 0.067$ Each additional point per game that a team scores is associated with a 6.7% increase in the average probability of making the playoffs. *That seems more reasonable, right?* Sure. The higher the points per game a team has, the higher their chances of making it into the playoffs. But keep reading.

If we were to plot our regression line on our original data, it would look like this:



This looks kind of strange, but maybe it's OK. Right?

The problem is this:

Remember the nature of our dependent variable: it is either a "0" for failure or a "1" for success. That allows us to interpret things as probabilities. But what if we get a predicted value SMALLER than 0 or GREATER than 1???

That is, what does a negative probability mean??

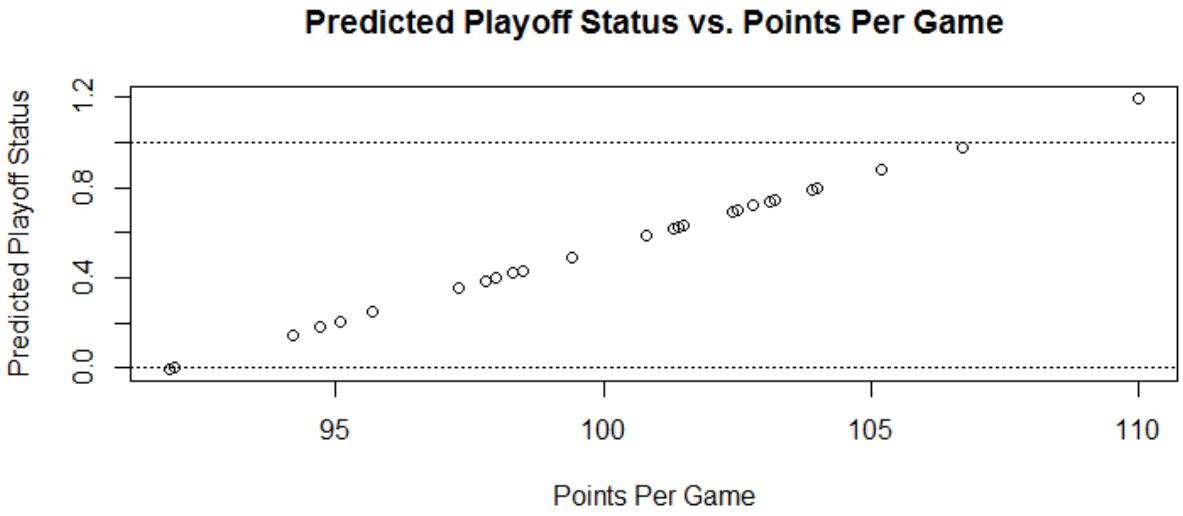
What does a probability greater than 100% mean??

Both are meaningless. They are flaws from the fact that we are trying to fit a straight line on a variable that only takes on TWO values!

Thus, we may get a regression line that makes IMPOSSIBLE predictions.

Can we see an example of an "impossible" prediction?

To see this, let's just plot the predictions of our model, with a reference line at 0 (meaning 0% probability of occurring or "failure") and a reference line at 1 (meaning 100% probability of occurring or "success").



We see that we have one prediction WELL ABOVE 1 or 100%, and we have a couple of predictions just lower than 0. These predictions are using points per game values that are within the range of the data but are giving nonsensical predictions.

For a "0, 1" variable we MUST make predictions that are BETWEEN 0 and 1!

Therefore, our linear probability model, while easy to interpret does NOT do a good at modeling "0, 1" variables realistically. We must look elsewhere.

OK, so what else can we do to prevent predictions from falling out the range of 0 to 1?

Simple: fit a model that is constrained between 0 and 1!

You're telling me that there are models that are specifically bounded by 0 and 1?

YES. And THAT is where logistic regression helps us.

II. The logistic regression model

While linear regression models things using a LINE, logistic regression models things using an S-shaped curve that is BOUNDED by 0 and 1. It's that simple!

Wait, what does that actually look like in terms of the model?

This is where I'll skip most of the math and jump right to the conclusion. You can read up on this on your own or in a more advanced course.

If we go back to our notation where π is the probability of making it into the playoffs, the logistic model that looks like an S-curve and is constrained to predict values between 0 and 1, looks like this:

Linear Model

$$\pi_{Playoff} = \beta_0 + \beta_1 PTS.G + \varepsilon$$

Logistic Model

$$\pi_{Playoff} = \frac{e^{\beta_0+\beta_1 PTS.G}}{1 + e^{\beta_0+\beta_1 PTS.G}} = \frac{1}{1 + e^{-(\beta_0+\beta_1 PTS.G)}}$$

Oh my gosh, what? That looks horrible.

Yeah, it's a mess. But, that legitimately gives predicted probabilities that are constrained between 0 and 1. The "e" is the exponential function, like you've seen before. Also, the error term goes away because we are predicting a probability, which is already random (see p. 461 in the book for more.)

Isn't there a nicer way to write it?

In order to make things more interpretable, that whole mess of an equation is rearranged, so that it becomes this final model:

The logistic model, as it is typically used

$$\ln\left(\frac{\pi_{Playoff}}{1 - \pi_{Playoff}}\right) = \beta_0 + \beta_1 PTS.G + \varepsilon$$

OK, that looks a little better, but what are we predicting now in this form?

Instead of predicting direct probabilities, we predict what's called **log-odds.**

What are log-odds?

Quick probability detour:

If the probability of success = π, then we define the odds of success as the probability of success divided by the probability of failure. If we take the natural logarithm of the odds, then we have the log-odds!

In summary:

	Formula	Explanation
Probability	π	The probability of success. Values between and including 0 and 1.
Odds	$\frac{\pi}{1-\pi}$	The odds of success. This is the probability of success divided by the probability of failure. Values between 0 and infinity.
Log-odds	$\ln\left(\frac{\pi}{1-\pi}\right)$	This is simply taking the natural logarithm of the value of the odds. Values range between negative infinity and positive infinity.

So, wait, we now predict log-odds, instead of probability. How do we interpret those?

For whatever value of the log-odds we predict, we can simply use the math to transform the log-odds prediction to an odds prediction, and then do more math to change it to a probability.

For example, if my model predicts the log-odds of making the playoffs to be 1.7, I can do the math to find the odds and the probability.

Can we see an example of that works?

Example:

If,

$$\log odds = \ln\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) = 1.7$$

Then to find the odds of success, we undo the natural logarithm with the exponential function:

$$\text{odds} = e^{\ln\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right)} = \frac{\hat{\pi}}{1-\hat{\pi}}$$

$$= e^{1.7} = 5.47$$

Thus, the odds of success are 5.47, or the probability of success is 5.47 times that of the probability of failure.

To convert this to a probability, we use the following math:

$$\text{Probablity } = \hat{\pi} = \frac{odds}{1+odds} = \frac{5.47}{1+5.47} = 0.85$$

Thus, the average probability of making the playoffs = 85%.

Is there a way to summarize these relationships?

	Probability	**Odds**	**Log-odds**
Probability Success > Probability Failure	$\hat{\pi} > 0.5$	Odds > 1 $\frac{\hat{\pi}}{1-\hat{\pi}} > 1$	Log-odds > 0 $\ln\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) > 0$
Probability Success = Probability Failure	$\hat{\pi} = 0.5$	Odds = 1 $\frac{\hat{\pi}}{1-\hat{\pi}} = 1$	Log-odds = 0 $\ln\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) = 0$
Probability Success < Probability Failure	$\hat{\pi} < 0.5$	Odds < 1 $\frac{\hat{\pi}}{1-\hat{\pi}} < 1$	Log-odds < 0 $\ln\left(\frac{\hat{\pi}}{1-\hat{\pi}}\right) < 0$

OK, can we actually run the logistic model now? How do we do that?

Fortunately, with the right code, R does all of the work for us!

To run a logistic regression, we use the "glm()" command and we add the option to specify that we're doing a logistic regression: `family = binomial(link = "logit")`

So, to run the model, we use:

```
> glm.1 = glm(PLAYOFF ~ 1 + PTS.G, data = nba15, family = binomial(link = "lo
git"))
```

And we can look at our results like usual:

```
> summary(glm.1)

Call:
glm(formula = PLAYOFF ~ 1 + PTS.G, family = binomial(link = "logit"),
    data = nba15)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.9425  -0.7916   0.2688   0.7342   1.4693
```

```
Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -38.3098    14.6018  -2.624  0.00870 **
PTS.G         0.3849     0.1462   2.633  0.00847 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 41.455  on 29  degrees of freedom
Residual deviance: 30.229  on 28  degrees of freedom
AIC: 34.229

Number of Fisher Scoring iterations: 4
```

This gives us an estimated model, in terms of log-odds of making the playoffs:

$$\ln\left(\frac{\hat{\pi}_{Playoff}}{1-\hat{\pi}_{Playoff}}\right) = -38.3 + 0.38 * PTS.G$$

III. Interpreting the logistic regression model

How do we interpret things?

First, noticed that we are given p-values like regular (but now on Z-scores instead of t-scores; we use Z-scores because we are predicting probabilities, which changes things). These function just like before. Therefore, we see that **PTS.G** is statistically significant at predicting **Playoff** status, p = 0.008

- $\widehat{\beta_0} = -38.3$ For a team that scores 0 points per game, the log-odds of making the playoffs is -38.3. The intercept still doesn't make sense.
- $\widehat{\beta_1} = 0.38$ Each additional point per game that a team scores is associated with a 0.38 increase in the average log-odds of making the playoffs.

Since we're in log-odds, that's about as good as we can do!!! Yes, we can convert this 0.38 to a probability, but since our model is CURVED, the change in probabilities is NOT constant over the range of our data. That is, the change in probabilities from one part of the curve is DIFFERENT than the change in probabilities for another part of the curve, as we look at changing values of points per game.

So, we can talk about log-odds, OR we can plug in different values for points per game, convert from log-odds to odds to probability, and see how the probability changes for different teams.

Is there a way to interpret the slope in some way OTHER than log-odds?

Yes! In short, if you apply the exponential, e^x, to the slope, then you transform the slope into something called the ODDS RATIO!

Oh gosh, what is an "odds ratio"?

It's exactly as it sounds: it's the RATIO of ODDS of success occurring (for two different events).

For example, if the odds of the Milwaukee Bucks making the playoffs were 0.5 and the odds of the Minnesota Timberwolves making the playoffs were 0.3, then the ODDS RATIO would be 0.5 / 0.3 = 1.67

In other words, the odds of the Bucks making the playoffs were 1.67 times HIGHER than Timberwolves.

Put another way: There is a 1.67-fold increase in the odds of making the playoffs.

So, how does an odds-ratio apply to a slope?

Let's do an example. First, take e^x of the slope of PTS.G, which was 0.38:

$$e^{0.38} = 1.46$$

Thus, we interpret the 1.46 as an odds-ratio. BUT, a ratio implies TWO things. So, what two things are we comparing?

We are comparing TWO teams where the DIFFERENCE between them is a ONE-unit difference in the explanatory variable (e.g., a ONE-point difference in PTS.G).

To interpret: For a one-point increase in PTS.G, there is a 1.46-fold increase in the odds of a team making the playoffs!

Can you show an example of how this odds-ratio works for the slope?

For example, suppose we have one team that scored 91 points per game vs. another team that scored 90 points per game. Thus, the odds that the first team makes the playoffs is 1.46 times higher than the odds that the second team makes the playoffs.

To get the odds that the first team makes the playoffs, plug in 91 for PTS.G, obtain the predicted log-odds from the model, and then use e^x on the result:

$$logodds\ (playoffs) = -38.3 + 0.38 * PTS.G$$

$$-38.3 + 0.38(91) = -3.72$$

$$e^{-3.72} = 0.024$$

Thus, the odds of the first team making the playoffs = 0.024

Next, repeat the above for a team scoring 90 points-per-game:

$$logodds\ (playoffs) = -38.3 + 0.38 * PTS.G$$

$$-38.3 + 0.38(90) = -4.1$$

$$e^{-4.1} = 0.017$$

Thus, the odds of the second team making the playoffs = 0.017

So, what's the RATIO of these two odds?

$$Odds\ Ratio = \frac{Odds\ for\ PTS.G = 91}{Odds\ for\ PTS.G = 90} = \frac{0.024}{0.017} = \mathbf{1.41}$$

Thus, the odds of first team (PTS.G = 91) making the playoffs is 1.41 times higher than the odds of the second team (PTS.G = 90).

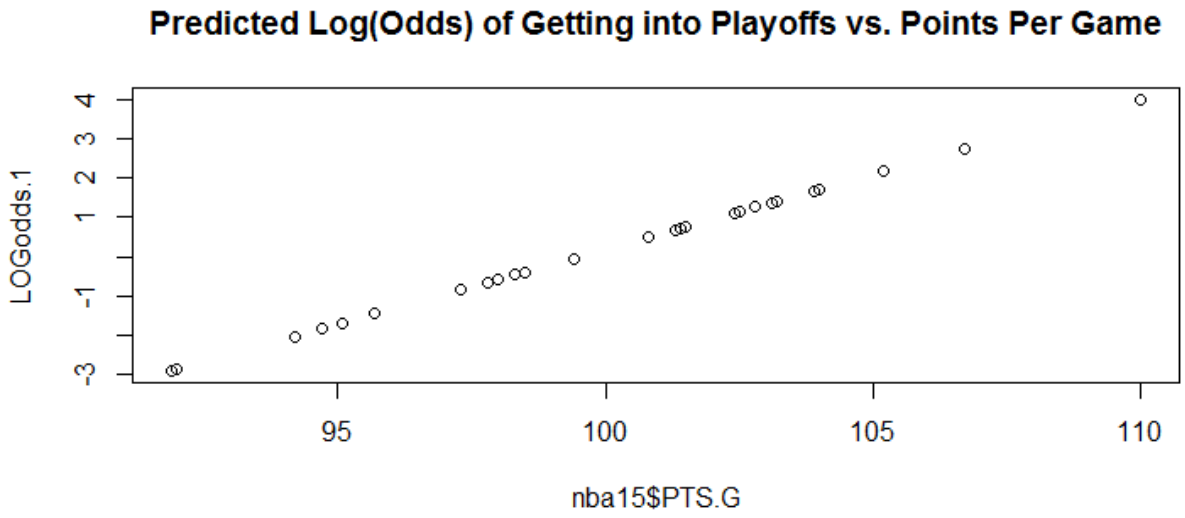
Other than rounding issues, this odds ratio of 1.41 matches the original slope when it's converted to an odds ratio of 1.46.

Ta-dah!

Couldn't we see this with some graphing?

YES! In fact, to show you how this works:

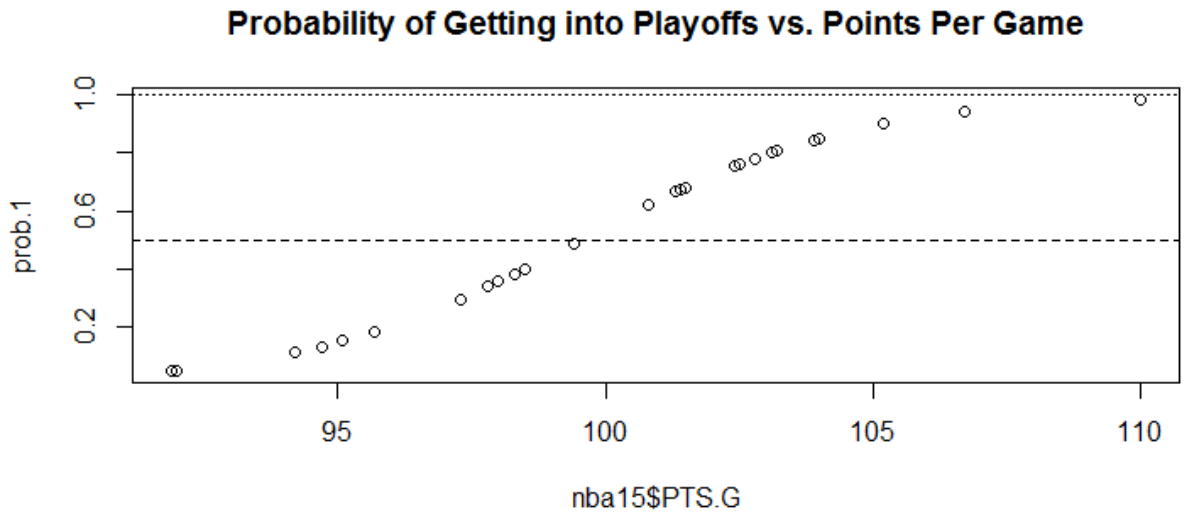
Here's a graph of the predicted log-odds. This will be a straight line, because of the right-hand side of the regression equation looks just like our usual linear regression:



So, this looks just like a regular regression line, except now our y-axis is log-odds of success, instead of probability of success.

What if we convert everything to probabilities, what will that look like?

If we convert everything to probabilities, it will look like this:



Wait, is that an S-curve?

Ta-dah! YES! Remember, our ORIGINAL logistic model wanted to constrain our predictions of probabilities to be between 0 and 1. By using logistic regression, we use an S-curve as our

model. Therefore, when we convert our predicted log-odds back to probabilities, it should look like the S-curve that we want!

I added a dotted line at y = 1 to show this (y = 0 is off of the plot here). As you can see, all predictions fall between 0 and 1 and are in an "S shape".

I added a dashed line at y = 0.5 to show the point at which the probability of making the playoffs is equal to the probability of not making the playoffs. We see that it becomes MORE likely to make the playoffs than not around 99 points per game.

IV. Quality of the logistic regression model

One last question, how do we measure error or "goodness of fit"?

This gets complicated.

The short answer is that we can use the "residual deviance". Let me re-paste the output:

```
Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -38.3098    14.6018  -2.624  0.00870 **
PTS.G         0.3849     0.1462   2.633  0.00847 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 41.455  on 29  degrees of freedom
Residual deviance: 30.229  on 28  degrees of freedom
AIC: 34.229

Number of Fisher Scoring iterations: 4
```

The residual deviance is a way of measuring "error", but it's a whole separate topic to explain what that is. So, residual deviance is a special kind of error. We want the residual deviance as small as possible. By comparison the "null deviance" is the "error" if we just used the average value of the response variable as our model, that is, if we only included the intercept in our model and NO explanatory variables. As you can see, including **PTS.G** reduced our "deviance" from 41 to 30, which means **PTS.G** did help us with our predictions.

Lastly, the blue highlighted statistic, called "AIC" or Akaike Information Criterion is another way to measure model fit when compared to other models. It only has meaning when compared to other models. If you run a bunch of models, the model with the smallest AIC value is the best value. In summary, AIC is a way to measure how much "information" a model loses compared to using the true best model (which we may never know). Thus, the smaller the information loss, the better, thus the smaller AIC the better, when comparing to other models.

How is this model actually estimated?

Ordinary least squares, like you've been using for a while tries to minimize the sum of the squared residuals.

Logistic regression uses something called "Maximum Likelihood". That find the values for the coefficients that MAXIMIZE the probability that the model is the "truest" model, given whatever data we have. It is a much different way of thinking about modeling.

What about assumptions?

There are assumptions, but there are varying perspectives on this.

<u>The MOST IMPORTANT assumption is the Independence Assumption.</u>

This assumption assumes the residuals or errors from the model are NOT RELATED to each other (knowing something about one error should NOT tell you anything about another error).

Working backward, this essentially assumes the OBSERVATIONS of the RESPONSE VARIABLE are independent from one another (unless we account for the explanatory variables in the model that explain how the response variable observations are related to each other).

The reason this assumption is so important is that the coefficients for the model are found by computing the JOINT PROBABILITY of a set of things from the data. Joint probability is REALLY MESSY to calculate if your things are related to each other. Thus, the underlying math is MUCH SIMPLER if we just assume our errors are unrelated: the joint probability calculation that helps the computer estimate the intercept, slope, etc., is way easier.

Loosely, the degree to which the Independence Assumption is violated is the degree to which you cannot trust the results of your model.

There are other assumptions, but this is just the start.

Logistic regression is cool and powerful, but it certainly is in a different world from linear regression. Thus, it takes a bit more work to use and understand it!