Chapter 1, Section 2 Solutions

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Exercise 1. Find a basis in the space of 3×2 matrices $M_{3\times 2}$.
Solution:
$\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 1\\0 & 0\\0 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 0\\1 & 0\\0 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 0\\0 & 1\\0 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 0\\0 & 0\\1 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 0\\0 & 0\\0 & 1\end{pmatrix}$
Exercise 2. True or false.
Solution:

1. **True**: *Any set containing a zero vector is linearly dependent.* It contains a nontrivial solution where the coefficient on $\mathbf{0}$ is any nonzero scalar.

2. **False**: *A basis must contain* $\mathbf{0}$. Because any collection of vectors containing $\mathbf{0}$ is linearly dependent, a basis *must not* contain $\mathbf{0}$.

3. **False**: *Subsets of linearly dependent sets are linearly dependent.* Counterexample: $\{1, -1\}$ is a linearly dependent subset of vectors in $\mathbb{R}$ (because $1 - 1 = 0$), but $\{1\}$ (and $\{-1\}$) is linearly dependent (and in fact a basis for $\mathbb{R}$).

4. **True**: *Subsets of linearly independent sets are linearly independent.* Let I be a linearly independent set of vectors and S be any subset of I. By way of contradiction, suppose there were some α_i not all 0, such that $\sum_{\mathbf{v}_i \in S} \alpha_i \mathbf{v}_i = \mathbf{0}$. Since $0\mathbf{v}_i = 0$ is always true (in particular, for v_i not in S), we would find that $\sum_{\mathbf{v}_i \in I} \alpha_i \mathbf{v}_i = \mathbf{0}$ has a nontrivial solution, contradicting the fact that S is linearly independent.

5. **False**: *if* $\alpha_1 v_1 + \cdots + \alpha_n v_n = \mathbf{0}$*, then all the scalars are* 0. In $\mathbb{R}$, $1(1) + 1(-1) = 0$, but $1 \neq 0$.

Exercise 3. Write a basis for the space of

1. 3×3 matrices

2. $n \times n$ matrices

3. $n \times n$ antisymmetric matrices

Solution: Define $e_{i,j}$ to be the matrix where $(e_{i,j})_{i,j} = 1$ and is 0 elsewhere. Then, for every $m \times n$ matrix A, $A_{i,j} = (\sum_{i,j} A_{i,j} e_{i,j})_{i,j}$, so the $e_{i,j}$ generate $M_{m\times n}$. And the $e_{i,j}$ are linearly independent, since if any of the $\alpha_{i,j}$ in $\sum_{i,j} \alpha_{i,j} e_{i,j}$ were nonzero, $(\sum_{i,j} A_{i,j} e_{i,j})_{i,j}$ would be nonzero. Therefore, the $e_{i,j}$ form a basis for $M_{m\times n}$.

This produces bases for 3×3 and $n \times n$ matrices as special cases.

We now move to $n \times n$ antisymmetric matrices.

For $1 < j < n$ and $i < j$, define $e_{i,j}$ by:

$$(e_{i,j})_{a,b} = \begin{cases} 1 & i = a, j = b \\ -1 & i = b, j = a \\ 0 & \text{Otherwise} \end{cases}$$

Let $A_{i,j}$ be an antisymmetric matrix, and set $\alpha_{i,j} = A_{i,j}$, $1 < j < n$ and $i < j$. By trichotomy:

- If $j - i > 0$, $A_{i,j} = (\sum_{i,j} \alpha_{i,j} e_{i,j})_{i,j}$.
- If $j - i < 0$, $A_{i,j} = -A_{j,i} = -(\sum_{i,j} \alpha_{i,j} e_{i,j})_{j,i} = (\sum_{i,j} \alpha_{i,j} e_{i,j})_{i,j}$.
- If $j - i = 0$, then because $A^T = -A$, we must have $A_{i,j} = -A_{j,i} = 0$. Since $e_{i,j} = 0$ whenever $j - i = 0$, these entries also match.

Thus, $A = \sum_{i,j} \alpha_{i,j} e_{i,j}$.

The $e_{i,j}$ are also linearly independent because $(\sum_{i,j} \alpha_{i,j} e_{i,j})_{i,j} = \alpha_{i,j} = 0$.

Exercise 4. Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a linearly independent system that is not generating. Show that it is possible to find a vector $\mathbf{v}_{r+1}$ such that $\mathbf{v}_1, \ldots, \mathbf{v}_{r+1}$ is linearly independent.

Solution: Because $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is not generating, there exists some vector $\mathbf{v} \in V$ such that, for every choice of scalars α_i, $\mathbf{v} \neq \sum_{i=1}^{r} \alpha_i v_i$. Set $\mathbf{v}_{r+1} = \mathbf{v}$

Now let $\sum_{i=1}^{r+1} \alpha_i \mathbf{v}_i = 0$. If $\alpha_{r+1} \neq 0$, we find that $\mathbf{v}_{r+1} = \sum_{i=1}^{r} -\frac{\alpha_i}{\alpha_{r+1}} \mathbf{v}_i$, a contradiction. Now $\alpha_{r+1} = 0$, so $\mathbf{0} = \sum_{i=1}^{r+1} \alpha_i \mathbf{v}_i = \sum_{i=1}^{r} \alpha_i \mathbf{v}_i$. Because $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent, the remaining α_i are all 0. Thus, $\mathbf{v}_1, \ldots, \mathbf{v}_{r+1}$ are linearly independent.

Exercise 5. Is it possible to have $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly dependent but $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3, \mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$ linearly independent?

Solution: No.

Solving for the $\mathbf{v}_i$ in terms of w_i, we attain:

- $2\mathbf{v}_1 = \mathbf{w}_1 - \mathbf{w}_2 + \mathbf{w}_3$
- $2\mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_3$
- $2\mathbf{v}_3 = -\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$

The $\mathbf{v}_i$ are linearly dependent, so they have a nontrivial solution $\sum_i \alpha_i \mathbf{v}_i = \mathbf{0}$. Substituting in the w_i, we attain

$$(\alpha_1 + \alpha_2 - \alpha_3)\mathbf{w}_1 + (-\alpha_1 + \alpha_2 + \alpha_3)\mathbf{w}_2 + (\alpha_1 - \alpha_2 + \alpha_3)\mathbf{w}_3 = \mathbf{0}$$

In order for this to have only the trivial solution, we would have to have $\alpha_1 = \alpha_2 = \alpha_3 = 0$, contradicting that this was a nontrivial solution (to $\sum_i \alpha_i \mathbf{v}_i = \mathbf{0}$.)

(I'm pretty certain there's a drastically more elegant way of solving this that I'm not seeing.)