5.3. $T_{\alpha+\beta} = T_\alpha T_\beta = T_\beta T_\alpha$

5.6. Define $T, T' : M^{\mathbb{R}}_{n\times n} \to \mathbb{R}$ with $T(X) = \text{trace}(AX)$ and $T'(X) = \text{trace}(XA)$. Both T and T' are linear transformations

$$\begin{aligned} T(\alpha_1 X_1 + \alpha_2 X_2) &= \text{trace}(\alpha_1 X_1 + \alpha_2 X_2) \\ &= \text{trace}(\alpha_1 X_1) + \text{trace}(\alpha_2 X_2) \\ &= \alpha_1 \text{trace}(X_1) + \alpha_2 \text{trace}(X_2) \\ &= \alpha_1 T(X_1) + \alpha_2 T(X_2) \end{aligned}$$

Over the standard basis in $M^{\mathbb{R}}_{n\times n}$ (a system of $e_{i,j}$ matrices of all entries zero but the (i, j)th as 1) transformations are $T(e_{i,j}) = \sum_k \sum_l a_{lk}(e_{i,j})_{kl} = a_{ji}(e_{i,j})_{ij}$, and $T'(e_{i,j}) = \sum_k \sum_l (e_{i,j})_{kl} a_{lk} = (e_{i,j})_{ij} a_{ji}$ It follows $T = T'$, or $\text{trace}(AX) = \text{trace}(XA)$. And, for $X = B$, $\text{trace}(AB) = \text{trace}(BA)$.

Alternatively,

$$\begin{aligned} \text{trace}(AB) &= \sum_i \sum_k a_{ik} b_{ki} \\ &= \begin{array}{l} a_{11}b_{11} + \ldots + a_{1n}b_{n1} \\ + a_{21}b_{12} + \ldots + a_{2n}b_{n2} \\ \quad\vdots \\ + a_{n1}b_{1n} + \ldots + a_{nn}b_{nn} \end{array} \\ &= \sum_k \sum_i b_{ki} a_{ik} \\ &= \text{trace}(BA) \end{aligned}$$