Exercise 5.1. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}.$$

(a) Mark all the products that are defined and give the dimension of the result: AB, BA, ABC, ABD, BC, BC^T, B^TC, DC, D^TC^T.

(b) Compute AB, $A(3B+C)$, B^TA, $A(BD)$, $(AB)D$.

Answer. Let A, B, C, D be as above.

(a) The sizes of the different matrices are as follows: $A_{2\times2}$, $B_{2\times3}$, $C_{2\times3}$, $D_{3\times1}$. We therefore have

- AB is a 2×3 matrix.
- BA is undefined.
- ABC is undefined.
- BC is undefined.
- BC^T is a 2×2 matrix.
- B^TC is a 3×3 matrix.
- DC is undefined.
- D^TC^T is a 1×2 matrix equal to $(CD)^T$.

(b) Let us compute

$$AB = \begin{pmatrix} 7 & 2 & -2 \\ 6 & 1 & 4 \end{pmatrix} \; ; \; 3B + C = \begin{pmatrix} 4 & -2 & 9 \\ 7 & 4 & -7 \end{pmatrix} \; ; \; A(3B+C) \begin{pmatrix} 18 & 6 & -5 \\ 19 & -2 & 20 \end{pmatrix}$$

Continuing with the computations,

$$B^T = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 2 & -2 \end{pmatrix} \; ; \; B^TA = \begin{pmatrix} 10 & 5 \\ 3 & 1 \\ -4 & 2 \end{pmatrix} \; ; \; BD = \begin{pmatrix} 0 \\ -6 \end{pmatrix}.$$

We shall now verify that $A(BD) = (AB)D$, as expected,

$$A(BD) = \begin{pmatrix} -12 \\ -6 \end{pmatrix} \; ; \; (AB)D = \begin{pmatrix} -12 \\ -6 \end{pmatrix}.$$

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Exercise 5.2. Let T_γ be the matrix of rotation by γ in $\mathbb{R}^2$. Check by matrix multiplication that $T_\gamma T_{-\gamma} = T_{-\gamma} T_\gamma = I$.

Answer. Recall that

$$T_\gamma = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}.$$

Since cos is an even function, and sin is an odd function we also have

$$T_{-\gamma} = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}.$$

$$
\begin{aligned}
T_\gamma T_{-\gamma} &= \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\gamma + \sin^2\gamma & \cos\gamma\sin\gamma - \sin\gamma\cos\gamma \\ \sin\gamma\cos\gamma - \cos\gamma\sin\gamma & \sin^2\gamma + \cos^2\gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Completely analogous computation shows that $T_{-\gamma}T_\gamma = I$ as well.

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Exercise 5.3. Multiply two rotation matrices T_α and T_β (it is a rare case when the multiplication is commutative, i.e. $T_\alpha T_\beta = T_\beta T_\alpha$, so the order is not essential). Deduce formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ from here.

Answer. The formula for the matrix representing rotation by γ is given in the previous answer. We have

$$
\begin{aligned}
T_\beta T_\alpha &= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \\
&= \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -\cos\beta\sin\alpha - \cos\alpha\sin\beta \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & -\sin\alpha\sin\beta + \cos\beta\cos\alpha \end{pmatrix}
\end{aligned}
$$

On the other hand $T_\beta T_\alpha$ corresponds to composition: first rotating by α, then rotating the result by β. Thus, it is equivalent to the rotation $T_{\alpha+\beta}$ (which also explains why the product is commutative $T_\alpha T_\beta = T_{\beta+\alpha}$). We therefore have

$$
T_\beta T_\alpha = T_{\alpha+\beta} = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}.
$$

Comparing the entries of the two matrices in our calculation we obtain the formulas

$$
\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta; \\
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.
\end{aligned}
$$

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Exercise 5.4. Find the matrix of the orthogonal projection in $\mathbb{R}^2$ onto the line $x_1 = -2x_2$.

Answer. Let us denote the transformation in question by T. We can decompose T as $T_{-\gamma}T_1T_\gamma$, where T_1 is the orthogonal projection onto x_1-axis

$$
[T_1] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
$$

and γ is the angle between $(1,0)^T$ and $(2,-1)^T$. The matrix of rotation is represented by

$$
[T_\gamma] = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \;\; ; \;\; [T_{-\gamma}] = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.
$$

Finally, we have

$$
T = T_{-\gamma}T_1T_\gamma = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}.
$$

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Exercise 5.5. Find linear transformations $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ such that $AB = \mathbf{0}$ but $BA \neq \mathbf{0}$.

Answer. We can represent these linear transformations by a 2×2 matrix in the standard basis:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \; ; \; B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

It is easy to verify that

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \; ; \; BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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Exercise 5.6. Prove that $\text{trace}(AB) = \text{trace}(BA)$.

Answer. Suppose $A_{m\times n}$ is an $m \times n$ matrix, and $B_{n\times m}$ is an $n \times m$ matrix. Then $C = AB$ is an $m \times m$ matrix whose (i, j)-entry is given by the formula

$$\sum_{k=1}^{n} A_{ik} B_{kj}.$$

In particular, the sum of the diagonal entries is

$$\sum_{i=1}^{m} C_{ii} = \sum_{i=1}^{m} \sum_{k=1}^{n} A_{ik} B_{ki} = \sum_{i=1}^{m} \sum_{k=1}^{n} B_{ki} A_{ik} = \sum_{k=1}^{n} \sum_{i=1}^{m} B_{ki} A_{ik},$$

which is the sum of the diagonal entries D_{kk} of the matrix $D = BA$.

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Exercise 5.7. Construct a non-zero matrix A such that $A^2 = \mathbf{0}$.

Answer. One can simply take

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

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Exercise 5.8. Find the matrix of the reflection through the line $y = -2x/3$. Perform all the multiplications.

Answer. We can rewrite $3y = -2x$ and decompose the reflection into $T = T_{-\gamma} T_1 T_\gamma$, where

$$[T_1] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is the reflection through the x-axis, and

$$[T_\gamma] = \begin{pmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix}$$

is the matrix representation of rotation by γ, the angle between $(1,0)^T$ and $(3,-2)^T$. We therefore have

$$T = \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} = \begin{pmatrix} \frac{9}{13} & -\frac{6}{13} \\ -\frac{6}{13} & \frac{4}{13} \end{pmatrix}.$$