

Introduction to Programming in Java

An Interdisciplinary Approach

SECOND EDITION

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◆ Addison-Wesley

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Preface

THE BASIS FOR EDUCATION IN THE last millennium was “reading, writing, and arithmetic”; now it is reading, writing, and *computing*. Learning to program is an essential part of the education of every student in the sciences and engineering. Beyond direct applications, it is the first step in understanding the nature of computer science’s undeniable impact on the modern world. This book aims to teach programming to those who need or want to learn it, in a scientific context.

Our primary goal is to *empower* students by supplying the experience and basic tools necessary to use computation effectively. Our approach is to teach students that composing a program is a natural, satisfying, and creative experience. We progressively introduce essential concepts, embrace classic applications from applied mathematics and the sciences to illustrate the concepts, and provide opportunities for students to write programs to solve engaging problems.

We use the Java programming language for all of the programs in this book—we refer to “Java” after “programming in the title to emphasize the idea that the book is about *fundamental concepts in programming*, not Java per se. This book teaches basic skills for computational problem solving that are applicable in many modern computing environments, and is a self-contained treatment intended for people with no previous experience in programming.

This book is an *interdisciplinary* approach to the traditional CS1 curriculum, in that we highlight the role of computing in other disciplines, from materials science to genomics to astrophysics to network systems. This approach emphasizes for students the essential idea that mathematics, science, engineering, and computing are intertwined in the modern world. While it is a CS1 textbook designed for any first-year college student, the book also can be used for self-study or as a supplement in a course that integrates programming with another field.

Coverage The book is organized around four stages of learning to program: basic elements, functions, object-oriented programming, and algorithms (with data structures). We provide the basic information readers need to build confidence in their ability to compose programs at each level before moving to the next level. An essential feature of our approach is the use of example programs that solve intriguing problems, supported with exercises ranging from self-study drills to challenging problems that call for creative solutions.

Basic elements include variables, assignment statements, built-in types of data, flow of control, arrays, and input/output, including graphics and sound.

Functions and modules are the student's first exposure to modular programming. We build upon familiarity with mathematical functions to introduce Java functions, and then consider the implications of programming with functions, including libraries of functions and recursion. We stress the fundamental idea of dividing a program into components that can be independently debugged, maintained, and reused.

Object-oriented programming is our introduction to data abstraction. We emphasize the concepts of a data type and their implementation using Java's class mechanism. We teach students how to *use*, *create*, and *design* data types. Modularity, encapsulation, and other modern programming paradigms are the central concepts of this stage.

Algorithms and data structures combine these modern programming paradigms with classic methods of organizing and processing data that remain effective for modern applications. We provide an introduction to classical algorithms for sorting and searching as well as fundamental data structures and their application, emphasizing the use of the scientific method to understand performance characteristics of implementations.

Applications in science and engineering are a key feature of the text. We motivate each programming concept that we address by examining its impact on specific applications. We draw examples from applied mathematics, the physical and biological sciences, and computer science itself, and include simulation of physical systems, numerical methods, data visualization, sound synthesis, image processing, financial simulation, and information technology. Specific examples include a treatment in the first chapter of Markov chains for web page ranks and case studies that address the percolation problem, n -body simulation, and the small-world phenomenon. These applications are an integral part of the text. They engage students in the material, illustrate the importance of the programming concepts, and provide persuasive evidence of the critical role played by computation in modern science and engineering.

Our primary goal is to teach the specific mechanisms and skills that are needed to develop effective solutions to any programming problem. We work with complete Java programs and encourage readers to use them. We focus on programming by individuals, not programming in the large.

Related texts This book is the second edition of our 2008 text that incorporates hundreds of improvements discovered during another decade of teaching the material, including, for example, a new treatment of hashing algorithms.

The four chapters in this book are identical to the first four chapters of our text *Computer Science: An Interdisciplinary Approach*. That book is a full introductory course on computer science that contains additional chapters on the theory of computing, machine-language programming, and machine architecture. We have published this book separately to meet the needs of people who are interested only in the Java programming content. We also have published a version of this book that is based on Python programming.

The chapters in this volume are suitable preparation for our book *Algorithms, Fourth Edition*, which is a thorough treatment of the most important algorithms in use today.

Use in the curriculum This book is suitable for a first-year college course aimed at teaching novices to program in the context of scientific applications. Taught from this book, any college student will learn to program in a familiar context. Students completing a course based on this book will be well prepared to apply their skills in later courses in their chosen major and to recognize when further education in computer science might be beneficial.

Instructors interested in a full-year course (or a fast-paced one-semester course with broader coverage) should instead consider adopting *Computer Science: An Interdisciplinary Approach*.

Prospective computer science majors, in particular, can benefit from learning to program in the context of scientific applications. A computer scientist needs the same basic background in the scientific method and the same exposure to the role of computation in science as does a biologist, an engineer, or a physicist.

Indeed, our interdisciplinary approach enables colleges and universities to teach prospective computer science majors and prospective majors in other fields in the *same* course. We cover the material prescribed by CS1, but our focus on applications brings life to the concepts and motivates students to learn them. Our interdisciplinary approach exposes students to problems in many different disciplines, helping them to choose a major more wisely.

Whatever the specific mechanism, the use of this book is best positioned early in the curriculum. This positioning allows us to leverage familiar material in high school mathematics and science. Moreover, students who learn to program early in their college curriculum will then be able to use computers more effectively when moving on to courses in their specialty. Like reading and writing, programming is certain to be an essential skill for any scientist or engineer. Students who have grasped the concepts in this book will continually develop that skill through a lifetime, reaping the benefits of exploiting computation to solve or to better understand the problems and projects that arise in their chosen field.

Prerequisites This book is suitable for typical first-year college students. In other words, we do not expect preparation beyond what is typically required for other entry-level science and mathematics courses.

Mathematical maturity is important. While we do not dwell on mathematical material, we do refer to the mathematics curriculum that students have taken in high school, including algebra, geometry, and trigonometry. Most students in our target audience automatically meet these requirements. Indeed, we take advantage of familiarity with this curriculum to introduce basic programming concepts.

Scientific curiosity is also an essential ingredient. Science and engineering students bring with them a sense of fascination with the ability of scientific inquiry to help explain what occurs in nature. We leverage this predilection with examples of simple programs that speak volumes about the natural world. We do not assume any specific knowledge beyond that provided by typical high school courses in mathematics, physics, biology, or chemistry.

Programming experience is not necessary, but also is not harmful. Teaching programming is our primary goal, so we assume no prior programming experience. Nevertheless, composing a program to solve a new problem is a challenging intellectual task, so students who have written numerous programs in high school can benefit from taking an introductory programming course based on this book. The book can support teaching students with varying backgrounds because the applications appeal to both novices and experts alike.

Experience using a computer is not necessary, but also is not at all a problem. College students use computers regularly—to communicate with friends and relatives, to listen to music, to process photos, and as part of many other activities. The realization that they can harness the power of their own computer in interesting and important ways is an exciting and lasting lesson.

Goals We cover the CS1 curriculum, but anyone who has taught an introductory programming course knows that expectations of instructors in later courses are typically high: Each instructor expects all students to be familiar with the computing environment and approach that he or she wants to use. For example, a physics professor might expect students to design a program over the weekend to run a simulation; a biology professor might expect students to be able to analyze genomes; or a computer science professor might expect knowledge of the details of a particular programming environment. Is it realistic to meet such diverse expectations? Is it realistic to offer a single introductory CS course for all students, as opposed to a different introductory course for each set of students?

With this book, and decades of experience at Princeton and other institutions that have adopted earlier versions, we answer these questions with a resounding *yes*. The most important reason to do so is that this approach encourages diversity. By keeping interesting applications at the forefront, we can keep advanced students engaged, and by avoiding classifying students at the beginning, we can ensure that every student who successfully masters this material is prepared for further study.

What can *teachers* of upper-level college courses expect of students who have completed a course based on this book?

This is a common introductory treatment of programming, which is analogous to commonly accepted introductory courses in mathematics, physics, biology, economics, or chemistry. *An Introduction to Programming in Java* strives to provide the basic preparation needed by all college students, while sending the clear message that there is much more to understand about computer science than just programming. Instructors teaching students who have studied from this book can expect that they will have the knowledge and experience necessary to enable them to effectively exploit computers in diverse applications.

What can *students* who have completed a course based on this book expect to accomplish in later courses?

Our message is that programming is not difficult to learn and that harnessing the power of the computer is rewarding. Students who master the material in this book are prepared to address computational challenges wherever they might appear later in their careers. They learn that modern programming environments, such as the one provided by Java, help open the door to any computational problem they might encounter later, and they gain the confidence to learn, evaluate, and use other computational tools. Students interested in computer science will be well prepared to pursue that interest; students in other fields will be ready to integrate computation into their studies.

Online lectures A complete set of studio-produced videos that can be used in conjunction with this text is available at

<http://www.informit.com/title/9780134493831>

As with traditional live lectures, the purpose is to *inform* and *inspire*, motivating students to study and learn from the text. Our experience is that student engagement with such online material is significantly better than with live lectures because of the ability to play the lectures at a chosen speed and to replay and review the lectures at any time.

Booksit An extensive body of other information that supplements this text may be found on the web at

<http://introcs.cs.princeton.edu/java>

For economy, we refer to this site as the *booksit* throughout. It contains material for instructors, students, and casual readers of the book. We briefly describe this material here, though, as all web users know, it is best surveyed by browsing. With a few exceptions to support testing, the material is all publicly available.

The booksit contains a condensed version of the text narrative for reference while online, hundreds of exercises and programming problems (some with solutions), hundreds of easily downloadable Java programs, real-world data sets, and our I/O libraries for processing text, graphics, and sound. It is the web presence associated with the book and is a living document that is accessed millions of times per year. It is an essential resource for everyone who owns this book and is critical to our goal of making computer science an integral component of the education of all college students.

One of the most important implications of the booksit is that it empowers teachers and students to use their own computers to teach and learn the material. Anyone with a computer and a browser can begin learning to program by following a few instructions on the booksit. The process is no more difficult than downloading a media player or a song.

For *teachers*, the booksit contains resources for teaching that (together with the book and the studio-produced videos) are sufficiently flexible to support many of the models for teaching that are emerging as teachers embrace technology in the 21st century. For example, at Princeton, our teaching style was for many years based on offering two lectures per week to a large audience, supplemented by two class sessions per week where students meet in small groups with instructors or teaching

assistants. More recently, we have moved to a model where students watch lectures online and we hold *class meetings* once a week in addition to the two class sessions. Other teachers may work completely online. Still others may use a “flipped” model involving enrichment of the lectures after students watch them.

For *students*, the booksite contains quick access to much of the material in the book, including source code, plus extra material to encourage self-learning. Solutions are provided for many of the book’s exercises, including complete program code and test data. There is a wealth of information associated with programming assignments, including suggested approaches, checklists, FAQs, and test data.

For *casual readers*, the booksite is a resource for accessing all manner of extra information associated with the book’s content. All of the booksite content provides web links and other routes to pursue more information about the topic under consideration. There is far more information accessible than any individual could fully digest, but our goal is to provide enough to whet any reader’s appetite for more information about the book’s content.

Acknowledgments This project has been under development since 1992, so far too many people have contributed to its success for us to acknowledge them all here. Special thanks are due to Anne Rogers, for helping to start the ball rolling; to Dave Hanson, Andrew Appel, and Chris van Wyk, for their patience in explaining data abstraction; and to Lisa Worthington and Donna Gabai, for being the first to truly relish the challenge of teaching this material to first-year students. We also gratefully acknowledge the efforts of /dev/126 ; the faculty, graduate students, and teaching staff who have dedicated themselves to teaching this material over the past 25 years here at Princeton University; and the thousands of undergraduates who have dedicated themselves to learning it.

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Elements of Programming

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OUR GOAL IN THIS CHAPTER IS to convince you that writing a program is easier than writing a piece of text, such as a paragraph or essay. Writing prose is difficult: we spend many years in school to learn how to do it. By contrast, just a few building blocks suffice to enable us to write programs that can help solve all sorts of fascinating, but otherwise unapproachable, problems. In this chapter, we take you through these building blocks, get you started on programming in Java, and study a variety of interesting programs. You will be able to express yourself (by writing programs) within just a few weeks. Like the ability to write prose, the ability to program is a lifetime skill that you can continually refine well into the future.

In this book, you will learn the *Java programming language*. This task will be much easier for you than, for example, learning a foreign language. Indeed, programming languages are characterized by only a few dozen vocabulary words and rules of grammar. Much of the material that we cover in this book could be expressed in the Python or C++ languages, or any of several other modern programming languages. We describe everything specifically in Java so that you can get started creating and running programs right away. On the one hand, we will focus on learning to program, as opposed to learning details about Java. On the other hand, part of the challenge of programming is knowing which details are relevant in a given situation. Java is widely used, so learning to program in this language will enable you to write programs on many computers (your own, for example). Also, learning to program in Java will make it easy for you to learn other languages, including lower-level languages such as C and specialized languages such as Matlab.

1.1 Your First Program

IN THIS SECTION, OUR PLAN IS to lead you into the world of Java programming by taking you through the basic steps required to get a simple program running. The *Java platform* (hereafter abbreviated *Java*) is a collection of applications, not unlike many of the other applications that you are accustomed to using (such as your word processor, email program, and web browser). As with any application, you need to be sure that Java is properly installed on your computer. It comes pre-loaded on many computers, or you can download it easily. You also need a text editor and a terminal application. Your first task is to find the instructions for installing such a Java programming environment on *your* computer by visiting

<http://introcs.cs.princeton.edu/java>

We refer to this site as the *booksite*. It contains an extensive amount of supplementary information about the material in this book for your reference and use while programming.

Programming in Java To introduce you to developing Java programs, we break the process down into three steps. To program in Java, you need to:

- *Create* a program by typing it into a file named, say, `MyProgram.java`.
- *Compile* it by typing `javac MyProgram.java` in a terminal window.
- *Execute* (or *run*) it by typing `java MyProgram` in the terminal window.

In the first step, you start with a blank screen and end with a sequence of typed characters on the screen, just as when you compose an email message or an essay. Programmers use the term *code* to refer to program text and the term *coding* to refer to the act of creating and editing the code. In the second step, you use a system application that *compiles* your program (translates it into a form more suitable for the computer) and puts the result in a file named `MyProgram.class`. In the third step, you transfer control of the computer from the system to your program (which returns control back to the system when finished). Many systems have several different ways to create, compile, and execute programs. We choose the sequence given here because it is the simplest to describe and use for small programs.

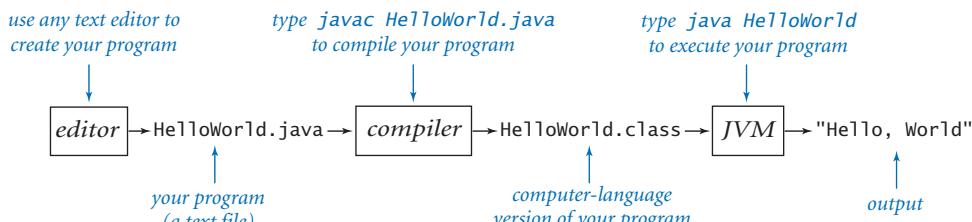
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Programs in this section

Creating a program. A Java program is nothing more than a sequence of characters, like a paragraph or a poem, stored in a file with a `.java` extension. To create one, therefore, you need simply define that sequence of characters, in the same way as you do for email or any other computer application. You can use any *text editor* for this task, or you can use one of the more sophisticated *integrated development environments* described on the booksite. Such environments are overkill for the sorts of programs we consider in this book, but they are not difficult to use, have many useful features, and are widely used by professionals.

Compiling a program. At first, it might seem that Java is designed to be best understood by the computer. To the contrary, the language is designed to be best understood by the programmer—that’s you. The computer’s language is far more primitive than Java. A *compiler* is an application that translates a program from the Java language to a language more suitable for execution on the computer. The compiler takes a file with a `.java` extension as input (your program) and produces a file with the same name but with a `.class` extension (the computer-language version). To use your Java compiler, type in a terminal window the `javac` command followed by the file name of the program you want to compile.

Executing (running) a program. Once you compile the program, you can execute (or run) it. This is the exciting part, where your program takes control of your computer (within the constraints of what Java allows). It is perhaps more accurate to say that your computer follows your instructions. It is even more accurate to say that a part of Java known as the *Java virtual machine* (JVM, for short) directs your computer to follow your instructions. To use the JVM to execute your program, type the `java` command followed by the program name in a terminal window.



Developing a Java program

Program 1.1.1 Hello, World

```
public class HelloWorld
{
    public static void main(String[] args)
    {
        // Prints "Hello, World" in the terminal window.
        System.out.println("Hello, World");
    }
}
```

This code is a Java program that accomplishes a simple task. It is traditionally a beginner’s first program. The box below shows what happens when you compile and execute the program. The terminal application gives a command prompt (% in this book) and executes the commands that you type (javac and then java in the example below). Our convention is to highlight in boldface the text that you type and display the results in regular face. In this case, the result is that the program prints the message Hello, World in the terminal window.

```
% javac HelloWorld.java
% java HelloWorld
Hello, World
```

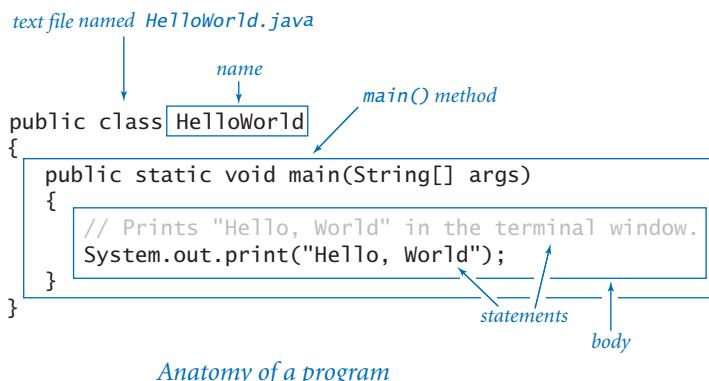
PROGRAM 1.1.1 is an example of a complete Java program. Its name is **HelloWorld**, which means that its code resides in a file named **HelloWorld.java** (by convention in Java). The program’s sole action is to print a message to the terminal window. For continuity, we will use some standard Java terms to describe the program, but we will not define them until later in the book: PROGRAM 1.1.1 consists of a single *class* named **HelloWorld** that has a single *method* named **main()**. (When referring to a method in the text, we use () after the name to distinguish it from other kinds of names.) Until SECTION 2.1, all of our classes will have this same structure. For the time being, you can think of “class” as meaning “program.”

The first line of a method specifies its name and other information; the rest is a sequence of *statements* enclosed in curly braces, with each statement typically followed by a semicolon. For the time being, you can think of “programming” as meaning “specifying a class name and a sequence of statements for its `main()` method,” with the heart of the program consisting of the sequence of statements in the `main()` method (its *body*). PROGRAM 1.1.1 contains two such statements:

- The first statement is a *comment*, which serves to document the program. In Java a single-line comment begins with two '/' characters and extends to the end of the line. In this book, we display comments in gray. Java ignores comments—they are present only for human readers of the program.
- The second statement is a *print statement*. It calls the method named `System.out.println()` to print a text message—the one specified between the matching double quotes—to the terminal window.

In the next two sections, you will learn about many different kinds of statements that you can use to make programs. For the moment, we will use only comments and print statements, like the ones in `HelloWorld`.

When you type `java` followed by a class name in your terminal window, the system calls the `main()` method that you defined in that class, and executes its statements in order, one by one. Thus, typing `java HelloWorld` causes the system to call the `main()` method in PROGRAM 1.1.1 and execute its two statements. The first statement is a comment, which Java ignores. The second statement prints the specified message to the terminal window.



Since the 1970s, it has been a tradition that a beginning programmer's first program should print `Hello, World`. So, you should type the code in PROGRAM 1.1.1 into a file, compile it, and execute it. By doing so, you will be following in the footsteps of countless others who have learned how to program. Also, you will be checking that you have a usable editor and terminal application. At first, accomplishing the task of printing something out in a terminal window might not seem very interesting; upon reflection, however, you will see that one of the most basic functions that we need from a program is its ability to tell us what it is doing.

For the time being, all our program code will be just like PROGRAM 1.1.1, except with a different sequence of statements in `main()`. Thus, you do not need to start with a blank page to write a program. Instead, you can

- Copy `HelloWorld.java` into a new file having a new program name of your choice, followed by `.java`.
- Replace `HelloWorld` on the first line with the new program name.
- Replace the comment and print statements with a different sequence of statements.

Your program is characterized by its sequence of statements and its name. Each Java program must reside in a file whose name matches the one after the word `class` on the first line, and it also must have a `.java` extension.

Errors. It is easy to blur the distinctions among editing, compiling, and executing programs. You should keep these processes separate in your mind when you are learning to program, to better understand the effects of the errors that inevitably arise.

You can fix or avoid most errors by carefully examining the program as you create it, the same way you fix spelling and grammatical errors when you compose an email message. Some errors, known as *compile-time* errors, are identified when you compile the program, because they prevent the compiler from doing the translation. Other errors, known as *run-time* errors, do not show up until you execute the program.

In general, errors in programs, also commonly known as *bugs*, are the bane of a programmer's existence: the error messages can be confusing or misleading, and the source of the error can be very hard to find. One of the first skills that you will learn is to identify errors; you will also learn to be sufficiently careful when coding, to avoid making many of them in the first place. You can find several examples of errors in the Q&A at the end of this section.

Program 1.1.2 Using a command-line argument

```
public class UseArgument
{
    public static void main(String[] args)
    {
        System.out.print("Hi, ");
        System.out.print(args[0]);
        System.out.println(". How are you?");
    }
}
```

This program shows the way in which we can control the actions of our programs: by providing an argument on the command line. Doing so allows us to tailor the behavior of our programs.

```
% javac UseArgument.java
% java UseArgument Alice
Hi, Alice. How are you?
% java UseArgument Bob
Hi, Bob. How are you?
```

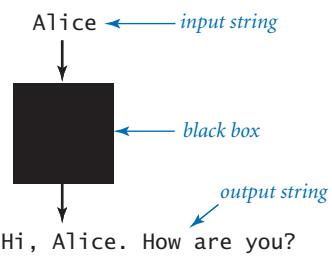
Input and output Typically, we want to provide *input* to our programs—that is, data that they can process to produce a result. The simplest way to provide input data is illustrated in `UseArgument` (PROGRAM 1.1.2). Whenever you execute the program `UseArgument`, it accepts the *command-line argument* that you type after the program name and prints it back out to the terminal window as part of the message. The result of executing this program depends on what you type after the program name. By executing the program with different command-line arguments, you produce different printed results. We will discuss in more detail the mechanism that we use to pass command-line arguments to our programs later, in SECTION 2.1. For now it is sufficient to understand that `args[0]` is the first command-line argument that you type after the program name, `args[1]` is the second, and so forth. Thus, you can use `args[0]` within your program’s body to represent the first string that you type on the command line when it is executed, as in `UseArgument`.

In addition to the `System.out.println()` method, `UseArgument` calls the `System.out.print()` method. This method is just like `System.out.println()`, but prints just the specified string (and not a newline character).

Again, accomplishing the task of getting a program to print back out what we type in to it may not seem interesting at first, but upon reflection you will realize that another basic function of a program is its ability to respond to basic information from the user to control what the program does. The simple model that `UseArgument` represents will suffice to allow us to consider Java's basic programming mechanism and to address all sorts of interesting computational problems.

Stepping back, we can see that `UseArgument` does neither more nor less than implement a function that maps a string of characters (the command-line argument) into another string of characters (the message printed back to the terminal window). When using it, we might think of our Java program as a black box that converts our input string to some output string.

This model is attractive because it is not only simple but also sufficiently general to allow completion, in principle, of any computational task. For example, the Java compiler itself is nothing more than a program that takes one string of characters as input (a `.java` file) and produces another string of characters as output (the corresponding `.class` file). Later, you will be able to write programs that accomplish a variety of interesting tasks (though we stop short of programs as complicated as a compiler). For the moment, we will live with various limitations on the size and type of the input and output to our programs; in SECTION 1.5, you will see how to incorporate more sophisticated mechanisms for program input and output. In particular, you will see that we can work with arbitrarily long input and output strings and other types of data such as sound and pictures.



A bird's-eye view of a Java program

Q&A

Q. Why Java?

A. The programs that we are writing are very similar to their counterparts in several other languages, so our choice of language is not crucial. We use Java because it is widely available, embraces a full set of modern abstractions, and has a variety of automatic checks for mistakes in programs, so it is suitable for learning to program. There is no perfect language, and you certainly will be programming in other languages in the future.

Q. Do I really have to type in the programs in the book to try them out? I believe that you ran them and that they produce the indicated output.

A. Everyone should type in and run `HelloWorld`. Your understanding will be greatly magnified if you also run `UseArgument`, try it on various inputs, and modify it to test different ideas of your own. To save some typing, you can find all of the code in this book (and much more) on the booksite. This site also has information about installing and running Java on your computer, answers to selected exercises, web links, and other extra information that you may find useful while programming.

Q. What is the meaning of the words `public`, `static`, and `void`?

A. These keywords specify certain properties of `main()` that you will learn about later in the book. For the moment, we just include these keywords in the code (because they are required) but do not refer to them in the text.

Q. What is the meaning of the `//`, `/*`, and `*/` character sequences in the code?

A. They denote *comments*, which are ignored by the compiler. A comment is either text in between `/*` and `*/` or at the end of a line after `//`. Comments are indispensable because they help other programmers to understand your code and even can help you to understand your own code in retrospect. The constraints of the book format demand that we use comments sparingly in our programs; instead we describe each program thoroughly in the accompanying text and figures. The programs on the booksite are commented to a more realistic degree.

Q. What are Java's rules regarding tabs, spaces, and newline characters?

A. Such characters are known as *whitespace* characters. Java compilers consider all whitespace in program text to be equivalent. For example, we could write `HelloWorld` as follows:

```
public class HelloWorld { public static void main ( String  
[] args) { System.out.println("Hello, World") ; } }
```

But we do normally adhere to spacing and indenting conventions when we write Java programs, just as we indent paragraphs and lines consistently when we write prose or poetry.

Q. What are the rules regarding quotation marks?

A. Material inside double quotation marks is an exception to the rule defined in the previous question: typically, characters within quotes are taken literally so that you can precisely specify what gets printed. If you put any number of successive spaces within the quotes, you get that number of spaces in the output. If you accidentally omit a quotation mark, the compiler may get very confused, because it needs that mark to distinguish between characters in the string and other parts of the program.

Q. What happens when you omit a curly brace or misspell one of the words, such as `public` or `static` or `void` or `main`?

A. It depends upon precisely what you do. Such errors are called *syntax errors* and are usually caught by the compiler. For example, if you make a program `Bad` that is exactly the same as `HelloWorld` except that you omit the line containing the first left curly brace (and change the program name from `HelloWorld` to `Bad`), you get the following helpful message:

```
% javac Bad.java  
Bad.java:1: error: '{' expected  
public class Bad  
^  
1 error
```

From this message, you might correctly surmise that you need to insert a left curly brace. But the compiler may not be able to tell you exactly which mistake you made, so the error message may be hard to understand. For example, if you omit the second left curly brace instead of the first one, you get the following message:

```
% javac Bad.java
Bad.java:3: error: ';' expected
    public static void main(String[] args)
                           ^
Bad.java:7: error: class, interface, or enum expected
}
 ^
2 errors
```

One way to get used to such messages is to intentionally introduce mistakes into a simple program and then see what happens. Whatever the error message says, you should treat the compiler as a friend, because it is just trying to tell you that something is wrong with your program.

Q. Which Java methods are available for me to use?

A. There are thousands of them. We introduce them to you in a deliberate fashion (starting in the next section) to avoid overwhelming you with choices.

Q. When I ran `UseArgument`, I got a strange error message. What's the problem?

A. Most likely, you forgot to include a command-line argument:

```
% java UseArgument
Hi, Exception in thread "main"
java.lang.ArrayIndexOutOfBoundsException: 0
        at UseArgument.main(UseArgument.java:6)
```

Java is complaining that you ran the program but did not type a command-line argument as promised. You will learn more details about array indices in SECTION 1.4. Remember this error message—you are likely to see it again. Even experienced programmers forget to type command-line arguments on occasion.

Exercises

1.1.1 Write a program that prints the Hello, World message 10 times.

1.1.2 Describe what happens if you omit the following in `HelloWorld.java`:

- a. `public`
- b. `static`
- c. `void`
- d. `args`

1.1.3 Describe what happens if you misspell (by, say, omitting the second letter) the following in `HelloWorld.java`:

- a. `public`
- b. `static`
- c. `void`
- d. `args`

1.1.4 Describe what happens if you put the double quotes in the print statement of `HelloWorld.java` on different lines, as in this code fragment:

```
System.out.println("Hello,  
    World");
```

1.1.5 Describe what happens if you try to execute `UseArgument` with each of the following command lines:

- a. `java UseArgument java`
- b. `java UseArgument @!&^%`
- c. `java UseArgument 1234`
- d. `java UseArgument.java Bob`
- e. `java UseArgument Alice Bob`

1.1.6 Modify `UseArgument.java` to make a program `UseThree.java` that takes three names as command-line arguments and prints a proper sentence with the names in the reverse of the order given, so that, for example, `java UseThree Alice Bob Carol` prints `Hi Carol, Bob, and Alice.`

1.2 Built-in Types of Data

WHEN PROGRAMMING IN JAVA, YOU MUST always be aware of the type of data that your program is processing. The programs in SECTION 1.1 process strings of characters, many of the programs in this section process numbers, and we consider numerous other types later in the book. Understanding the distinctions among them is so important that we formally define the idea: a *data type* is a *set of values* and a *set of operations* defined on those values. You are familiar with various types of numbers, such as integers and real numbers, and with operations defined on them, such as addition and multiplication. In mathematics, we are accustomed to thinking of sets of numbers as being infinite; in computer programs we have to work with a finite number of possibilities. Each operation that we perform is well defined *only* for the finite set of values in an associated data type.

There are eight *primitive* types of data in Java, mostly for different kinds of numbers. Of the eight primitive types, we most often use these: `int` for integers; `double` for real numbers; and `boolean` for true–false values. Other data types are available in Java libraries: for example, the programs in SECTION 1.1 use the type `String` for strings of characters. Java treats the `String` type differently from other types because its usage for input and output is essential. Accordingly, it shares some characteristics of the primitive types; for example, some of its operations are built into the Java language. For clarity, we refer to primitive types and `String` collectively as *built-in* types. For the time being, we concentrate on programs that are based on computing with built-in types. Later, you will learn about Java library data types and building your own data types. Indeed, programming in Java often centers on building data types, as you shall see in CHAPTER 3.

After defining basic terms, we consider several sample programs and code fragments that illustrate the use of different types of data. These code fragments do not do much real computing, but you will soon see similar code in longer programs. Understanding data types (values and operations on them) is an essential step in beginning to program. It sets the stage for us to begin working with more intricate programs in the next section. Every program that you write will use code like the tiny fragments shown in this section.

1.2.1	String concatenation	20
1.2.2	Integer multiplication and division	23
1.2.3	Quadratic formula.	25
1.2.4	Leap year	28
1.2.5	Casting to get a random integer . .	34

Programs in this section

<i>type</i>	<i>set of values</i>	<i>common operators</i>	<i>sample literal values</i>
<code>int</code>	integers	<code>+ - * / %</code>	99 12 2147483647
<code>double</code>	floating-point numbers	<code>+ - * /</code>	3.14 2.5 6.022e23
<code>boolean</code>	boolean values	<code>&& !</code>	<code>true false</code>
<code>char</code>	characters		<code>'A' '1' '%' '\n'</code>
<code>String</code>	sequences of characters	<code>+</code>	<code>"AB" "Hello" "2.5"</code>

Basic built-in data types

Terminology To talk about data types, we need to introduce some terminology. To do so, we start with the following code fragment:

```
int a, b, c;
a = 1234;
b = 99;
c = a + b;
```

The first line is a *declaration statement* that declares the names of three *variables* using the *identifiers* `a`, `b`, and `c` and their type to be `int`. The next three lines are *assignment statements* that change the values of the variables, using the *literals* `1234` and `99`, and the *expression* `a + b`, with the end result that `c` has the value `1333`.

Literals. A *literal* is a Java-code representation of a data-type value. We use sequences of digits such as `1234` or `99` to represent values of type `int`; we add a decimal point, as in `3.14159` or `2.71828`, to represent values of type `double`; we use the keywords `true` or `false` to represent the two values of type `boolean`; and we use sequences of characters enclosed in matching quotes, such as `"Hello, World"`, to represent values of type `String`.

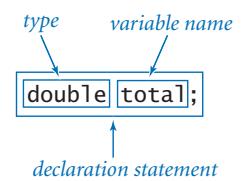
Operators. An *operator* is a Java-code representation of a data-type operation. Java uses `+` and `*` to represent addition and multiplication for integers and floating-point numbers; Java uses `&&`, `||`, and `!` to represent boolean operations; and so forth. We will describe the most commonly used operators on built-in types later in this section.

Identifiers. An *identifier* is a Java-code representation of a name (such as for a variable). Each identifier is a sequence of letters, digits, underscores, and currency symbols, the first of which is not a digit. For example, the sequences of characters

abc, Ab\$, abc123, and a_b are all legal Java identifiers, but Ab*, 1abc, and a+b are not. Identifiers are case sensitive, so Ab, ab, and AB are all different names. Certain *reserved words*—such as `public`, `static`, `int`, `double`, `String`, `true`, `false`, and `null`—are special, and you cannot use them as identifiers.

Variables. A *variable* is an entity that holds a data-type value, which we can refer to by name. In Java, each variable has a specific type and stores one of the possible values from that type. For example, an `int` variable can store either the value 99 or 1234 but not 3.14159 or "Hello, World". Different variables of the same type may store the same value. Also, as the name suggests, the value of a variable may *change* as a computation unfolds. For example, we use a variable named `sum` in several programs in this book to keep the running sum of a sequence of numbers. We create variables using *declaration statements* and compute with them in *expressions*, as described next.

Declaration statements. To create a variable in Java, you use a *declaration statement*, or just *declaration* for short. A declaration includes a type followed by a variable name. Java reserves enough memory to store a data-type value of the specified type, and associates the variable name with that area of memory, so that it can access the value when you use the variable in later code. For economy, you can declare several variables of the same type in a single declaration statement.

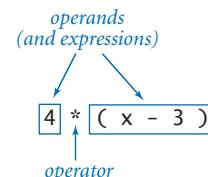


Anatomy of a declaration

Variable naming conventions. Programmers typically follow stylistic conventions when naming things. In this book, our convention is to give each variable a meaningful name that consists of a lowercase letter followed by lowercase letters, uppercase letters, and digits. We use uppercase letters to mark the words of a multi-word variable name. For example, we use the variable names `i`, `x`, `y`, `sum`, `isLeapYear`, and `outDegrees`, among many others. Programmers refer to this naming style as *camel case*.

Constant variables. We use the oxymoronic term *constant variable* to describe a variable whose value does not change during the execution of a program (or from one execution of the program to the next). In this book, our convention is to give each constant variable a name that consists of an uppercase letter followed by uppercase letters, digits, and underscores. For example, we might use the constant variable names `SPEED_OF_LIGHT` and `DARK_RED`.

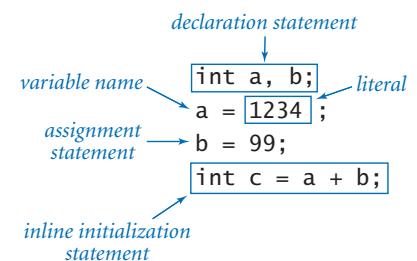
Expressions. An *expression* is a combination of literals, variables, and operations that Java *evaluates* to produce a value. For primitive types, expressions often look just like mathematical formulas, using *operators* to specify data-type operations to be performed on one or more *operands*. Most of the operators that we use are *binary operators* that take exactly two operands, such as $x - 3$ or $5 * x$. Each operand can be any expression, perhaps within parentheses. For example, we can write $4 * (x - 3)$ or $5 * x - 6$ and Java will understand what we mean. An expression is a directive to perform a sequence of operations; the expression is a representation of the resulting value.



Anatomy of an expression

Operator precedence. An expression is shorthand for a sequence of operations: in which order should the operators be applied? Java has natural and well defined *precedence* rules that fully specify this order. For arithmetic operations, multiplication and division are performed before addition and subtraction, so that $a - b * c$ and $a - (b * c)$ represent the same sequence of operations. When arithmetic operators have the same precedence, the order is determined by *left associativity*, so that $a - b - c$ and $(a - b) - c$ represent the same sequence of operations. You can use parentheses to override the rules, so you can write $a - (b - c)$ if that is what you want. You might encounter in the future some Java code that depends subtly on precedence rules, but we use parentheses to avoid such code in this book. If you are interested, you can find full details on the rules on the booksite.

Assignment statements. An assignment statement associates a data-type value with a variable. When we write $c = a + b$ in Java, we are not expressing mathematical equality, but are instead expressing an *action*: set the value of the variable c to be the value of a plus the value of b . It is true that the value of c is mathematically equal to the value of $a + b$ immediately after the assignment statement has been executed, but the point of the statement is to change (or initialize) the value of c . The left-hand side of an assignment statement must be a single variable; the right-hand side can be any expression that produces a value of a compatible type. So, for example, both $1234 = a;$ and $a + b = b + a;$ are invalid statements in Java. In short, *the meaning of = is decidedly not the same as in mathematical equations*.



Using a primitive data type

Inline initialization. Before you can use a variable in an expression, you must first declare the variable and assign to it an initial value. Failure to do either results in a compile-time error. For economy, you can combine a declaration statement with an assignment statement in a construct known as an *inline initialization* statement. For example, the following code declares two variables `a` and `b`, and initializes them to the values 1234 and 99, respectively:

```
int a = 1234;
int b = 99;
```

Most often, we declare and initialize a variable in this manner *at the point of its first use* in our program.

Tracing changes in variable values. As a final check on your understanding of the purpose of assignment statements, convince yourself that the following code *exchanges* the values of `a` and `b` (assume that `a` and `b` are `int` variables):

```
int t = a;
a = b;
b = t;
```

To do so, use a time-honored method of examining program behavior: study a table of the variable values after each statement (such a table is known as a *trace*).

	a	b	t
int a, b;	<i>undefined</i>	<i>undefined</i>	
a = 1234;	1234	<i>undefined</i>	
b = 99;	1234	99	
int t = a;	1234	99	1234
a = b;	99	99	1234
b = t;	99	1234	1234

Your first trace

Type safety. Java requires you to declare the type of every variable. This enables Java to check for type mismatch errors at compile time and alert you to potential bugs in your program. For example, you cannot assign a `double` value to an `int` variable, multiply a `String` with a `boolean`, or use an uninitialized variable within an expression. This situation is analogous to making sure that quantities have the proper units in a scientific application (for example, it does not make sense to add a quantity measured in inches to another measured in pounds).

NEXT, WE CONSIDER THESE DETAILS FOR the basic built-in types that you will use most often (strings, integers, floating-point numbers, and true–false values), along with sample code illustrating their use. To understand how to use a data type, you need to know not just its defined set of values, but also which operations you can perform, the language mechanism for invoking the operations, and the conventions for specifying literals.

Characters and strings The `char` type represents individual alphanumeric characters or symbols, like the ones that you type. There are 2^{16} different possible `char` values, but we usually restrict attention to the ones that represent letters, numbers, symbols, and whitespace characters such as tab and newline. You can specify a `char` literal by enclosing a character within single quotes; for example, '`a`' represents the letter `a`. For tab, newline, backslash, single quote, and double quote, we use the special *escape sequences* `\t`, `\n`, `\\"`, `\'`, and `\\"`, respectively. The characters are encoded as 16-bit integers using an encoding scheme known as *Unicode*, and there are also escape sequences for specifying special characters not found on your keyboard (see the booksite). We usually do not perform any operations directly on characters other than assigning values to variables.

The `String` type represents sequences of characters. You can specify a `String` literal by enclosing a sequence of characters within double quotes, such as "`Hello, World`". The `String` data type is *not* a primitive type, but Java sometimes treats it like one. For example, the *concatenation* operator (`+`) takes two `String` operands and produces a third `String` that is formed by appending the characters of the second operand to the characters of the first operand.

The concatenation operation (along with the ability to declare `String` variables and to use them in expressions and assignment statements) is sufficiently powerful to allow us to attack some nontrivial computing tasks. As an example, `Ruler` (PROGRAM 1.2.1) computes a table of values of the *ruler function* that describes the relative lengths of the marks on a ruler. One noteworthy feature of this computation is that it illustrates how easy it is to craft a short program that produces a huge amount of output. If you extend this program in the obvious way to print five lines, six lines, seven lines, and so forth, you will see that

each time you add two statements to this program, you double the size of the output. Specifically, if the program prints n lines, the n th line contains $2^n - 1$ numbers. For example, if you were to add statements in this way so that the program prints 30 lines, it would print more than 1 *billion* numbers.

expression	value
<code>"Hi, " + "Bob"</code>	<code>"Hi, Bob"</code>
<code>"1" + " 2 " + "1"</code>	<code>"1 2 1"</code>
<code>"1234" + " " + "99"</code>	<code>"1234 + 99"</code>
<code>"1234" + "99"</code>	<code>"123499"</code>
<i>Typical String expressions</i>	

<i>values</i>	characters
<i>typical literals</i>	<code>'a'</code> <code>'\n'</code>
	<i>Java's built-in char data type</i>

<i>values</i>	sequences of characters
<i>typical literals</i>	<code>"Hello, World"</code> <code>" " * "</code>
<i>operation</i>	concatenate
<i>operator</i>	<code>+</code>

Java's built-in String data type

Program 1.2.1 String concatenation

```
public class Ruler
{
    public static void main(String[] args)
    {
        String ruler1 = "1";
        String ruler2 = ruler1 + " 2 " + ruler1;
        String ruler3 = ruler2 + " 3 " + ruler2;
        String ruler4 = ruler3 + " 4 " + ruler3;
        System.out.println(ruler1);
        System.out.println(ruler2);
        System.out.println(ruler3);
        System.out.println(ruler4);
    }
}
```

This program prints the relative lengths of the subdivisions on a ruler. The n th line of output is the relative lengths of the marks on a ruler subdivided in intervals of $1/2^n$ of an inch. For example, the fourth line of output gives the relative lengths of the marks that indicate intervals of one-sixteenth of an inch on a ruler.

```
% javac Ruler.java
% java Ruler
1
1 2 1
1 2 1 3 1 2 1
1 2 1 3 1 2 1 4 1 2 1 3 1 2 1
```



The ruler function for $n = 4$

Our most frequent use (by far) of the concatenation operation is to put together results of computation for output with `System.out.println()`. For example, we could simplify `UseArgument` (PROGRAM 1.1.2) by replacing its three statements in `main()` with this single statement:

```
System.out.println("Hi, " + args[0] + ". How are you?");
```

We have considered the `String` type first precisely because we need it for output (and command-line arguments) in programs that process not only strings but other types of data as well. Next we consider two convenient mechanisms in Java for converting numbers to strings and strings to numbers.

Converting numbers to strings for output. As mentioned at the beginning of this section, Java's built-in `String` type obeys special rules. One of these special rules is that you can easily convert a value of any type to a `String` value: whenever we use the `+` operator with a `String` as one of its operands, Java automatically converts the other operand to a `String`, producing as a result the `String` formed from the characters of the first operand followed by the characters of the second operand. For example, the result of these two code fragments

```
String a = "1234";           String a = "1234";
String b = "99";             int b = 99;
String c = a + b;           String c = a + b;
```

are both the same: they assign to `c` the value "123499". We use this automatic conversion liberally to form `String` values for use with `System.out.print()` and `System.out.println()`. For example, we can write statements like this one:

```
System.out.println(a + " + " + b + " = " + c);
```

If `a`, `b`, and `c` are `int` variables with the values 1234, 99, and 1333, respectively, then this statement prints the string 1234 + 99 = 1333.

Converting strings to numbers for input. Java also provides library methods that convert the strings that we type as command-line arguments into numeric values for primitive types. We use the Java library methods `Integer.parseInt()` and `Double.parseDouble()` for this purpose. For example, typing `Integer.parseInt("123")` in program text is equivalent to typing the `int` literal 123. If the user types 123 as the first command-line argument, then the code `Integer.parseInt(args[0])` converts the `String` value "123" into the `int` value 123. You will see several examples of this usage in the programs in this section.

WITH THESE MECHANISMS, OUR VIEW OF each Java program as a black box that takes string arguments and produces string results is still valid, but we can now interpret those strings as numbers and use them as the basis for meaningful computations.

Integers The `int` type represents integers (natural numbers) between -2^{31} ($-2^{31}-1$) and $2^{31}-1$ ($2^{31}-1$). These bounds derive from the fact that integers are represented in binary with 32 binary digits; there are 2^{32} possible values. (The term *binary digit* is omnipresent in computer science, and we nearly always use the abbreviation *bit*: a bit is either 0 or 1.) The range of possible `int` values is asymmetric because zero is included with the positive values. You can see the Q&A at the end of this section for more details about number representation, but in the present context it suffices to know that an `int` is one of the finite set of values in the range just given. You can specify an `int` literal with a sequence of the decimal digits 0 through 9 (that, when interpreted as decimal numbers, fall within the defined range). We use `ints` frequently because they naturally arise when we are implementing programs.

Standard arithmetic operators for addition/subtraction (+ and -), multiplication (*), division (/), and remainder (%) for the `int` data type are built into Java. These operators take two `int` operands and produce an `int` result, with one significant exception—division or remainder by zero is not allowed. These operations are defined as in grade school (keeping in mind that all results must be integers): given two `int` values `a` and `b`, the value of `a / b` is the number of times `b` goes into `a` *with the fractional part discarded*, and the value of `a % b` is the remainder

that you get when you divide `a` by `b`. For example, the value of `17 / 3` is 5, and the value of `17 % 3` is 2. The `int` results that we get from arithmetic operations are just what we expect, except that if the result is too large to fit into `int`'s 32-bit representation, then it will be truncated in a well-defined manner. This situation is known

expression	value	comment
99	99	integer literal
+99	99	positive sign
-99	-99	negative sign
5 + 3	8	addition
5 - 3	2	subtraction
5 * 3	15	multiplication
5 / 3	1	no fractional part
5 % 3	2	remainder
1 / 0		run-time error
3 * 5 - 2	13	* has precedence
3 + 5 / 2	5	/ has precedence
3 - 5 - 2	-4	left associative
(3 - 5) - 2	-4	better style
3 - (5 - 2)	0	unambiguous

Typical int expressions

<i>values</i>	integers between -2^{31} and $2^{31}-1$					
<i>typical literals</i>	1234 99 0 1000000					
<i>operations</i>	<i>sign</i>	<i>add</i>	<i>subtract</i>	<i>multiply</i>	<i>divide</i>	<i>remainder</i>
<i>operators</i>	+	-	+	-	*	/

Java's built-in `int` data type

Program 1.2.2 Integer multiplication and division

```
public class IntOps
{
    public static void main(String[] args)
    {
        int a = Integer.parseInt(args[0]);
        int b = Integer.parseInt(args[1]);
        int p = a * b;
        int q = a / b;
        int r = a % b;
        System.out.println(a + " * " + b + " = " + p);
        System.out.println(a + " / " + b + " = " + q);
        System.out.println(a + " % " + b + " = " + r);
        System.out.println(a + " = " + q + " * " + b + " + " + r);
    }
}
```

Arithmetic for integers is built into Java. Most of this code is devoted to the task of getting the values in and out; the actual arithmetic is in the simple statements in the middle of the program that assign values to *p*, *q*, and *r*.

```
% javac IntOps.java
% java IntOps 1234 99
1234 * 99 = 122166
1234 / 99 = 12
1234 % 99 = 46
1234 = 12 * 99 + 46
```

as *overflow*. In general, we have to take care that such a result is not misinterpreted by our code. For the moment, we will be computing with small numbers, so you do not have to worry about these boundary conditions.

PROGRAM 1.2.2 illustrates three basic operations (multiplication, division, and remainder) for manipulating integers,. It also demonstrates the use of `Integer.parseInt()` to convert `String` values on the command line to `int` values, as well as the use of automatic type conversion to convert `int` values to `String` values for output.

Three other built-in types are different representations of integers in Java. The `long`, `short`, and `byte` types are the same as `int` except that they use 64, 16, and 8 bits respectively, so the range of allowed values is accordingly different. Programmers use `long` when working with huge integers, and the other types to save space. You can find a table with the maximum and minimum values for each type on the booksite, or you can figure them out for yourself from the numbers of bits.

Floating-point numbers The `double` type represents *floating-point* numbers, for use in scientific and commercial applications. The internal representation is like scientific notation, so that we can compute with numbers in a huge range. We use floating-point numbers to represent real numbers, but they are decidedly not the same as real numbers! There are infinitely many real numbers, but we can represent only a finite number of floating-point numbers in any digital computer representation. Floating-point numbers do approximate real numbers sufficiently well that we can use them in applications, but we often need to cope with the fact that we cannot always do exact computations.

You can specify a `double` literal with a sequence of digits with a decimal point. For example, the literal `3.14159` represents a six-digit approximation to π . Alternatively, you specify a `double` literal with a notation like scientific notation: the literal `6.022e23` represents the number 6.022×10^{23} . As with integers, you can use these conventions to type floating-point literals in your programs or to provide floating-point numbers as string arguments on the command line.

The arithmetic operators `+`, `-`, `*`, and `/` are defined for `double`. Beyond these built-in operators, the Java `Math` library defines the square root function, trigonometric functions, logarithm/exponential functions, and other common functions for floating-point numbers. To use one of these functions in an expression, you type the name of the function followed by its argument in parentheses. For ex-

<i>expression</i>	<i>value</i>
<code>3.141 + 2.0</code>	5.141
<code>3.141 - 2.0</code>	1.111
<code>3.141 / 2.0</code>	1.5705
<code>5.0 / 3.0</code>	1.6666666666666667
<code>10.0 % 3.141</code>	0.577
<code>1.0 / 0.0</code>	<code>Infinity</code>
<code>Math.sqrt(2.0)</code>	1.4142135623730951
<code>Math.sqrt(-1.0)</code>	<code>NaN</code>

Typical double expressions

<i>values</i> <i>typical literals</i> <i>operations</i> <i>operators</i>	real numbers (specified by IEEE 754 standard) <code>3.14159</code> <code>6.022e23</code> <code>2.0</code> <code>1.4142135623730951</code> <code>add</code> <code>subtract</code> <code>multiply</code> <code>divide</code> <code>+</code> <code>-</code> <code>*</code> <code>/</code>
<i>Java's built-in double data type</i>	

Program 1.2.3 Quadratic formula

```
public class Quadratic
{
    public static void main(String[] args)
    {
        double b = Double.parseDouble(args[0]);
        double c = Double.parseDouble(args[1]);
        double discriminant = b*b - 4.0*c;
        double d = Math.sqrt(discriminant);
        System.out.println((-b + d) / 2.0);
        System.out.println((-b - d) / 2.0);
    }
}
```

This program prints the roots of the polynomial $x^2 + bx + c$, using the quadratic formula. For example, the roots of $x^2 - 3x + 2$ are 1 and 2 since we can factor the equation as $(x - 1)(x - 2)$; the roots of $x^2 - x - 1$ are ϕ and $1 - \phi$, where ϕ is the golden ratio; and the roots of $x^2 + x + 1$ are not real numbers.

% javac Quadratic.java % java Quadratic -3.0 2.0 2.0 1.0	% java Quadratic -1.0 -1.0 1.618033988749895 -0.6180339887498949 % java Quadratic 1.0 1.0 NaN NaN
---	--

ample, the code `Math.sqrt(2.0)` evaluates to a `double` value that is approximately the square root of 2. We discuss the mechanism behind this arrangement in more detail in SECTION 2.1 and more details about the `Math` library at the end of this section.

When working with floating-point numbers, one of the first things that you will encounter is the issue of *precision*. For example, printing `5.0/2.0` results in `2.5` as expected, but printing `5.0/3.0` results in `1.6666666666666667`. In SECTION 1.5, you will learn Java's mechanism for controlling the number of significant digits that you see in output. Until then, we will work with the Java default output format.

The result of a calculation can be one of the special values `Infinity` (if the number is too large to be represented) or `NaN` (if the result of the calculation is undefined). Though there are myriad details to consider when calculations involve these values, you can use `double` in a natural way and begin to write Java programs instead of using a calculator for all kinds of calculations. For example, PROGRAM 1.2.3 shows the use of `double` values in computing the roots of a quadratic equation using the quadratic formula. Several of the exercises at the end of this section further illustrate this point.

As with `long`, `short`, and `byte` for integers, there is another representation for real numbers called `float`. Programmers sometimes use `float` to save space when precision is a secondary consideration. The `double` type is useful for about 15 significant digits; the `float` type is good for only about 7 digits. We do not use `float` in this book.

Booleans The `boolean` type represents truth values from logic. It has just two values: `true` and `false`. These are also the two possible `boolean` literals. Every `boolean` variable has one of these two values, and every `boolean` operation has operands and a result that takes on just one of these two values. This simplicity is deceiving—`boolean` values lie at the foundation of computer science.

The most important operations defined for `booleans` are `and` (`&&`), `or` (`||`), and `not` (`!`), which have familiar definitions:

- `a && b` is `true` if both operands are `true`, and `false` if either is `false`.
- `a || b` is `false` if both operands are `false`, and `true` if either is `true`.
- `!a` is `true` if `a` is `false`, and `false` if `a` is `true`.

Despite the intuitive nature of these definitions, it is worthwhile to fully specify each possibility for each operation in tables known as *truth tables*. The `not` function has only one operand: its value for each of the two possible values of the operand is

<i>values</i>	<i>true or false</i>	
<i>literals</i>	<code>true</code>	<code>false</code>
<i>operations</i>	<code>and</code>	<code>or</code>
<i>operators</i>	<code>&&</code>	<code> </code>

Java's built-in boolean data type

<code>a</code>	<code>!a</code>	<code>a</code>	<code>b</code>	<code>a && b</code>	<code>a b</code>
<code>true</code>	<code>false</code>	<code>false</code>	<code>false</code>	<code>false</code>	<code>false</code>
<code>false</code>	<code>true</code>	<code>false</code>	<code>true</code>	<code>false</code>	<code>true</code>
		<code>true</code>	<code>false</code>	<code>false</code>	<code>true</code>
		<code>true</code>	<code>true</code>	<code>true</code>	<code>true</code>

Truth-table definitions of boolean operations

specified in the second column. The *and* and *or* functions each have two operands: there are four different possibilities for operand values, and the values of the functions for each possibility are specified in the right two columns.

We can use these operators with parentheses to develop arbitrarily complex expressions, each of which specifies a well-defined boolean function. Often the same function appears in different guises. For example, the expressions `(a && b)` and `!(!a || !b)` are equivalent.

a	b	<code>a && b</code>	<code>!a</code>	<code>!b</code>	<code>!a !b</code>	<code>!(!a !b)</code>
false	false	false	true	true	true	false
false	true	false	true	false	true	false
true	false	false	false	true	true	false
true	true	true	false	false	false	true

Truth-table proof that `a && b` and `!(!a || !b)` are identical

The study of manipulating expressions of this kind is known as *Boolean logic*. This field of mathematics is fundamental to computing: it plays an essential role in the design and operation of computer hardware itself, and it is also a starting point for the theoretical foundations of computation. In the present context, we are interested in boolean expressions because we use them to control the behavior of our programs. Typically, a particular condition of interest is specified as a boolean expression, and a piece of program code is written to execute one set of statements if that expression is `true` and a different set of statements if the expression is `false`. The mechanics of doing so are the topic of SECTION 1.3.

Comparisons Some *mixed-type* operators take operands of one type and produce a result of another type. The most important operators of this kind are the comparison operators `==`, `!=`, `<`, `<=`, `>`, and `>=`, which all are defined for each primitive numeric type and produce a boolean result. Since operations are defined only with respect to data types, each of these symbols stands for many operations, one for each data type. It is required that both operands be of the same type.

non-negative discriminant?

`(b*b - 4.0*a*c) >= 0.0`

beginning of a century?

`(year % 100) == 0`

legal month?

`(month >= 1) && (month <= 12)`

Typical comparison expressions

Program 1.2.4 Leap year

```
public class LeapYear
{
    public static void main(String[] args)
    {
        int year = Integer.parseInt(args[0]);
        boolean isLeapYear;
        isLeapYear = (year % 4 == 0);
        isLeapYear = isLeapYear && (year % 100 != 0);
        isLeapYear = isLeapYear || (year % 400 == 0);
        System.out.println(isLeapYear);
    }
}
```

This program tests whether an integer corresponds to a leap year in the Gregorian calendar. A year is a leap year if it is divisible by 4 (2004), unless it is divisible by 100 in which case it is not (1900), unless it is divisible by 400 in which case it is (2000).

```
% javac LeapYear.java
% java LeapYear 2004
true
% java LeapYear 1900
false
% java LeapYear 2000
true
```

Even without going into the details of number representation, it is clear that the operations for the various types are quite different. For example, it is one thing to compare two `ints` to check that `(2 <= 2)` is `true`, but quite another to compare two `doubles` to check whether `(2.0 <= 0.002e3)` is `true`. Still, these operations are well defined and useful to write code that tests for conditions such as `(b*b - 4.0*a*c) >= 0.0`, which is frequently needed, as you will see.

The comparison operations have lower precedence than arithmetic operators and higher precedence than boolean operators, so you do not need the parentheses in an expression such as `(b*b - 4.0*a*c) >= 0.0`, and you could write an expression such as `month >= 1 && month <= 12` without parentheses to test whether the value of the `int` variable `month` is between 1 and 12. (It is better style to use the parentheses, however.)

Comparison operations, together with boolean logic, provide the basis for decision making in Java programs. PROGRAM 1.2.4 is an example of their use, and you can find other examples in the exercises at the end of this section. More importantly, in SECTION 1.3 we will see the role that boolean expressions play in more sophisticated programs.

<i>operator</i>	<i>meaning</i>	<i>true</i>	<i>false</i>
<code>==</code>	<i>equal</i>	<code>2 == 2</code>	<code>2 == 3</code>
<code>!=</code>	<i>not equal</i>	<code>3 != 2</code>	<code>2 != 2</code>
<code><</code>	<i>less than</i>	<code>2 < 13</code>	<code>2 < 2</code>
<code><=</code>	<i>less than or equal</i>	<code>2 <= 2</code>	<code>3 <= 2</code>
<code>></code>	<i>greater than</i>	<code>13 > 2</code>	<code>2 > 13</code>
<code>>=</code>	<i>greater than or equal</i>	<code>3 >= 2</code>	<code>2 >= 3</code>

Comparisons with int operands and a boolean result

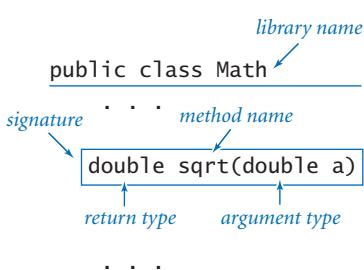
Library methods and APIs As we have seen, many programming tasks involve using Java library methods in addition to the built-in operators. The number of available library methods is vast. As you learn to program, you will learn to use more and more library methods, but it is best at the beginning to restrict your attention to a relatively small set of methods. In this chapter, you have already used some of Java's methods for printing, for converting data from one type to another, and for computing mathematical functions (the Java Math library). In later chapters, you will learn not just how to use other methods, but how to create and use your own methods.

For convenience, we will consistently summarize the library methods that you need to know how to use in tables like this one:

<code>void System.out.print(String s)</code>	<i>print s</i>
<code>void System.out.println(String s)</code>	<i>print s, followed by a newline</i>
<code>void System.out.println()</code>	<i>print a newline</i>

Note: Any type of data can be used as argument (and will be automatically converted to String).

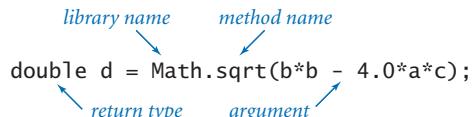
Java library methods for printing strings to the terminal



Anatomy of a method signature

Such a table is known as an *application programming interface (API)*. Each method is described by a line in the API that specifies the information you need to know to use the method. The code in the tables is *not* the code that you type to use the method; it is known as the method's *signature*. The signature specifies the type of the arguments, the method name, and the type of the result that the method computes (the *return value*). In your code, you can call a method by typing its name followed by arguments, enclosed in parentheses and separated by commas. When Java executes your program, we say that it *calls* (or *evaluates*) the method with the given arguments and that the method *returns* a value. A method call is an expression, so you can use a method call in the same way that you use variables and literals to build up more complicated expressions. For example, you can write expressions like `Math.sin(x) * Math.cos(y)` and so on. An argument is also an expression, so you can write code like `Math.sqrt(b*b - 4.0*a*c)` and Java knows what you mean—it evaluates the argument expression and passes the resulting value to the method.

The API tables on the facing page show some of the commonly used methods in Java's Math library, along with the Java methods we have seen for printing text to the terminal window and for converting strings to primitive types. The following table shows several examples of calls that use these library methods:



Using a library method

method call	library	return type	value
<code>Integer.parseInt("123")</code>	<code>Integer</code>	<code>int</code>	123
<code>Double.parseDouble("1.5")</code>	<code>Double</code>	<code>double</code>	1.5
<code>Math.sqrt(5.0*5.0 - 4.0*4.0)</code>	<code>Math</code>	<code>double</code>	3.0
<code>Math.log(Math.E)</code>	<code>Math</code>	<code>double</code>	1.0
<code>Math.random()</code>	<code>Math</code>	<code>double</code>	random in [0, 1)
<code>Math.round(3.14159)</code>	<code>Math</code>	<code>long</code>	3
<code>Math.max(1.0, 9.0)</code>	<code>Math</code>	<code>double</code>	9.0

Typical calls to Java library methods

```
public class Math
```

double abs(double a)	<i>absolute value of a</i>
double max(double a, double b)	<i>maximum of a and b</i>
double min(double a, double b)	<i>minimum of a and b</i>

Note 1: abs(), max(), and min() are defined also for int, long, and float.

double sin(double theta)	<i>sine of theta</i>
double cos(double theta)	<i>cosine of theta</i>
double tan(double theta)	<i>tangent of theta</i>

Note 2: Angles are expressed in radians. Use toDegrees() and toRadians() to convert.

Note 3: Use asin(), acos(), and atan() for inverse functions.

double exp(double a)	<i>exponential (e^a)</i>
double log(double a)	<i>natural log ($\log_e a$, or $\ln a$)</i>
double pow(double a, double b)	<i>raise a to the bth power (a^b)</i>
long round(double a)	<i>round a to the nearest integer</i>
double random()	<i>random number in [0, 1)</i>
double sqrt(double a)	<i>square root of a</i>
double E	<i>value of e (constant)</i>
double PI	<i>value of π (constant)</i>

See booksite for other available functions.

Excerpts from Java's Math library

void System.out.print(String s)	<i>print s</i>
void System.out.println(String s)	<i>print s, followed by a newline</i>
void System.out.println()	<i>print a newline</i>

Java library methods for printing strings to the terminal

int Integer.parseInt(String s)	<i>convert s to an int value</i>
double Double.parseDouble(String s)	<i>convert s to a double value</i>
long Long.parseLong(String s)	<i>convert s to a long value</i>

Java library methods for converting strings to primitive types

With three exceptions, the methods on the previous page are *pure*—given the same arguments, they always return the same value, without producing any observable *side effect*. The method `Math.random()` is impure because it returns potentially a different value each time it is called; the methods `System.out.print()` and `System.out.println()` are impure because they produce side effects—printing strings to the terminal. In APIs, we use a verb phrase to describe the behavior of a method that produces side effects; otherwise, we use a noun phrase to describe the return value. The keyword `void` designates a method that does not return a value (and whose main purpose is to produce side effects).

The `Math` library also defines the constant values `Math.PI` (for π) and `Math.E` (for e), which you can use in your programs. For example, the value of `Math.sin(Math.PI/2)` is 1.0 and the value of `Math.log(Math.E)` is 1.0 (because `Math.sin()` takes its argument in radians and `Math.log()` implements the natural logarithm function).

THESE APIs ARE TYPICAL OF THE online documentation that is the standard in modern programming. The extensive online documentation of the Java APIs is routinely used by professional programmers, and it is available to you (if you are interested) directly from the Java website or through our booksite. You do not need to go to the online documentation to understand the code in this book or to write similar code, because we present and explain in the text all of the library methods that we use in APIs like these and summarize them in the endpapers. More important, in CHAPTERS 2 AND 3 you will learn in this book how to develop your own APIs and to implement methods for your own use.

Type conversion One of the primary rules of modern programming is that you should always be aware of the type of data that your program is processing. Only by knowing the type can you know precisely which set of values each variable can have, which literals you can use, and which operations you can perform. For example, suppose that you wish to compute the average of the four integers 1, 2, 3, and 4. Naturally, the expression `(1 + 2 + 3 + 4) / 4` comes to mind, but it produces the `int` value 2 instead of the `double` value 2.5 because of type conversion conventions. The problem stems from the fact that the operands are `int` values but it is natural to expect a `double` value for the result, so conversion from `int` to `double` is necessary at some point. There are several ways to do so in Java.

Implicit type conversion. You can use an `int` value wherever a `double` value is expected, because Java automatically converts integers to doubles when appropriate. For example, `11*0.25` evaluates to `2.75` because `0.25` is a `double` and both operands need to be of the same type; thus, `11` is converted to a `double` and then the result of dividing two `double`s is a `double`. As another example, `Math.sqrt(4)` evaluates to `2.0` because `4` is converted to a `double`, as expected by `Math.sqrt()`, which then returns a `double` value. This kind of conversion is called *automatic promotion* or *coercion*. Automatic promotion is appropriate because your intent is clear and it can be done with no loss of information. In contrast, a conversion that might involve loss of information (for example, assigning a `double` value to an `int` variable) leads to a compile-time error.

Explicit cast. Java has some built-in type conversion conventions for primitive types that you can take advantage of when you are aware that you might lose information. You have to make your intention to do so explicit by using a device called a *cast*. You cast an expression from one primitive type to another by prepending the desired type name within parentheses. For example, the expression `(int) 2.71828` is a cast from `double` to `int` that produces an `int` with value `2`. The conversion methods defined for casts throw away information in a reasonable way (for a full list, see the booksite). For example, casting a floating-point number to an integer discards the fractional part by rounding toward zero. `RandomInt` (PROGRAM 1.2.5) is an example that uses a cast for a practical computation.

Casting has higher precedence than arithmetic operations—any cast is applied to the value that immediately follows it. For example, if we write `int value = (int) 11 * 0.25`, the cast is no help: the literal `11` is already an integer, so the cast `(int)` has no effect. In this example, the compiler produces a possible loss of precision error message because there would be a loss

<i>expression</i>	<i>expression type</i>	<i>expression value</i>
<code>(1 + 2 + 3 + 4) / 4.0</code>	<code>double</code>	<code>2.5</code>
<code>Math.sqrt(4)</code>	<code>double</code>	<code>2.0</code>
<code>"1234" + 99</code>	<code>String</code>	<code>"123499"</code>
<code>11 * 0.25</code>	<code>double</code>	<code>2.75</code>
<code>(int) 11 * 0.25</code>	<code>double</code>	<code>2.75</code>
<code>11 * (int) 0.25</code>	<code>int</code>	<code>0</code>
<code>(int) (11 * 0.25)</code>	<code>int</code>	<code>2</code>
<code>(int) 2.71828</code>	<code>int</code>	<code>2</code>
<code>Math.round(2.71828)</code>	<code>long</code>	<code>3</code>
<code>(int) Math.round(2.71828)</code>	<code>int</code>	<code>3</code>
<code>Integer.parseInt("1234")</code>	<code>int</code>	<code>1234</code>

Typical type conversions

Program 1.2.5 Casting to get a random integer

```
public class RandomInt
{
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        double r = Math.random(); // uniform between 0.0 and 1.0
        int value = (int) (r * n); // uniform between 0 and n-1
        System.out.println(value);
    }
}
```

This program uses the Java method `Math.random()` to generate a random number r between 0.0 (inclusive) and 1.0 (exclusive); then multiplies r by the command-line argument n to get a random number greater than or equal to 0 and less than n ; then uses a cast to truncate the result to be an integer $value$ between 0 and $n-1$.

```
% javac RandomInt.java
% java RandomInt 1000
548
% java RandomInt 1000
141
% java RandomInt 1000000
135032
```

of precision in converting the resulting value (2.75) to an `int` for assignment to `value`. The error is helpful because the intended computation for this code is likely `(int) (11 * 0.25)`, which has the value 2, not 2.75.

Explicit type conversion. You can use a method that takes an argument of one type (the value to be converted) and produces a result of another type. We have already used the `Integer.parseInt()` and `Double.parseDouble()` library methods to convert `String` values to `int` and `double` values, respectively. Many other methods are available for conversion among other types. For example, the library

method `Math.round()` takes a `double` argument and returns a `long` result: the nearest integer to the argument. Thus, for example, `Math.round(3.14159)` and `Math.round(2.71828)` are both of type `long` and have the same value (3). If you want to convert the result of `Math.round()` to an `int`, you must use an explicit cast.

BEGINNING PROGRAMMERS TEND TO FIND TYPE conversion to be an annoyance, but experienced programmers know that paying careful attention to data types is a key to success in programming. It may also be a key to avoiding failure: in a famous incident in 1996, a French rocket exploded in midair because of a type-conversion problem. While a bug in your program may not cause an explosion, it is well worth your while to take the time to understand what type conversion is all about. After you have written just a few programs, you will see that an understanding of data types will help you not only compose compact code but also make your intentions explicit and avoid subtle bugs in your programs.

Summary A *data type* is a set of values and a set of operations on those values. Java has eight primitive data types: `boolean`, `char`, `byte`, `short`, `int`, `long`, `float`, and `double`. In Java code, we use operators and expressions like those in familiar mathematical expressions to invoke the operations associated with each type. The `boolean` type is used for computing with the logical values `true` and `false`; the `char` type is the set of character values that we type; and the other six numeric types are used for computing with numbers. In this book, we most often use `boolean`, `int`, and `double`; we do not use `short` or `float`. Another data type that we use frequently, `String`, is not primitive, but Java has some built-in facilities for `Strings` that are like those for primitive types.

When programming in Java, we have to be aware that every operation is defined only in the context of its data type (so we may need type conversions) and that all types can have only a finite number of values (so we may need to live with imprecise results).

The `boolean` type and its operations—`&&`, `||`, and `!`—are the basis for logical decision making in Java programs, when used in conjunction with the mixed-type comparison operators `==`, `!=`, `<`, `>`, `<=`, and `>=`. Specifically, we use `boolean` expressions to control Java’s conditional (`if`) and loop (`for` and `while`) constructs, which we will study in detail in the next section.

The numeric types and Java's libraries give us the ability to use Java as an extensive mathematical calculator. We write arithmetic expressions using the built-in operators `+`, `-`, `*`, `/`, and `%` along with Java methods from the `Math` library.

Although the programs in this section are quite rudimentary by the standards of what we will be able to do after the next section, this class of programs is quite useful in its own right. You will use primitive types and basic mathematical functions extensively in Java programming, so the effort that you spend now in understanding them will certainly be worthwhile.

Q&A (*Strings*)

Q. How does Java store strings internally?

A. Strings are sequences of characters that are encoded with Unicode, a modern standard for encoding text. Unicode supports more than 100,000 different characters, including more than 100 different languages plus mathematical and musical symbols.

Q. Can you use < and > to compare `String` values?

A. No. Those operators are defined only for primitive-type values.

Q. How about == and !=?

A. Yes, but the result may not be what you expect, because of the meanings these operators have for nonprimitive types. For example, there is a distinction between a `String` and its value. The expression "abc" == "ab" + x is `false` when x is a `String` with value "c" because the two operands are stored in different places in memory (even though they have the same value). This distinction is essential, as you will learn when we discuss it in more detail in SECTION 3.1.

Q. How can I compare two strings like words in a book index or dictionary?

A. We defer discussion of the `String` data type and associated methods until SECTION 3.1, where we introduce object-oriented programming. Until then, the string concatenation operation suffices.

Q. How can I specify a string literal that is too long to fit on a single line?

A. You can't. Instead, divide the string literal into independent string literals and concatenate them together, as in the following example:

```
String dna = "ATGCCGCCACAGCTGGTCTAACCGGACTCTG" +
    "AAGTCGGAAATTACACCTGTTAG";
```

Q&A (*Integers*)

Q. How does Java store integers internally?

A. The simplest representation is for small positive integers, where the *binary number system* is used to represent each integer with a fixed amount of computer memory.

Q. What's the binary number system?

A. In the *binary number system*, we represent an integer as a sequence of *bits*. A bit is a single binary (base 2) digit—either 0 or 1—and is the basis for representing information in computers. In this case the bits are coefficients of powers of 2. Specifically, the sequence of bits $b_n b_{n-1} \dots b_2 b_1 b_0$ represents the integer

$$b_n 2^n + b_{n-1} 2^{n-1} + \dots + b_2 2^2 + b_1 2^1 + b_0 2^0$$

For example, 1100011 represents the integer

$$99 = 1 \cdot 64 + 1 \cdot 32 + 0 \cdot 16 + 0 \cdot 8 + 0 \cdot 4 + 1 \cdot 2 + 1 \cdot 1$$

The more familiar *decimal number system* is the same except that the digits are between 0 and 9 and we use powers of 10. Converting a number to binary is an interesting computational problem that we will consider in the next section. Java uses 32 bits to represent `int` values. For example, the decimal integer 99 might be represented with the 32 bits 00000000000000000000000000001100011.

Q. How about negative numbers?

A. Negative numbers are handled with a convention known as *two's complement*, which we need not consider in detail. This is why the range of `int` values in Java is -2^{31} to $2^{31} - 1$. One surprising consequence of this representation is that `int` values can become negative when they get large and *overflow* (exceed $2^{31} - 1$). If you have not experienced this phenomenon, see EXERCISE 1.2.10. A safe strategy is to use the `int` type when you know the integer values will be fewer than ten digits and the `long` type when you think the integer values might get to be ten digits or more.

Q. It seems wrong that Java should just let `ints` overflow and give bad values. Shouldn't Java automatically check for overflow?

A. Yes, this issue is a contentious one among programmers. The short answer for now is that the lack of such checking is one reason such types are called *primitive* data types. A little knowledge can go a long way in avoiding such problems. Again, it is fine to use the `int` type for small numbers, but when values run into the billions, you cannot.

Q. What is the value of `Math.abs(-2147483648)?`

A. -2147483648. This strange (but true) result is a typical example of the effects of integer overflow and two's complement representation.

Q. What do the expressions `1 / 0` and `1 % 0` evaluate to in Java?

A. Each generates a run-time *exception*, for division by zero.

Q. What is the result of division and remainder for negative integers?

A. The quotient `a / b` rounds toward 0; the remainder `a % b` is defined such that $(a / b) * b + a \% b$ is always equal to `a`. For example, `-14 / 3` and `14 / -3` are both `-4`, but `-14 % 3` is `-2` and `14 % -3` is `2`. Some other languages (including Python) have different conventions when dividing by negative integers.

Q. Why is the value of `10 ^ 6` not 1000000 but 12?

A. The `^` operator is not an exponentiation operator, which you must have been thinking. Instead, it is the *bitwise exclusive or* operator, which is seldom what you want. Instead, you can use the literal `1e6`. You could also use `Math.pow(10, 6)` but doing so is wasteful if you are raising 10 to a known power.

Q&A (Floating-Point Numbers)

Q. Why is the type for real numbers named `double`?

A. The decimal point can “float” across the digits that make up the real number. In contrast, with integers the (implicit) decimal point is fixed after the least significant digit.

Q. How does Java store floating-point numbers internally?

A. Java follows the IEEE 754 standard, which supported in hardware by most modern computer systems. The standard specifies that a floating-point number is stored using three fields: sign, mantissa, and exponent. If you are interested, see the booksite for more details. The IEEE 754 standard also specifies how special floating-point values—positive zero, negative zero, positive infinity, negative infinity, and NaN (not a number)—should be handled. In particular, floating-point arithmetic never leads to a run-time exception. For example, the expression `-0.0/3.0` evaluates to `-0.0`, the expression `1.0/0.0` evaluates to positive infinity, and `Math.sqrt(-2.0)` evaluates to NaN.

Q. Fifteen digits for floating-point numbers certainly seems enough to me. Do I really need to worry much about precision?

A. Yes, because you are used to mathematics based on real numbers with infinite precision, whereas the computer always deals with finite approximations. For example, the expression `(0.1 + 0.1 == 0.2)` evaluates to `true` but the expression `(0.1 + 0.1 + 0.1 == 0.3)` evaluates to `false`! Pitfalls like this are not at all unusual in scientific computing. Novice programmers should avoid comparing two floating-point numbers for equality.

Q. How can I initialize a `double` variable to NaN or infinity?

A. Java has built-in constants available for this purpose: `Double.NaN`, `Double.POSITIVE_INFINITY`, and `Double.NEGATIVE_INFINITY`.

Q. Are there functions in Java’s `Math` library for other trigonometric functions, such as cosecant, secant, and cotangent?

A. No, but you could use `Math.sin()`, `Math.cos()`, and `Math.tan()` to compute them. Choosing which functions to include in an API is a tradeoff between the convenience of having every function that you need and the annoyance of having to find one of the few that you need in a long list. No choice will satisfy all users, and the Java designers have many users to satisfy. Note that there are plenty of redundancies even in the APIs that we have listed. For example, you could use `Math.sin(x)/Math.cos(x)` instead of `Math.tan(x)`.

Q. It is annoying to see all those digits when printing a `double`. Can we arrange `System.out.println()` to print just two or three digits after the decimal point?

A. That sort of task involves a closer look at the method used to convert from `double` to `String`. The Java library function `System.out.printf()` is one way to do the job, and it is similar to the basic printing method in the C programming language and many modern languages, as discussed in SECTION 1.5. Until then, we will live with the extra digits (which is not all bad, since doing so helps us to get used to the different primitive types of numbers).

Q&A (Variables and Expressions)

Q. What happens if I forget to declare a variable?

A. The compiler complains when you refer to that variable in an expression. For example, IntOpsBad is the same as PROGRAM 1.2.2 except that the variable p is not declared (to be of type int).

```
% javac IntOpsBad.java
IntOpsBad.java:7: error: cannot find symbol
    p = a * b;
           ^
symbol:   variable p
location: class IntOpsBad
IntOpsBad.java:10: error: cannot find symbol
    System.out.println(a + " * " + b + " = " + p);
                           ^
symbol:   variable p
location: class IntOpsBad
2 errors
```

The compiler says that there are two errors, but there is really just one: the declaration of p is missing. If you forget to declare a variable that you use often, you will get quite a few error messages. A good strategy is to correct the *first* error and check that correction before addressing later ones.

Q. What happens if I forget to initialize a variable?

A. The compiler checks for this condition and will give you a `variable might not have been initialized` error message if you try to use the variable in an expression before you have initialized it.

Q. Is there a difference between the = and == operators?

A. Yes, they are quite different! The first is an assignment operator that changes the value of a variable, and the second is a comparison operator that produces a boolean result. Your ability to understand this answer is a sure test of whether you understood the material in this section. Think about how you might explain the difference to a friend.

Q. Can you compare a `double` to an `int`?

A. Not without doing a type conversion, but remember that Java usually does the requisite type conversion automatically. For example, if `x` is an `int` with the value 3, then the expression `(x < 3.1)` is `true`—Java converts `x` to `double` (because 3.1 is a `double` literal) before performing the comparison.

Q. Will the statement `a = b = c = 17;` assign the value 17 to the three integer variables `a`, `b`, and `c`?

A. Yes. It works because an assignment statement in Java is also an expression (that evaluates to its right-hand side) and the assignment operator is right associative. As a matter of style, we do not use such *chained assignments* in this book.

Q. Will the expression `(a < b < c)` test whether the values of three integer variables `a`, `b`, and `c` are in strictly ascending order?

A. No, it will not compile because the expression `a < b` produces a `boolean` value, which would then be compared to an `int` value. Java does not support *chained comparisons*. Instead, you need to write `(a < b && b < c)`.

Q. Why do we write `(a && b)` and not `(a & b)`?

A. Java also has an `&` operator that you may encounter if you pursue advanced programming courses.

Q. What is the value of `Math.round(6.022e23)`?

A. You should get in the habit of typing in a tiny Java program to answer such questions yourself (and trying to understand why your program produces the result that it does).

Q. I've heard Java referred to as a *statically typed* language. What does this mean?

A. Static typing means that the type of every variable and expression is known at compile time. Java also verifies and enforces type constraints at compile time; for example, your program will not compile if you attempt to store a value of type `double` in a variable of type `int` or call `Math.sqrt()` with a `String` argument.

Exercises

1.2.1 Suppose that `a` and `b` are `int` variables. What does the following sequence of statements do?

```
int t = a; b = t; a = b;
```

1.2.2 Write a program that uses `Math.sin()` and `Math.cos()` to check that the value of $\cos^2 \theta + \sin^2 \theta$ is approximately 1 for any θ entered as a command-line argument. Just print the value. Why are the values not always exactly 1?

1.2.3 Suppose that `a` and `b` are `boolean` variables. Show that the expression

```
(!(a && b) && (a || b)) || ((a && b) || !(a || b))
```

evaluates to `true`.

1.2.4 Suppose that `a` and `b` are `int` variables. Simplify the following expression:
 $(!(a < b) \&\& !(a > b))$.

1.2.5 The *exclusive or* operator `^` for `boolean` operands is defined to be `true` if they are different, `false` if they are the same. Give a truth table for this function.

1.2.6 Why does `10/3` give 3 and not `3.3333333333`?

Solution. Since both 10 and 3 are integer literals, Java sees no need for type conversion and uses integer division. You should write `10.0/3.0` if you mean the numbers to be `double` literals. If you write `10/3.0` or `10.0/3`, Java does implicit conversion to get the same result.

1.2.7 What does each of the following print?

- a. `System.out.println(2 + "bc");`
- b. `System.out.println(2 + 3 + "bc");`
- c. `System.out.println((2+3) + "bc");`
- d. `System.out.println("bc" + (2+3));`
- e. `System.out.println("bc" + 2 + 3);`

Explain each outcome.

1.2.8 Explain how to use PROGRAM 1.2.3 to find the square root of a number.

1.2.9 What does each of the following print?

- a. `System.out.println('b');`
- b. `System.out.println('b' + 'c');`
- c. `System.out.println((char) ('a' + 4));`

Explain each outcome.

1.2.10 Suppose that a variable `a` is declared as `int a = 2147483647` (or equivalently, `Integer.MAX_VALUE`). What does each of the following print?

- a. `System.out.println(a);`
- b. `System.out.println(a+1);`
- c. `System.out.println(2-a);`
- d. `System.out.println(-2-a);`
- e. `System.out.println(2*a);`
- f. `System.out.println(4*a);`

Explain each outcome.

1.2.11 Suppose that a variable `a` is declared as `double a = 3.14159`. What does each of the following print?

- a. `System.out.println(a);`
- b. `System.out.println(a+1);`
- c. `System.out.println(8/(int) a);`
- d. `System.out.println(8/a);`
- e. `System.out.println((int) (8/a));`

Explain each outcome.

1.2.12 Describe what happens if you write `sqrt` instead of `Math.sqrt` in PROGRAM 1.2.3.

1.2.13 Evaluate the expression `(Math.sqrt(2) * Math.sqrt(2) == 2)`.

1.2.14 Write a program that takes two positive integers as command-line arguments and prints `true` if either evenly divides the other.

1.2.15 Write a program that takes three positive integers as command-line arguments and prints `false` if any one of them is greater than or equal to the sum of the other two and `true` otherwise. (*Note:* This computation tests whether the three numbers could be the lengths of the sides of some triangle.)

1.2.16 A physics student gets unexpected results when using the code

```
double force = G * mass1 * mass2 / r * r;
```

to compute values according to the formula $F = Gm_1m_2 / r^2$. Explain the problem and correct the code.

1.2.17 Give the value of the variable `a` after the execution of each of the following sequences of statements:

<code>int a = 1;</code>	<code>boolean a = true;</code>	<code>int a = 2;</code>
<code>a = a + a;</code>	<code>a = !a;</code>	<code>a = a * a;</code>
<code>a = a + a;</code>	<code>a = !a;</code>	<code>a = a * a;</code>
<code>a = a + a;</code>	<code>a = !a;</code>	<code>a = a * a;</code>

1.2.18 Write a program that takes two integer command-line arguments `x` and `y` and prints the Euclidean distance from the point (x, y) to the origin $(0, 0)$.

1.2.19 Write a program that takes two integer command-line arguments `a` and `b` and prints a random integer between `a` and `b`, inclusive.

1.2.20 Write a program that prints the sum of two random integers between 1 and 6 (such as you might get when rolling dice).

1.2.21 Write a program that takes a `double` command-line argument `t` and prints the value of $\sin(2t) + \sin(3t)$.

1.2.22 Write a program that takes three `double` command-line arguments x_0 , v_0 , and t and prints the value of $x_0 + v_0t - gt^2/2$, where g is the constant 9.80665. (*Note:* This value is the displacement in meters after t seconds when an object is thrown straight up from initial position x_0 at velocity v_0 meters per second.)

1.2.23 Write a program that takes two integer command-line arguments `m` and `d` and prints `true` if day `d` of month `m` is between 3/20 and 6/20, `false` otherwise.

Creative Exercises

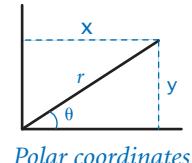
1.2.24 *Continuously compounded interest.* Write a program that calculates and prints the amount of money you would have after t years if you invested P dollars at an annual interest rate r (compounded continuously). The desired value is given by the formula Pe^{rt} .

1.2.25 *Wind chill.* Given the temperature T (in degrees Fahrenheit) and the wind speed v (in miles per hour), the National Weather Service defines the effective temperature (the wind chill) as follows:

$$w = 35.74 + 0.6215 T + (0.4275 T - 35.75) v^{0.16}$$

Write a program that takes two double command-line arguments `temperature` and `velocity` and prints the wind chill. Use `Math.pow(a, b)` to compute a^b . Note: The formula is not valid if T is larger than 50 in absolute value or if v is larger than 120 or less than 3 (you may assume that the values you get are in that range).

1.2.26 *Polar coordinates.* Write a program that converts from Cartesian to polar coordinates. Your program should accept two double command-line arguments `x` and `y` and print the polar coordinates r and θ . Use the method `Math.atan2(y, x)` to compute the arctangent value of y/x that is in the range from $-\pi$ to π .



1.2.27 *Gaussian random numbers.* Write a program `RandomGaussian` that prints a random number r drawn from the *Gaussian distribution*. One way to do so is to use the *Box–Muller* formula

$$r = \sin(2 \pi v) (-2 \ln u)^{1/2}$$

where u and v are real numbers between 0 and 1 generated by the `Math.random()` method.

1.2.28 *Order check.* Write a program that takes three double command-line arguments `x`, `y`, and `z` and prints `true` if the values are strictly ascending or descending ($x < y < z$ or $x > y > z$), and `false` otherwise.

1.2.29 *Day of the week.* Write a program that takes a date as input and prints the day of the week that date falls on. Your program should accept three `int` command-line arguments: `m` (month), `d` (day), and `y` (year). For `m`, use 1 for January, 2 for February, and so forth. For output, print 0 for Sunday, 1 for Monday, 2 for Tuesday, and so forth. Use the following formulas, for the Gregorian calendar:

$$\begin{aligned}y_0 &= y - (14 - m) / 12 \\x &= y_0 + y_0 / 4 - y_0 / 100 + y_0 / 400 \\m_0 &= m + 12 \times ((14 - m) / 12) - 2 \\d_0 &= (d + x + (31 \times m_0) / 12) \% 7\end{aligned}$$

Example: On which day of the week did February 14, 2000 fall?

$$\begin{aligned}y_0 &= 2000 - 1 = 1999 \\x &= 1999 + 1999 / 4 - 1999 / 100 + 1999 / 400 = 2483 \\m_0 &= 2 + 12 \times 1 - 2 = 12 \\d_0 &= (14 + 2483 + (31 \times 12) / 12) \% 7 = 2500 \% 7 = 1\end{aligned}$$

Answer: Monday.

1.2.30 *Uniform random numbers.* Write a program that prints five uniform random numbers between 0 and 1, their average value, and their minimum and maximum values. Use `Math.random()`, `Math.min()`, and `Math.max()`.

1.2.31 *Mercator projection.* The *Mercator projection* is a conformal (angle-preserving) projection that maps latitude φ and longitude λ to rectangular coordinates (x, y) . It is widely used—for example, in nautical charts and in the maps that you print from the web. The projection is defined by the equations $x = \lambda - \lambda_0$ and $y = 1/2 \ln((1 + \sin \varphi) / (1 - \sin \varphi))$, where λ_0 is the longitude of the point in the center of the map. Write a program that takes λ_0 and the latitude and longitude of a point from the command line and prints its projection.

1.2.32 *Color conversion.* Several different formats are used to represent color. For example, the primary format for LCD displays, digital cameras, and web pages, known as the *RGB format*, specifies the level of red (R), green (G), and blue (B) on an integer scale from 0 to 255. The primary format for publishing books and magazines, known as the *CMYK format*, specifies the level of cyan (C), magenta

(M), yellow (Y), and black (K) on a real scale from 0.0 to 1.0. Write a program `RGBtoCMYK` that converts RGB to CMYK. Take three integers—`r`, `g`, and `b`—from the command line and print the equivalent CMYK values. If the RGB values are all 0, then the CMY values are all 0 and the K value is 1; otherwise, use these formulas:

$$\begin{aligned} w &= \max(r / 255, g / 255, b / 255) \\ c &= (w - (r / 255)) / w \\ m &= (w - (g / 255)) / w \\ y &= (w - (b / 255)) / w \\ k &= 1 - w \end{aligned}$$

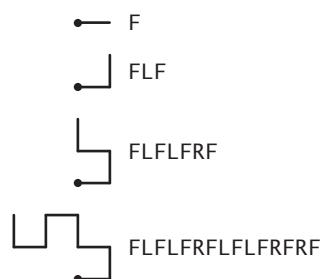
1.2.33 Great circle. Write a program `GreatCircle` that takes four double command-line arguments—`x1`, `y1`, `x2`, and `y2`—(the latitude and longitude, in degrees, of two points on the earth) and prints the great-circle distance between them. The great-circle distance (in nautical miles) is given by the following equation:

$$d = 60 \arccos(\sin(x_1) \sin(x_2) + \cos(x_1) \cos(x_2) \cos(y_1 - y_2))$$

Note that this equation uses degrees, whereas Java’s trigonometric functions use radians. Use `Math.toRadians()` and `Math.toDegrees()` to convert between the two. Use your program to compute the great-circle distance between Paris (48.87° N and -2.33° W) and San Francisco (37.8° N and 122.4° W).

1.2.34 Three-sort. Write a program that takes three integer command-line arguments and prints them in ascending order. Use `Math.min()` and `Math.max()`.

1.2.35 Dragon curves. Write a program to print the instructions for drawing the dragon curves of order 0 through 5. The instructions are strings of F, L, and R characters, where F means “draw line while moving 1 unit forward,” L means “turn left,” and R means “turn right.” A dragon curve of order n is formed when you fold a strip of paper in half n times, then unfold to right angles. The key to solving this problem is to note that a curve of order n is a curve of order $n-1$ followed by an L followed by a curve of order $n-1$ traversed in reverse order, and then to figure out a similar description for the reverse curve.



Dragon curves of order 0, 1, 2, and 3

1.3 Conditionals and Loops

IN THE PROGRAMS THAT WE HAVE examined to this point, each of the statements in the program is executed once, in the order given. Most programs are more complicated because the sequence of statements and the number of times each is executed can vary. We use the term *control flow* to refer to statement sequencing in a program.

In this section, we introduce statements that allow us to change the control flow, using logic about the values of program variables. This feature is an essential component of programming.

Specifically, we consider Java statements that implement *conditionals*, where some other statements may or may not be executed depending on certain conditions, and *loops*, where some other statements may be executed multiple times, again depending on certain conditions. As you will see in this section, conditionals and loops truly harness the power of the computer and will equip you to write programs to accomplish a broad variety of tasks that you could not contemplate attempting without a computer.

If statements Most computations require different actions for different inputs. One way to express these differences in Java is the `if` statement:

```
if (<boolean expression>) { <statements> }
```

This description introduces a formal notation known as a *template* that we will use to specify the format of Java constructs. We put within angle brackets (`< >`) a construct that we have already defined, to indicate that we can use any instance of that construct where specified. In this case, `<boolean expression>` represents an expression that evaluates to a boolean value, such as one involving a comparison operation, and `<statements>` represents a *statement block* (a sequence of Java statements). This latter construct is familiar to you: the body of `main()` is such a sequence. If the sequence is a single statement, the curly braces are optional. It is possible to make formal definitions of `<boolean expression>` and `<statements>`, but we refrain from going into that level of detail. The meaning of an `if` statement

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Programs in this section

is self-explanatory: the statement(s) in the sequence are to be executed if and only if the expression is `true`.

As a simple example, suppose that you want to compute the absolute value of an `int` value `x`. This statement does the job:

```
if (x < 0) x = -x;
```

(More precisely, it replaces `x` with the absolute value of `x`.) As a second simple example, consider the following statement:

```
if (x > y)
{
    int t = x;
    x = y;
    y = t;
}
```

This code puts the smaller of the two `int` values in `x` and the larger of the two values in `y`, by exchanging the values in the two variables if necessary.

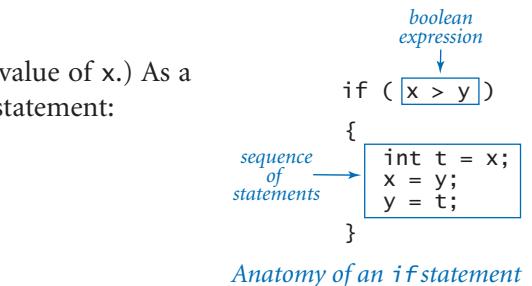
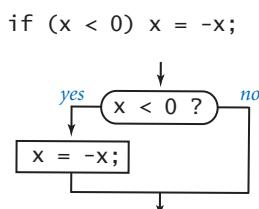
You can also add an `else` clause to an `if` statement, to express the concept of executing either one statement (or sequence of statements) or another, depending on whether the boolean expression is `true` or `false`, as in the following template:

```
if (<boolean expression>) <statements T>
else                      <statements F>
```

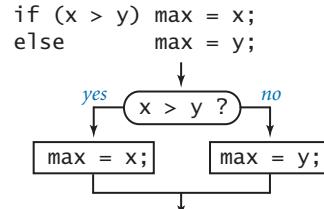
As a simple example of the need for an `else` clause, consider the following code, which assigns the maximum of two `int` values to the variable `max`:

```
if (x > y) max = x;
else        max = y;
```

One way to understand control flow is to visualize it with a diagram called a *flowchart*. Paths through the flowchart correspond to flow-of-control paths in the



Anatomy of an `if` statement



Flowchart examples (`if` statements)

program. In the early days of computing, when programmers used low-level languages and difficult-to-understand flows of control, flowcharts were an essential part of programming. With modern languages, we use flowcharts just to understand basic building blocks like the `if` statement.

The accompanying table contains some examples of the use of `if` and `if-else` statements. These examples are typical of simple calculations you might need in programs that you write. Conditional statements are an essential part of programming. Since the *semantics* (meaning) of statements like these is similar to their meanings as natural-language phrases, you will quickly grow used to them.

PROGRAM 1.3.1 is another example of the use of the `if-else` statement, in this case for the task of simulating a fair coin flip. The body of the program is a single statement, like the ones in the table, but it is worth special attention because it introduces an interesting philosophical issue that is worth contemplating: can a computer program produce *random* values? Certainly not, but a program *can* produce numbers that have many of the properties of random numbers.

<i>absolute value</i>	<code>if (x < 0) x = -x;</code>
<i>put the smaller value in x and the larger value in y</i>	<code>if (x > y) { int t = x; x = y; y = t; }</code>
<i>maximum of x and y</i>	<code>if (x > y) max = x; else max = y;</code>
<i>error check for division operation</i>	<code>if (den == 0) System.out.println("Division by zero"); else System.out.println("Quotient = " + num/den);</code>
<i>error check for quadratic formula</i>	<code>double discriminant = b*b - 4.0*c; if (discriminant < 0.0) { System.out.println("No real roots"); } else { System.out.println((-b + Math.sqrt(discriminant))/2.0); System.out.println((-b - Math.sqrt(discriminant))/2.0); }</code>

Typical examples of using if and if-else statements

Program 1.3.1 Flipping a fair coin

```
public class Flip
{
    public static void main(String[] args)
    { // Simulate a fair coin flip.
        if (Math.random() < 0.5) System.out.println("Heads");
        else                        System.out.println("Tails");
    }
}
```

This program uses `Math.random()` to simulate a fair coin flip. Each time you run it, it prints either `Heads` or `Tails`. A sequence of flips will have many of the same properties as a sequence that you would get by flipping a fair coin, but it is not a truly random sequence.

```
% java Flip
Heads
% java Flip
Tails
% java Flip
Tails
```

While loops Many computations are inherently repetitive. The basic Java construct for handling such computations has the following format:

```
while (<boolean expression>) { <statements> }
```

The `while` statement has the same form as the `if` statement (the only difference being the use of the keyword `while` instead of `if`), but the meaning is quite different. It is an instruction to the computer to behave as follows: if the boolean expression is `false`, do nothing; if the boolean expression is `true`, execute the sequence of statements (just as with an `if` statement) but then check the expression again, execute the sequence of statements again if the expression is `true`, and *continue* as long as the expression is `true`. We refer to the statement block in a loop as the *body* of the loop. As with the `if` statement, the curly braces are optional if a `while` loop body has just one statement. The `while` statement is equivalent to a sequence of identical `if` statements:

```

if (<boolean expression>) { <statements> }
if (<boolean expression>) { <statements> }
if (<boolean expression>) { <statements> }
...

```

At some point, the code in one of the statements must change something (such as the value of some variable in the boolean expression) to make the boolean expression *false*, and then the sequence is broken.

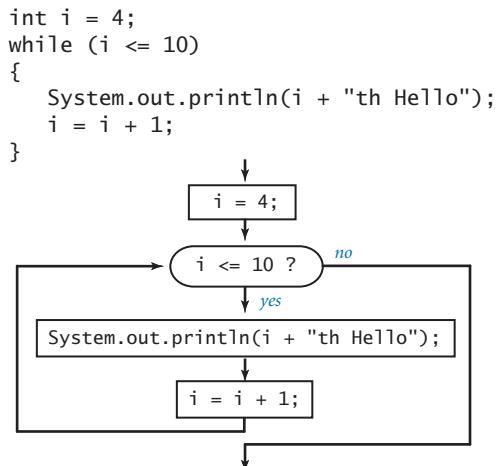
A common programming paradigm involves maintaining an integer value that keeps track of the number of times a loop iterates. We start at some initial value, and then increment the value by 1 each time through the loop, testing whether it exceeds a predetermined maximum before deciding to continue. TenHello (PROGRAM 1.3.2) is a simple example of this paradigm that uses a `while` statement. The key to the computation is the statement

```
i = i + 1;
```

As a mathematical equation, this statement is nonsense, but as a Java assignment statement it makes perfect sense: it says to compute the value $i + 1$ and then assign the result to the variable i . If the value of i was 4 before the statement, it becomes 5 afterward; if it was 5, it becomes 6; and so forth. With the initial condition in TenHello that the value of i starts at 4, the statement block is ex-

ecuted seven times until the sequence is broken, when the value of i becomes 11.

Using the `while` loop is barely worthwhile for this simple task, but you will soon be addressing tasks where you will need to specify that statements be repeated far too many times to contemplate doing it without loops. There is a profound difference between programs with `while` statements and programs without them, because `while` statements allow us to specify a potentially unlimited number of statements to be executed in a program. In particular, the `while` statement allows us to specify lengthy



Flowchart example (`while` statement)

initialization is a separate statement
loop-continuation condition
braces are optional when body is a single statement

```

int power = 1;
while (power <= n/2)
{
    power = 2*power;
}

```

body

Anatomy of a `while` loop

Program 1.3.2 Your first while loop

```
public class TenHello
{
    public static void main(String[] args)
    { // Print 10 Hellos.
        System.out.println("1st Hello");
        System.out.println("2nd Hello");
        System.out.println("3rd Hello");
        int i = 4;
        while (i <= 10)
        { // Print the ith Hello.
            System.out.println(i + "th Hello");
            i = i + 1;
        }
    }
}
```

This program uses a `while` loop for the simple, repetitive task of printing the output shown below. After the third line, the lines to be printed differ only in the value of the index counting the line printed, so we define a variable `i` to contain that index. After initializing the value of `i` to 4, we enter into a `while` loop where we use the value of `i` in the `System.out.println()` statement and increment it each time through the loop. After printing 10th Hello, the value of `i` becomes 11 and the loop terminates.

% java TenHello	i	i <= 10	output
1st Hello	4	true	4th Hello
2nd Hello	5	true	5th Hello
3rd Hello	6	true	6th Hello
4th Hello	7	true	7th Hello
5th Hello	8	true	8th Hello
6th Hello	9	true	9th Hello
7th Hello	10	true	10th Hello
8th Hello	11	false	
9th Hello			
10th Hello			

Trace of java TenHello

computations in short programs. This ability opens the door to writing programs for tasks that we could not contemplate addressing without a computer. But there is also a price to pay: as your programs become more sophisticated, they become more difficult to understand.

`PowersOfTwo` (PROGRAM 1.3.3) uses a `while` loop to print out a table of the powers of 2. Beyond the loop control counter `i`, it maintains a variable `power` that holds the powers of 2 as it computes them. The loop body contains three statements: one to print the current power of 2, one to compute the next (multiply the current one by 2), and one to increment the loop control counter.

There are many situations in computer science where it is useful to be familiar with powers of 2. You should know at least the first 10 values in this table and you should note that 2^{10} is about 1 thousand, 2^{20} is about 1 million, and 2^{30} is about 1 billion.

`PowersOfTwo` is the prototype for many useful computations. By varying the computations that change the accumulated value and the way that the loop control variable is incremented, we can print out tables of a variety of functions (see EXERCISE 1.3.12).

It is worthwhile to carefully examine the behavior of programs that use loops by studying a *trace* of the program. For example, a trace of the operation of `PowersOfTwo` should show the value of each variable before each iteration of the loop and the value of the boolean expression that controls the loop. Tracing the operation of a loop can be very tedious, but it is often worthwhile to run a trace because it clearly exposes what a program is doing.

`PowersOfTwo` is nearly a self-tracing program, because it prints the values of its variables each time through the loop. Clearly, you can make any program produce a trace of itself by adding appropriate `System.out.println()` statements. Modern program-

<code>i</code>	<code>power</code>	<code>i <= n</code>
0	1	true
1	2	true
2	4	true
3	8	true
4	16	true
5	32	true
6	64	true
7	128	true
8	256	true
9	512	true
10	1024	true
11	2048	true
12	4096	true
13	8192	true
14	16384	true
15	32768	true
16	65536	true
17	131072	true
18	262144	true
19	524288	true
20	1048576	true
21	2097152	true
22	4194304	true
23	8388608	true
24	16777216	true
25	33554432	true
26	67108864	true
27	134217728	true
28	268435456	true
29	536870912	true
30	1073741824	false

Program 1.3.3 Computing powers of 2

```

public class PowersOfTwo
{
    public static void main(String[] args)
    { // Print the first n powers of 2.
        int n = Integer.parseInt(args[0]);
        int power = 1;
        int i = 0;
        while (i <= n)
        { // Print ith power of 2.
            System.out.println(i + " " + power);
            power = 2 * power;
            i = i + 1;
        }
    }
}

```

<code>n</code> <code>i</code> <code>power</code>	<i>loop termination value</i> <i>loop control counter</i> <i>current power of 2</i>
--	---

This program takes an integer command-line argument `n` and prints a table of the powers of 2 that are less than or equal to 2^n . Each time through the loop, it increments the value of `i` and doubles the value of `power`. We show only the first three and the last three lines of the table; the program prints $n+1$ lines.

```

% java PowersOfTwo 5
0 1
1 2
2 4
3 8
4 16
5 32
% java PowersOfTwo 29
0 1
1 2
2 4
...
27 134217728
28 268435456
29 536870912

```

ming environments provide sophisticated tools for tracing, but this tried-and-true method is simple and effective. You certainly should add print statements to the first few loops that you write, to be sure that they are doing precisely what you expect.

There is a hidden trap in `PowersOfTwo`, because the largest integer in Java's `int` data type is $2^{31} - 1$ and the program does not test for that possibility. If you invoke it with `java PowersOfTwo 31`, you may be surprised by the last line of output printed:

```
...
1073741824
-2147483648
```

The variable `power` becomes too large and takes on a negative value because of the way Java represents integers. The maximum value of an `int` is available for us to use as `Integer.MAX_VALUE`. A better version of PROGRAM 1.3.3 would use this value to test for overflow and print an error message if the user types too large a value, though getting such a program to work properly for all inputs is trickier than you might think. (For a similar challenge, see EXERCISE 1.3.16.)

As a more complicated example, suppose that we want to compute the largest power of 2 that is less than or equal to a given positive integer `n`. If `n` is 13, we want the result 8; if `n` is 1000, we want the result 512; if `n` is 64, we want the result 64; and so forth. This computation is simple to perform with a `while` loop:

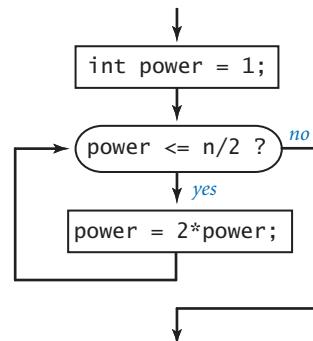
```
int power = 1;
while (power <= n/2)
    power = 2*power;
```

It takes some thought to convince yourself that this simple piece of code produces the desired result. You can do so by making these observations:

- `power` is always a power of 2.
- `power` is never greater than `n`.
- `power` increases each time through the loop, so the loop must terminate.
- After the loop terminates, `2*power` is greater than `n`.

Reasoning of this sort is often important in understanding how `while` loops work. Even though many of the loops you will write will be much simpler than this one, you should be sure to convince yourself that each loop you write will behave as you expect.

The logic behind such arguments is the same whether the loop iterates just a few times, as in `TenHello`, or dozens of times, as in `PowersOfTwo`, or millions of

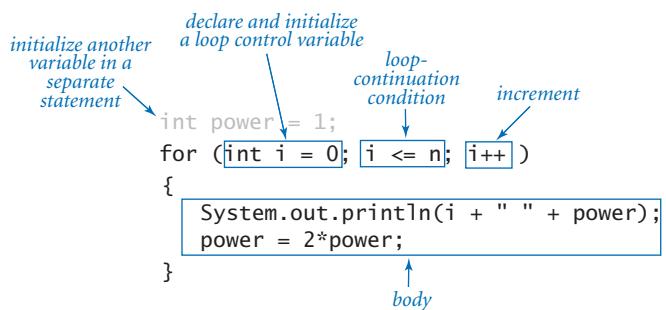


Flowchart for the statements

```
int power = 1;
while (power <= n/2)
    power = 2*power;
```

times, as in several examples that we will soon consider. That leap from a few tiny cases to a huge computation is profound. When writing loops, understanding how the values of the variables change each time through the loop (and checking that understanding by adding statements to trace their values and running for a small number of iterations) is essential. Having done so, you can confidently remove those training wheels and truly unleash the power of the computer.

For loops As you will see, the `while` loop allows us to write programs for all manner of applications. Before considering more examples, we will look at an alternate Java construct that allows us even more flexibility when writing programs with loops. This alternate notation is not fundamentally different from the basic `while` loop, but it is widely used because it often allows us to write more compact and more readable programs than if we used only `while` statements.



Anatomy of a `for` loop (that prints powers of 2)

For notation. Many loops follow this scheme: initialize an index variable to some value and then use a `while` loop to test a loop-continuation condition involving the index variable, where the last statement in the `while` loop increments the index variable. You can express such loops directly with Java's `for` notation:

```
for (<initialize>; <boolean expression>; <increment>)
{
    <statements>
}
```

This code is, with only a few exceptions, equivalent to

```
<initialize>;
while (<boolean expression>)
{
    <statements>
    <increment>;
}
```

Your Java compiler might even produce identical results for the two loops. In truth, *<initialize>* and *<increment>* can be more complicated statements, but we nearly always use `for` loops to support this typical initialize-and-increment programming idiom. For example, the following two lines of code are equivalent to the corresponding lines of code in `TenHello`s (PROGRAM 1.3.2):

```
for (int i = 4; i <= 10; i = i + 1)
    System.out.println(i + "th Hello");
```

Typically, we work with a slightly more compact version of this code, using the shorthand notation discussed next.

Compound assignment idioms. Modifying the value of a variable is something that we do so often in programming that Java provides a variety of shorthand notations for the purpose. For example, the following four statements all increment the value of `i` by 1:

```
i = i+1;    i++;    ++i;    i += 1;
```

You can also say `i--` or `--i` or `i == 1` or `i = i-1` to decrement that value of `i` by 1. Most programmers use `i++` or `i--` in `for` loops, though any of the others would do. The `++` and `--` constructs are normally used for integers, but the *compound assignment* constructs are useful operations for any arithmetic operator in any primitive numeric type. For example, you can say `power *= 2` or `power += power` instead of `power = 2*power`. All of these idioms are provided for notational convenience, nothing more. These shortcuts came into widespread use with the C programming language in the 1970s and have become standard. They have survived the test of time because they lead to compact, elegant, and easily understood programs. When you learn to write (and to read) programs that use them, you will be able to transfer that skill to programming in numerous modern languages, not just Java.

Scope. The *scope* of a variable is the part of the program that can refer to that variable by name. Generally the scope of a variable comprises the statements that follow the declaration in the same block as the declaration. For this purpose, the code in the `for` loop header is considered to be in the same block as the `for` loop body. Therefore, the `while` and `for` formulations of loops are not quite equivalent: in a typical `for` loop, the incrementing variable is *not* available for use in later statements; in the corresponding `while` loop, it is. This distinction is often a reason to use a `while` loop instead of a `for` loop.

CHOOSING AMONG DIFFERENT FORMULATIONS OF THE same computation is a matter of each programmer's taste, as when a writer picks from among synonyms or chooses between using active and passive voice when composing a sentence. You will not find good hard-and-fast rules on how to write a program any more than you will find such rules on how to compose a paragraph. Your goal should be to find a style that suits you, gets the computation done, and can be appreciated by others.

The accompanying table includes several code fragments with typical examples of loops used in Java code. Some of these relate to code that you have already seen; others are new code for straightforward computations. To cement your understanding of loops in Java, write some loops for similar computations of your own invention, or do some of the early exercises at the end of this section. There is no substitute for the experience gained by running code that you create yourself, and it is imperative that you develop an understanding of how to write Java code that uses loops.

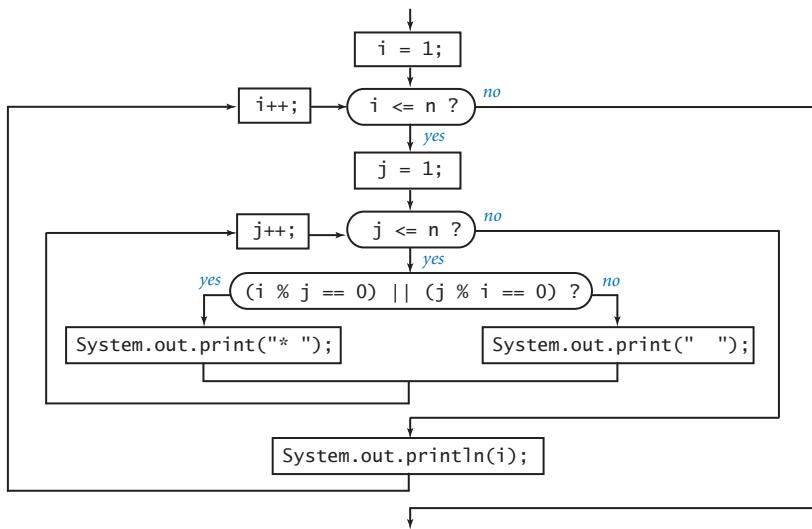
<i>compute the largest power of 2 less than or equal to n</i>	<pre>int power = 1; while (power <= n/2) power = 2*power; System.out.println(power);</pre>
<i>compute a finite sum ($1 + 2 + \dots + n$)</i>	<pre>int sum = 0; for (int i = 1; i <= n; i++) sum += i; System.out.println(sum);</pre>
<i>compute a finite product ($n! = 1 \times 2 \times \dots \times n$)</i>	<pre>int product = 1; for (int i = 1; i <= n; i++) product *= i; System.out.println(product);</pre>
<i>print a table of function values</i>	<pre>for (int i = 0; i <= n; i++) System.out.println(i + " " + 2*Math.PI*i/n);</pre>
<i>compute the ruler function (see PROGRAM 1.2.1)</i>	<pre>String ruler = "1"; for (int i = 2; i <= n; i++) ruler = ruler + " " + i + " " + ruler; System.out.println(ruler);</pre>

Typical examples of using for and while loops

Nesting The `if`, `while`, and `for` statements have the same status as assignment statements or any other statements in Java; that is, we can use them wherever a statement is called for. In particular, we can use one or more of them in the body of another statement to make *compound statements*. As a first example, `DivisorPattern` (PROGRAM 1.3.4) has a `for` loop whose body contains a `for` loop (whose body is an `if-else` statement) and a `print` statement. It prints a pattern of asterisks where the i th row has an asterisk in each position corresponding to divisors of i (the same holds true for the columns).

To emphasize the nesting, we use indentation in the program code. We refer to the i loop as the *outer* loop and the j loop as the *inner* loop. The inner loop iterates all the way through for each iteration of the outer loop. As usual, the best way to understand a new programming construct like this is to study a trace.

`DivisorPattern` has a complicated control flow, as you can see from its flowchart. A diagram like this illustrates the importance of using a limited number of simple control flow structures in programming. With nesting, you can compose loops and conditionals to build programs that are easy to understand even though they may have a complicated control flow. A great many useful computations can be accomplished with just one or two levels of nesting. For example, many programs in this book have the same general structure as `DivisorPattern`.



Flowchart for `DivisorPattern`

Program 1.3.4 Your first nested loops

```
public class DivisorPattern
{
    public static void main(String[] args)
    { // Print a square that visualizes divisors.
        int n = Integer.parseInt(args[0]);
        for (int i = 1; i <= n; i++)
        { // Print the ith line.
            for (int j = 1; j <= n; j++)
            { // Print the jth element in the ith line.
                if ((i % j == 0) || (j % i == 0))
                    System.out.print("* ");
                else
                    System.out.print("  ");
            }
            System.out.println(i);
        }
    }
}
```

n	number of rows and columns
i	row index
j	column index

This program takes an integer command-line argument *n* and uses nested `for` loops to print an *n*-by-*n* table with an asterisk in row *i* and column *j* if either *i* divides *j* or *j* divides *i*. The loop control variables *i* and *j* control the computation.

% java DivisorPattern 3	<i>i</i>	<i>j</i>	<i>i</i> % <i>j</i>	<i>j</i> % <i>i</i>	output
* * * 1	1	1	0	0	*
* * 2	1	2	1	0	*
* * * 3	1	3	1	0	*
% java DivisorPattern 12					
* * * * * * * * * * * 1	1				
* * * * * * * * * * * 2	2	1	0	1	*
* * * * * * * * * * 3	2	2	0	0	*
* * * * * * * * 4	2	3	2	1	
* * * * * * 5					
* * * * * 6					
* * * * 7	3	1	0	1	*
* * * * 8	3	2	1	2	
* * * * 9	3	3	0	0	*
* * * * 10					
* * * * 11					
* * * * 12					

Trace of java DivisorPattern 3

As a second example of nesting, consider the following program fragment, which a tax preparation program might use to compute income tax rates:

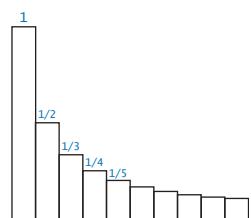
```
if      (income <      0) rate = 0.00;
else if (income <  8925) rate = 0.10;
else if (income < 36250) rate = 0.15;
else if (income < 87850) rate = 0.23;
else if (income < 183250) rate = 0.28;
else if (income < 398350) rate = 0.33;
else if (income < 400000) rate = 0.35;
else                      rate = 0.396;
```

In this case, a number of `if` statements are nested to test from among a number of mutually exclusive possibilities. This construct is a special one that we use often. Otherwise, it is best to use curly braces to resolve ambiguities when nesting `if` statements. This issue and more examples are addressed in the Q&A and exercises.

Applications The ability to program with loops immediately opens up the full world of computation. To emphasize this fact, we next consider a variety of examples. These examples all involve working with the types of data that we considered in SECTION 1.2, but rest assured that the same mechanisms serve us well for any computational application. The sample programs are carefully crafted, and by studying them, you will be prepared to write your own programs containing loops.

The examples that we consider here involve computing with numbers. Several of our examples are tied to problems faced by mathematicians and scientists throughout the past several centuries. While computers have existed for only 70 years or so, many of the computational methods that we use are based on a rich mathematical tradition tracing back to antiquity.

Finite sum. The computational paradigm used by `PowersOfTwo` is one that you will use frequently. It uses two variables—one as an index that controls a loop and the other to accumulate a computational result. `HarmonicNumber` (PROGRAM 1.3.5) uses the same paradigm to evaluate the finite sum $H_n = 1 + 1/2 + 1/3 + \dots + 1/n$. These numbers, which are known as the *harmonic numbers*, arise frequently in discrete mathematics. Harmonic numbers are the discrete analog of the logarithm. They also approximate the area under the curve $y = 1/x$. You can use PROGRAM 1.3.5 as a model for computing the values of other finite sums (see EXERCISE 1.3.18).



Program 1.3.5 Harmonic numbers

```
public class HarmonicNumber
{
    public static void main(String[] args)
    { // Compute the nth harmonic number.
        int n = Integer.parseInt(args[0]);
        double sum = 0.0;
        for (int i = 1; i <= n; i++)
        { // Add the ith term to the sum.
            sum += 1.0/i;
        }
        System.out.println(sum);
    }
}
```

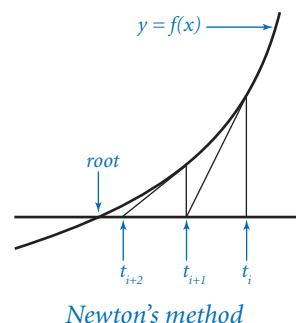
n	number of terms in sum
i	loop index
sum	cumulated sum

This program takes an integer command-line argument n and computes the value of the n th harmonic number. The value is known from mathematical analysis to be about $\ln(n) + 0.57721$ for large n . Note that $\ln(1,000,000) + 0.57721 \approx 14.39272$.

```
% java HarmonicNumber 2
1.5
% java HarmonicNumber 10
2.9289682539682538
```

```
% java HarmonicNumber 10000
7.485470860550343
% java HarmonicNumber 1000000
14.392726722864989
```

Computing the square root. How are functions in Java's Math library, such as `Math.sqrt()`, implemented? `Sqrt` (PROGRAM 1.3.6) illustrates one technique. To compute the square root of a positive number, it uses an iterative computation that was known to the Babylonians more than 4,000 years ago. It is also a special case of a general computational technique that was developed in the 17th century by Isaac Newton and Joseph Raphson and is widely known as *Newton's method*. Under generous conditions on a given function $f(x)$, Newton's method is an effective way to find roots



Program 1.3.6 Newton's method

```

public class Sqrt
{
    public static void main(String[] args)
    {
        double c = Double.parseDouble(args[0]);
        double EPSILON = 1e-15;
        double t = c;
        while (Math.abs(t - c/t) > EPSILON * t)
        { // Replace t by the average of t and c/t.
            t = (c/t + t) / 2.0;
        }
        System.out.println(t);
    }
}

```

c	argument
EPSILON	error tolerance
t	estimate of square root of c

This program takes a positive floating-point number c as a command-line argument and computes the square root of c to 15 decimal places of accuracy, using Newton's method (see text).

	<i>iteration</i>	<i>t</i>	c/t
% java Sqrt 2.0		2.000000000000000	1.0
1.414213562373095			
% java Sqrt 2544545	1	1.500000000000000	1.333333333333333
1595.1630010754388	2	1.4166666666666665	1.4117647058823530
	3	1.4142156862745097	1.4142114384748700
	4	1.4142135623746899	1.4142135623715002
	5	1.4142135623730950	1.4142135623730951

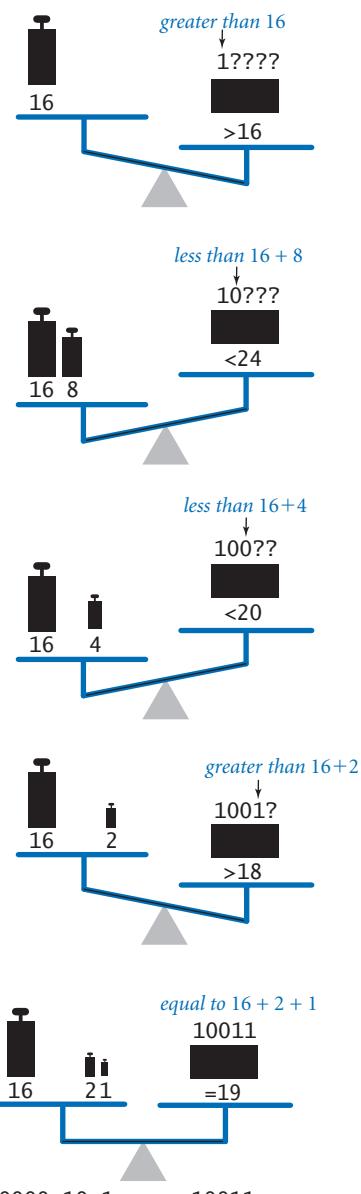
Trace of java Sqrt 2.0

(values of x for which the function is 0). Start with an initial estimate, t_0 . Given the estimate t_i , compute a new estimate by drawing a line tangent to the curve $y=f(x)$ at the point $(t_i, f(t_i))$ and set t_{i+1} to the x -coordinate of the point where that line hits the x -axis. Iterating this process, we get closer to the root.

Computing the square root of a positive number c is equivalent to finding the positive root of the function $f(x) = x^2 - c$. For this special case, Newton's method amounts to the process implemented in `Sqrt` (see EXERCISE 1.3.19). Start with the estimate $t = c$. If t is equal to c / t , then t is equal to the square root of c , so the computation is complete. If not, refine the estimate by replacing t with the average of t and c / t . With Newton's method, we get the value of the square root of 2 accurate to 15 decimal places in just 5 iterations of the loop.

Newton's method is important in scientific computing because the same iterative approach is effective for finding the roots of a broad class of functions, including many for which analytic solutions are not known (and for which the Java Math library is no help). Nowadays, we take for granted that we can find whatever values we need of mathematical functions; before computers, scientists and engineers had to use tables or compute values by hand. Computational techniques that were developed to enable calculations by hand needed to be very efficient, so it is not surprising that many of those same techniques are effective when we use computers. Newton's method is a classic example of this phenomenon. Another useful approach for evaluating mathematical functions is to use Taylor series expansions (see EXERCISE 1.3.37 and EXERCISE 1.3.38).

Number conversion. `Binary` (PROGRAM 1.3.7) prints the binary (base 2) representation of the decimal number typed as the command-line argument. It is based on decomposing a number into a sum of powers of 2. For example, the binary representation of 19 is 10011, which is the same as saying that $19 = 16 + 2 + 1$. To compute the binary representation of n , we consider the powers of 2 less than or equal to n in decreasing order to determine which belong in the binary decomposition (and therefore correspond to a 1 bit in the binary



Scale analog to binary conversion

Program 1.3.7 Converting to binary

```

public class Binary
{
    public static void main(String[] args)
    { // Print binary representation of n.
        int n = Integer.parseInt(args[0]);
        int power = 1;
        while (power <= n/2)
            power *= 2;
        // Now power is the largest power of 2 <= n.

        while (power > 0)
        { // Cast out powers of 2 in decreasing order.
            if (n < power) { System.out.print(0); }
            else           { System.out.print(1); n -= power; }
            power /= 2;
        }
        System.out.println();
    }
}

```

n | integer to convert
power | current power of 2

This program takes a positive integer n as a command-line argument and prints the binary representation of n, by casting out powers of 2 in decreasing order (see text).

```

% java Binary 19
10011
% java Binary 100000000
101111101011110000100000000

```

representation). The process corresponds precisely to using a balance scale to weigh an object, using weights whose values are powers of 2. First, we find the largest weight not heavier than the object. Then, considering the weights in decreasing order, we add each weight to test whether the object is lighter. If so, we remove the weight; if not, we leave the weight and try the next one. Each weight corresponds to

<i>n</i>	<i>binary representation</i>	<i>power</i>	<i>power > 0</i>	<i>binary representation</i>	<i>n < power</i>	<i>output</i>
19	10011	16	true	10000	false	1
3	0011	8	true	1000	true	0
3	011	4	true	100	true	0
3	01	2	true	10	false	1
1	1	1	true	1	false	1
0		0	false			

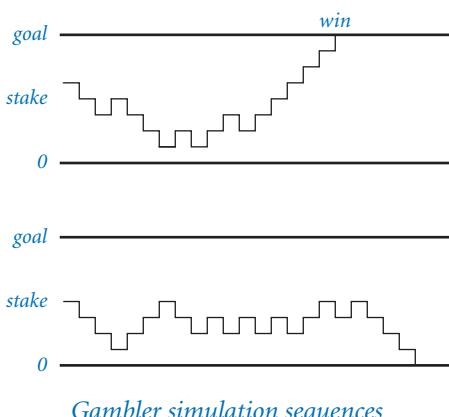
Trace of casting-out-powers-of-2 loop for java Binary 19

a bit in the binary representation of the weight of the object; leaving a weight corresponds to a 1 bit in the binary representation of the object's weight, and removing a weight corresponds to a 0 bit in the binary representation of the object's weight.

In `Binary`, the variable `power` corresponds to the current weight being tested, and the variable `n` accounts for the excess (unknown) part of the object's weight (to simulate leaving a weight on the balance, we just subtract that weight from `n`). The value of `power` decreases through the powers of 2. When it is larger than `n`, `Binary` prints 0; otherwise, it prints 1 and subtracts `power` from `n`. As usual, a trace (of the values of `n`, `power`, `n < power`, and the output bit for each loop iteration) can be very useful in helping you to understand the program. Read from top to bottom in the rightmost column of the trace, the output is 10011, the binary representation of 19.

Converting data from one representation to another is a frequent theme in writing computer programs. Thinking about conversion emphasizes the distinction between an abstraction (an integer like the number of hours in a day) and a representation of that abstraction (24 or 11000). The irony here is that the computer's representation of an integer is actually based on its binary representation.

Simulation. Our next example is different in character from the ones we have been considering, but it is representative of a common situation where we use computers to simulate what might happen in the real world so that we can make informed decisions. The specific example that we consider now is from a thoroughly studied class of problems known as *gambler's ruin*. Suppose that a gambler makes a series of fair \$1 bets, starting with some given initial stake. The gambler always goes broke eventually, but when we set other limits on the game, various questions



arise. For example, suppose that the gambler decides ahead of time to walk away after reaching a certain goal. What are the chances that the gambler will win? How many bets might be needed to win or lose the game? What is the maximum amount of money that the gambler will have during the course of the game?

`Gambler` (PROGRAM 1.3.8) is a simulation that can help answer these questions. It does a sequence of trials, using `Math.random()` to simulate the sequence of bets, continuing until either the gambler is broke or the goal is reached, and keeping track of the number of times the gambler reaches the goal and the number of bets. After running the experi-

ment for the specified number of trials, it averages and prints the results. You might wish to run this program for various values of the command-line arguments, not necessarily just to plan your next trip to the casino, but to help you think about the following questions: Is the simulation an accurate reflection of what would happen in real life? How many trials are needed to get an accurate answer? What are the computational limits on performing such a simulation? Simulations are widely used in applications in economics, science, and engineering, and questions of this sort are important in any simulation.

In the case of `Gambler`, we are verifying classical results from probability theory, which say that *the probability of success is the ratio of the stake to the goal* and that *the expected number of bets is the product of the stake and the desired gain* (the difference between the goal and the stake). For example, if you go to Monte Carlo to try to turn \$500 into \$2,500, you have a reasonable (20%) chance of success, but you should expect to make a million \$1 bets! If you try to turn \$1 into \$1,000, you have a 0.1% chance and can expect to be done (ruined, most likely) in about 999 bets.

Simulation and analysis go hand-in-hand, each validating the other. In practice, the value of simulation is that it can suggest answers to questions that might be too difficult to resolve with analysis. For example, suppose that our gambler, recognizing that there will never be enough time to make a million bets, decides ahead of time to set an upper limit on the number of bets. How much money can the gambler expect to take home in that case? You can address this question with an easy change to PROGRAM 1.3.8 (see EXERCISE 1.3.26), but addressing it with mathematical analysis is not so easy.

Program 1.3.8 Gambler's ruin simulation

```

public class Gambler
{
    public static void main(String[] args)
    { // Run trials experiments that start with
      // $stake and terminate on $0 or $goal.
      int stake = Integer.parseInt(args[0]);
      int goal = Integer.parseInt(args[1]);
      int trials = Integer.parseInt(args[2]);
      int bets = 0;
      int wins = 0;
      for (int t = 0; t < trials; t++)
      { // Run one experiment.
        int cash = stake;
        while (cash > 0 && cash < goal)
        { // Simulate one bet.
          bets++;
          if (Math.random() < 0.5) cash++;
          else                         cash--;
        } // Cash is either 0 (ruin) or $goal (win).
        if (cash == goal) wins++;
      }
      System.out.println(100*wins/trials + "% wins");
      System.out.println("Avg # bets: " + bets/trials);
    }
}

```

stake	<i>initial stake</i>
goal	<i>walkaway goal</i>
trials	<i>number of trials</i>
bets	<i>bet count</i>
wins	<i>win count</i>
cash	<i>cash on hand</i>

This program takes three integers command-line arguments *stake*, *goal*, and *trials*. The inner *while* loop in this program simulates a gambler with *\$stake* who makes a series of \$1 bets, continuing until going broke or reaching *\$goal*. The running time of this program is proportional to *trials* times the average number of bets. For example, the third command below causes nearly 100 million random numbers to be generated.

```

% java Gambler 10 20 1000
50% wins
Avg # bets: 100
% java Gambler 10 20 1000
51% wins
Avg # bets: 98

```

```

% java Gambler 50 250 100
19% wins
Avg # bets: 11050
% java Gambler 500 2500 100
21% wins
Avg # bets: 998071

```

Factoring. A *prime number* is an integer greater than 1 whose only positive divisors are 1 and itself. The prime factorization of an integer is the multiset of primes whose product is the integer. For example, $3,757,208 = 2 \times 2 \times 2 \times 7 \times 13 \times 13 \times 397$. **Factors** (PROGRAM 1.3.9) computes the prime factorization of any given positive integer. In contrast to many of the other programs that we have seen (which we could do in a few minutes with a calculator or even a pencil and paper), this computation would not be feasible without a computer. How would you go about trying to find the factors of a number like 287994837222311? You might find the factor 17 quickly, but even with a calculator it would take you quite a while to find 1739347.

Although **Factors** is compact, it certainly will take some thought to convince yourself that it produces the desired result for any given integer. As usual, following a trace that shows the values of the variables at the beginning of each iteration of the outer `for` loop is a good way to understand the computation. For the case where the initial value of `n` is 3757208, the inner `while` loop iterates three times

<i>factor</i>	<i>n</i>	<i>output</i>
2	3757208	2 2 2
3	469651	
4	469651	
5	469651	
6	469651	
7	469651	7
8	67093	
9	67093	
10	67093	
11	67093	
12	67093	
13	67093	13 13
14	397	
15	397	
16	397	
17	397	
18	397	
19	397	
20	397	

when `factor` is 2, to remove the three factors of 2; then zero times when `factor` is 3, 4, 5, and 6, since none of those numbers divides 469651; and so forth. Tracing the program for a few example inputs reveals its basic operation. To convince ourselves that the program will behave as expected for all inputs, we reason about what we expect each of the loops to do. The `while` loop prints and removes from `n` all factors of `factor`, but the key to understanding the program is to see that the following fact holds at the beginning of each iteration of the `for` loop: `n` has no factors between 2 and `factor-1`. Thus, if `factor` is not prime, it will not divide `n`; if `factor` is prime, the `while` loop will do its job. Once we know that `n` has no divisors less than or equal to `factor`, we also know that it has no factors greater than `n/factor`, so we need look no further when `factor` is greater than `n/factor`.

In a more naïve implementation, we might simply have used the condition (`factor < n`) to terminate the `for` loop. Even given the blinding speed of modern computers, such a decision would have a dramatic effect on the size of the numbers that we could factor. EXERCISE 1.3.28 encourages you to experiment with the program to learn the ef-

Program 1.3.9 Factoring integers

```

public class Factors
{
    public static void main(String[] args)
    { // Print the prime factorization of n.
        long n = Long.parseLong(args[0]);
        for (long factor = 2; factor <= n/factor; factor++)
        { // Test potential factor.
            while (n % factor == 0)
            { // Cast out and print factor.
                n /= factor;
                System.out.print(factor + " ");
            } // Any factor of n must be greater than factor.
        }
        if (n > 1) System.out.print(n);
        System.out.println();
    }
}

```

n unfactored part
factor potential factor

This program takes a positive integer n as a command-line argument and prints the prime factorization of n. The code is simple, but it takes some thought to convince yourself that it is correct (see text).

```
% java Factors 3757208
2 2 2 7 13 13 397
```

```
% java Factors 287994837222311
17 1739347 9739789
```

fectiveness of this simple change. On a computer that can do billions of operations per second, we could factor numbers on the order of 10^9 in a few seconds; with the (`factor <= n/factor`) test, we can factor numbers on the order of 10^{18} in a comparable amount of time. Loops give us the ability to solve difficult problems, but they also give us the ability to construct simple programs that run slowly, so we must always be cognizant of performance.

In modern applications in cryptography, there are important situations where we wish to factor truly huge numbers (with, say, hundreds or thousands of digits). Such a computation is prohibitively difficult even *with* the use of a computer.

Other conditional and loop constructs To more fully cover the Java language, we consider here four more control-flow constructs. You need not think about using these constructs for every program that you write, because you are likely to encounter them much less frequently than the `if`, `while`, and `for` statements. You certainly do not need to worry about using these constructs until you are comfortable using `if`, `while`, and `for`. You might encounter one of them in a program in a book or on the web, but many programmers do not use them at all and we rarely use any of them outside this section.

Break statements. In some situations, we want to immediately exit a loop without letting it run to completion. Java provides the `break` statement for this purpose. For example, the following code is an effective way to test whether a given integer $n > 1$ is prime:

```
int factor;
for (factor = 2; factor <= n/factor; factor++)
    if (n % factor == 0) break;
if (factor > n/factor)
    System.out.println(n + " is prime");
```

There are two different ways to leave this loop: either the `break` statement is executed (because `factor` divides `n`, so `n` is not prime) or the loop-continuation condition is not satisfied (because no `factor` with `factor <= n/factor` was found that divides `n`, which implies that `n` is prime). Note that we have to declare `factor` outside the `for` loop instead of in the initialization statement so that its scope extends beyond the loop.

Continue statements. Java also provides a way to skip to the next iteration of a loop: the `continue` statement. When a `continue` statement is executed within the body of a `for` loop, the flow of control transfers directly to the increment statement for the next iteration of the loop.

Switch statements. The `if` and `if-else` statements allow one or two alternatives in directing the flow of control. Sometimes, a computation naturally suggests more than two mutually exclusive alternatives. We could use a sequence or a chain of `if-else` statements (as in the tax rate calculation discussed earlier in this section), but the Java `switch` statement provides a more direct solution. Let us move right to a typical example. Rather than printing an `int` variable `day` in a program that works with days of the weeks (such as a solution to EXERCISE 1.2.29), it is easier to use a `switch` statement, as follows:

```
switch (day)
{
    case 0: System.out.println("Sun"); break;
    case 1: System.out.println("Mon"); break;
    case 2: System.out.println("Tue"); break;
    case 3: System.out.println("Wed"); break;
    case 4: System.out.println("Thu"); break;
    case 5: System.out.println("Fri"); break;
    case 6: System.out.println("Sat"); break;
}
```

When you have a program that seems to have a long and regular sequence of `if` statements, you might consider consulting the booksite and using a `switch` statement, or using an alternate approach described in SECTION 1.4.

Do-while loops. Another way to write a loop is to use the template

```
do { <statements> } while (<boolean expression>);
```

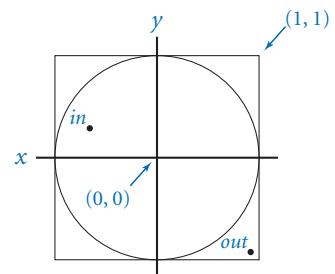
The meaning of this statement is the same as

```
while (<boolean expression>) { <statements> }
```

except that the first test of the boolean condition is omitted. If the boolean condition initially holds, there is no difference. For an example in which `do-while` is useful, consider the problem of generating points that are randomly distributed in the unit disk. We can use `Math.random()` to generate x - and y -coordinates independently to get points that are randomly distributed in the 2-by-2 square centered on the origin. Most points fall within the unit disk, so we just reject those that do not. We always want to generate at least one point, so a `do-while` loop is ideal for this computation. The following code sets x and y such that the point (x, y) is randomly distributed in the unit disk:

```
do
{ // Scale x and y to be random in (-1, 1).
    x = 2.0*Math.random() - 1.0;
    y = 2.0*Math.random() - 1.0;
} while (x*x + y*y > 1.0);
```

Since the area of the disk is π and the area of the square is 4, the expected number of times the loop is iterated is $4/\pi$ (about 1.27).



Infinite loops Before you write programs that use loops, you need to think about the following issue: what if the loop-continuation condition in a `while` loop is always satisfied? With the statements that you have learned so far, one of two bad things could happen, both of which you need to learn to cope with.

First, suppose that such a loop calls `System.out.println()`. For example, if the loop-continuation condition in `TenHello`s were (`i > 3`) instead of (`i <= 10`), it would always be true. What happens? Nowadays, we use *print* as an abstraction to mean *display in a terminal window* and the result of attempting to display an unlimited number of lines in a terminal window is dependent on operating-system conventions. If your system is set up to have *print* mean *print characters on a piece of paper*,

you might run out of paper or have to unplug the printer. In a terminal window, you need a *stop printing* operation. Before running programs with loops on your own, you make sure that you know what to do to “pull the plug” on an infinite loop of `System.out.println()` calls and then test out the strategy by making the change to `TenHello`s indicated above and trying to stop it. On most systems, `<Ctrl-C>` means *stop the current program*, and should do the job.

Second, *nothing* might happen. If your program has an infinite loop that does not produce any output, it will spin through the loop and you will see no results at all. When you find yourself in such a situation, you can inspect the loops to make sure that the loop exit condition always happens, but the problem may not be easy to identify. One way to locate such a bug is to insert calls to `System.out.println()` to produce a trace. If these calls fall within an infinite loop, this strategy reduces the problem to the case discussed in the previous paragraph, but the output might give you a clue about what to do.

You might not know (or it might not matter) whether a loop is infinite or just very long. Even `BadHello`s eventually would terminate after printing more than 1 billion lines because of integer overflow. If you invoke PROGRAM 1.3.8 with arguments such as `java Gambler 100000 200000 100`, you may not want to wait for the answer. You will learn to be aware of and to estimate the running time of your programs.

Why not have Java detect infinite loops and warn us about them? You might be surprised to know that it is not possible to do so, in general. This counterintuitive fact is one of the fundamental results of theoretical computer science.

```
public class BadHello
{
    int i = 4;
    while (i > 3)
    {
        System.out.println
            (i + "th Hello");
        i = i + 1;
    }
    ...
}

% java BadHello
1st Hello
2nd Hello
3rd Hello
5th Hello
6th Hello
7th Hello
...

```

An infinite loop

Summary For reference, the accompanying table lists the programs that we have considered in this section. They are representative of the kinds of tasks we can address with short programs composed of `if`, `while`, and `for` statements processing built-in types of data. These types of computations are an appropriate way to become familiar with the basic Java flow-of-control constructs.

To learn how to use conditionals and loops, you must practice writing and debugging programs with `if`, `while`, and `for` statements. The exercises at the end of this section provide many opportunities for you to begin this process. For each exercise, you will write a Java program, then run and test it. All programmers know that it is unusual to have a program work as planned the first time it is run, so you will want to have an understanding of your program and an expectation of what it should do, step by step. At first, use explicit traces to check your understanding and expectation. As you gain experience, you will find yourself thinking in terms of what a trace might produce as you compose your loops. Ask yourself the following kinds of questions: What will be the values of the variables after the loop iterates the first time? The second time? The final time? Is there any way this program could get stuck in an infinite loop?

Loops and conditionals are a giant step in our ability to compute: `if`, `while`, and `for` statements take us from simple straight-line programs to arbitrarily complicated flow of control. In the next several chapters, we will take more giant steps that will allow us to process large amounts of input data and allow us to define and process types of data other than simple numeric types. The `if`, `while`, and `for` statements of this section will play an essential role in the programs that we consider as we take these steps.

<i>program</i>	<i>description</i>
Flip	<i>simulate a coin flip</i>
TenHello	<i>your first loop</i>
PowersOfTwo	<i>compute and print a table of values</i>
DivisorPattern	<i>your first nested loop</i>
Harmonic	<i>compute finite sum</i>
Sqrt	<i>classic iterative algorithm</i>
Binary	<i>basic number conversion</i>
Gambler	<i>simulation with nested loops</i>
Factors	<i>while loop within a for loop</i>

Summary of programs in this section

Q&A

Q. What is the difference between `=` and `==`?

A. We repeat this question here to remind you to be sure not to use `=` when you mean `==` in a boolean expression. The expression `(x = y)` assigns the value of `y` to `x`, whereas the expression `(x == y)` tests whether the two variables currently have the same values. In some programming languages, this difference can wreak havoc in a program and be difficult to detect, but Java's type safety usually will come to the rescue. For example, if we make the mistake of typing `(cash = goal)` instead of `(cash == goal)` in PROGRAM 1.3.8, the compiler finds the bug for us:

```
javac Gambler.java
Gambler.java:18: incompatible types
  found : int
  required: boolean
    if (cash = goal) wins++;
           ^
1 error
```

Be careful about writing `if (x = y)` when `x` and `y` are `boolean` variables, since this will be treated as an assignment statement, which assigns the value of `y` to `x` and evaluates to the truth value of `y`. For example, you should write `if (!isPrime)` instead of `if (isPrime = false)`.

Q. So I need to pay attention to using `==` instead of `=` when writing loops and conditionals. Is there something else in particular that I should watch out for?

A. Another common mistake is to forget the braces in a loop or conditional with a multi-statement body. For example, consider this version of the code in `Gambler`:

```
for (int t = 0; t < trials; t++)
    for (cash = stake; cash > 0 && cash < goal; bets++)
        if (Math.random() < 0.5) cash++;
        else
            cash--;
    if (cash == goal) wins++;
```

The code appears correct, but it is dysfunctional because the second `if` is outside both `for` loops and gets executed just once. Many programmers always use braces to delimit the body of a loop or conditional precisely to avoid such insidious bugs.

Q. Anything else?

A. The third classic pitfall is ambiguity in nested `if` statements:

```
if <expr1> if <expr2> <stmtA> else <stmtB>
```

In Java, this is equivalent to

```
if <expr1> { if <expr2> <stmtA> else <stmtB> }
```

even if you might have been thinking

```
if <expr1> { if <expr2> <stmtA> } else <stmtB>
```

Again, using explicit braces to delimit the body is a good way to avoid this pitfall.

Q. Are there cases where I must use a `for` loop but not a `while`, or vice versa?

A. No. Generally, you should use a `for` loop when you have an initialization, an increment, and a loop continuation test (if you do not need the loop control variable outside the loop). But the equivalent `while` loop still might be fine.

Q. What are the rules on where we declare the loop-control variables?

A. Opinions differ. In older programming languages, it was required that all variables be declared at the beginning of a block, so many programmers are in this habit and a lot of code follows this convention. But it makes a great deal of sense to declare variables where they are first used, particularly in `for` loops, when it is normally the case that the variable is not needed outside the loop. However, it is not uncommon to need to test (and therefore declare) the loop-control variable outside the loop, as in the primality-testing code we considered as an example of the `break` statement.

Q. What is the difference between `++i` and `i++`?

A. As statements, there is no difference. In expressions, both increment `i`, but `++i` has the value after the increment and `i++` the value before the increment. In this book, we avoid statements like `x = ++i` that have the side effect of changing variable values. So, it is safe to not worry much about this distinction and just use `i++`

in `for` loops and as a statement. When we do use `++i` in this book, we will call attention to it and say why we are using it.

Q. In a `for` loop, `<initialize>` and `<increment>` can be statements more complicated than declaring, initializing, and updating a loop-control variable. How can I take advantage of this ability?

A. The `<initialize>` and `<increment>` can be *sequences* of statements, separated by commas. This notation allows for code that initializes and modifies other variables besides the loop-control variable. In some cases, this ability leads to compact code. For example, the following two lines of code could replace the last eight lines in the body of the `main()` method in `PowersOfTwo` (PROGRAM 1.3.3):

```
for (int i = 0, power = 1; i <= n; i++, power *= 2)
    System.out.println(i + " " + power);
```

Such code is rarely necessary and better avoided, particularly by beginners.

Q. Can I use a `double` variable as a loop-control variable in a `for` loop?

A. It is legal, but generally bad practice to do so. Consider the following loop:

```
for (double x = 0.0; x <= 1.0; x += 0.1)
    System.out.println(x + " " + Math.sin(x));
```

How many times does it iterate? The number of iterations depends on an equality test between `double` values, which may not always give the result that you expect because of floating-point precision.

Q. Anything else tricky about loops?

A. Not all parts of a `for` loop need to be filled in with code. The initialization statement, the boolean expression, the increment statement, and the loop body can each be omitted. It is generally better style to use a `while` statement than null statements in a `for` loop. In the code in this book, we avoid such *empty statements*.

```

int power = 1;
while (power <= n/2)           empty increment
    power *= 2;                statement
                                ↓
for (int power = 1; power <= n/2; )
    power *= 2;

for (int power = 1; power <= n/2; power *= 2)
; ← empty loop body

```

Three equivalent loops

Exercises

1.3.1 Write a program that takes three integer command-line arguments and prints `equal` if all three are equal, and `not equal` otherwise.

1.3.2 Write a more general and more robust version of `Quadratic` (PROGRAM 1.2.3) that prints the roots of the polynomial $ax^2 + bx + c$, prints an appropriate message if the discriminant is negative, and behaves appropriately (avoiding division by zero) if a is zero.

1.3.3 What (if anything) is wrong with each of the following statements?

- a. `if (a > b) then c = 0;`
- b. `if a > b { c = 0; }`
- c. `if (a > b) c = 0;`
- d. `if (a > b) c = 0 else b = 0;`

1.3.4 Write a code fragment that prints `true` if the `double` variables `x` and `y` are both strictly between 0 and 1, and `false` otherwise.

1.3.5 Write a program `RollLoadedDie` that prints the result of rolling a loaded die such that the probability of getting a 1, 2, 3, 4, or 5 is $1/8$ and the probability of getting a 6 is $3/8$.

1.3.6 Improve your solution to EXERCISE 1.2.25 by adding code to check that the values of the command-line arguments fall within the ranges of validity of the formula, and by also adding code to print out an error message if that is not the case.

1.3.7 Suppose that `i` and `j` are both of type `int`. What is the value of `j` after each of the following statements is executed?

- a. `for (i = 0, j = 0; i < 10; i++) j += i;`
- b. `for (i = 0, j = 1; i < 10; i++) j += j;`
- c. `for (j = 0; j < 10; j++) j += j;`
- d. `for (i = 0, j = 0; i < 10; i++) j += j++;`

1.3.8 Rewrite `TenHello`s to make a program `Hello`s that takes the number of lines to print as a command-line argument. You may assume that the argument is less than 1000. Hint: Use `i % 10` and `i % 100` to determine when to use `st`, `nd`, `rd`, or `th` for printing the `i`th Hello.

1.3.9 Write a program that, using one `for` loop and one `if` statement, prints the integers from 1,000 to 2,000 with five integers per line. Hint: Use the `%` operation.

1.3.10 Write a program that takes an integer command-line argument `n`, uses `Math.random()` to print `n` uniform random values between 0 and 1, and then prints their average value (see EXERCISE 1.2.30).

1.3.11 Describe what happens when you try to print a ruler function (see the table on page 57) with a value of `n` that is too large, such as 100.

1.3.12 Write a program `FunctionGrowth` that prints a table of the values $\log n$, n , $n \log_e n$, n^2 , n^3 , and 2^n for $n = 16, 32, 64, \dots, 2,048$. Use tabs (`\t` characters) to align columns.

1.3.13 What are the values of `m` and `n` after executing the following code?

```
int n = 123456789;
int m = 0;
while (n != 0)
{
    m = (10 * m) + (n % 10);
    n = n / 10;
}
```

1.3.14 What does the following code fragment print?

```
int f = 0, g = 1;
for (int i = 0; i <= 15; i++)
{
    System.out.println(f);
    f = f + g;
    g = f - g;
}
```

Solution. Even an expert programmer will tell you that the only way to understand a program like this is to trace it. When you do, you will find that it prints the values 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 134, 233, 377, and 610. These numbers are the first sixteen of the famous *Fibonacci sequence*, which are defined by the following formulas: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$.

1.3.15 How many lines of output does the following code fragment produce?

```
for (int i = 0; i < 999; i++);
{ System.out.println("Hello"); }
```

Solution. One. Note the spurious semicolon at the end of the first line.

1.3.16 Write a program that takes an integer command-line argument n and prints all the positive powers of 2 less than or equal to n . Make sure that your program works properly for all values of n .

1.3.17 Expand your solution to EXERCISE 1.2.24 to print a table giving the total amount of money you would have after t years for $t = 0$ to 25.

1.3.18 Unlike the harmonic numbers, the sum $1/1^2 + 1/2^2 + \dots + 1/n^2$ does converge to a constant as n grows to infinity. (Indeed, the constant is $\pi^2/6$, so this formula can be used to estimate the value of π .) Which of the following for loops computes this sum? Assume that n is an `int` variable initialized to 1000000 and `sum` is a `double` variable initialized to 0.0.

- a. `for (int i = 1; i <= n; i++) sum += 1 / (i*i);`
- b. `for (int i = 1; i <= n; i++) sum += 1.0 / i*i;`
- c. `for (int i = 1; i <= n; i++) sum += 1.0 / (i*i);`
- d. `for (int i = 1; i <= n; i++) sum += 1 / (1.0*i*i);`

1.3.19 Show that PROGRAM 1.3.6 implements Newton's method for finding the square root of c . *Hint:* Use the fact that the slope of the tangent to a (differentiable) function $f(x)$ at $x = t$ is $f'(t)$ to find the equation of the tangent line, and then use that equation to find the point where the tangent line intersects the x -axis to show that you can use Newton's method to find a root of any function as follows: at each iteration, replace the estimate t by $t - f(t) / f'(t)$.

1.3.20 Using Newton's method, develop a program that takes two integer command-line arguments n and k and prints the k th root of n (*Hint:* See EXERCISE 1.3.19).

1.3.21 Modify `Binary` to get a program `Kary` that takes two integer command-line arguments i and k and converts i to base k . Assume that i is an integer in Java's `long` data type and that k is an integer between 2 and 16. For bases greater than 10, use the letters A through F to represent the 11th through 16th digits, respectively.

1.3.22 Write a code fragment that puts the binary representation of a positive integer n into a `String` variable `s`.

Solution. Java has a built-in method `Integer.toBinaryString(n)` for this job, but the point of the exercise is to see how such a method might be implemented. Working from PROGRAM 1.3.7, we get the solution

```
String s = "";
int power = 1;
while (power <= n/2) power *= 2;
while (power > 0)
{
    if (n < power) { s += 0; }
    else           { s += 1; n -= power; }
    power /= 2;
}
```

A simpler option is to work from right to left:

```
String s = "";
for (int i = n; i > 0; i /= 2)
    s = (i % 2) + s;
```

Both of these methods are worthy of careful study.

1.3.23 Write a version of `Gambler` that uses two nested `while` loops or two nested `for` loops instead of a `while` loop inside a `for` loop.

1.3.24 Write a program `GamblerPlot` that traces a gambler's ruin simulation by printing a line after each bet in which one asterisk corresponds to each dollar held by the gambler.

1.3.25 Modify `Gambler` to take an extra command-line argument that specifies the (fixed) probability that the gambler wins each bet. Use your program to try to learn how this probability affects the chance of winning and the expected number of bets. Try a value of p close to 0.5 (say, 0.48).

1.3.26 Modify `Gambler` to take an extra command-line argument that specifies the number of bets the gambler is willing to make, so that there are three possible

ways for the game to end: the gambler wins, loses, or runs out of time. Add to the output to give the expected amount of money the gambler will have when the game ends. *Extra credit:* Use your program to plan your next trip to Monte Carlo.

1.3.27 Modify `Factors` to print just one copy each of the prime divisors.

1.3.28 Run quick experiments to determine the impact of using the termination condition (`factor <= n/factor`) instead of (`factor < n`) in `Factors` in PROGRAM 1.3.9. For each method, find the largest n such that when you type in an n -digit number, the program is sure to finish within 10 seconds.

1.3.29 Write a program `Checkerboard` that takes an integer command-line argument n and uses a loop nested within a loop to print out a two-dimensional n -by- n checkerboard pattern with alternating spaces and asterisks.

1.3.30 Write a program `GreatestCommonDivisor` that finds the greatest common divisor (gcd) of two integers using *Euclid's algorithm*, which is an iterative computation based on the following observation: if x is greater than y , then if y divides x , the gcd of x and y is y ; otherwise, the gcd of x and y is the same as the gcd of $x \% y$ and y .

1.3.31 Write a program `RelativelyPrime` that takes an integer command-line argument n and prints an n -by- n table such that there is an * in row i and column j if the gcd of i and j is 1 (i and j are relatively prime) and a space in that position otherwise.

1.3.32 Write a program `PowersOfK` that takes an integer command-line argument k and prints all the positive powers of k in the Java `long` data type. *Note:* The constant `Long.MAX_VALUE` is the value of the largest integer in `long`.

1.3.33 Write a program that prints the coordinates of a random point (a, b, c) on the surface of a sphere. To generate such a point, use *Marsaglia's method*: Start by picking a random point (x, y) in the unit disk using the method described at the end of this section. Then, set a to $2x\sqrt{1-x^2-y^2}$, b to $2\sqrt{1-x^2-y^2}$, and c to $1-2(x^2+y^2)$.

Creative Exercises

1.3.34 Ramanujan's taxi. Srinivasa Ramanujan was an Indian mathematician who became famous for his intuition for numbers. When the English mathematician G. H. Hardy came to visit him one day, Hardy remarked that the number of his taxi was 1729, a rather dull number. To which Ramanujan replied, “No, Hardy! No, Hardy! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways.” Verify this claim by writing a program that takes an integer command-line argument n and prints all integers less than or equal to n that can be expressed as the sum of two cubes in two different ways. In other words, find distinct positive integers a, b, c , and d such that $a^3 + b^3 = c^3 + d^3$. Use four nested for loops.

1.3.35 Checksum. The International Standard Book Number (ISBN) is a 10-digit code that uniquely specifies a book. The rightmost digit is a checksum digit that can be uniquely determined from the other 9 digits, from the condition that $d_1 + 2d_2 + 3d_3 + \dots + 10d_{10}$ must be a multiple of 11 (here d_i denotes the i th digit from the right). The checksum digit d_1 can be any value from 0 to 10. The ISBN convention is to use the character 'X' to denote 10. As an example, the checksum digit corresponding to 020131452 is 5 since 5 is the only value of x between 0 and 10 for which

$$10 \cdot 0 + 9 \cdot 2 + 8 \cdot 0 + 7 \cdot 1 + 6 \cdot 3 + 5 \cdot 1 + 4 \cdot 4 + 3 \cdot 5 + 2 \cdot 2 + 1 \cdot x$$

is a multiple of 11. Write a program that takes a 9-digit integer as a command-line argument, computes the checksum, and prints the ISBN number.

1.3.36 Counting primes. Write a program `PrimeCounter` that takes an integer command-line argument n and finds the number of primes less than or equal to n . Use it to print out the number of primes less than or equal to 10 million. *Note:* If you are not careful, your program may not finish in a reasonable amount of time!

1.3.37 2D random walk. A two-dimensional random walk simulates the behavior of a particle moving in a grid of points. At each step, the random walker moves north, south, east, or west with probability equal to 1/4, independent of previous moves. Write a program `RandomWalker` that takes an integer command-line argument n and estimates how long it will take a random walker to hit the boundary of a $2n$ -by- $2n$ square centered at the starting point.

1.3.38 Exponential function. Assume that x is a positive variable of type `double`. Write a code fragment that uses the Taylor series expansion to set the value of `sum` to $e^x = 1 + x + x^2/2! + x^3/3! + \dots$.

Solution. The purpose of this exercise is to get you to think about how a library function like `Math.exp()` might be implemented in terms of elementary operators. Try solving it, then compare your solution with the one developed here.

We start by considering the problem of computing one term. Suppose that x and `term` are variables of type `double` and n is a variable of type `int`. The following code fragment sets `term` to $x^n / n!$ using the direct method of having one loop for the numerator and another loop for the denominator, then dividing the results:

```
double num = 1.0, den = 1.0;
for (int i = 1; i <= n; i++) num *= x;
for (int i = 1; i <= n; i++) den *= i;
double term = num/den;
```

A better approach is to use just a single `for` loop:

```
double term = 1.0;
for (i = 1; i <= n; i++) term *= x/i;
```

Besides being more compact and elegant, the latter solution is preferable because it avoids inaccuracies caused by computing with huge numbers. For example, the two-loop approach breaks down for values like $x = 10$ and $n = 100$ because $100!$ is too large to represent as a `double`.

To compute e^x , we nest this `for` loop within another `for` loop:

```
double term = 1.0;
double sum = 0.0;
for (int n = 1; sum != sum + term; n++)
{
    sum += term;
    term = 1.0;
    for (int i = 1; i <= n; i++) term *= x/i;
}
```

The number of times the loop iterates depends on the relative values of the next term and the accumulated sum. Once the value of the sum stops changing, we leave

the loop. (This strategy is more efficient than using the loop-continuation condition (`term > 0`) because it avoids a significant number of iterations that do not change the value of the sum.) This code is effective, but it is inefficient because the inner `for` loop recomputes all the values it computed on the previous iteration of the outer `for` loop. Instead, we can make use of the term that was added in on the previous loop iteration and solve the problem with a single `for` loop:

```
double term = 1.0;
double sum = 0.0;
for (int n = 1; sum != sum + term; n++)
{
    sum += term;
    term *= x/n;
}
```

1.3.39 Trigonometric functions. Write two programs, `Sin` and `Cos`, that compute the sine and cosine functions using their Taylor series expansions $\sin x = x - x^3/3! + x^5/5! - \dots$ and $\cos x = 1 - x^2/2! + x^4/4! - \dots$.

1.3.40 Experimental analysis. Run experiments to determine the relative costs of `Math.exp()` and the methods from EXERCISE 1.3.38 for computing e^x : the direct method with nested `for` loops, the improvement with a single `for` loop, and the latter with the loop-continuation condition (`term > 0`). Use trial-and-error with a command-line argument to determine how many times your computer can perform each computation in 10 seconds.

1.3.41 Pepys problem. In 1693 Samuel Pepys asked Isaac Newton which is more likely: getting 1 at least once when rolling a fair die six times or getting 1 at least twice when rolling it 12 times. Write a program that could have provided Newton with a quick answer.

1.3.42 Game simulation. In the game show *Let's Make a Deal*, a contestant is presented with three doors. Behind one of them is a valuable prize. After the contestant chooses a door, the host opens one of the other two doors (never revealing the prize, of course). The contestant is then given the opportunity to switch to the other unopened door. Should the contestant do so? Intuitively, it might seem that the

contestant's initial choice door and the other unopened door are equally likely to contain the prize, so there would be no incentive to switch. Write a program `MonteHall` to test this intuition by simulation. Your program should take a command-line argument `n`, play the game `n` times using each of the two strategies (switch or do not switch), and print the chance of success for each of the two strategies.

1.3.43 Median-of-5. Write a program that takes five distinct integers as command-line arguments and prints the median value (the value such that two of the other integers are smaller and two are larger). *Extra credit:* Solve the problem with a program that compares values fewer than 7 times for any given input.

1.3.44 Sorting three numbers. Suppose that the variables `a`, `b`, `c`, and `t` are all of the type `int`. Explain why the following code puts `a`, `b`, and `c` in ascending order:

```
if (a > b) { t = a; a = b; b = t; }
if (a > c) { t = a; a = c; c = t; }
if (b > c) { t = b; b = c; c = t; }
```

1.3.45 Chaos. Write a program to study the following simple model for population growth, which might be applied to study fish in a pond, bacteria in a test tube, or any of a host of similar situations. We suppose that the population ranges from 0 (extinct) to 1 (maximum population that can be sustained). If the population at time t is x , then we suppose the population at time $t + 1$ to be $rx(1 - x)$, where the argument r , known as the *fecundity parameter*, controls the rate of growth. Start with a small population—say, $x = 0.01$ —and study the result of iterating the model, for various values of r . For which values of r does the population stabilize at $x = 1 - 1/r$? Can you say anything about the population when r is 3.5? 3.8? 5?

1.3.46 Euler's sum-of-powers conjecture. In 1769 Leonhard Euler formulated a generalized version of Fermat's Last Theorem, conjecturing that at least n n th powers are needed to obtain a sum that is itself an n th power, for $n > 2$. Write a program to disprove Euler's conjecture (which stood until 1967), using a quintuply nested loop to find four positive integers whose 5th power sums to the 5th power of another positive integer. That is, find a , b , c , d , and e such that $a^5 + b^5 + c^5 + d^5 = e^5$. Use the `long` data type.

1.4 Arrays

IN THIS SECTION, WE INTRODUCE YOU to the idea of a *data structure* and to your first data structure, the *array*. The primary purpose of an array is to facilitate storing and manipulating large quantities of data.

Arrays play an essential role in many data processing tasks. They also correspond to vectors and matrices, which are widely used in science and in scientific programming. We will consider basic properties of arrays in Java, with many examples illustrating why they are useful.

A *data structure* is a way to organize data in a computer (usually to save time or space). Data structures play an essential role in computer programming—indeed, CHAPTER 4 of this book is devoted to the study of classic data structures of all sorts.

A *one-dimensional array* (or *array*) is a data structure that stores a sequence of values, all of the same type. We refer to the components of an array as its *elements*. We use *indexing* to refer to the array elements: If we have n elements in an array, we think of the elements as being numbered from 0 to $n-1$ so that we can unambiguously specify an element with an integer index in this range.

A *two-dimensional array* is an array of one-dimensional arrays. Whereas the elements of a one-dimensional array are indexed by a single integer, the elements of a two-dimensional array are indexed by a pair of integers: the first index specifies the row, and the second index specifies the column.

Often, when we have a large amount of data to process, we first put all of the data into one or more arrays. Then we use indexing to refer to individual elements and to process the data. We might have exam scores, stock prices, nucleotides in a DNA strand, or characters in a book. Each of these examples involves a large number of values that are all of the same type. We consider such applications when we discuss input/output in SECTION 1.5 and in the case study that is the subject of SECTION 1.6. In this section, we expose the basic properties of arrays by considering examples where our programs first populate arrays with values computed from experimental studies and then process them.

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Programs in this section

a	a[0]
	a[1]
	a[2]
	a[3]
	a[4]
	a[5]
	a[6]
	a[7]

An array

Arrays in Java

Making an array in a Java program involves three distinct steps:

- Declare the array.
- Create the array.
- Initialize the array elements.

To declare an array, you need to specify a name and the type of data it will contain. To create it, you need to specify its *length* (the number of elements). To initialize it, you need to assign a value to each of its elements. For example, the following code makes an array of *n* elements, each of type `double` and initialized to 0.0:

```
double[] a;                      // declare the array
a = new double[n];                // create the array
for (int i = 0; i < n; i++)      // initialize the array
    a[i] = 0.0;
```

The first statement is the *array declaration*. It is just like a declaration of a variable of the corresponding primitive type except for the square brackets following the type name, which specify that we are declaring an array. The second statement *creates the array*; it uses the keyword `new` to allocate memory to store the specified number of elements. This action is unnecessary for variables of a primitive type, but it is needed for all other types of data in Java (see SECTION 3.1). The `for` loop assigns the value 0.0 to each of the *n* array elements. We refer to an array element by putting its index in square brackets after the array name: the code `a[i]` refers to element *i* of array `a[]`. (In the text, we use the notation `a[]` to indicate that variable `a` is an array, but we do not use `a[]` in Java code.)

The obvious advantage of using arrays is to define many variables without explicitly naming them. For example, if you wanted to process eight variables of type `double`, you could declare them with

```
double a0, a1, a2, a3, a4, a5, a6, a7;
```

and then refer to them as `a0`, `a1`, `a2`, and so forth. Naming dozens of individual variables in this way is cumbersome and naming millions is untenable. Instead, with arrays, you can declare *n* variables with the statement `double[] a = new double[n]` and refer to them as `a[0]`, `a[1]`, `a[2]`, and so forth. Now, it is easy to define dozens or millions of variables. Moreover, since you can use a variable (or other expression computed at run time) as an array index, you can process arbitrarily many elements in a single loop, as we do above. You should think of each array element as an individual variable, which you can use in an expression or as the left-hand side of an assignment statement.

As our first example, we use arrays to represent *vectors*. We consider vectors in detail in SECTION 3.3; for the moment, think of a vector as a sequence of real numbers. The *dot product* of two vectors (of the same length) is the sum of the products of their corresponding elements. The dot product of two vectors that are represented as one-dimensional arrays $x[]$ and $y[]$, each of length 3, is the expression $x[0]*y[0] + x[1]*y[1] + x[2]*y[2]$. More generally, if each array is of length n , then the following code computes their dot product:

<i>i</i>	$x[i]$	$y[i]$	$x[i]*y[i]$	<i>sum</i>
				0.00
0	0.30	0.50	0.15	0.15
1	0.60	0.10	0.06	0.21

The simplicity of coding such computations makes the use of arrays the natural choice for all kinds of applications.

Trace of dot product computation

THE TABLE ON THE FACING PAGE has many examples of array-processing code, and we will consider even more examples later in the book, because arrays play a central role in processing data in many applications. Before considering more sophisticated examples, we describe a number of important characteristics of programming with arrays.

Zero-based indexing. The first element of an array $a[]$ is $a[0]$, the second element is $a[1]$, and so forth. It might seem more natural to you to refer to the first element as $a[1]$, the second element as $a[2]$, and so forth, but starting the indexing with 0 has some advantages and has emerged as the convention used in most modern programming languages. Misunderstanding this convention often leads to *off-by one-errors* that are notoriously difficult to avoid and debug, so be careful!

Array length. Once you create an array in Java, its length is fixed. One reason that you need to explicitly create arrays at run time is that the Java compiler cannot always know how much space to reserve for the array at compile time (because its length may not be known until run time). You do not need to explicitly allocate memory for variables of type `int` or `double` because their size is fixed, and known at compile time. You can use the code `a.length` to refer to the length of an array $a[]$. Note that the last element of an array $a[]$ is always $a[a.length-1]$. For convenience, we often keep the array length in an integer variable n .

<i>create an array with random values</i>	<pre>double[] a = new double[n]; for (int i = 0; i < n; i++) a[i] = Math.random();</pre>
<i>print the array values, one per line</i>	<pre>for (int i = 0; i < n; i++) System.out.println(a[i]);</pre>
<i>find the maximum of the array values</i>	<pre>double max = Double.NEGATIVE_INFINITY; for (int i = 0; i < n; i++) if (a[i] > max) max = a[i];</pre>
<i>compute the average of the array values</i>	<pre>double sum = 0.0; for (int i = 0; i < n; i++) sum += a[i]; double average = sum / n;</pre>
<i>reverse the values within an array</i>	<pre>for (int i = 0; i < n/2; i++) { double temp = a[i]; a[i] = a[n-1-i]; a[n-1-i] = temp; }</pre>
<i>copy a sequence of values to another array</i>	<pre>double[] b = new double[n]; for (int i = 0; i < n; i++) b[i] = a[i];</pre>

Typical array-processing code (for an array `a[]` of `n` `double` values)

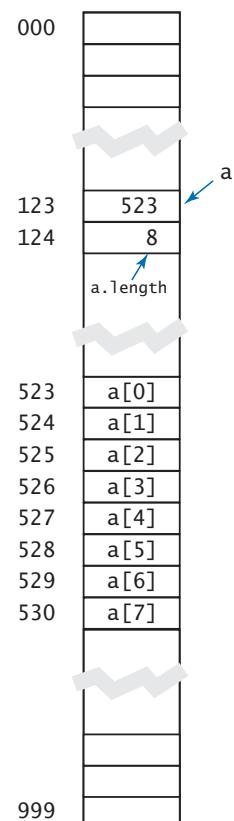
Default array initialization. For economy in code, we often take advantage of Java's *default array initialization* to declare, create, and initialize an array in a single statement. For example, the following statement is equivalent to the code at the top of page 91:

```
double[] a = new double[n];
```

The code to the left of the equals sign constitutes the declaration; the code to the right constitutes the creation. The `for` loop is unnecessary in this case because Java automatically initializes array elements of any primitive type to zero (for numeric types) or `false` (for the type `boolean`). Java automatically initializes array elements of type `String` (and other nonprimitive types) to `null`, a special value that you will learn about in CHAPTER 3.

Memory representation. Arrays are fundamental data structures in that they have a direct correspondence with memory systems on virtually all computers. The elements of an array are stored consecutively in memory, so that it is easy to quickly access any array value. Indeed, we can view memory itself as a giant array. On modern computers, memory is implemented in hardware as a sequence of memory locations, each of which can be quickly accessed with an appropriate index. When referring to computer memory, we normally refer to a location's index as its *address*. It is convenient to think of the name of the array—say, `a`—as storing the memory address of the first element of the array `a[0]`. For the purposes of illustration, suppose that the computer's memory is organized as 1,000 values, with addresses from 000 to 999. (This simplified model bypasses the fact that array elements can occupy differing amounts of memory depending on their type, but you can ignore such details for the moment.) Now, suppose that an array of eight elements is stored in memory locations 523 through 530. In such a situation, Java would store the memory address (index) of the first array element somewhere else in memory, along with the array length. We refer to the address as a *pointer* and think of it as *pointing to* the referenced memory location. When we specify `a[i]`, the compiler generates code that accesses the desired value by adding the index `i` to the memory address of the array `a[]`. For example, the Java code `a[4]` would generate machine code that finds the value at memory location $523 + 4 = 527$. Accessing element `i` of an array is an efficient operation because it simply requires adding two integers and then referencing memory—just two elementary operations.

Memory allocation. When you use the keyword `new` to create an array, Java reserves sufficient space in memory to store the specified number of elements. This process is called *memory allocation*. The same process is required for all variables that you use in a program (but you do not use the keyword `new` with variables of primitive types because Java knows how much memory to allocate). We call attention to it now because it is your responsibility to create an array before accessing any of its elements. If you fail to adhere to this rule, you will get an *uninitialized variable* error at compile time.



Memory representation

Bounds checking. As already indicated, you must be careful when programming with arrays. It is your responsibility to use valid indices when referring to an array element. If you have created an array of length n and use an index whose value is less than 0 or greater than $n-1$, your program will terminate with an `ArrayIndexOutOfBoundsException` at run time. (In many programming languages, such *buffer overflow* conditions are not checked by the system. Such unchecked errors can and do lead to debugging nightmares, but it is also not uncommon for such an error to go unnoticed and remain in a finished program. You might be surprised to know that such a mistake can be exploited by a hacker to take control of a system, even your personal computer, to spread viruses, steal personal information, or wreak other malicious havoc.) The error messages provided by Java may seem annoying to you at first, but they are small price to pay to have a more secure program.

Setting array values at compile time. When we have a small number of values that we want to keep in array, we can declare, create, and initialize the array by listing the values between curly braces, separated by commas. For example, we might use the following code in a program that processes playing cards:

```
String[] SUITS = { "Clubs", "Diamonds", "Hearts", "Spades" };

String[] RANKS =
{
    "2", "3", "4", "5", "6", "7", "8", "9", "10",
    "Jack", "Queen", "King", "Ace"
};
```

Now, we can use the two arrays to print a random card name, such as Queen of Clubs, as follows:

```
int i = (int) (Math.random() * RANKS.length);
int j = (int) (Math.random() * SUITS.length);
System.out.println(RANKS[i] + " of " + SUITS[j]);
```

This code uses the idiom introduced in SECTION 1.2 to generate random indices and then uses the indices to pick strings out of the two arrays. Whenever the values of all array elements are known (and the length of the array is not too large), it makes sense to use this method of initializing the array—just put all the values in curly braces on the right-hand side of the equals sign in the array declaration. Doing so implies array creation, so the `new` keyword is not needed.

Setting array values at run time. A more typical situation is when we wish to compute the values to be stored in an array. In this case, we can use an array name with indices in the same way we use a variable names on the left-hand side of an assignment statement. For example, we might use the following code to initialize an array of length 52 that represents a deck of playing cards, using the two arrays just defined:

```
String[] deck = new String[RANKS.length * SUITS.length];
for (int i = 0; i < RANKS.length; i++)
    for (int j = 0; j < SUITS.length; j++)
        deck[SUITS.length*i + j] = RANKS[i] + " of " + SUITS[j];
```

After this code has been executed, if you were to print the contents of `deck[]` in order from `deck[0]` through `deck[51]`, you would get

```
2 of Clubs
2 of Diamonds
2 of Hearts
2 of Spades
3 of Clubs
3 of Diamonds
...
Ace of Hearts
Ace of Spades
```

Exchanging two values in an array. Frequently, we wish to exchange the values of two elements in an array. Continuing our example with playing cards, the following code exchanges the cards at indices `i` and `j` using the same idiom that we traced as our first example of the use of assignment statements in SECTION 1.2:

```
String temp = deck[i];
deck[i] = deck[j];
deck[j] = temp;
```

For example, if we were to use this code with `i` equal to 1 and `j` equal to 4 in the `deck[]` array of the previous example, it would leave `3 of Clubs` in `deck[1]` and `2 of Diamonds` in `deck[4]`. You can also verify that the code leaves the array unchanged when `i` and `j` are equal. So, when we use this code, we are assured that we are perhaps changing the *order* of the values in the array but not the *set* of values in the array.

Shuffling an array. The following code shuffles the values in our deck of cards:

```
int n = deck.length;
for (int i = 0; i < n; i++)
{
    int r = i + (int) (Math.random() * (n-i));
    String temp = deck[i];
    deck[i] = deck[r];
    deck[r] = temp;
}
```

Proceeding from left to right, we pick a random card from `deck[i]` through `deck[n-1]` (each card equally likely) and exchange it with `deck[i]`. This code is more sophisticated than it might seem: First, we ensure that the cards in the deck after the shuffle are the same as the cards in the deck before the shuffle by using the exchange idiom. Second, we ensure that the shuffle is random by choosing uniformly from the cards not yet chosen.

Sampling without replacement. In many situations, we want to draw a random sample from a set such that each member of the set appears at most once in the sample. Drawing numbered ping-pong balls from a basket for a lottery is an example of this kind of sample, as is dealing a hand from a deck of cards. Sample (PROGRAM 1.4.1) illustrates how to sample, using the basic operation underlying shuffling. It takes two command-line arguments `m` and `n` and creates a *permutation* of length `n` (a rearrangement of the integers from 0 to `n-1`) whose first `m` elements

<i>i</i>	<i>r</i>	<i>perm[]</i>															
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	9	9	1	2	3	4	5	6	7	8	0	10	11	12	13	14	15
1	5	9	5	2	3	4	1	6	7	8	0	10	11	12	13	14	15
2	13	9	5	13	3	4	1	6	7	8	0	10	11	12	2	14	15
3	5	9	5	13	1	4	3	6	7	8	0	10	11	12	2	14	15
4	11	9	5	13	1	11	3	6	7	8	0	10	4	12	2	14	15
5	8	9	5	13	1	11	8	6	7	3	0	10	4	12	2	14	15
		9	5	13	1	11	8	6	7	3	0	10	4	12	2	14	15

Trace of java Sample 6 16

Program 1.4.1 Sampling without replacement

```

public class Sample
{
    public static void main(String[] args)
    { // Print a random sample of m integers
        // from 0 ... n-1 (no duplicates).
        int m = Integer.parseInt(args[0]);
        int n = Integer.parseInt(args[1]);
        int[] perm = new int[n];
        // Initialize perm[].
        for (int j = 0; j < n; j++)
            perm[j] = j;
        // Take sample.
        for (int i = 0; i < m; i++)
        { // Exchange perm[i] with a random element to its right.
            int r = i + (int) (Math.random() * (n-i));
            int t = perm[r];
            perm[r] = perm[i];
            perm[i] = t;
        }
        // Print sample.
        for (int i = 0; i < m; i++)
            System.out.print(perm[i] + " ");
        System.out.println();
    }
}

```

m | sample size
 n | range
 $\text{perm}[]$ | permutation of 0 to $n-1$

This program takes two command-line arguments m and n and produces a sample of m of the integers from 0 to $n-1$. This process is useful not just in state and local lotteries, but in scientific applications of all sorts. If the first argument is equal to the second, the result is a random permutation of the integers from 0 to $n-1$. If the first argument is greater than the second, the program will terminate with an `ArrayOutOfBoundsException`.

```

% java Sample 6 16
9 5 13 1 11 8
% java Sample 10 1000
656 488 298 534 811 97 813 156 424 109
% java Sample 20 20
6 12 9 8 13 19 0 2 4 5 18 1 14 16 17 3 7 11 10 15

```

comprise a random sample. The accompanying trace of the contents of the `perm[]` array at the end of each iteration of the main loop (for a run where the values of `m` and `n` are 6 and 16, respectively) illustrates the process.

If the values of `r` are chosen such that each value in the given range is equally likely, then the elements `perm[0]` through `perm[m-1]` are a uniformly random sample at the end of the process (even though some values might move multiple times) because each element in the sample is assigned a value uniformly at random from those values not yet sampled. One important reason to explicitly compute the permutation is that we can use it to print a random sample of *any* array by using the elements of the permutation as indices into the array. Doing so is often an attractive alternative to actually rearranging the array because it may need to be in order for some other reason (for instance, a company might wish to draw a random sample from a list of customers that is kept in alphabetical order).

To see how this trick works, suppose that we wish to draw a random poker hand from our `deck[]` array, constructed as just described. We use the code in `Sample` with `n = 52` and `m = 5` and replace `perm[i]` with `deck[perm[i]]` in the `System.out.print()` statement (and change it to `println()`), resulting in output such as the following:

```
3 of Clubs
Jack of Hearts
6 of Spades
Ace of Clubs
10 of Diamonds
```

Sampling like this is widely used as the basis for statistical studies in polling, scientific research, and many other applications, whenever we want to draw conclusions about a large population by analyzing a small random sample.

Precomputed values. One simple application of arrays is to save values that you have computed for later use. As an example, suppose that you are writing a program that performs calculations using small values of the harmonic numbers (see PROGRAM 1.3.5). An efficient approach is to save the values in an array, as follows:

```
double[] harmonic = new double[n];
for (int i = 1; i < n; i++)
    harmonic[i] = harmonic[i-1] + 1.0/i;
```

Then you can just use the code `harmonic[i]` to refer to the i th harmonic number. Precomputing values in this way is an example of a *space–time tradeoff*: by investing in space (to save the values), we save time (since we do not need to recompute them). This method is not effective if we need values for huge n , but it is very effective if we need values for small n many different times.

Simplifying repetitive code. As an example of another simple application of arrays, consider the following code fragment, which prints the name of a month given its number (1 for January, 2 for February, and so forth):

```
if      (m == 1) System.out.println("Jan");
else if (m == 2) System.out.println("Feb");
else if (m == 3) System.out.println("Mar");
else if (m == 4) System.out.println("Apr");
else if (m == 5) System.out.println("May");
else if (m == 6) System.out.println("Jun");
else if (m == 7) System.out.println("Jul");
else if (m == 8) System.out.println("Aug");
else if (m == 9) System.out.println("Sep");
else if (m == 10) System.out.println("Oct");
else if (m == 11) System.out.println("Nov");
else if (m == 12) System.out.println("Dec");
```

We could also use a `switch` statement, but a much more compact alternative is to use an array of strings, consisting of the names of each month:

```
String[] MONTHS =
{
    "", "Jan", "Feb", "Mar", "Apr", "May", "Jun",
    "Jul", "Aug", "Sep", "Oct", "Nov", "Dec"
};
System.out.println(MONTHS[m]);
```

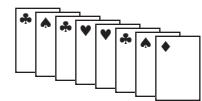
This technique would be especially useful if you needed to access the name of a month by its number in several different places in your program. Note that we intentionally waste one slot in the array (element 0) to make `MONTHS[1]` correspond to January, as required.

WITH THESE BASIC DEFINITIONS AND EXAMPLES out of the way, we can now consider two applications that both address interesting classical problems and illustrate the fundamental importance of arrays in efficient computation. In both cases, the idea of using an expression to index into an array plays a central role and enables a computation that would not otherwise be feasible.

Coupon collector Suppose that you have a deck of cards and you turn up cards uniformly at random (with replacement) one by one. How many cards do you need to turn up before you have seen one of each suit? How many cards do you need to turn up before seeing one of each value? These are examples of the famous *coupon collector* problem. In general, suppose that a trading card company issues trading cards with n different possible cards: how many do you have to collect before you have all n possibilities, assuming that each possibility is equally likely for each card that you collect?

Coupon collecting is no toy problem. For example, scientists often want to know whether a sequence that arises in nature has the same characteristics as a random sequence. If so, that fact might be of interest; if not, further investigation may be warranted to look for patterns that might be of importance. For example, such tests are used by scientists to decide which parts of genomes are worth studying. One effective test for whether a sequence is truly random is the *coupon collector test*: compare the number of elements that need to be examined before all values are found against the corresponding number for a uniformly random sequence. `CouponCollector` (PROGRAM 1.4.2) is an example program that simulates this process and illustrates the utility of arrays. It takes a command-line argument n and generates a sequence of random integers between 0 and $n-1$ using the code (`int`) (`Math.random() * n`)—see PROGRAM 1.2.5. Each integer represents a card: for each card, we want to know if we have seen that card before. To maintain that knowledge, we use an array `isCollected[]`, which uses the card as an index; `isCollected[i]` is `true` if we have seen a card i and `false` if we have not. When we get a new card that is represented by the integer r , we check whether we have seen it before by accessing `isCollected[r]`. The computation consists of keeping count of the number of distinct cards seen and the number of cards generated, and printing the latter when the former reaches n .

As usual, the best way to understand a program is to consider a trace of the values of its variables for a typical run. It is easy to add code to `CouponCollector` that produces a trace that gives the values of the variables at the



Coupon collection

<i>r</i>	<i>isCollected[]</i>						<i>distinct</i>	<i>count</i>
	0	1	2	3	4	5		
	F	F	F	F	F	F	0	0
2	F	F	T	F	F	F	1	1
0	T	F	T	F	F	F	2	2
4	T	F	T	F	T	F	3	3
0	T	F	T	F	T	F	3	4
1	T	T	T	F	T	F	4	5
2	T	T	T	F	T	F	4	6
5	T	T	T	F	T	T	5	7
0	T	T	T	F	T	T	5	8
1	T	T	T	F	T	T	5	9
3	T	T	T	T	T	T	6	10

*Trace for a typical run of
java CouponCollector 6*

Program 1.4.2 Coupon collector simulation

```

public class CouponCollector
{
    public static void main(String[] args)
    {
        // Generate random values in [0..n) until finding each one.
        int n = Integer.parseInt(args[0]);
        boolean[] isCollected = new boolean[n];
        int count = 0;
        int distinct = 0;

        while (distinct < n)
        {
            // Generate another coupon.
            int r = (int) (Math.random() * n);
            count++;
            if (!isCollected[r])
            {
                distinct++;                                n
                isCollected[r] = true;                      isCollected[i]
            }
        } // n distinct coupons found.                  count
        System.out.println(count);                     distinct
    }
}

```

coupon values
(0 to n-1)
has coupon *i*
been collected?
coupons
distinct coupons
random coupon

This program takes an integer command-line argument *n* and simulates coupon collection by generating random numbers between 0 and *n*-1 until getting every possible value.

```

% java CouponCollector 1000
6583
% java CouponCollector 1000
6477
% java CouponCollector 1000000
12782673

```

end of the `while` loop. In the accompanying figure, we use `F` for the value `false` and `T` for the value `true` to make the trace easier to follow. Tracing programs that use large arrays can be a challenge: when you have an array of length n in your program, it represents n variables, so you have to list them all. Tracing programs that use `Math.random()` also can be a challenge because you get a different trace every time you run the program. Accordingly, we check relationships among variables carefully. Here, note that `distinct` always is equal to the number of `true` values in `isCollected[]`.

Without arrays, we could not contemplate simulating the coupon collector process for huge n ; with arrays, it is easy to do so. We will see many examples of such processes throughout the book.

Sieve of Eratosthenes Prime numbers play an important role in mathematics and computation, including cryptography. A *prime number* is an integer greater than 1 whose only positive divisors are 1 and itself. The prime counting function $\pi(n)$ is the number of primes less than or equal to n . For example, $\pi(25) = 9$ since the first nine primes are 2, 3, 5, 7, 11, 13, 17, 19, and 23. This function plays a central role in number theory.

One approach to counting primes is to use a program like `Factors` (PROGRAM 1.3.9). Specifically, we could modify the code in `Factors` to set a boolean variable to `true` if a given number is prime and `false` otherwise (instead of printing out factors), then enclose that code in a loop that increments a counter for each prime number. This approach is effective for small n , but becomes too slow as n grows.

`PrimeSieve` (PROGRAM 1.4.3) takes a command-line integer n and computes the prime count using a technique known as the *Sieve of Eratosthenes*. The program uses a boolean array `isPrime[]` to record which integers are prime. The goal is to set `isPrime[i]` to `true` if the integer i is prime, and to `false` otherwise. The sieve works as follows: Initially, set all array elements to `true`, indicating that no factors of any integer have yet been found. Then, repeat the following steps as long as $i \leq n/i$:

- Find the next smallest integer i for which no factors have been found.
- Leave `isPrime[i]` as `true` since i has no smaller factors.
- Set the `isPrime[]` elements for all multiples of i to `false`.

When the nested `for` loop ends, `isPrime[i]` is `true` if and only if integer i is prime. With one more pass through the array, we can count the number of primes less than or equal to n .

Program 1.4.3 Sieve of Eratosthenes

```

public class PrimeSieve
{
    public static void main(String[] args)
    { // Print the number of primes <= n.
        int n = Integer.parseInt(args[0]);
        boolean[] isPrime = new boolean[n+1];
        for (int i = 2; i <= n; i++)
            isPrime[i] = true;

        for (int i = 2; i <= n/i; i++)
        { if (isPrime[i])
            { // Mark multiples of i as nonprime.
                for (int j = i; j <= n/i; j++)
                    isPrime[i * j] = false;
            }
        }

        // Count the primes.
        int primes = 0;
        for (int i = 2; i <= n; i++)
            if (isPrime[i]) primes++;
        System.out.println(primes);
    }
}

```

n	<i>argument</i>
isPrime[i]	<i>is i prime?</i>
primes	<i>prime counter</i>

This program takes an integer command-line argument n and computes the number of primes less than or equal to n . To do so, it computes a boolean array with $\text{isPrime}[i]$ set to `true` if i is prime, and to `false` otherwise. First, it sets to `true` all array elements to indicate that no numbers are initially known to be nonprime. Then it sets to `false` array elements corresponding to indices that are known to be nonprime (multiples of known primes). If $\text{a}[i]$ is still `true` after all multiples of smaller primes have been set to `false`, then we know i to be prime. The termination test in the second `for` loop is $i \leq n/i$ instead of the naive $i \leq n$ because any number with no factor less than n/i has no factor greater than n/i , so we do not have to look for such factors. This improvement makes it possible to run the program for large n .

```

% java PrimeSieve 25
9
% java PrimeSieve 100
25
% java PrimeSieve 1000000000
50847534

```

As usual, it is easy to add code to print a trace. For programs such as `PrimeSieve`, you have to be a bit careful—it contains a nested `for-if-for`, so you have to pay attention to the curly braces to put the print code in the correct place. Note that we stop when $i > n/i$, as we did for `Factors`.

<i>i</i>	<i>isPrime[]</i>																							
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	
3	T	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	
5	T	T	F	T	F	T	F	F	F	T	F	T	F	F	T	F	T	F	F	T	F	T	F	
	T	T	F	T	F	T	F	F	F	T	F	T	F	F	T	F	T	F	F	F	T	F	F	

Trace of java PrimeSieve 25

With `PrimeSieve`, we can compute $\pi(n)$ for large n , limited primarily by the maximum array length allowed by Java. This is another example of a space–time tradeoff. Programs like `PrimeSieve` play an important role in helping mathematicians to develop the theory of numbers, which has many important applications.

Two-dimensional arrays In many applications, a convenient way to store information is to use a table of numbers organized in a rectangle and refer to *rows* and *columns* in the table. For example, a teacher might need to maintain a table with rows corresponding to students and columns corresponding to exams, a scientist might need to maintain a table of experimental data with rows corresponding to experiments and columns corresponding to various outcomes, or a programmer might want to prepare an image for display by setting a table of pixels to various grayscale values or colors.

The mathematical abstraction corresponding to such tables is a *matrix*; the corresponding Java construct is a *two-dimensional array*. You are likely to have already encountered many applications of matrices and two-dimensional arrays, and you will certainly encounter many others in science, engineering, and computing applications, as we will demonstrate with examples throughout this book. As with vectors and one-dimensional arrays, many of the most important applications involve processing large amounts of data, and we defer considering those applications until we introduce input and output, in SECTION 1.5.

Extending Java array constructs to handle two-dimensional arrays is straightforward. To refer to the element in row *i* and column *j* of a two-dimensional array `a[][]`, we use the notation `a[i][j]`; to declare a two-dimensional array, we add another pair of square brackets; and to create the array, we specify the number of rows followed by the number of columns after the type name (both within square brackets), as follows:

```
double[][] a = new double[m][n];
```

We refer to such an array as an *m-by-n* array. By convention, the first dimension is the number of rows and the second is the number of columns. As with one-dimensional arrays, Java initializes all elements in arrays of numbers to zero and in boolean arrays to `false`.

Default initialization. Default initialization of two-dimensional arrays is useful because it masks more code than for one-dimensional arrays. The following code is equivalent to the single-line create-and-initialize idiom that we just considered:

99	85	98
98	57	78
92	77	76
94	32	11
99	34	22
90	46	54
76	59	88
92	66	89
97	71	24
89	29	38

↑
row 1 → ↓
 column 2

Anatomy of a
two-dimensional array

```

double[][] a;
a = new double[m][n];
for (int i = 0; i < m; i++)
{ // Initialize the ith row.
    for (int j = 0; j < n; j++)
        a[i][j] = 0.0;
}

```

This code is superfluous when initializing the elements of a two-dimensional array to zero, but the nested `for` loops are needed to initialize the elements to some other value(s). As you will see, this code is a model for the code that we use to access or modify each element of a two-dimensional array.

Output. We use nested `for` loops for many two-dimensional array-processing operations. For example, to print an m -by- n array in the tabular format, we can use the following code:

```

for (int i = 0; i < m; i++)
{ // Print the ith row.
    for (int j = 0; j < n; j++)
        System.out.print(a[i][j] + " ");
    System.out.println();
}

```

If desired, we could add code to embellish the output with row and column indices (see EXERCISE 1.4.6). Java programmers typically tabulate two-dimensional arrays with row indices running top to bottom from 0 and column indices running left to right from 0.

Memory representation. Java represents a two-dimensional array as an array of arrays. That is, a two-dimensional array with m rows and n columns is actually an array of length m , each element of which is a one-dimensional array of length n . In a two-dimensional Java array `a[][]`, you can use the code `a[i]` to refer to row i (which is a one-dimensional array), but there is no corresponding way to refer to column j .

a[0][0]	a[0][1]	a[0][2]
a[1][0]	a[1][1]	a[1][2]
a[2][0]	a[2][1]	a[2][2]
a[3][0]	a[3][1]	a[3][2]
a[4][0]	a[4][1]	a[4][2]
a[5][0]	a[5][1]	a[5][2]
a[6][0]	a[6][1]	a[6][2]
a[7][0]	a[7][1]	a[7][2]
a[8][0]	a[8][1]	a[8][2]
a[9][0]	a[9][1]	a[9][2]

A 10-by-3 array

Setting values at compile time. The Java method for initializing an array of values at compile time follows immediately from the representation. A two-dimensional array is an array of rows, each row initialized as a one-dimensional array. To initialize a two-dimensional array, we enclose in curly braces a list of terms to initialize the rows, separated by commas. Each term in the list is itself a list: the values for the array elements in the row, enclosed in curly braces and separated by commas.

Spreadsheets. One familiar use of arrays is a *spreadsheet* for maintaining a table of numbers. For example, a teacher with m students and n test grades for each student might maintain an $(m+1)$ -by- $(n+1)$ array, reserving the last column for each student's average grade and the last row for the average test grades. Even though we typically do such computations within specialized applications, it is worthwhile to study the underlying code as an introduction to array processing. To compute the average grade for each student (average values for each row), sum the elements for each row and divide by n . The row-by-row order in which this code processes the matrix elements is known as *row-major* order. Similarly, to compute the average test grade (average values for each column), sum the elements for each column and divide by m . The column-by-column order in which this code processes the matrix elements is known as *column-major* order.

				row averages in column n
				$\frac{92 + 77 + 74}{3}$
				81.0
$m = 10$	99.0	85.0	98.0	94.0
	98.0	57.0	79.0	78.0
	92.0	77.0	74.0	81.0
	94.0	62.0	81.0	79.0
	99.0	94.0	92.0	95.0
	80.0	76.5	67.0	74.5
	76.0	58.5	90.5	75.0
	92.0	66.0	91.0	83.0
	97.0	70.5	66.5	78.0
	89.0	89.5	81.0	86.5
				column averages in row m
				$\frac{85 + 57 + \dots + 89.5}{10}$

```
double[][] a =
{
    { 99.0, 85.0, 98.0, 0.0 },
    { 98.0, 57.0, 79.0, 0.0 },
    { 92.0, 77.0, 74.0, 0.0 },
    { 94.0, 62.0, 81.0, 0.0 },
    { 99.0, 94.0, 92.0, 0.0 },
    { 80.0, 76.5, 67.0, 0.0 },
    { 76.0, 58.5, 90.5, 0.0 },
    { 92.0, 66.0, 91.0, 0.0 },
    { 97.0, 70.5, 66.5, 0.0 },
    { 89.0, 89.5, 81.0, 0.0 },
    { 0.0, 0.0, 0.0, 0.0 }
};
```

*Compile-time initialization of a
of an 11-by-4 double array*

Compute row averages

```
for (int i = 0; i < m; i++)
{ // Compute average for row i
    double sum = 0.0;
    for (int j = 0; j < n; j++)
        sum += a[i][j];
    a[i][n] = sum / n;
}
```

Compute column averages

```
for (int j = 0; j < n; j++)
{ // Compute average for column j
    double sum = 0.0;
    for (int i = 0; i < m; i++)
        sum += a[i][j];
    a[m][j] = sum / m;
}
```

Typical spreadsheet calculations

Matrix operations. Typical applications in science and engineering involve representing matrices as two-dimensional arrays and then implementing various mathematical operations with matrix operands. Again, even though such processing is often done within specialized applications, it is worthwhile for you to understand the underlying computation. For example, you can *add* two n -by- n matrices as follows:

```
double[][] c = new double[n][n];
for (int i = 0; i < n; i++)
    for (int j = 0; j < n; j++)
        c[i][j] = a[i][j] + b[i][j];
```

Similarly, you can *multiply* two matrices. You may have learned matrix multiplication, but if you do not recall or are not familiar with it, the Java code below for multiplying two square matrices is essentially the same as the mathematical definition. Each element $c[i][j]$ in the product of $a[] []$ and $b[] []$ is computed by taking the dot product of row i of $a[] []$ with column j of $b[] []$.

```
double[][] c = new double[n][n];
for (int i = 0; i < n; i++)
{
    for (int j = 0; j < n; j++)
    {
        // Dot product of row i and column j.
        for (int k = 0; k < n; k++)
            c[i][j] += a[i][k]*b[k][j];
    }
}
```

a[] []	.70 .20 .10 .30 .60 .10 .50 .10 .40	a[1][2]
b[] []	.20 .30 .50 .10 .20 .10 .10 .30 .40	b[1][2]
c[] []	.90 .50 .60 .40 .80 .20 .60 .40 .80	c[1][2]

Matrix addition

a[] []	b[] []	column 2	c[] []
.70 .20 .10 .30 .60 .10 .50 .10 .40	.20 .30 .50 .10 .20 .10 .10 .30 .40		.17 .28 .41 .13 .24 .25 .15 .29 .42

$$\begin{aligned} c[1][2] &= 0.3 * 0.5 \\ &+ 0.6 * 0.1 \\ &+ 0.1 * 0.4 \\ &= 0.25 \end{aligned}$$

Matrix multiplication

Special cases of matrix multiplication. Two special cases of matrix multiplication are important. These special cases occur when one of the dimensions of one of the matrices is 1, so it may be viewed as a vector. We have *matrix–vector multiplication*, where we multiply an m -by- n matrix by a *column vector* (an n -by-1 matrix) to get an m -by-1 column vector result (each element in the result is the dot product of the corresponding row in the matrix with the operand vector). The second case is *vector–matrix multiplication*, where we multiply a *row vector* (a 1-by- m matrix) by an m -by- n matrix to get a 1-by- n row vector result (each element in the result is the dot product of the operand vector with the corresponding column in the matrix).

These operations provide a succinct way to express numerous matrix calculations. For example, the row-average computation for such a spreadsheet with m rows and n columns is equivalent to a matrix–vector multiplication where the column vector has n elements all equal to $1/n$. Similarly, the column-average computation in such a spreadsheet is equivalent to a vector–matrix multiplication where the row vector has m elements all equal to $1/m$. We return to vector–matrix multiplication in the context of an important application at the end of this chapter.

Matrix–vector multiplication $a[]\cdot x[] = b[]$

```
for (int i = 0; i < m; i++)
{ // Dot product of row i and x[].
    for (int j = 0; j < n; j++)
        b[i] += a[i][j]*x[j];
}
```

$a[]\cdot x[]$	$b[]$
99 85 98	94
98 57 78	77
92 77 76	81
94 32 11	45
99 34 22	51
90 46 54	63
76 59 88	74
92 66 89	82
97 71 24	64
89 29 38	52

\leftarrow *row averages*

Vector–matrix multiplication $y[]\cdot a[] = c[]$

```
for (int j = 0; j < n; j++)
{ // Dot product of y[] and column j.
    for (int i = 0; i < m; i++)
        c[j] += y[i]*a[i][j];
}
```

$y[] = [.1 .1 .1 .1 .1 .1 .1 .1 .1]$

$a[]\cdot y[]$
99 85 98
98 57 78
92 77 76
94 32 11
99 34 22
90 46 54
76 59 88
92 66 89
97 71 24
89 29 38

\leftarrow *column averages*

Matrix–vector and vector–matrix multiplication

Ragged arrays. There is actually no requirement that all rows in a two-dimensional array have the same length—an array with rows of nonuniform length is known as a *ragged array* (see EXERCISE 1.4.34 for an example application). The possibility of ragged arrays creates the need for more care in crafting array-processing code. For example, this code prints the contents of a ragged array:

```
for (int i = 0; i < a.length; i++)
{
    for (int j = 0; j < a[i].length; j++)
        System.out.print(a[i][j] + " ");
    System.out.println();
}
```

This code tests your understanding of Java arrays, so you should take the time to study it. In this book, we normally use square or rectangular arrays, whose dimension are given by the variable `m` or `n`. Code that uses `a[i].length` in this way is a clear signal to you that an array is ragged.

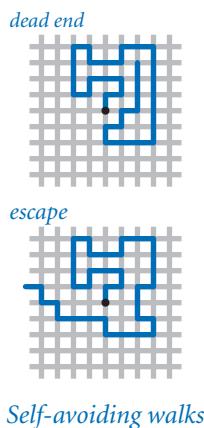
Multidimensional arrays. The same notation extends to allow us to write code using arrays that have any number of dimensions. For instance, we can declare and initialize a three-dimensional array with the code

```
double[][][] a = new double[n][n][n];
```

and then refer to an element with code like `a[i][j][k]`, and so forth.

TWO-DIMENSIONAL ARRAYS PROVIDE A NATURAL REPRESENTATION for matrices, which are omnipresent in science, mathematics, and engineering. They also provide a natural way to organize large amounts of data—a key component in spreadsheets and many other computing applications. Through Cartesian coordinates, two- and three-dimensional arrays also provide the basis for models of the physical world. We consider their use in all three arenas throughout this book.

Example: self-avoiding random walks Suppose that you leave your dog in the middle of a large city whose streets form a familiar grid pattern. We assume that there are n north–south streets and n east–west streets all regularly spaced and fully intersecting in a pattern known as a *lattice*. Trying to escape the city, the dog makes a random choice of which way to go at each intersection, but knows by scent to avoid visiting any place previously visited. But it is possible for the dog to get stuck in a dead end where there is no choice but to revisit some intersection. What is the chance that this will happen? This amusing problem is a simple example of a famous model known as the *self-avoiding random walk*, which has important scientific applications in the study of polymers and in statistical mechanics, among many others. For example, you can see that this process models a chain of material growing a bit at a time, until no growth is possible. To better understand such processes, scientists seek to understand the properties of self-avoiding walks.



Self-avoiding walks

The dog’s escape probability is certainly dependent on the size of the city. In a tiny 5-by-5 city, it is easy to convince yourself that the dog is certain to escape. But what are the chances of escape when the city is large? We are also interested in other parameters. For example, how long is the dog’s path, on the average? How often does the dog come within one block of escaping? These sorts of properties are important in the various applications just mentioned.

`SelfAvoidingWalk` (PROGRAM 1.4.4) is a simulation of this situation that uses a two-dimensional boolean array, where each element represents an intersection. The value `true` indicates that the dog has visited the intersection; `false` indicates that the dog has not visited the intersection. The path starts in the center and takes random steps to places not yet visited until getting stuck or escaping at a boundary. For simplicity, the code is written so that if a random choice is made to go to a spot that has already been visited, it takes no action, trusting that some subsequent random choice will find a new place (which is assured because the code explicitly tests for a dead end and terminates the loop in that case).

Note that the code depends on Java initializing all of the array elements to `false` for each experiment. It also exhibits an important programming technique where we code the loop exit test in the `while` statement as a *guard* against an illegal statement in the body of the loop. In this case, the `while` loop-continuation condition serves as a guard against an out-of-bounds array access within the loop. This corresponds to checking whether the dog has escaped. Within the loop, a successful dead-end test results in a `break` out of the loop.

Program 1.4.4 Self-avoiding random walks

```

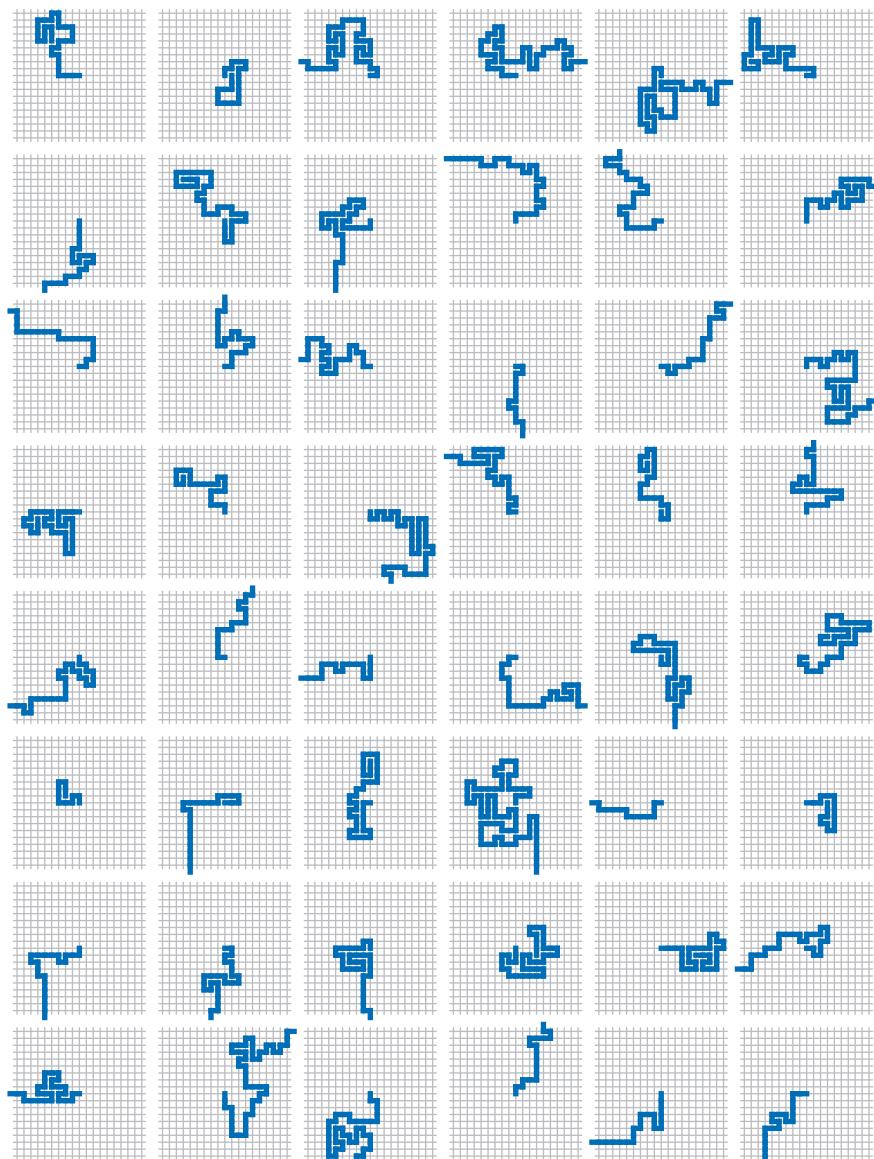
public class SelfAvoidingWalk
{
    public static void main(String[] args)
    {
        // Do trials random self-avoiding
        // walks in an n-by-n lattice.
        int n = Integer.parseInt(args[0]);
        int trials = Integer.parseInt(args[1]);
        int deadEnds = 0;
        for (int t = 0; t < trials; t++)
        {
            boolean[][] a = new boolean[n][n];
            int x = n/2, y = n/2;
            while (x > 0 && x < n-1 && y > 0 && y < n-1)
            {
                // Check for dead end and make a random move.
                a[x][y] = true;
                if (a[x-1][y] && a[x+1][y] && a[x][y-1] && a[x][y+1])
                { deadEnds++; break; }
                double r = Math.random();
                if (r < 0.25) { if (!a[x+1][y]) x++; }
                else if (r < 0.50) { if (!a[x-1][y]) x--; }
                else if (r < 0.75) { if (!a[x][y+1]) y++; }
                else if (r < 1.00) { if (!a[x][y-1]) y--; }
            }
            System.out.println(100*deadEnds/trials + "% dead ends");
        }
    }
}

```

<code>n</code> <code>trials</code> <code>deadEnds</code> <code>a[][]</code> <code>x, y</code> <code>r</code>	<i>lattice size</i> <i># trials</i> <i># trials resulting in a dead end</i> <i>intersections visited</i> <i>current position</i> <i>random number in (0, 1)</i>
---	--

This program takes command-line arguments `n` and `trials` and computes `trials` self-avoiding walks in an `n`-by-`n` lattice. For each walk, it creates a boolean array, starts the walk in the center, and continues until either a dead end or a boundary is reached. The result of the computation is the percentage of dead ends. Increasing the number of experiments increases the precision.

<pre>% java SelfAvoidingWalk 5 100 0% dead ends</pre>	<pre>% java SelfAvoidingWalk 5 1000 0% dead ends</pre>
<pre>% java SelfAvoidingWalk 20 100 36% dead ends</pre>	<pre>% java SelfAvoidingWalk 20 1000 32% dead ends</pre>
<pre>% java SelfAvoidingWalk 40 100 80% dead ends</pre>	<pre>% java SelfAvoidingWalk 40 1000 70% dead ends</pre>
<pre>% java SelfAvoidingWalk 80 100 98% dead ends</pre>	<pre>% java SelfAvoidingWalk 80 1000 95% dead ends</pre>



Self-avoiding random walks in a 21-by-21 grid

As you can see from the sample runs on the facing page, the unfortunate truth is that your dog is nearly certain to get trapped in a dead end in a large city. If you are interested in learning more about self-avoiding walks, you can find several suggestions in the exercises. For example, the dog is virtually certain to escape in the three-dimensional version of the problem. While this is an intuitive result that is confirmed by our tests, the development of a mathematical model that explains the behavior of self-avoiding walks is a famous open problem; despite extensive research, no one knows a succinct mathematical expression for the escape probability, the average length of the path, or any other important parameter.

Summary Arrays are the fourth basic element (after assignments, conditionals, and loops) found in virtually every programming language, completing our coverage of basic Java constructs. As you have seen with the sample programs that we have presented, you can write programs that can solve all sorts of problems using just these constructs.

Arrays are prominent in many of the programs that we consider, and the basic operations that we have discussed here will serve you well in addressing many programming tasks. When you are not using arrays explicitly (and you are sure to do so frequently), you will be using them implicitly, because all computers have a memory that is conceptually equivalent to an array.

The fundamental ingredient that arrays add to our programs is a potentially huge increase in the size of a program’s *state*. The state of a program can be defined as the information you need to know to understand what a program is doing. In a program without arrays, if you know the values of the variables and which statement is the next to be executed, you can normally determine what the program will do next. When we trace a program, we are essentially tracking its state. When a program uses arrays, however, there can be too huge a number of values (each of which might be changed in each statement) for us to effectively track them all. This difference makes writing programs with arrays more of a challenge than writing programs without them.

Arrays directly represent vectors and matrices, so they are of direct use in computations associated with many basic problems in science and engineering. Arrays also provide a succinct notation for manipulating a potentially huge amount of data in a uniform way, so they play a critical role in any application that involves processing large amounts of data, as you will see throughout this book.

Q&A

Q. Some Java programmers use `int a[]` instead of `int[] a` to declare arrays. What's the difference?

A. In Java, both are legal and essentially equivalent. The former is how arrays are declared in C. The latter is the preferred style in Java since the type of the variable `int[]` more clearly indicates that it is an *array* of integers.

Q. Why do array indices start at 0 instead of 1?

A. This convention originated with machine-language programming, where the address of an array element would be computed by adding the index to the address of the beginning of an array. Starting indices at 1 would entail either a waste of space at the beginning of the array or a waste of time to subtract the 1.

Q. What happens if I use a negative integer to index an array?

A. The same thing as when you use an index that is too large. Whenever a program attempts to index an array with an index that is not between 0 and the array length minus 1, Java will issue an `ArrayIndexOutOfBoundsException`.

Q. Must the entries in an array initializer be literals?

A. No. The entries in an array initializer can be arbitrary expressions (of the specified type), even if their values are not known at compile time. For example, the following code fragment initializes a two-dimensional array using a command-line argument `theta`:

```
double theta = Double.parseDouble(args[0]);
double[][] rotation =
{
    { Math.cos(theta), -Math.sin(theta) },
    { Math.sin(theta), Math.cos(theta) },
};
```

Q. Is there a difference between an array of characters and a `String`?

A. Yes. For example, you can change the individual characters in a `char[]` but not in a `String`. We will consider strings in detail in SECTION 3.1.

Q. What happens when I compare two arrays with `(a == b)`?

A. The expression evaluates to `true` if and only if `a[]` and `b[]` refer to the same array (memory address), not if they store the same sequence of values. Unfortunately, this is rarely what you want. Instead, you can use a loop to compare the corresponding elements.

Q. What happens when I use an array in an assignment statement like `a = b`?

A. The assignment statement makes the variable `a` refer to the same array as `b`—it does not copy the values from the array `b` to the array `a`, as you might expect. For example, consider the following code fragment:

```
int[] a = { 1, 2, 3, 4 };
int[] b = { 5, 6, 7, 8 };
a = b;
a[0] = 9;
```

After the assignment statement `a = b`, we have `a[0]` equal to 5, `a[1]` equal to 6, and so forth, as expected. That is, the arrays correspond to the same sequence of values. However, they are not *independent* arrays. For example, after the last statement, not only is `a[0]` equal to 9, but `b[0]` is equal to 9 as well. This is one of the key differences between primitive types (such as `int` and `double`) and nonprimitive types (such as arrays). We will revisit this subtle (but fundamental) distinction in more detail when we consider passing arrays to functions in SECTION 2.1 and reference types in SECTION 3.1.

Q. If `a[]` is an array, why does `System.out.println(a)` print something like `@f62373`, instead of the sequence of values in the array?

A. Good question. It prints the memory address of the array (as a hexadecimal integer), which, unfortunately, is rarely what you want.

Q. Which other pitfalls should I watch out for when using arrays?

A. It is very important to remember that Java *automatically* initializes arrays when you create them, so that *creating an array takes time proportional to its length*.

Exercises

1.4.1 Write a program that declares, creates, and initializes an array `a[]` of length 1000 and accesses `a[1000]`. Does your program compile? What happens when you run it?

1.4.2 Describe and explain what happens when you try to compile a program with the following statement:

```
int n = 1000;
int[] a = new int[n*n*n*n];
```

1.4.3 Given two vectors of length `n` that are represented with one-dimensional arrays, write a code fragment that computes the *Euclidean distance* between them (the square root of the sums of the squares of the differences between corresponding elements).

1.4.4 Write a code fragment that reverses the order of the values in a one-dimensional string array. Do not create another array to hold the result. *Hint:* Use the code in the text for exchanging the values of two elements.

1.4.5 What is wrong with the following code fragment?

```
int[] a;
for (int i = 0; i < 10; i++)
    a[i] = i * i;
```

1.4.6 Write a code fragment that prints the contents of a two-dimensional boolean array, using `*` to represent `true` and a space to represent `false`. Include row and column indices.

1.4.7 What does the following code fragment print?

```
int[] a = new int[10];
for (int i = 0; i < 10; i++)
    a[i] = 9 - i;
for (int i = 0; i < 10; i++)
    a[i] = a[a[i]];
for (int i = 0; i < 10; i++)
    System.out.println(a[i]);
```

1.4.8 Which values does the following code put in the array `a[]`?

```
int n = 10;
int[] a = new int[n];
a[0] = 1;
a[1] = 1;
for (int i = 2; i < n; i++)
    a[i] = a[i-1] + a[i-2];
```

1.4.9 What does the following code fragment print?

```
int[] a = { 1, 2, 3 };
int[] b = { 1, 2, 3 };
System.out.println(a == b);
```

1.4.10 Write a program `Deal` that takes an integer command-line argument `n` and prints `n` poker hands (five cards each) from a shuffled deck, separated by blank lines.

1.4.11 Write a program `HowMany` that takes a variable number of command-line arguments and prints how many there are.

1.4.12 Write a program `DiscreteDistribution` that takes a variable number of integer command-line arguments and prints the integer `i` with probability proportional to the `i`th command-line argument.

1.4.13 Write code fragments to create a two-dimensional array `b[][]` that is a copy of an existing two-dimensional array `a[][]`, under each of the following assumptions:

- a. `a[][]` is square
- b. `a[][]` is rectangular
- c. `a[][]` may be ragged

Your solution to `b` should work for `a`, and your solution to `c` should work for both `b` and `a`, and your code should get progressively more complicated.

1.4.14 Write a code fragment to print the *transposition* (rows and columns exchanged) of a square two-dimensional array. For the example spreadsheet array in the text, you code would print the following:

99	98	92	94	99	90	76	92	97	89
85	57	77	32	34	46	59	66	71	29
98	78	76	11	22	54	88	89	24	38

1.4.15 Write a code fragment to transpose a square two-dimensional array *in place* without creating a second array.

1.4.16 Write a program that takes an integer command-line argument n and creates an n -by- n boolean array $a[][]$ such that $a[i][j]$ is `true` if i and j are relatively prime (have no common factors), and `false` otherwise. Use your solution to EXERCISE 1.4.6 to print the array. *Hint:* Use sieving.

1.4.17 Modify the spreadsheet code fragment in the text to compute a *weighted* average of the rows, where the weights of each exam score are in a one-dimensional array `weights[]`. For example, to assign the last of the three exams in our example to be twice the weight of the first two, you would use

```
double[] weights = { 0.25, 0.25, 0.50 };
```

Note that the weights should sum to 1.

1.4.18 Write a code fragment to multiply two *rectangular* matrices that are not necessarily square. *Note:* For the dot product to be well defined, the number of columns in the first matrix must be equal to the number of rows in the second matrix. Print an error message if the dimensions do not satisfy this condition.

1.4.19 Write a program that multiplies two square *boolean* matrices, using the `or` operation instead of `+` and the `and` operation instead of `*`.

1.4.20 Modify `SelfAvoidingWalk` (PROGRAM 1.4.4) to calculate and print the average length of the paths as well as the dead-end probability. Keep separate the average lengths of escape paths and dead-end paths.

1.4.21 Modify `SelfAvoidingWalk` to calculate and print the average area of the smallest axis-aligned rectangle that encloses the dead-end paths.

Creative Exercises

1.4.22 *Dice simulation.* The following code computes the exact probability distribution for the sum of two dice:

```
int[] frequencies = new int[13];
for (int i = 1; i <= 6; i++)
    for (int j = 1; j <= 6; j++)
        frequencies[i+j]++;
double[] probabilities = new double[13];
for (int k = 1; k <= 12; k++)
    probabilities[k] = frequencies[k] / 36.0;
```

The value `probabilities[k]` is the probability that the dice sum to `k`. Run experiments that validate this calculation by simulating `n` dice throws, keeping track of the frequencies of occurrence of each value when you compute the sum of two uniformly random integers between 1 and 6. How large does `n` have to be before your empirical results match the exact results to three decimal places?

1.4.23 *Longest plateau.* Given an array of integers, find the length and location of the longest contiguous sequence of equal values for which the values of the elements just before and just after this sequence are smaller.

1.4.24 *Empirical shuffle check.* Run computational experiments to check that our shuffling code works as advertised. Write a program `ShuffleTest` that takes two integer command-line arguments `m` and `n`, does `n` shuffles of an array of length `m` that is initialized with `a[i] = i` before each shuffle, and prints an `m`-by-`m` table such that row `i` gives the number of times `i` wound up in position `j` for all `j`. All values in the resulting array should be close to `n / m`.

1.4.25 *Bad shuffling.* Suppose that you choose a random integer between 0 and `n-1` in our shuffling code instead of one between `i` and `n-1`. Show that the resulting order is *not* equally likely to be one of the $n!$ possibilities. Run the test of the previous exercise for this version.

1.4.26 *Music shuffling.* You set your music player to shuffle mode. It plays each of the n songs before repeating any. Write a program to estimate the likelihood that you will not hear any sequential pair of songs (that is, song 3 does not follow song 2, song 10 does not follow song 9, and so on).

1.4.27 Minima in permutations. Write a program that takes an integer command-line argument n , generates a random permutation, prints the permutation, and prints the number of left-to-right minima in the permutation (the number of times an element is the smallest seen so far). Then write a program that takes two integer command-line arguments m and n , generates m random permutations of length n , and prints the average number of left-to-right minima in the permutations generated. *Extra credit:* Formulate a hypothesis about the number of left-to-right minima in a permutation of length n , as a function of n .

1.4.28 Inverse permutation. Write a program that reads in a permutation of the integers 0 to $n-1$ from n command-line arguments and prints the inverse permutation. (If the permutation is in an array $a[]$, its *inverse* is the array $b[]$ such that $a[b[i]] = b[a[i]] = i$.) Be sure to check that the input is a valid permutation.

1.4.29 Hadamard matrix. The n -by- n Hadamard matrix $H(n)$ is a boolean matrix with the remarkable property that any two rows differ in exactly $n / 2$ values. (This property makes it useful for designing error-correcting codes.) $H(1)$ is a 1-by-1 matrix with the single element `true`, and for $n > 1$, $H(2n)$ is obtained by aligning four copies of $H(n)$ in a large square, and then inverting all of the values in the lower right n -by- n copy, as shown in the following examples (with `T` representing `true` and `F` representing `false`, as usual).

$H(1)$	$H(2)$	$H(4)$
<code>T</code>	<code>T T</code>	<code>T T T T</code>
	<code>T F</code>	<code>T F T F</code>
		<code>T T F F</code>
		<code>T F F T</code>

Write a program that takes an integer command-line argument n and prints $H(n)$. Assume that n is a power of 2.

1.4.30 *Rumors.* Alice is throwing a party with n other guests, including Bob. Bob starts a rumor about Alice by telling it to one of the other guests. A person hearing this rumor for the first time will immediately tell it to one other guest, chosen uniformly at random from all the people at the party except Alice and the person from whom they heard it. If a person (including Bob) hears the rumor for a second time, he or she will not propagate it further. Write a program to estimate the probability that everyone at the party (except Alice) will hear the rumor before it stops propagating. Also calculate an estimate of the expected number of people to hear the rumor.

1.4.31 *Counting primes.* Compare PrimeSieve with the method that we used to demonstrate the break statement, at the end of SECTION 1.3. This is a classic example of a space–time tradeoff: PrimeSieve is fast, but requires a boolean array of length n ; the other approach uses only two integer variables, but is substantially slower. Estimate the magnitude of this difference by finding the value of n for which this second approach can complete the computation in about the same time as `java PrimeSeive 1000000`.

1.4.32 *Minesweeper.* Write a program that takes three command-line arguments m , n , and p and produces an m -by- n boolean array where each element is occupied with probability p . In the minesweeper game, occupied cells represent bombs and empty cells represent safe cells. Print out the array using an asterisk for bombs and a period for safe cells. Then, create an integer two-dimensional array with the number of neighboring bombs (above, below, left, right, or diagonal).

* * . . .	* * 1 0 0
.	3 3 2 0 0
. * . . .	1 * 1 0 0

Write your code so that you have as few special cases as possible to deal with, by using an $(m+2)$ -by- $(n+2)$ boolean array.

1.4.33 *Find a duplicate.* Given an integer array of length n , with each value between 1 and n , write a code fragment to determine whether there are any duplicate values. You may not use an extra array (but you do not need to preserve the contents of the given array.)

1.4.34 *Self-avoiding walk length.* Suppose that there is no limit on the size of the grid. Run experiments to estimate the average path length.

1.4.35 *Three-dimensional self-avoiding walks.* Run experiments to verify that the dead-end probability is 0 for a three-dimensional self-avoiding walk and to compute the average path length for various values of n .

1.4.36 *Random walkers.* Suppose that n random walkers, starting in the center of an n -by- n grid, move one step at a time, choosing to go left, right, up, or down with equal probability at each step. Write a program to help formulate and test a hypothesis about the number of steps taken before all cells are touched.

1.4.37 *Bridge hands.* In the game of bridge, four players are dealt hands of 13 cards each. An important statistic is the distribution of the number of cards in each suit in a hand. Which is the most likely, 5–3–3–2, 4–4–3–2, or 4–3–3–3?

1.4.38 *Birthday problem.* Suppose that people enter an empty room until a pair of people share a birthday. On average, how many people will have to enter before there is a match? Run experiments to estimate the value of this quantity. Assume birthdays to be uniform random integers between 0 and 364.

1.4.39 *Coupon collector.* Run experiments to validate the classical mathematical result that the expected number of coupons needed to collect n values is approximately $n H_n$, where H_n is the n th harmonic number. For example, if you are observing the cards carefully at the blackjack table (and the dealer has enough decks randomly shuffled together), you will wait until approximately 235 cards are dealt, on average, before seeing every card value.

1.4.40 Riffle shuffle. Compose a program to rearrange a deck of n cards using the Gilbert–Shannon–Reeds model of a riffle shuffle. First, generate a random integer r according to a *binomial distribution*: flip a fair coin n times and let r be the number of heads. Now, divide the deck into two piles: the first r cards and the remaining $n - r$ cards. To complete the shuffle, repeatedly take the top card from one of the two piles and put it on the bottom of a new pile. If there are n_1 cards remaining in the first pile and n_2 cards remaining in the second pile, choose the next card from the first pile with probability $n_1 / (n_1 + n_2)$ and from the second pile with probability $n_2 / (n_1 + n_2)$. Investigate how many riffle shuffles you need to apply to a deck of 52 cards to produce a (nearly) uniformly shuffled deck.

1.4.41 Binomial distribution. Write a program that takes an integer command-line argument n and creates a two-dimensional ragged array $a[][]$ such that $a[n][k]$ contains the probability that you get exactly k heads when you toss a fair coin n times. These numbers are known as the *binomial distribution*: if you multiply each element in row i by 2^n , you get the *binomial coefficients*—the coefficients of x^k in $(x+1)^n$ —arranged in *Pascal's triangle*. To compute them, start with $a[n][0] = 0.0$ for all n and $a[1][1] = 1.0$, then compute values in successive rows, left to right, with $a[n][k] = (a[n-1][k] + a[n-1][k-1]) / 2.0$.

<i>Pascal's triangle</i>	<i>binomial distribution</i>
1	1
1 1	1/2 1/2
1 2 1	1/4 1/2 1/4
1 3 3 1	1/8 3/8 3/8 1/8
1 4 6 4 1	1/16 1/4 3/8 1/4 1/16

1.5 Input and Output

IN THIS SECTION WE EXTEND THE set of simple abstractions (command-line arguments and standard output) that we have been using as the interface between our Java programs and the outside world to include *standard input*, *standard drawing*, and *standard audio*. Standard input makes it convenient for us to write programs that process arbitrary amounts of input and to interact with our programs; standard drawing makes it possible for us to work with graphical representations of images, freeing us from having to encode everything as text; and standard audio adds sound. These extensions are easy to use, and you will find that they bring you to yet another new world of programming.

The abbreviation *I/O* is universally understood to mean *input/output*, a collective term that refers to the mechanisms by which programs communicate with the outside world. Your computer’s operating system controls the physical devices that are connected to your computer. To implement the standard I/O abstractions, we use libraries of methods that interface to the operating system.

You have already been accepting arguments from the command line and printing strings in a terminal window; the purpose of this section is to provide you with a much richer set of tools for processing and presenting data. Like the `System.out.print()` and `System.out.println()` methods that you have been using, these methods do not implement pure mathematical functions—their purpose is to cause some *side effect*, either on an input device or an output device. Our prime concern is using such devices to get information into and out of our programs.

An essential feature of standard I/O mechanisms is that there is no limit on the amount of input or output, from the point of view of the program. Your programs can consume input or produce output indefinitely.

One use of standard I/O mechanisms is to connect your programs to *files* on your computer’s external storage. It is easy to connect standard input, standard output, standard drawing, and standard audio to files. Such connections make it easy to have your Java programs save or load results to files for archival purposes or for later reference by other programs or other applications.

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Programs in this section

Bird's-eye view The conventional model that we have been using for Java programming has served us since SECTION 1.1. To build context, we begin by briefly reviewing the model.

A Java program takes input strings from the command line and prints a string of characters as output. By default, both *command-line arguments* and *standard output* are associated with the application that takes commands (the one in which you have been typing the `java` and `javac` commands). We use the generic term *terminal window* to refer to this application. This model has proved to be a convenient and direct way for us to interact with our programs and data.

Command-line arguments. This mechanism, which we have been using to provide input values to our programs, is a standard part of Java programming. All of our classes have a `main()` method that takes a `String` array `args[]` as its argument. That array is the sequence of command-line arguments that we type, provided to Java by the operating system. By convention, both Java and the operating system process the arguments as strings, so if we intend for an argument to be a number, we use a method such as `Integer.parseInt()` or `Double.parseDouble()` to convert it from `String` to the appropriate type.

Standard output. To print output values in our programs, we have been using the system methods `System.out.println()` and `System.out.print()`. Java puts the results of a program's sequence of these method calls into the form of an abstract stream of characters known as *standard output*. By default, the operating system connects standard output to the terminal window. All of the output in our programs so far has been appearing in the terminal window.

For reference, and as a starting point, `RandomSeq` (PROGRAM 1.5.1) is a program that uses this model. It takes a command-line argument `n` and produces an output sequence of `n` random numbers between 0 and 1.

NOW WE ARE GOING TO COMPLEMENT command-line arguments and standard output with three additional mechanisms that address their limitations and provide us with a far more useful programming model. These mechanisms give us a new bird's-eye view of a Java program in which the program converts a standard input stream and a sequence of command-line arguments into a standard output stream, a standard drawing, and a standard audio stream.

Program 1.5.1 Generating a random sequence

```
public class RandomSeq
{
    public static void main(String[] args)
    { // Print a random sequence of n real values in [0, 1)
        int n = Integer.parseInt(args[0]);
        for (int i = 0; i < n; i++)
            System.out.println(Math.random());
    }
}
```

This program illustrates the conventional model that we have been using so far for Java programming. It takes a command-line argument n and prints n random numbers between 0.0 and 1.0. From the program's point of view, there is no limit on the length of the output sequence.

```
% java RandomSeq 1000000
0.2498362534343327
0.5578468691774513
0.5702167639727175
0.32191774192688727
0.6865902823177537
...
```

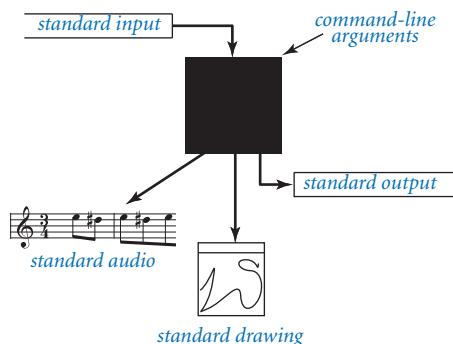
Standard input. Our class `StdIn` is a library that implements a standard input abstraction to complement the standard output abstraction. Just as you can print a value to standard output at any time during the execution of your program, so you can read a value from a standard input stream at any time.

Standard drawing. Our class `StdDraw` allows you to create drawings with your programs. It uses a simple graphics model that allows you to create drawings consisting of points and lines in a window on your computer. `StdDraw` also includes facilities for text, color, and animation.

Standard audio. Our class `StdAudio` allows you to create sound with your programs. It uses a standard format to convert arrays of numbers into sound.

To USE BOTH COMMAND-LINE ARGUMENTS AND standard output, you have been using built-in Java facilities. Java also has built-in facilities that support abstractions like standard input, standard drawing, and standard audio, but they are somewhat more complicated to use, so we have developed a simpler interface to them in our `StdIn`, `StdDraw`, and `StdAudio` libraries. To logically complete our programming model, we also include a `StdOut` library. To use these libraries, you must make `StdIn.java`, `StdOut.java`, `StdDraw.java`, and `StdAudio.java` available to Java (see the Q&A at the end of this section for details).

The standard input and standard output abstractions date back to the development of the UNIX operating system in the 1970s and are found in some form on all modern systems. Although they are primitive by comparison to various mechanisms developed since then, modern programmers still depend on them as a reliable way to connect data to programs. We have developed for this book standard drawing and standard audio in the same spirit as these earlier abstractions to provide you with an easy way to produce visual and aural output.



A bird's-eye view of a Java program (revisited)

Standard output Java's `System.out.print()` and `System.out.println()` methods implement the basic standard output abstraction that we need. Nevertheless, to treat standard input and standard output in a uniform manner (and to provide a few technical improvements), starting in this section and continuing through the rest of the book, we use similar methods that are defined in our `StdOut` library. `StdOut.print()` and `StdOut.println()` are nearly the same as the Java methods that you have been using (see the booksite for a discussion of the differences, which need not concern you now). The `StdOut.printf()` method is a main topic of this section and will be of interest to you now because it gives you more control over the appearance of the output. It was a feature of the C language of the early 1970s that still survives in modern languages because it is so useful.

Since the first time that we printed `double` values, we have been distracted by excessive precision in the printed output. For example, when we use `System.out.print(Math.PI)` we get the output `3.141592653589793`, even though we might prefer to see `3.14` or `3.14159`. The `print()` and `println()`

<code>public class StdOut</code>	
<code>void print(String s)</code>	<i>print s to standard output</i>
<code>void println(String s)</code>	<i>print s and a newline to standard output</i>
<code>void println()</code>	<i>print a newline to standard output</i>
<code>void printf(String format, ...)</code>	<i>print the arguments to standard output, as specified by the format string format</i>

API for our library of static methods for standard output

methods present each number to up to 15 decimal places even when we would be happy with only a few. The `printf()` method is more flexible. For example, it allows us to specify the number of decimal places when converting floating-point numbers to strings for output. We can write `StdOut.printf("%7.5f", Math.PI)` to get `3.14159`, and we can replace `System.out.print(t)` with

```
StdOut.printf("The square root of %.1f is %.6f", c, t);
```

in `Newton` (PROGRAM 1.3.6) to get output like

```
The square root of 2.0 is 1.414214
```

Next, we describe the meaning and operation of these statements, along with extensions to handle the other built-in types of data.

Formatted printing basics. In its simplest form, `printf()` takes two arguments. The first argument is called the *format string*. It contains a *conversion specification* that describes how the second argument is to be converted to a string for output. A conversion specification has the form `%w.pc`, where `w` and `p` are integers and `c` is a character, to be interpreted as follows:

- `w` is the *field width*, the number of characters that should be written. If the number of characters to be written exceeds (or equals) the field width, then the field width is ignored; otherwise, the output is padded with spaces on the left. A negative field width indicates that the output instead should be padded with spaces on the right.
- `.p` is the *precision*. For floating-point numbers, the precision is the number of digits that should be written after the decimal point; for strings, it is the number of characters of the string that should be printed. The precision is not used with integers.

- c is the *conversion code*. The conversion codes that we use most frequently are d (for decimal values from Java's integer types), f (for floating-point values), e (for floating-point values using scientific notation), s (for string values), and b (for boolean values).

The field width and precision can be omitted, but every specification must have a conversion code.

The most important thing to remember about using `printf()` is that *the conversion code and the type of the corresponding argument must match*. That is, Java must be able to convert from the type of the argument to the type required by the conversion code. Every type of data can be converted to `String`, but if you write `StdOut.printf("%12d", Math.PI)` or `StdOut.printf("%4.2f", 512)`, you will get an `IllegalFormatConversionException` run-time error.

Format string. The format string can contain characters in addition to those for the conversion specification. The conversion specification is replaced by the argument value (converted to a string as specified) and all remaining characters are passed through to the output. For example, the statement

```
StdOut.printf("PI is approximately %.2f.\n", Math.PI);
```

prints the line

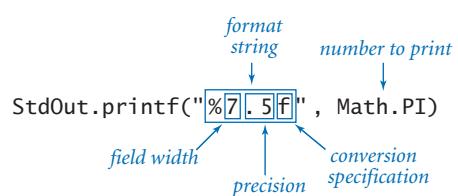
PI is approximately 3.14.

Note that we need to explicitly include the newline character `\n` in the format string to print a new line with `printf()`.

Multiple arguments. The `printf()` method can take more than two arguments. In this case, the format string will have an additional conversion specification for each additional argument, perhaps separated by other characters to pass through to the output. For example, if you were making payments on a loan, you might use code whose inner loop contains the statements

```
String formats = "%3s $%6.2f $%7.2f $%5.2f\n";
StdOut.printf(formats, month[i], pay, balance, interest);
```

to print the second and subsequent lines in a table like this (see EXERCISE 1.5.13):



Anatomy of a formatted print statement

	payment	balance	interest
Jan	\$299.00	\$9742.67	\$41.67
Feb	\$299.00	\$9484.26	\$40.59
Mar	\$299.00	\$9224.78	\$39.52
	...		

Formatted printing is convenient because this sort of code is much more compact than the string-concatenation code that we have been using to create output strings. We have described only the basic options; see the booksite for more details.

<i>type</i>	<i>code</i>	<i>typical literal</i>	<i>sample format strings</i>	<i>converted string values for output</i>
int	d	512	"%14d" "%-14d"	"512" "-512"
double	f	1595.1680010754388	"%14.2f" "%.7f" "%14.4e"	"1595.1680011" "1.5952e+03"
String	s	"Hello, World"	"%14s" "%-14s" "%-14.5s"	"Hello, World" "Hello, World " "Hello "
boolean	b	true	"%b"	"true"

Format conventions for printf() (see the booksite for many other options)

Standard input Our StdIn library takes data from a *standard input stream* that may be empty or may contain a sequence of values separated by whitespace (spaces, tabs, newline characters, and the like). Each value is a string or a value from one of Java's primitive types. One of the key features of the standard input stream is that your program *consumes* values when it reads them. Once your program has read a value, it cannot back up and read it again. This assumption is restrictive, but it reflects the physical characteristics of some input devices. The API for StdIn appears on the facing page. The methods fall into one of four categories:

- Those for reading individual values, one at a time
- Those for reading lines, one at a time
- Those for reading characters, one at a time
- Those for reading a sequence of values of the same type

Generally, it is best not to mix functions from the different categories in the same program. These methods are largely self-documenting (the names describe their effect), but their precise operation is worthy of careful consideration, so we will consider several examples in detail.

```
public class StdIn
```

methods for reading individual tokens from standard input

boolean isEmpty()	<i>is standard input empty (or only whitespace)?</i>
int readInt()	<i>read a token, convert it to an int, and return it</i>
double readDouble()	<i>read a token, convert it to a double, and return it</i>
boolean readBoolean()	<i>read a token, convert it to a boolean, and return it</i>
String readString()	<i>read a token and return it as a String</i>

methods for reading characters from standard input

boolean hasNextChar()	<i>does standard input have any remaining characters?</i>
char readChar()	<i>read a character from standard input and return it</i>

methods for reading lines from standard input

boolean hasNextLine()	<i>does standard input have a next line?</i>
String readLine()	<i>read the rest of the line and return it as a String</i>

methods for reading the rest of standard input

int[] readAllInts()	<i>read all remaining tokens and return them as an int array</i>
double[] readAllDoubles()	<i>read all remaining tokens and return them as a double array</i>
boolean[] readAllBooleans()	<i>read all remaining tokens and return them as a boolean array</i>
String[] readAllStrings()	<i>read all remaining tokens and return them as a String array</i>
String[] readAllLines()	<i>read all remaining lines and return them as a String array</i>
String readAll()	<i>read the rest of the input and return it as a String</i>

Note 1: A token is a maximal sequence of non-whitespace characters.

Note 2: Before reading a token, any leading whitespace is discarded.

Note 3: Analogous methods are available for reading values of type byte, short, long, and float.

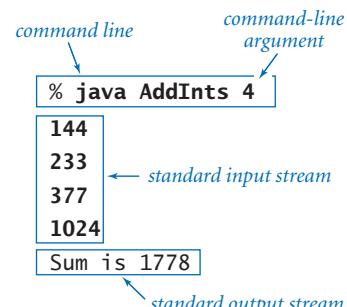
Note 4: Each method that reads input throws a run-time exception if it cannot read in the next value, either because there is no more input or because the input does not match the expected type.

API for our library of static methods for standard input

Typing input. When you use the `java` command to invoke a Java program from the command line, you actually are doing three things: (1) issuing a command to start executing your program, (2) specifying the command-line arguments, and (3) beginning to define the standard input stream. The string of characters that you type in the terminal window after the command line *is* the standard input stream. When you type characters, you are interacting with your program. The program *waits* for you to type characters in the terminal window.

For example, consider the program `AddInts`, which takes a command-line argument `n`, then reads `n` numbers from standard input, adds them, and prints the result to standard output. When you type `java AddInts 4`, after the program takes the command-line argument, it calls the method `StdIn.readInt()` and waits for you to type an integer. Suppose that you want 144 to be the first value. As you type 1, then 4, and then 4, nothing happens, because `StdIn` does not know that you are done typing the integer. But when you then type <Return> to signify the end of your integer, `StdIn.readInt()` immediately returns the value 144, which your program adds to `sum` and then calls `StdIn.readInt()` again. Again, nothing happens until you type the second value: if you type 2, then 3, then 3, and then <Return> to end the number, `StdIn.readInt()` returns the value 233, which your program again adds to `sum`. After you have typed four numbers in this way, `AddInts` expects no more input and prints the sum, as desired.

```
public class AddInts
{
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        int sum = 0;
        for (int i = 0; i < n; i++)
        {
            int value = StdIn.readInt();
            sum += value;
        }
        StdOut.println("Sum is " + sum);
    }
}
```



Anatomy of a command

Input format. If you type abc or 12.2 or true when `StdIn.readInt()` is expecting an `int`, it will respond with an `InputMismatchException`. The format for each type is essentially the same as you have been using to specify literals within Java programs. For convenience, `StdIn` treats strings of consecutive whitespace characters as identical to one space and allows you to delimit your numbers with such strings. It does not matter how many spaces you put between numbers, or whether you enter numbers on one line or separate them with tab characters or spread them out over several lines, (except that your terminal application processes standard input one line at a time, so it will wait until you type <Return> before sending all of the numbers on that line to standard input). You can mix values of different types in an input stream, but whenever the program expects a value of a particular type, the input stream must have a value of that type.

Interactive user input. `TwentyQuestions` (PROGRAM 1.5.2) is a simple example of a program that interacts with its user. The program generates a random integer and then gives clues to a user trying to guess the number. (As a side note, by using *binary search*, you can always get to the answer in at most 20 questions. See SECTION 4.2.) The fundamental difference between this program and others that we have written is that the user has the ability to change the control flow *while* the program is executing. This capability was very important in early applications of computing, but we rarely write such programs nowadays because modern applications typically take such input through the graphical user interface, as discussed in CHAPTER 3. Even a simple program like `TwentyQuestions` illustrates that writing programs that support user interaction is potentially very difficult because you have to plan for all possible user inputs.

Program 1.5.2 Interactive user input

```

public class TwentyQuestions
{
    public static void main(String[] args)
    { // Generate a number and answer questions
        // while the user tries to guess the value.
        int secret = 1 + (int) (Math.random() * 1000000);
        StdOut.print("I'm thinking of a number ");
        StdOut.println("between 1 and 1,000,000");
        int guess = 0;
        while (guess != secret)
        { // Solicit one guess and provide one answer.
            StdOut.print("What's your guess? ");
            guess = StdIn.readInt();
            if (guess == secret) StdOut.println("You win!");
            if (guess < secret) StdOut.println("Too low ");
            if (guess > secret) StdOut.println("Too high");
        }
    }
}

```

secret *secret value*
guess *user's guess*

This program plays a simple guessing game. You type numbers, each of which is an implicit question (“Is this the number?”) and the program tells you whether your guess is too high or too low. You can always get it to print You win! with fewer than 20 questions. To use this program, you StdIn and StdOut must be available to Java (see the first Q&A at the end of this section).

```

% java TwentyQuestions
I'm thinking of a number between 1 and 1,000,000
What's your guess? 500000
Too high
What's your guess? 250000
Too low
What's your guess? 375000
Too high
What's your guess? 312500
Too high
What's your guess? 300500
Too low
...

```

Processing an arbitrary-size input stream. Typically, input streams are finite: your program marches through the input stream, consuming values until the stream is empty. But there is no restriction of the size of the input stream, and some programs simply process all the input presented to them. `Average` (PROGRAM 1.5.3) is an example that reads in a sequence of floating-point numbers from standard input and prints their average. It illustrates a key property of using an input stream: the length of the stream is not known to the program. We type all the numbers that we have, and then the program averages them. Before reading each number, the program uses the method `StdIn.isEmpty()` to check whether there are any more numbers in the input stream. How do we signal that we have no more data to type? By convention, we type a special sequence of characters known as the *end-of-file* sequence. Unfortunately, the terminal applications that we typically encounter on modern operating systems use different conventions for this critically important sequence. In this book, we use `<Ctrl-D>` (many systems require `<Ctrl-D>` to be on a line by itself); the other widely used convention is `<Ctrl-Z>` on a line by itself. `Average` is a simple program, but it represents a profound new capability in programming: with standard input, we can write programs that process an unlimited amount of data. As you will see, writing such programs is an effective approach for numerous data-processing applications.

STANDARD INPUT IS A SUBSTANTIAL STEP up from the command-line-arguments model that we have been using, for two reasons, as illustrated by `TwentyQuestions` and `Average`. First, we can interact with our program—with command-line arguments, we can provide data to the program only *before* it begins execution. Second, we can read in large amounts of data—with command-line arguments, we can enter only values that fit on the command line. Indeed, as illustrated by `Average`, the amount of data can be potentially unlimited, and many programs are made simpler by that assumption. A third *raison d'être* for standard input is that your operating system makes it possible to change the source of standard input, so that you do not have to type all the input. Next, we consider the mechanisms that enable this possibility.

Program 1.5.3 Averaging a stream of numbers

```

public class Average
{
    public static void main(String[] args)
    { // Average the numbers on standard input.
        double sum = 0.0;
        int n = 0;
        while (!StdIn.isEmpty())
        { // Read a number from standard input and add to sum.
            double value = StdIn.readDouble();
            sum += value;
            n++;
        }
        double average = sum / n;
        StdOut.println("Average is " + average);
    }
}

```

n	<i>count of numbers read</i>
sum	<i>cumulated sum</i>

This program reads in a sequence of floating-point numbers from standard input and prints their average on standard output (provided that the sum does not overflow). From its point of view, there is no limit on the size of the input stream. The commands on the right below use redirection and piping (discussed in the next subsection) to provide 100,000 numbers to average.

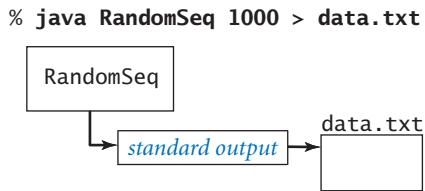
<pre>% java Average 10.0 5.0 6.0 3.0 7.0 32.0 <Ctrl-D> Average is 10.5</pre>	<pre>% java RandomSeq 100000 > data.txt % java Average < data.txt Average is 0.5010473676174824 % java RandomSeq 100000 java Average Average is 0.5000499417963857</pre>
--	--

Redirection and piping For many applications, typing input data as a standard input stream from the terminal window is untenable because our program’s processing power is then limited by the amount of data that we can type (and our typing speed). Similarly, we often want to save the information printed on the standard output stream for later use. To address such limitations, we next focus on the idea that standard input is an *abstraction*—the program expects to read data from an input stream but it has no dependence on the *source* of that input stream. Standard output is a similar abstraction. The power of these abstractions derives from our ability (through the operating system) to specify various other sources for standard input and standard output, such as a file, the network, or another program. All modern operating systems implement these mechanisms.

Redirecting standard output to a file. By adding a simple directive to the command that invokes a program, we can *redirect* its standard output stream to a file, either for permanent storage or for input to another program at a later time. For example,

```
% java RandomSeq 1000 > data.txt
```

specifies that the standard output stream is not to be printed in the terminal window, but instead is to be written to a text file named `data.txt`. Each call to `System.out.print()` or `System.out.println()` appends text at the end of that file. In this example, the end result is a file that contains 1,000 random values. No output appears in the terminal window: it goes directly into the file named after the `>` symbol. Thus, we can save away information for later retrieval. Note that we do not have to change `RandomSeq` (PROGRAM 1.5.1) in any way for this mechanism to work—it uses the standard output abstraction and is unaffected by our use of a different implementation of that abstraction. You can use redirection to save output from any program that you write. Once you have expended a significant amount of effort to obtain a result, you often want to save the result for later reference. In a modern system, you can save some information by using cut-and-paste or some similar mechanism that is provided by the operating system, but cut-and-paste is inconvenient for large amounts of data. By contrast, redirection is specifically designed to make it easy to handle large amounts of data.



Redirecting standard output to a file

Program 1.5.4 A simple filter

```
public class RangeFilter
{
    public static void main(String[] args)
    { // Filter out numbers not between lo and hi.
        int lo = Integer.parseInt(args[0]);
        int hi = Integer.parseInt(args[1]);
        while (!StdIn.isEmpty())
        { // Process one number.
            int value = StdIn.readInt();
            if (value >= lo && value <= hi)
                StdOut.print(value + " ");
        }
        StdOut.println();
    }
}
```

lo *lower bound of range*
 hi *upper bound of range*
 value *current number*

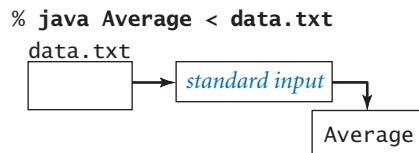
This filter copies to the output stream the numbers from the input stream that fall inside the range given by the command-line arguments. There is no limit on the length of the streams.

```
% java RangeFilter 100 400
358 1330 55 165 689 1014 3066 387 575 843 203 48 292 877 65 998
358 165 387 203 292
<Ctrl-D>
```

Redirecting from a file to standard input. Similarly, we can redirect the standard input stream so that `StdIn` reads data from a file instead of the terminal window:

```
% java Average < data.txt
```

This command reads a sequence of numbers from the file `data.txt` and computes their average value. Specifically, the `<` symbol is a directive that tells the operating system to implement the standard input stream by reading from the text file `data.txt` instead of waiting for the user to type something into



Redirecting from a file to standard input

the terminal window. When the program calls `StdIn.readDouble()`, the operating system reads the value from the file. The file `data.txt` could have been created by any application, not just a Java program—many applications on your computer can create text files. This facility to redirect from a file to standard input enables us to create *data-driven code* where you can change the data processed by a program without having to change the program at all. Instead, you can keep data in files and write programs that read from the standard input stream.

Connecting two programs. The most flexible way to implement the standard input and standard output abstractions is to specify that they are implemented by our own programs! This mechanism is called *piping*. For example, the command

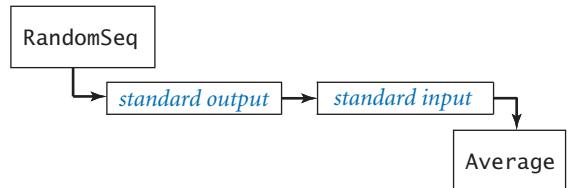
```
% java RandomSeq 1000 | java Average
```

specifies that the standard output stream for `RandomSeq` and the standard input stream for `Average` are the *same* stream. The effect is as if `RandomSeq` were typing the numbers it generates into the terminal window while `Average` is running. This example also has the same effect as the following sequence of commands:

```
% java RandomSeq 1000 > data.txt  
% java Average < data.txt
```

In this case, the file `data.txt` is not created. This difference is profound, because it removes another limitation on the size of the input and output streams that we can process. For example, you could replace 1000 in the example with 1000000000, even though you might not have the space to save a billion numbers on our computer (you, however, do need the *time* to process them). When `RandomSeq` calls `System.out.println()`, a string is added to the end of the stream; when `Average` calls `StdIn.readInt()`, a string is removed from the beginning of the stream. The timing of precisely what happens is up to the operating system: it might run `RandomSeq` until it produces some output, and then run `Average` to consume that output, or it might run `Average` until it needs some input, and then run `RandomSeq` until it produces the needed input. The end result is the same, but your programs are freed from worrying about such details because they work solely with the standard input and standard output abstractions.

```
% java RandomSeq 1000 | java Average
```



Piping the output of one program to the input of another

Filters. Piping, a core feature of the original Unix system of the early 1970s, still survives in modern systems because it is a simple abstraction for communicating among disparate programs. Testimony to the power of this abstraction is that many Unix programs are still being used today to process files that are thousands or millions of times larger than imagined by the programs' authors. We can communicate with other Java programs via method calls, but standard input and standard output allow us to communicate with programs that were written at another time and, perhaps, in another language. With standard input and standard output, we are agreeing on a simple interface to the outside world.

For many common tasks, it is convenient to think of each program as a *filter* that converts a standard input stream to a standard output stream in some way, with piping as the command mechanism to connect programs together. For example, `RangeFilter` (PROGRAM 1.5.4) takes two command-line arguments and prints on standard output those numbers from standard input that fall within the specified range. You might imagine standard input to be measurement data from some instrument, with the filter being used to throw away data outside the range of interest for the experiment at hand.

Several standard filters that were designed for Unix still survive (sometimes with different names) as commands in modern operating systems. For example, the `sort` filter puts the lines on standard input in sorted order:

```
% java RandomSeq 6 | sort
0.035813305516568916
0.14306638757584322
0.348292877655532103
0.5761644592016527
0.7234592733392126
0.9795908813988247
```

We discuss sorting in SECTION 4.2. A second useful filter is `grep`, which prints the lines from standard input that match a given pattern. For example, if you type

```
% grep lo < RangeFilter.java
```

you get the result

```
// Filter out numbers not between lo and hi.
int lo = Integer.parseInt(args[0]);
if (value >= lo && value <= hi)
```

Programmers often use tools such as `grep` to get a quick reminder of variable names or language usage details. A third useful filter is `more`, which reads data from standard input and displays it in your terminal window one screenful at a time. For example, if you type

```
% java RandomSeq 1000 | more
```

you will see as many numbers as fit in your terminal window, but `more` will wait for you to hit the space bar before displaying each succeeding screenful. The term *filter* is perhaps misleading: it was meant to describe programs like `RangeFilter` that write some subsequence of standard input to standard output, but it is now often used to describe any program that reads from standard input and writes to standard output.

Multiple streams. For many common tasks, we want to write programs that take input from multiple sources and/or produce output intended for multiple destinations. In SECTION 3.1 we discuss our `Out` and `In` libraries, which generalize `StdOut` and `StdIn` to allow for multiple input and output streams. These libraries include provisions for redirecting these streams not only to and from files, but also from web pages.

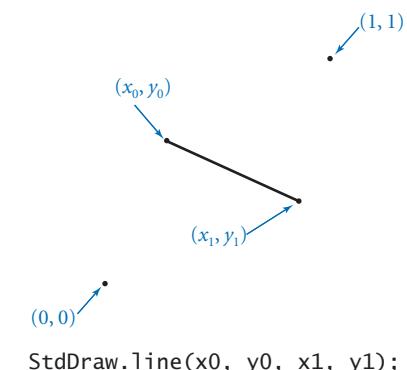
PROCESSING LARGE AMOUNTS OF INFORMATION PLAYS an essential role in many applications of computing. A scientist may need to analyze data collected from a series of experiments, a stock trader may wish to analyze information about recent financial transactions, or a student may wish to maintain collections of music and movies. In these and countless other applications, data-driven programs are the norm. Standard output, standard input, redirection, and piping provide us with the capability to address such applications with our Java programs. We can collect data into files on our computer through the web or any of the standard devices and use redirection and piping to connect data to our programs. Many (if not most) of the programming examples that we consider throughout this book have this ability.

Standard drawing Up to this point, our input/output abstractions have focused exclusively on text strings. Now we introduce an abstraction for producing drawings as output. This library is easy to use and allows us to take advantage of a visual medium to work with far more information than is possible with mere text.

As with StdIn and StdOut, our standard drawing abstraction is implemented in a library StdDraw that you will need to make available to Java (see the first Q&A at the end of this section). Standard drawing is very simple. We imagine an abstract drawing device capable of drawing lines and points on a two-dimensional canvas. The device is capable of responding to the commands that our programs issue in the form of calls to methods in StdDraw such as the following:

```
public class StdDraw {(basic drawing commands)
    void line(double x0, double y0, double x1, double y1)
    void point(double x, double y)
```

Like the methods for standard input and standard output, these methods are nearly self-documenting: StdDraw.line() draws a straight line segment connecting the point (x_0, y_0) with the point (x_1, y_1) whose coordinates are given as arguments. StdDraw.point() draws a spot centered on the point (x, y) whose coordinates are given as arguments. The default scale is the unit square (all x - and y -coordinates between 0 and 1). StdDraw displays the canvas in a window on your computer's screen, with black lines and points on a white background. The window includes a menu option to save your drawing to a file, in a format suitable for publishing on paper or on the web.



Your first drawing. The HelloWorld equivalent for graphics programming with StdDraw is to draw an equilateral triangle with a point inside. To form the triangle, we draw three line segments: one from the point $(0, 0)$ at the lower-left corner to the point $(1, 0)$, one from that point to the third point at $(1/2, \sqrt{3}/2)$, and one from that point back to $(0, 0)$. As a final flourish, we draw a spot in the middle of the triangle. Once you have successfully compiled and run Triangle, you are off and

running to write your own programs that draw figures composed of line segments and points. This ability literally adds a new dimension to the output that you can produce.

When you use a computer to create drawings, you get immediate feedback (the drawing) so that you can refine and improve your program quickly. With a computer program, you can create drawings that you could not contemplate making by hand. In particular, instead of viewing our data as merely numbers, we can use pictures, which are far more expressive. We will consider other graphics examples after we discuss a few other drawing commands.

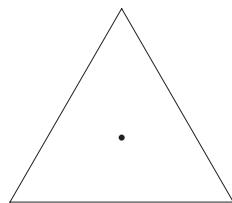
Control commands. The default canvas size is 512-by-512 pixels; if you want to change it, call `setCanvasSize()` before any drawing commands. The default coordinate system for standard drawing is the unit square, but we often want to draw plots at different scales. For example, a typical situation is to use coordinates in some range for the x -coordinate, or the y -coordinate, or both. Also, we often want to draw line segments of different thickness and points of different size from the standard. To accommodate these needs, `StdDraw` has the following methods:

`public class StdDraw (basic control commands)`

<code>void setCanvasSize(int w, int h)</code>	<i>create canvas in screen window of width w and height h (in pixels)</i>
<code>void setXscale(double x0, double x1)</code>	<i>reset x-scale to $(x0, x1)$</i>
<code>void setYscale(double y0, double y1)</code>	<i>reset y-scale to $(y0, y1)$</i>
<code>void setPenRadius(double radius)</code>	<i>set pen radius to $radius$</i>

Note: Methods with the same names but no arguments reset to default values the unit square for the x - and y -scales, 0.002 for the pen radius.

```
public class Triangle
{
    public static void main(String[] args)
    {
        double t = Math.sqrt(3.0)/2.0;
        StdDraw.line(0.0, 0.0, 1.0, 0.0);
        StdDraw.line(1.0, 0.0, 0.5, t);
        StdDraw.line(0.5, t, 0.0, 0.0);
        StdDraw.point(0.5, t/3.0);
    }
}
```

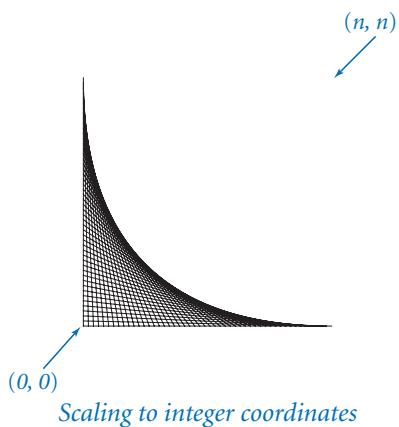


Your first drawing

```

int n = 50;
StdDraw.setXscale(0, n);
StdDraw.setYscale(0, n);
for (int i = 0; i <= n; i++)
    StdDraw.line(0, n-i, i, 0);

```



For example, the two-call sequence

```

StdDraw.setXscale(x0, x1);
StdDraw.setYscale(y0, y1);

```

sets the drawing coordinates to be within a *bounding box* whose lower-left corner is at (x_0, y_0) and whose upper-right corner is at (x_1, y_1) . Scaling is the simplest of the transformations commonly used in graphics. In the applications that we consider in this chapter, we use it in a straightforward way to match our drawings to our data.

The pen is circular, so that lines have rounded ends, and when you set the pen radius to r and draw a point, you get a circle of radius r . The default pen radius is 0.002 and is not affected by coordinate scaling. This default is about 1/500 the width of the default window, so that if you draw 200 points equally spaced along a horizontal or vertical line, you will be able to see individual circles, but if you draw 250 such points, the result will look like a line. When you issue the command `StdDraw.setPenRadius(0.01)`, you are saying that you want the thickness of the line segments and the size of the points to be five times the 0.002 standard.

Filtering data to standard drawing. One of the simplest applications of standard drawing is to plot data, by filtering it from standard input to standard drawing. `PlotFilter` (PROGRAM 1.5.5) is such a filter: it reads from standard input a sequence of points defined by (x, y) coordinates and draws a spot at each point. It adopts the convention that the first four numbers on standard input specify the bounding box, so that it can scale the plot without having to make an extra pass through all the points to determine the scale. The graphical representation of points plotted in this way is far more expressive (and far more compact) than the numbers themselves. The image that is produced by PROGRAM 1.5.5 makes it far easier for us to infer properties of the points (such as, for example, clustering of population centers when plotting points that represent city locations) than does a list of the coordinates. Whenever we are processing data that represents the physical world, a visual image is likely to be one of the most meaningful ways that we can use to display output. `PlotFilter` illustrates how easily you can create such an image.

Program 1.5.5 Standard input-to-drawing filter

```

public class PlotFilter
{
    public static void main(String[] args)
    {
        // Scale as per first four values.
        double x0 = StdIn.readDouble();
        double y0 = StdIn.readDouble();
        double x1 = StdIn.readDouble();
        double y1 = StdIn.readDouble();
        StdDraw.setXscale(x0, x1);
        StdDraw.setYscale(y0, y1);

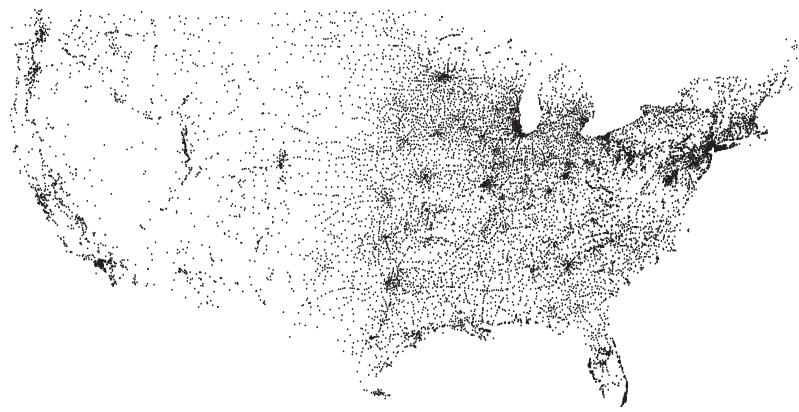
        // Read the points and plot to standard drawing.
        while (!StdIn.isEmpty())
        {
            double x = StdIn.readDouble();
            double y = StdIn.readDouble();
            StdDraw.point(x, y);
        }
    }
}

```

x0	<i>left bound</i>
y0	<i>bottom bound</i>
x1	<i>right bound</i>
y1	<i>top bound</i>
x, y	<i>current point</i>

This program reads a sequence of points from standard input and plots them to standard drawing. (By convention, the first four numbers are the minimum and maximum x- and y-coordinates.) The file USA.txt contains the coordinates of 13,509 cities in the United States

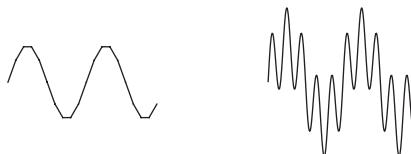
```
% java PlotFilter < USA.txt
```



Plotting a function graph. Another important use of standard drawing is to plot experimental data or the values of a mathematical function. For example, suppose that we want to plot values of the function $y = \sin(4x) + \sin(20x)$ in the interval $[0, \pi]$. Accomplishing this task is a prototypical example of *sampling*: there are an infinite number of points in the interval but we have to make do with evaluating the function at a finite number of such points. We sample the function by choosing a set of x -values, then computing y -values by evaluating the function at each of these x -value. Plotting the function by connecting successive points with lines produces what is known as a *piecewise linear approximation*.

The simplest way to proceed is to evenly space the x -values. First, we decide ahead of time on a sample size, then we space the x -values by the interval size divided by the sample size. To make sure that the values we plot fall in the visible canvas, we scale the x -axis corresponding to the interval and the y -axis corresponding to the maximum and minimum values of the function within the interval. The smoothness of the curve depends on properties of the function and the size of the sample. If the sample size is too small, the rendition of the function may not be at all accurate (it might not be very smooth, and it might miss major fluctuations); if the sample is too large, producing the plot may be time-consuming, since some functions are time-consuming to compute. (In SECTION 2.4, we will look at a method for plotting a smooth curve without using an excessive number of points.) You can use this same technique to plot the function graph of any function you choose. That is, you can decide on an x -interval where you want to plot the function, compute function values evenly spaced within that interval, determine and set the y -scale, and draw the line segments.

```
double[] x = new double[n+1];
double[] y = new double[n+1];
for (int i = 0; i <= n; i++)
    x[i] = Math.PI * i / n;
for (int i = 0; i <= n; i++)
    y[i] = Math.sin(4*x[i]) + Math.sin(20*x[i]);
StdDraw.setXscale(0, Math.PI);
StdDraw.setYscale(-2.0, 2.0);
for (int i = 1; i <= n; i++)
    StdDraw.line(x[i-1], y[i-1], x[i], y[i]);
```

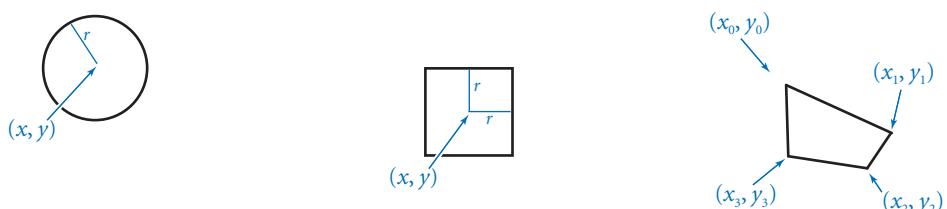
 $n = 20$ $n = 200$ 

Plotting a function graph

Outline and filled shapes. StdDraw also includes methods to draw circles, squares, rectangles, and arbitrary polygons. Each shape defines an outline. When the method name is the name of a shape, that outline is traced by the drawing pen. When the name begins with `filled`, the named shape is filled solid, not traced. As usual, we summarize the available methods in an API:

```
public class StdDraw (shapes)
    void circle(double x, double y, double radius)
    void filledCircle(double x, double y, double radius)
    void square(double x, double y, double r)
    void filledSquare(double x, double y, double r)
    void rectangle(double x, double y, double r1, double r2)
    void filledRectangle(double x, double y, double r1, double r2)
    void polygon(double[] x, double[] y)
    void filledPolygon(double[] x, double[] y)
```

The arguments for `circle()` and `filledCircle()` define a circle of radius r centered at (x, y) ; the arguments for `square()` and `filledSquare()` define a square of side length $2r$ centered at (x, y) ; the arguments for `rectangle()` and `filledRectangle()` define a rectangle of width $2r_1$ and height $2r_2$, centered at (x, y) ; and the arguments for `polygon()` and `filledPolygon()` define a sequence of points that are connected by line segments, including one from the last point to the first point.



```
StdDraw.circle(x, y, r);      StdDraw.square(x, y, r);      double[] x = {x0, x1, x2, x3};
                               double[] y = {y0, y1, y2, y3};  
StdDraw.polygon(x, y);
```

Text and color. Occasionally, you may wish to annotate or highlight various elements in your drawings. StdDraw has a method for drawing text, another for setting parameters associated with text, and another for changing the color of the ink in the pen. We make scant use of these features in this book, but they can be very useful, particularly for drawings on your computer screen. You will find many examples of their use on the booksite.

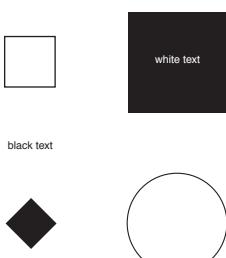
```
public class StdDraw (text and color commands)
    void text(double x, double y, String s)
    void setFont(Font font)
    void setPenColor(Color color)
```

In this code, `Font` and `Color` are nonprimitive types that you will learn about in SECTION 3.1. Until then, we leave the details to `StdDraw`. The available pen colors are `BLACK`, `BLUE`, `CYAN`, `DARK_GRAY`, `GRAY`, `GREEN`, `LIGHT_GRAY`, `MAGENTA`, `ORANGE`, `PINK`, `RED`, `WHITE`, `YELLOW`, and `BOOK_BLUE`, all of which are defined as constants within `StdDraw`. For example, the call `StdDraw.setPenColor(StdDraw.GRAY)` changes

```
StdDraw.square(.2, .8, .1);
StdDraw.filledSquare(.8, .8, .2);
StdDraw.circle(.8, .2, .2);
double[] xd = { .1, .2, .3, .2 };
double[] yd = { .2, .3, .2, .1 };
StdDraw.filledPolygon(xd, yd);
StdDraw.text(.2, .5, "black text");
StdDraw.setPenColor(StdDraw.WHITE);
StdDraw.text(.8, .8, "white text");
```

the pen to use gray ink. The default ink color is `BLACK`. The default font in `StdDraw` suffices for most of the drawings that you need (you can find information on using other fonts on the booksite). For example, you might wish to use these methods to annotate function graphs to highlight relevant values, and you might find it useful to develop similar methods to annotate other parts of your drawings.

Shapes, color, and text are basic tools that you can use to produce a dizzying variety of images, but you should use them sparingly. Use of such artifacts usually presents a design challenge, and our `StdDraw` commands are crude by the standards of modern graphics libraries, so that you are likely to need an extensive number of calls to them to produce the beautiful images that you may imagine. By comparison, using color or labels to help focus on important information in drawings is often worthwhile, as is using color to represent data values.



Shape and text examples

Double buffering and computer animations. StdDraw supports a powerful computer graphics feature known as *double buffering*. When double buffering is enabled by calling `enableDoubleBuffering()`, all drawing takes place on the *offscreen canvas*. The offscreen canvas is not displayed; it exists only in computer memory. Only when you call `show()` does your drawing get copied from the offscreen canvas to the *onscreen canvas*, where it is displayed in the standard drawing window. You can think of double buffering as collecting all of the lines, points, shapes, and text that you tell it to draw, and then drawing them all simultaneously, upon request. Double buffering enables you to precisely control *when* the drawing takes place.

One reason to use double buffering is for efficiency when performing a large number of drawing commands. Incrementally displaying a complex drawing *while* it is being created can be intolerably inefficient on many computer systems. For example, you can dramatically speed up PROGRAM 1.5.5 by adding a call to `enableDoubleBuffering()` before the `while` loop and a call to `show()` after the `while` loop. Now, the points appear all at once (instead of one at a time).

Our most important use of double buffering is to produce *computer animations*, where we create the illusion of motion by rapidly displaying static drawings. Such effects can provide compelling and dynamic visualizations of scientific phenomenon. We can produce animations by repeating the following four steps:

- Clear the offscreen canvas.
- Draw objects on the offscreen canvas.
- Copy the offscreen canvas to the onscreen canvas.
- Wait for a short while.

In support of the first and last of these steps, StdDraw provides three additional methods. The `clear()` methods clear the canvas, either to white or to a specified color. To control the apparent speed of an animation, the `pause()` method takes an argument `dt` and tells StdDraw to wait for `dt` milliseconds before processing additional commands.

```
public class StdDraw (advanced control commands)
```

<code>void enableDoubleBuffering()</code>	<i>enable double buffering</i>
<code>void disableDoubleBuffering()</code>	<i>disable double buffering</i>
<code>void show()</code>	<i>copy the offscreen canvas to the onscreen canvas</i>
<code>void clear()</code>	<i>clear the canvas to white (default)</i>
<code>void clear(Color color)</code>	<i>clear the canvas to color color</i>
<code>void pause(double dt)</code>	<i>pause dt milliseconds</i>

Bouncing ball. The “Hello, World” program for animation is to produce a black ball that appears to move around on the canvas, bouncing off the boundary according to the laws of elastic collision. Suppose that the ball is at position (r_x, r_y) and we want to create the impression of moving it to a nearby position, say, $(r_x + 0.01, r_y + 0.02)$. We do so in four steps:

- Clear the offscreen canvas to white.
- *Draw a black ball at the new position on the offscreen canvas.*
- Copy the offscreen canvas to the onscreen canvas.
- Wait for a short while.

To create the illusion of movement, we iterate these steps for a whole sequence of positions of the ball (one that will form a straight line, in this case). Without double buffering, the image of the ball will rapidly flicker between black and white instead of creating a smooth animation.

`BouncingBall` (PROGRAM 1.5.6) implements these steps to create the illusion of a ball moving in the 2-by-2 box centered at the origin. The current position of the ball is (r_x, r_y) , and we compute the new position at each step by adding v_x to r_x and v_y to r_y . Since (v_x, v_y) is the fixed distance that the ball moves in each time unit, it represents the *velocity*. To keep the ball in the standard drawing window, we simulate the effect of the ball bouncing off the walls according to the laws of elastic collision. This effect is easy to implement: when the ball hits a vertical wall, we change the velocity in the x -direction from v_x to $-v_x$, and when the ball hits a horizontal wall, we change the velocity in the y -direction from v_y to $-v_y$. Of course, you have to download the code from the booksite and run it on your computer to see motion. To make the image clearer on the printed page, we modified `BouncingBall` to use a gray background that also shows the track of the ball as it moves (see EXERCISE 1.5.34).

STANDARD DRAWING COMPLETES OUR PROGRAMMING MODEL by adding a “picture is worth a thousand words” component. It is a natural abstraction that you can use to better open up your programs to the outside world. With it, you can easily produce the function graphs and visual representations of data that are commonly used in science and engineering. We will put it to such uses frequently throughout this book. Any time that you spend now working with the sample programs on the last few pages will be well worth the investment. You can find many useful examples on the booksite and in the exercises, and you are certain to find some outlet for your creativity by using `StdDraw` to meet various challenges. Can you draw an n -pointed star? Can you make our bouncing ball actually bounce (by adding gravity)? You may be surprised at how easily you can accomplish these and other tasks.

Program 1.5.6 Bouncing ball

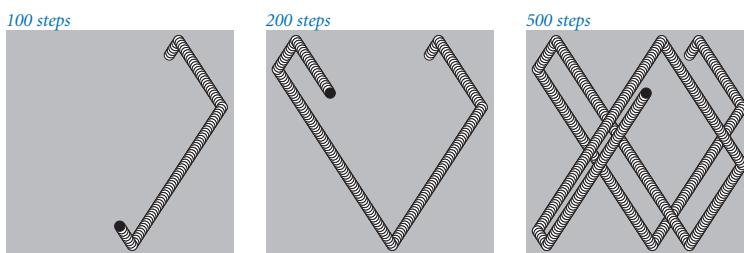
```

public class BouncingBall
{
    public static void main(String[] args)
    { // Simulate the motion of a bouncing ball.
        StdDraw.setScale(-1.0, 1.0);
        StdDraw.setYscale(-1.0, 1.0);
        StdDraw.enableDoubleBuffering();
        double rx = 0.480, ry = 0.860;
        double vx = 0.015, vy = 0.023;
        double radius = 0.05;
        while(true)
        { // Update ball position and draw it.
            if (Math.abs(rx + vx) + radius > 1.0) vx = -vx;
            if (Math.abs(ry + vy) + radius > 1.0) vy = -vy;
            rx += vx;
            ry += vy;
            StdDraw.clear();
            StdDraw.filledCircle(rx, ry, radius);
            StdDraw.show();
            StdDraw.pause(20);
        }
    }
}

```

rx, ry	position
vx, vy	velocity
dt	wait time
radius	ball radius

This program simulates the motion of a bouncing ball in the box with coordinates between -1 and $+1$. The ball bounces off the boundary according to the laws of inelastic collision. The 20-millisecond wait for `StdDraw.pause()` keeps the black image of the ball persistent on the screen, even though most of the ball's pixels alternate between black and white. The images below, which show the track of the ball, are produced by a modified version of this code (see EXERCISE 1.5.34).



This API table summarizes the StdDraw methods that we have considered:

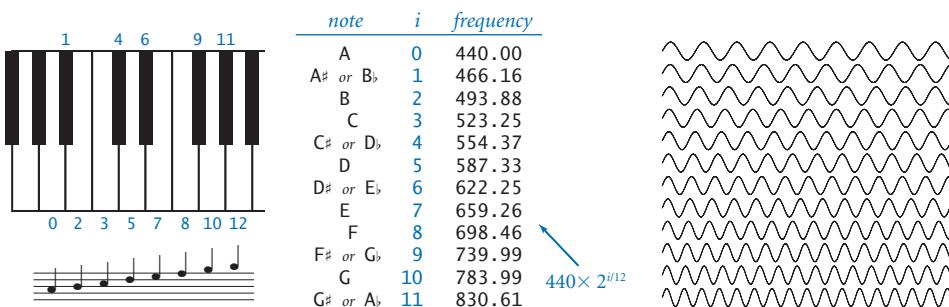
<code>public class StdDraw</code>	
<hr/>	
<i>drawing commands</i>	
<code>void line(double x0, double y0, double x1, double y1)</code>	
<code>void point(double x, double y)</code>	
<code>void circle(double x, double y, double radius)</code>	
<code>void filledCircle(double x, double y, double radius)</code>	
<code>void square(double x, double y, double radius)</code>	
<code>void filledSquare(double x, double y, double radius)</code>	
<code>void rectangle(double x, double y, double r1, double r2)</code>	
<code>void filledRectangle(double x, double y, double r1, double r2)</code>	
<code>void polygon(double[] x, double[] y)</code>	
<code>void filledPolygon(double[] x, double[] y)</code>	
<code>void text(double x, double y, String s)</code>	
<hr/>	
<i>control commands</i>	
<code>void setXscale(double x0, double x1)</code>	<i>reset x-scale to (x0, x1)</i>
<code>void setYscale(double y0, double y1)</code>	<i>reset y-scale to (y0, y1)</i>
<code>void setPenRadius(double radius)</code>	<i>set pen radius to radius</i>
<code>void setPenColor(Color color)</code>	<i>set pen color to color</i>
<code>void setFont(Font font)</code>	<i>set text font to font</i>
<code>void setCanvasSize(int w, int h)</code>	<i>set canvas size to w-by-h</i>
<code>void enableDoubleBuffering()</code>	<i>enable double buffering</i>
<code>void disableDoubleBuffering()</code>	<i>disable double buffering</i>
<code>void show()</code>	<i>copy the offscreen canvas to the onscreen canvas</i>
<code>void clear(Color color)</code>	<i>clear the canvas to color color</i>
<code>void pause(int dt)</code>	<i>pause dt milliseconds</i>
<code>void save(String filename)</code>	<i>save to a .jpg or .png file</i>

Note: Methods with the same names but no arguments reset to default values.

API for our library of static methods for standard drawing

Standard audio As a final example of a basic abstraction for output, we consider StdAudio, a library that you can use to play, manipulate, and synthesize sound. You probably have used your computer to process music. Now you can write programs to do so. At the same time, you will learn some concepts behind a venerable and important area of computer science and scientific computing: *digital signal processing*. We will merely scratch the surface of this fascinating subject, but you may be surprised at the simplicity of the underlying concepts.

Concert A. Sound is the perception of the vibration of molecules—in particular, the vibration of our eardrums. Therefore, oscillation is the key to understanding sound. Perhaps the simplest place to start is to consider the musical note *A* above middle *C*, which is known as *concert A*. This note is nothing more than a sine wave, scaled to oscillate at a frequency of 440 times per second. The function $\sin(t)$ repeats itself once every 2π units, so if we measure t in seconds and plot the function $\sin(2\pi t \times 440)$, we get a curve that oscillates 440 times per second. When you play an *A* by plucking a guitar string, pushing air through a trumpet, or causing a small cone to vibrate in a speaker, this sine wave is the prominent part of the sound that you hear and recognize as concert *A*. We measure frequency in *hertz* (cycles per second). When you double or halve the frequency, you move up or down one octave on the scale. For example, 880 hertz is one octave above concert *A* and 110 hertz is two octaves below concert *A*. For reference, the frequency range of human hearing is about 20 to 20,000 hertz. The amplitude (*y*-value) of a sound corresponds to the volume. We plot our curves between -1 and $+1$ and assume that any devices that record and play sound will scale as appropriate, with further scaling controlled by you when you turn the volume knob.



Notes, numbers, and waves

Other notes. A simple mathematical formula characterizes the other notes on the chromatic scale. There are 12 notes on the chromatic scale, evenly spaced on a logarithmic (base 2) scale. We get the i th note above a given note by multiplying its frequency by the $(i/12)$ th power of 2. In other words, the frequency of each note in the chromatic scale is precisely the frequency of the previous note in the scale multiplied by the twelfth root of 2 (about 1.06). This information suffices to create music! For example, to play the tune *Frère Jacques*, play each of the notes *A B C# A* by producing sine waves of the appropriate frequency for about half a second each, and then repeat the pattern. The primary method in the `StdAudio` library, `StdAudio.play()`, allows you to do exactly this.

Sampling. For digital sound, we represent a curve by sampling it at regular intervals, in precisely the same manner as when we plot function graphs. We sample sufficiently often that we have an accurate representation of the curve—a widely used sampling rate for digital sound is 44,100 samples per second. For concert *A*, that rate corresponds to plotting each cycle of the sine wave by sampling it at about 100 points. Since we sample at regular intervals, we only need to compute the y -coordinates of the sample points. It is that simple: *we represent sound as an array of real numbers* (between -1 and $+1$). The method `StdAudio.play()` takes an array as its argument and plays the sound represented by that array on your computer.

For example, suppose that you want to play concert *A* for 10 seconds. At 44,100 samples per second, you need a `double` array of length 441,001. To fill in the array, use a `for` loop that samples the function $\sin(2\pi t \times 440)$ at $t = 0/44,100, 1/44,100, 2/44,100, 3/44,100, \dots, 441,000/44,100$. Once we fill the array with these values, we are ready for `StdAudio.play()`, as in the following code:

```

int SAMPLING_RATE = 44100;           // samples per second
int hz = 440;                      // concert A
double duration = 10.0;             // ten seconds
int n = (int) (SAMPLING_RATE * duration);
double[] a = new double[n+1];
for (int i = 0; i <= n; i++)
    a[i] = Math.sin(2 * Math.PI * i * hz / SAMPLING_RATE);
StdAudio.play(a);

```

This code is the “Hello, World” of digital audio. Once you use it to get your computer to play this note, you can write code to play other notes and make music! The difference between creating sound and plotting an oscillating curve is nothing

more than the output device. Indeed, it is instructive and entertaining to send the same numbers to both standard drawing and standard audio (see EXERCISE 1.5.27).

Saving to a file. Music can take up a lot of space on your computer. At 44,100 samples per second, a four-minute song corresponds to $4 \times 60 \times 44100 = 10,584,000$ numbers. Therefore, it is common to represent the numbers corresponding to a song in a binary format that uses less space than the string-of-digits representation that we use for standard input and output. Many such formats have been developed in recent years—`StdAudio` uses the `.wav` format. You can find some information about the `.wav` format on the booksite, but you do not need to know the details, because `StdAudio` takes care of the conversions for you. Our standard library for audio allows you to read `.wav` files, write `.wav` files, and convert `.wav` files to arrays of `double` values for processing.

`PlayThatTune` (PROGRAM 1.5.7) is an example that shows how you can use `StdAudio` to turn your computer into a musical instrument. It takes notes from standard input, indexed on the chromatic scale from concert *A*, and plays them on standard audio. You can imagine all sorts of extensions on this basic scheme, some of which are addressed in the exercises.

WE INCLUDE STANDARD AUDIO IN OUR basic arsenal of programming tools because sound processing is one important application of scientific computing that is certainly familiar to you. Not only has the commercial application of digital signal processing had a phenomenal impact on modern society, but the science and engineering behind it combine physics and computer science in interesting ways. We will study more components of digital signal processing in some detail later in the book. (For example, you will learn in SECTION 2.1 how to create sounds that are more musical than the pure sounds produced by `PlayThatTune`.)

1/40 second (various sample rates)

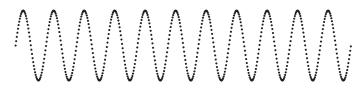
5,512 samples/second, 137 samples



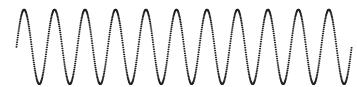
11,025 samples/second, 275 samples



22,050 samples/second, 551 samples

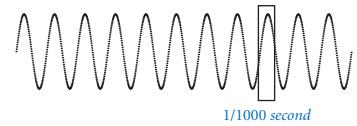


44,100 samples/second, 1,102 samples

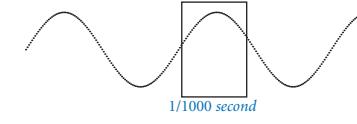


44,100 samples/second (various times)

1/40 second, 1,102 samples



1/200 second, 220 samples



1/1,000 second, 44 samples



Sampling a sine wave

Program 1.5.7 Digital signal processing

```

public class PlayThatTune
{
    public static void main(String[] args)
    { // Read a tune from StdIn and play it.
        int SAMPLING_RATE = 44100;
        while (!StdIn.isEmpty())
        { // Read and play one note.
            int pitch = StdIn.readInt();
            double duration = StdIn.readDouble();
            double hz = 440 * Math.pow(2, pitch / 12.0);
            int n = (int) (SAMPLING_RATE * duration);
            double[] a = new double[n+1];
            for (int i = 0; i <= n; i++)
                a[i] = Math.sin(2*Math.PI * i * hz / SAMPLING_RATE);
            StdAudio.play(a);
        }
    }
}

```

<p><code>pitch</code></p> <p><code>duration</code></p> <p><code>hz</code></p> <p><code>n</code></p> <p><code>a[]</code></p>	<p><i>distance from A</i></p> <p><i>note play time</i></p> <p><i>frequency</i></p> <p><i>number of samples</i></p> <p><i>sampled sine wave</i></p>
---	--

This data-driven program turns your computer into a musical instrument. It reads notes and durations from standard input and plays a pure tone corresponding to each note for the specified duration on standard audio. Each note is specified as a pitch (distance from concert A). After reading each note and duration, the program creates an array by sampling a sine wave of the specified frequency and duration at 44,100 samples per second, and plays it using `StdAudio.play()`.

```
% more elise.txt
7 0.25
6 0.25
7 0.25
6 0.25
7 0.25
2 0.25
5 0.25
3 0.25
0 0.50
```



```
% java PlayThatTune < elise.txt
```

The API table below summarizes the methods in StdAudio:

public class StdAudio	
void play(String filename)	<i>play the given .wav file</i>
void play(double[] a)	<i>play the given sound wave</i>
void play(double x)	<i>play sample for 1/44,100 second</i>
void save(String filename, double[] a)	<i>save to a .wav file</i>
double[] read(String filename)	<i>read from a .wav file</i>

API for our library of static methods for standard audio

Summary I/O is a compelling example of the power of abstraction because standard input, standard output, standard drawing, and standard audio can be tied to different physical devices at different times without making any changes to programs. Although devices may differ dramatically, we can write programs that can do I/O without depending on the properties of specific devices. From this point forward, we will use methods from StdOut, StdIn, StdDraw, and/or StdAudio in nearly every program in this book. For economy, we collectively refer to these libraries as `Std*`. One important advantage of using such libraries is that you can switch to new devices that are faster, are cheaper, or hold more data without changing your program at all. In such a situation, the details of the connection are a matter to be resolved between your operating system and the `Std*` implementations. On modern systems, new devices are typically supplied with software that resolves such details automatically both for the operating system and for Java.

Q&A

Q. How can I make `StdIn`, `StdOut`, `StdDraw`, and `StdAudio` available to Java?

A. If you followed the step-by-step instructions on the booksite for installing Java, these libraries should already be available to Java. Alternatively, you can copy the files `StdIn.java`, `StdOut.java`, `StdDraw.java`, and `StdAudio.java` from the booksite and put them in the same directory as the programs that use them.

Q. What does the error message `Exception in thread "main" java.lang.NoClassDefFoundError: StdIn` mean?

A. The library `StdIn` is not available to Java.

Q. Why are we not using the standard Java libraries for input, graphics, and sound?

A. We *are* using them, but we prefer to work with simpler abstract models. The Java libraries behind `StdIn`, `StdDraw`, and `StdAudio` are built for production programming, and the libraries and their APIs are a bit unwieldy. To get an idea of what they are like, look at the code in `StdIn.java`, `StdDraw.java`, and `StdAudio.java`.

Q. So, let me get this straight. If I use the format `%2.4f` for a `double` value, I get two digits before the decimal point and four digits after, right?

A. No, that specifies 4 digits after the decimal point. The first value is the width of the whole field. You want to use the format `%7.2f` to specify 7 characters in total, 4 before the decimal point, the decimal point itself, and 2 digits after the decimal point.

Q. Which other conversion codes are there for `printf()`?

A. For integer values, there is `o` for octal and `x` for hexadecimal. There are also numerous formats for dates and times. See the booksite for more information.

Q. Can my program reread data from standard input?

A. No. You get only one shot at it, in the same way that you cannot undo a `println()` command.

Q. What happens if my program attempts to read data from standard input after it is exhausted?

A. You will get an error. `StdIn.isEmpty()` allows you to avoid such an error by checking whether there is more input available.

Q. Why does `StdDraw.square(x, y, r)` draw a square of width $2 \cdot r$ instead of r ?

A. This makes it consistent with the function `StdDraw.circle(x, y, r)`, in which the third argument is the radius of the circle, not the diameter. In this context, r is the radius of the biggest circle that can fit inside the square.

Q. My terminal window hangs at the end of a program using `StdAudio`. How can I avoid having to use `<Ctrl-C>` to get a command prompt?

A. Add a call to `System.exit(0)` as the last line in `main()`. Don't ask why.

Q. Can I use negative integers to specify notes below concert A when making input files for `PlayThatTune`?

A. Yes. Actually, our choice to put concert A at 0 is arbitrary. A popular standard, known as the *MIDI Tuning Standard*, starts numbering at the C five octaves below concert A. By that convention, concert A is 69 and you do not need to use negative numbers.

Q. Why do I hear weird results on standard audio when I try to sonify a sine wave with a frequency of 30,000 hertz (or more)?

A. The *Nyquist frequency*, defined as one-half the sampling frequency, represents the highest frequency that can be reproduced. For standard audio, the sampling frequency is 44,100 hertz, so the Nyquist frequency is 22,050 hertz.

Exercises

1.5.1 Write a program that reads in integers (as many as the user enters) from standard input and prints the maximum and minimum values.

1.5.2 Modify your program from the previous exercise to insist that the integers must be positive (by prompting the user to enter positive integers whenever the value entered is not positive).

1.5.3 Write a program that takes an integer command-line argument n , reads n floating-point numbers from standard input, and prints their *mean* (average value) and *sample standard deviation* (square root of the sum of the squares of their differences from the average, divided by $n-1$).

1.5.4 Extend your program from the previous exercise to create a filter that reads n floating-point numbers from standard input, and prints those that are further than 1.5 standard deviations from the mean.

1.5.5 Write a program that reads in a sequence of integers and prints both the integer that appears in a longest consecutive run and the length of that run. For example, if the input is 1 2 2 1 5 1 1 7 7 7 7 1 1, then your program should print `Longest run: 4 consecutive 7s.`

1.5.6 Write a filter that reads in a sequence of integers and prints the integers, removing repeated values that appear consecutively. For example, if the input is 1 2 2 1 5 1 1 7 7 7 7 1 1 1 1 1 1 1, your program should print 1 2 1 5 1 7 1.

1.5.7 Write a program that takes an integer command-line argument n , reads in $n-1$ distinct integers between 1 and n , and determines the missing value.

1.5.8 Write a program that reads in positive floating-point numbers from standard input and prints their geometric and harmonic means. The *geometric mean* of n positive numbers x_1, x_2, \dots, x_n is $(x_1 \times x_2 \times \dots \times x_n)^{1/n}$. The *harmonic mean* is $n / (1/x_1 + 1/x_2 + \dots + 1/x_n)$. Hint: For the geometric mean, consider taking logarithms to avoid overflow.

1.5.9 Suppose that the file `input.txt` contains the two strings F and F. What does the following command do (see EXERCISE 1.2.35)?

```
% java Dragon < input.txt | java Dragon | java Dragon

public class Dragon
{
    public static void main(String[] args)
    {
        String dragon = StdIn.readString();
        String nogard = StdIn.readString();
        StdOut.print(dragon + "L" + nogard);
        StdOut.print(" ");
        StdOut.print(dragon + "R" + nogard);
        StdOut.println();
    }
}
```

1.5.10 Write a filter `TenPerLine` that reads from standard input a sequence of integers between 0 and 99 and prints them back, 10 integers per line, with columns aligned. Then write a program `RandomIntSeq` that takes two integer command-line arguments `m` and `n` and prints `n` random integers between 0 and `m-1`. Test your programs with the command `java RandomIntSeq 200 100 | java TenPerLine`.

1.5.11 Write a program that reads in text from standard input and prints the number of words in the text. For the purpose of this exercise, a word is a sequence of non-whitespace characters that is surrounded by whitespace.

1.5.12 Write a program that reads in lines from standard input with each line containing a name and two integers and then uses `printf()` to print a table with a column of the names, the integers, and the result of dividing the first by the second, accurate to three decimal places. You could use a program like this to tabulate batting averages for baseball players or grades for students.

1.5.13 Write a program that prints a table of the monthly payments, remaining principal, and interest paid for a loan, taking three numbers as command-line arguments: the number of years, the principal, and the interest rate (see EXERCISE 1.2.24).

1.5.14 Which of the following *require* saving all the values from standard input (in an array, say), and which could be implemented as a filter using only a fixed number of variables? For each, the input comes from standard input and consists of n real numbers between 0 and 1.

- Print the maximum and minimum numbers.
- Print the sum of the squares of the n numbers.
- Print the average of the n numbers.
- Print the median of the n numbers.
- Print the percentage of numbers greater than the average.
- Print the n numbers in increasing order.
- Print the n numbers in random order.

1.5.15 Write a program that takes three double command-line arguments x , y , and z , reads from standard input a sequence of point coordinates (x_i, y_i, z_i) , and prints the coordinates of the point closest to (x, y, z) . Recall that the square of the distance between (x, y, z) and (x_i, y_i, z_i) is $(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2$. For efficiency, do not use `Math.sqrt()`.

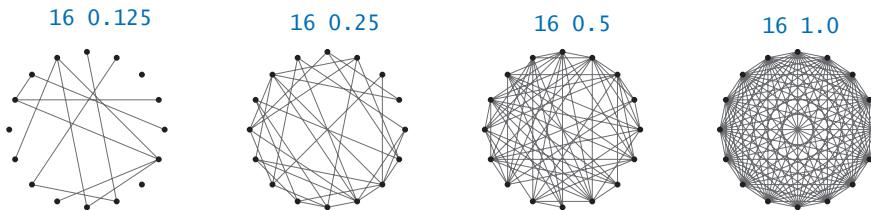
1.5.16 Given the positions and masses of a sequence of objects, write a program to compute their center-of-mass, or *centroid*. The centroid is the average position of the n objects, weighted by mass. If the positions and masses are given by (x_i, y_i, m_i) , then the centroid (x, y, m) is given by

$$\begin{aligned}m &= m_1 + m_2 + \dots + m_n \\x &= (m_1 x_1 + \dots + m_n x_n) / m \\y &= (m_1 y_1 + \dots + m_n y_n) / m\end{aligned}$$

1.5.17 Write a program that reads in a sequence of real numbers between -1 and $+1$ and prints their average magnitude, average power, and the number of zero crossings. The *average magnitude* is the average of the absolute values of the data values. The *average power* is the average of the squares of the data values. The number of *zero crossings* is the number of times a data value transitions from a strictly negative number to a strictly positive number, or vice versa. These three statistics are widely used to analyze digital signals.

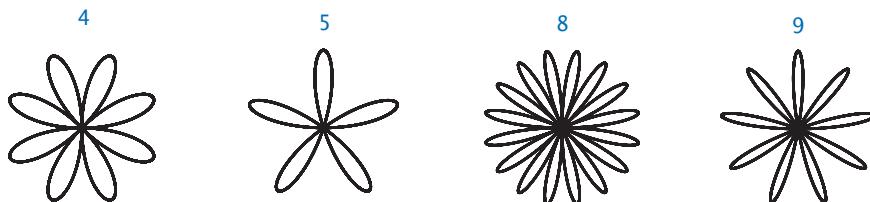
1.5.18 Write a program that takes an integer command-line argument n and plots an n -by- n checkerboard with red and black squares. Color the lower-left square red.

1.5.19 Write a program that takes as command-line arguments an integer n and a floating-point number p (between 0 and 1), plots n equally spaced points on the circumference of a circle, and then, with probability p for each pair of points, draws a gray line connecting them.



1.5.20 Write code to draw hearts, spades, clubs, and diamonds. To draw a heart, draw a filled diamond, then attach two filled semicircles to the upper left and upper right sides.

1.5.21 Write a program that takes an integer command-line argument n and plots a rose with n petals (if n is odd) or $2n$ petals (if n is even), by plotting the polar coordinates (r, θ) of the function $r = \sin(n\theta)$ for θ ranging from 0 to 2π radians.

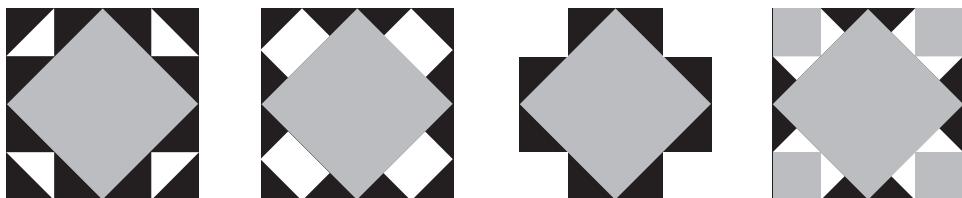


1.5.22 Write a program that takes a string command-line argument s and displays it in banner style on the screen, moving from left to right and wrapping back to the beginning of the string as the end is reached. Add a second command-line argument to control the speed.

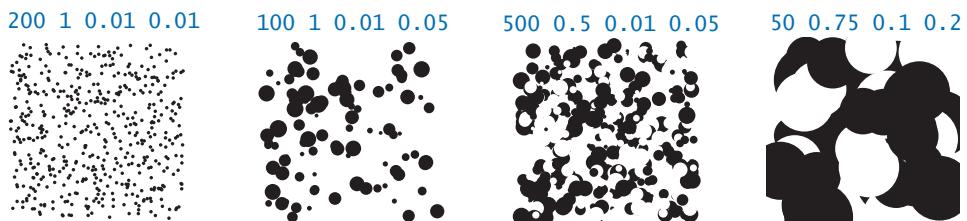
1.5.23 Modify `PlayThatTune` to take additional command-line arguments that control the volume (multiply each sample value by the volume) and the tempo (multiply each note's duration by the tempo).

1.5.24 Write a program that takes the name of a `.wav` file and a playback rate r as command-line arguments and plays the file at the given rate. First, use `StdAudio.read()` to read the file into an array $a[]$. If $r = 1$, play $a[]$; otherwise, create a new array $b[]$ of approximate size r times the length of $a[]$. If $r < 1$, populate $b[]$ by *sampling* from the original; if $r > 1$, populate $b[]$ by *interpolating* from the original. Then play $b[]$.

1.5.25 Write programs that uses `StdDraw` to create each of the following designs.



1.5.26 Write a program `Circles` that draws filled circles of random radii at random positions in the unit square, producing images like those below. Your program should take four command-line arguments: the number of circles, the probability that each circle is black, the minimum radius, and the maximum radius.



Creative Exercises

1.5.27 *Visualizing audio.* Modify `PlayThatTune` to send the values played to standard drawing, so that you can watch the sound waves as they are played. You will have to experiment with plotting multiple curves in the drawing canvas to synchronize the sound and the picture.

1.5.28 *Statistical polling.* When collecting statistical data for certain political polls, it is very important to obtain an unbiased sample of registered voters. Assume that you have a file with n registered voters, one per line. Write a filter that prints a uniformly random sample of size m (see PROGRAM 1.4.1).

1.5.29 *Terrain analysis.* Suppose that a terrain is represented by a two-dimensional grid of elevation values (in meters). A *peak* is a grid point whose four neighboring cells (left, right, up, and down) have strictly lower elevation values. Write a program `Peaks` that reads a terrain from standard input and then computes and prints the number of peaks in the terrain.

1.5.30 *Histogram.* Suppose that the standard input stream is a sequence of `double` values. Write a program that takes an integer n and two real numbers lo and hi as command-line arguments and uses `StdDraw` to plot a histogram of the count of the numbers in the standard input stream that fall in each of the n intervals defined by dividing (lo, hi) into n equal-sized intervals.

1.5.31 *Spirographs.* Write a program that takes three `double` command-line arguments R , r , and a and draws the resulting *spirograph*. A spirograph (technically, an epicycloid) is a curve formed by rolling a circle of radius r around a larger fixed circle of radius R . If the pen offset from the center of the rolling circle is $(r+a)$, then the equation of the resulting curve at time t is given by

$$\begin{aligned}x(t) &= (R + r) \cos(t) - (r + a) \cos((R + r)t/r) \\y(t) &= (R + r) \sin(t) - (r + a) \sin((R + r)t/r)\end{aligned}$$

Such curves were popularized by a best-selling toy that contains discs with gear teeth on the edges and small holes that you could put a pen in to trace spirographs.

1.5.32 Clock. Write a program that displays an animation of the second, minute, and hour hands of an analog clock. Use the method `StdDraw.pause(1000)` to update the display roughly once per second.

1.5.33 Oscilloscope. Write a program that simulates the output of an oscilloscope and produces *Lissajous patterns*. These patterns are named after the French physicist, Jules A. Lissajous, who studied the patterns that arise when two mutually perpendicular periodic disturbances occur simultaneously. Assume that the inputs are sinusoidal, so that the following parametric equations describe the curve:

$$\begin{aligned}x(t) &= A_x \sin(w_x t + \theta_x) \\y(t) &= A_y \sin(w_y t + \theta_y)\end{aligned}$$

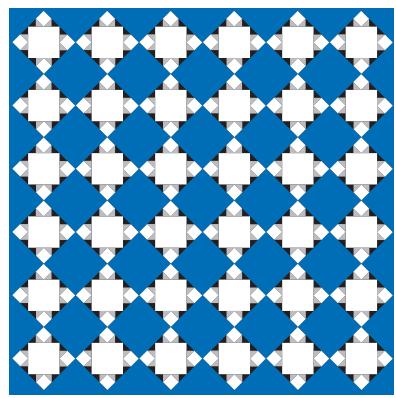
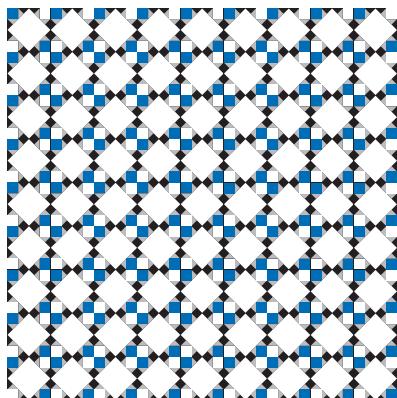
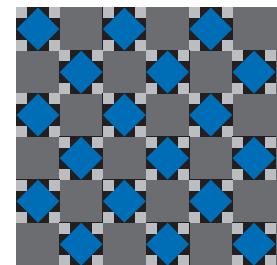
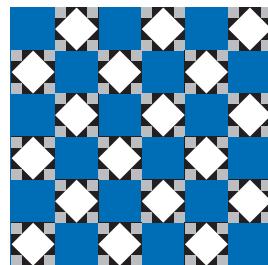
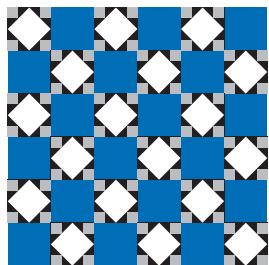
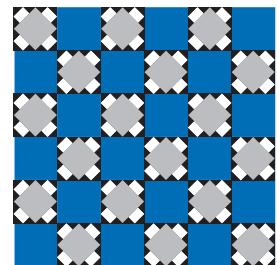
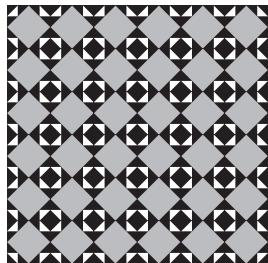
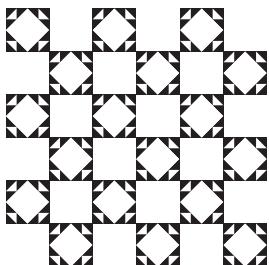
Take the six arguments A_x , w_x , θ_x , A_y , w_y , and θ_y from the command line.

1.5.34 Bouncing ball with tracks. Modify `BouncingBall` to produce images like the ones shown in the text, which show the track of the ball on a gray background.

1.5.35 Bouncing ball with gravity. Modify `BouncingBall` to incorporate gravity in the vertical direction. Add calls to `StdAudio.play()` to add a sound effect when the ball hits a wall and a different sound effect when it hits the floor.

1.5.36 Random tunes. Write a program that uses `StdAudio` to play random tunes. Experiment with keeping in key, assigning high probabilities to whole steps, repetition, and other rules to produce reasonable melodies.

1.5.37 Tile patterns. Using your solution to EXERCISE 1.5.25, write a program `TilePattern` that takes an integer command-line argument n and draws an n -by- n pattern, using the tile of your choice. Add a second command-line argument that adds a checkerboard option. Add a third command-line argument for color selection. Using the patterns on the facing page as a starting point, design a tile floor. Be creative! *Note:* These are all designs from antiquity that you can find in many ancient (and modern) buildings.



1.6 Case Study: Random Web Surfer

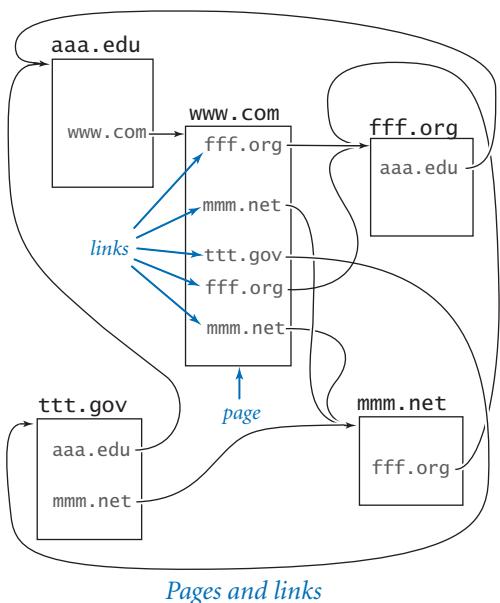
COMMUNICATING ACROSS THE WEB HAS BECOME an integral part of everyday life. This communication is enabled in part by scientific studies of the structure of the web, a subject of active research since its inception. We next consider a simple model of the web that has proved to be a particularly successful approach to understanding some of its properties. Variants of this model are widely used and have been a key factor in the explosive growth of search applications on the web.

The model is known as the *random surfer* model, and is simple to describe. We consider the web to be a fixed set of *web pages*, with each page containing a fixed set of *hyperlinks*, and each link a reference to some other page. (For brevity, we use the terms *pages* and *links*.) We study what happens to a web surfer who randomly moves from page to page, either by typing a page name into the address bar or by clicking a link on the current page.

The mathematical model that underlies the link structure of the web is known as the *graph*, which we will consider in detail at the end of the book (in SECTION 4.5).

We defer discussion about processing graphs until then. Instead, we concentrate on calculations associated with a natural and well-studied probabilistic model that accurately describes the behavior of the random surfer.

The first step in studying the random surfer model is to formulate it more precisely. The crux of the matter is to specify what it means to randomly move from page to page. The following intuitive *90–10 rule* captures both methods of moving to a new page: Assume that 90% of the time the random surfer clicks a random link on the current page (each link chosen with equal probability) and that 10% of the time the random surfer goes directly to a random page (all pages on the web chosen with equal probability).



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1.6.3 Mixing a Markov chain	182

Programs in this section

You can immediately see that this model has flaws, because you know from your own experience that the behavior of a real web surfer is not quite so simple:

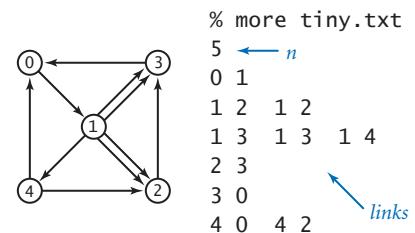
- No one chooses links or pages with equal probability.
- There is no real potential to surf directly to each page on the web.
- The 90–10 (or any fixed) breakdown is just a guess.
- It does not take the back button or bookmarks into account.

Despite these flaws, the model is sufficiently rich that computer scientists have learned a great deal about properties of the web by studying it. To appreciate the model, consider the small example on the previous page. Which page do you think the random surfer is most likely to visit?

Each person using the web behaves a bit like the random surfer, so understanding the fate of the random surfer is of intense interest to people building web infrastructure and web applications. The model is a tool for understanding the experience of each of the billions of web users. In this section, you will use the basic programming tools from this chapter to study the model and its implications.

Input format We want to be able to study the behavior of the random surfer on various graphs, not just one example. Consequently, we want to write *data-driven code*, where we keep data in files and write programs that read the data from standard input. The first step in this approach is to define an *input format* that we can use to structure the information in the input files. We are free to define any convenient input format.

Later in the book, you will learn how to read web pages in Java programs (SECTION 3.1) and to convert from names to numbers (SECTION 4.4) as well as other techniques for efficient graph processing. For now, we assume that there are n web pages, numbered from 0 to $n-1$, and we represent links with ordered pairs of such numbers, the first specifying the page containing the link and the second specifying the page to which it refers. Given these conventions, a straightforward input format for the random surfer problem is an input stream consisting of an integer (the value of n) followed by a sequence of pairs of integers (the representations of all the links). StdIn treats all sequences of whitespace characters as a single delimiter, so we are free to either put one link per line or arrange them several to a line.



Random surfer input format

Transition matrix We use a two-dimensional matrix, which we refer to as the *transition matrix*, to completely specify the behavior of the random surfer. With n web pages, we define an n -by- n matrix such that the value in row i and column j is the probability that the random surfer moves to page j when on page i . Our first task is to write code that can create such a matrix for any given input. By the 90–10 rule, this computation is not difficult. We do so in three steps:

- Read n , and then create arrays `counts[][]` and `outDegrees[]`.
- Read the links and accumulate counts so that `counts[i][j]` counts the links from i to j and `outDegrees[i]` counts the links from i to anywhere.
- Use the 90–10 rule to compute the probabilities.

The first two steps are elementary, and the third is not much more difficult: multiply `counts[i][j]` by $0.90/\text{outDegree}[i]$ if there is a link from i to j (take a random link with probability 0.9), and then add $0.10/n$ to each element (go to a random page with probability 0.1). `Transition` (PROGRAM 1.6.1) performs this calculation: it is a filter that reads a graph from standard input and prints the associated transition matrix to standard output.

The transition matrix is significant because each row represents a *discrete probability distribution*—the elements fully specify the behavior of the random surfer’s next move, giving the probability of surfing to each page. Note in particular that the elements sum to 1 (the surfer always goes somewhere).

The output of `Transition` defines another file format, one for matrices: the numbers of rows and columns followed by the values of the matrix elements, in row-major order. Now, we can write programs that read and process transition matrices.

input graph	link counts	outdegrees																				
	<table border="0"> <tr><td>5</td><td></td></tr> <tr><td>0 1</td><td>[0 1 0 0 0]</td><td>[1]</td></tr> <tr><td>1 2 1 2</td><td>[0 0 2 2 1]</td><td>[5]</td></tr> <tr><td>1 3 1 3 1 4</td><td>[0 0 0 1 0]</td><td>[1]</td></tr> <tr><td>2 3</td><td>[1 0 0 0 0]</td><td>[1]</td></tr> <tr><td>3 0</td><td>[1 0 1 0 0]</td><td>[2]</td></tr> <tr><td>4 0 4 2</td><td></td><td></td></tr> </table>	5		0 1	[0 1 0 0 0]	[1]	1 2 1 2	[0 0 2 2 1]	[5]	1 3 1 3 1 4	[0 0 0 1 0]	[1]	2 3	[1 0 0 0 0]	[1]	3 0	[1 0 1 0 0]	[2]	4 0 4 2			
5																						
0 1	[0 1 0 0 0]	[1]																				
1 2 1 2	[0 0 2 2 1]	[5]																				
1 3 1 3 1 4	[0 0 0 1 0]	[1]																				
2 3	[1 0 0 0 0]	[1]																				
3 0	[1 0 1 0 0]	[2]																				
4 0 4 2																						
leap probabilities	link probabilities	transition matrix																				
$\begin{bmatrix} .02 & .02 & .02 & .02 & .02 \\ .02 & .02 & .02 & .02 & .02 \\ .02 & .02 & .02 & .02 & .02 \\ .02 & .02 & .02 & .02 & .02 \\ .02 & .02 & .02 & .02 & .02 \end{bmatrix}$	$\begin{bmatrix} 0 & .90 & 0 & 0 & 0 \\ 0 & 0 & .36 & .36 & .18 \\ 0 & 0 & 0 & .90 & 0 \\ .90 & 0 & 0 & 0 & 0 \\ .45 & 0 & .45 & 0 & 0 \end{bmatrix}$	$=$ $\begin{bmatrix} .02 & .92 & .02 & .02 & .02 \\ .02 & .02 & .38 & .38 & .20 \\ .02 & .02 & .02 & .92 & .02 \\ .92 & .02 & .02 & .02 & .02 \\ .47 & .02 & .47 & .02 & .02 \end{bmatrix}$																				

Transition matrix computation

Program 1.6.1 Computing the transition matrix

```

public class Transition
{
    public static void main(String[] args)
    {
        int n = StdIn.readInt();
        int[][] counts = new int[n][n];
        int[] outDegrees = new int[n];
        while (!StdIn.isEmpty())
        { // Accumulate link counts.
            int i = StdIn.readInt();
            int j = StdIn.readInt();
            outDegrees[i]++;
            counts[i][j]++;
        }

        StdOut.println(n + " " + n);
        for (int i = 0; i < n; i++)
        { // Print probability distribution for row i.
            for (int j = 0; j < n; j++)
            { // Print probability for row i and column j.
                double p = 0.9*counts[i][j]/outDegrees[i] + 0.1/n;
                StdOut.printf("%.5f", p);
            }
            StdOut.println();
        }
    }
}

```

n $\text{counts}[i][j]$ $\text{outDegrees}[i]$ p	<i>number of pages</i> <i>count of links from page i to page j</i> <i>count of links from page i to anywhere</i> <i>transition probability</i>
---	--

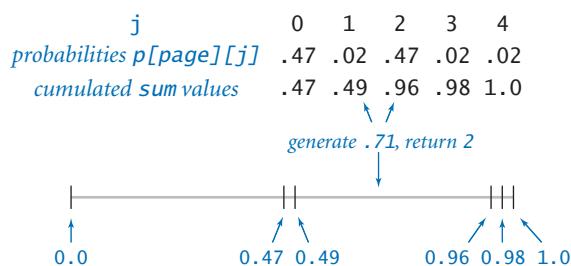
This program is a filter that reads links from standard input and produces the corresponding transition matrix on standard output. First it processes the input to count the outlinks from each page. Then it applies the 90–10 rule to compute the transition matrix (see text). It assumes that there are no pages that have no outlinks in the input (see EXERCISE 1.6.3).

```
% more tinyG.txt
5
0 1
1 2  1 2
1 3  1 3  1 4
2 3
3 0
4 0  4 2
```

```
% java Transition < tinyG.txt
5 5
0.02000 0.92000 0.02000 0.02000 0.02000
0.02000 0.02000 0.38000 0.38000 0.20000
0.02000 0.02000 0.02000 0.92000 0.02000
0.92000 0.02000 0.02000 0.02000 0.02000
0.47000 0.02000 0.47000 0.02000 0.02000
```

Simulation Given the transition matrix, simulating the behavior of the random surfer involves surprisingly little code, as you can see in `RandomSurfer` (PROGRAM 1.6.2). This program reads a transition matrix from standard input and surfs according to the rules, starting at page 0 and taking the number of moves as a command-line argument. It counts the number of times that the surfer visits each page. Dividing that count by the number of moves yields an estimate of the probability that a random surfer winds up on the page. This probability is known as the page's *rank*. In other words, `RandomSurfer` computes an estimate of all page ranks.

One random move. The key to the computation is the random move, which is specified by the transition matrix. We maintain a variable `page` whose value is the current location of the surfer. Row `page` of the matrix gives, for each `j`, the probability that the surfer next goes to `j`. In other words, when the surfer is at `page`, our task is to generate a random integer between 0 and `n-1` according to the distribution



Generating a random integer from a discrete distribution

given by row `page` in the transition matrix. How can we accomplish this task? We use a technique known as *roulette-wheel selection*. We use `Math.random()` to generate a random number `r` between 0 and 1, but how does that help us get to a random page? One way to answer this question is to think of the probabilities in row `page` as defining a set of n intervals in $(0, 1)$, with each probability corresponding to an interval length. Then

our random variable `r` falls into one of the intervals, with probability precisely specified by the interval length. This reasoning leads to the following code:

```
double sum = 0.0;
for (int j = 0; j < n; j++)
{ // Find interval containing r.
    sum += p[page][j];
    if (r < sum) { page = j; break; }
}
```

The variable `sum` tracks the endpoints of the intervals defined in row `page`, and the `for` loop finds the interval containing the random value `r`. For example, suppose that the surfer is at page 4 in our example. The transition probabilities are 0.47, 0.02, 0.47, 0.02, and 0.02, and `sum` takes on the values 0.0, 0.47, 0.49, 0.96,

Program 1.6.2 Simulating a random surfer

```

public class RandomSurfer
{
    public static void main(String[] args)
    { // Simulate random surfer.
        int trials = Integer.parseInt(args[0]);
        int n = StdIn.readInt();
        StdIn.readInt();

        // Read transition matrix.
        double[][] p = new double[n][n];
        for (int i = 0; i < n; i++)
            for (int j = 0; j < n; j++)
                p[i][j] = StdIn.readDouble();

        int page = 0;
        int[] freq = new int[n];
        for (int t = 0; t < trials; t++)
        { // Make one random move to next page.
            double r = Math.random();
            double sum = 0.0;
            for (int j = 0; j < n; j++)
            { // Find interval containing r.
                sum += p[page][j];
                if (r < sum) { page = j; break; }
            }
            freq[page]++;
        }

        for (int i = 0; i < n; i++) // Print page ranks.
            StdOut.printf("%8.5f", (double) freq[i] / trials);
        StdOut.println();
    }
}

```

<code>trials</code> <code>n</code> <code>page</code> <code>p[i][j]</code> <code>freq[i]</code>	<i>number of moves</i> <i>number of pages</i> <i>current page</i> <i>probability that the surfer moves from page i to page j</i> <i>number of times the surfer hits page i</i>
--	---

This program uses a transition matrix to simulate the behavior of a random surfer. It takes the number of moves as a command-line argument, reads the transition matrix, performs the indicated number of moves as prescribed by the matrix, and prints the relative frequency of hitting each page. The key to the computation is the random move to the next page (see text).

```

% java Transition < tinyG.txt | java RandomSurfer 100
0.24000 0.23000 0.16000 0.25000 0.12000
% java Transition < tinyG.txt | java RandomSurfer 1000000
0.27324 0.26568 0.14581 0.24737 0.06790

```

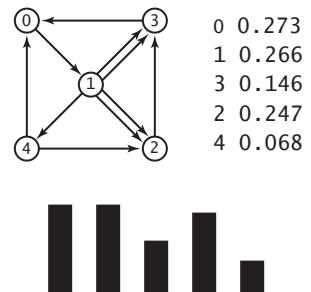
0.98, and 1.0. These values indicate that the probabilities define the five intervals $(0, 0.47)$, $(0.47, 0.49)$, $(0.49, 0.96)$, $(0.96, 0.98)$, and $(0.98, 1)$, one for each page. Now, suppose that `Math.random()` returns the value 0.71. We increment j from 0 to 1 to 2 and stop there, which indicates that 0.71 is in the interval $(0.49, 0.96)$, so we send the surfer to page 2. Then, we perform the same computation start at page 2, and the random surfer is off and surfing. For large n , we can use *binary search* to substantially speed up this computation (see EXERCISE 4.2.38). Typically, we are interested in speeding up the search in this situation because we are likely to need a huge number of random moves, as you will see.

Markov chains. The random process that describes the surfer's behavior is known as a *Markov chain*, named after the Russian mathematician Andrey Markov, who developed the concept in the early 20th century. Markov chains are widely applicable and well studied, and they have many remarkable and useful properties. For example, you may have wondered why `RandomSurfer` starts the random surfer at page 0—you might have expected a random choice. A basic limit theorem for Markov chains says that the surfer could start *anywhere*, because the probability that a random surfer eventually winds up on any particular page is the same for all starting pages! No matter where the surfer starts, the process eventually stabilizes to a point where further surfing provides no further information. This phenomenon is known as *mixing*. Though this phenomenon is perhaps counterintuitive at first, it explains coherent behavior in a situation that might seem chaotic. In the present context, it captures the idea that the web looks pretty much the same to everyone after surfing for a sufficiently long time. However, not all Markov chains have this mixing property. For example, if we eliminate the random leap from our model, certain configurations of web pages can present problems for the surfer. Indeed, there exist on the web sets of pages known as *spider traps*, which are designed to attract incoming links but have no outgoing links. Without the random leap, the surfer could get stuck in a spider trap. The primary purpose of the 90–10 rule is to guarantee mixing and eliminate such anomalies.

Page ranks. The `RandomSurfer` simulation is straightforward: it loops for the indicated number of moves, randomly surfing through the graph. Because of the mixing phenomenon, increasing the number of iterations gives increasingly accurate estimates of the probability that the surfer lands on each page (the page ranks). How do the results compare with your intuition when you first thought about the question? You might have guessed that page 4 was the lowest-ranked page, but did

you think that pages 0 and 1 would rank higher than page 3? If we want to know which page is the highest rank, we need more precision and more accuracy. `RandomSurfer` needs 10^n moves to get answers precise to n decimal places and many more moves for those answers to stabilize to an accurate value. For our example, it takes tens of thousands of iterations to get answers accurate to two decimal places and millions of iterations to get answers accurate to three places (see EXERCISE 1.6.5). The end result is that page 0 beats page 1 by 27.3% to 26.6%. That such a tiny difference would appear in such a small problem is quite surprising: if you guessed that page 0 is the most likely spot for the surfer to end up, you were lucky!

Accurate page rank estimates for the web are valuable in practice for many reasons. First, using them to put in order the pages that match the search criteria for web searches proved to be vastly more in line with people's expectations than previous methods. Next, this measure of confidence and reliability led to the investment of huge amounts of money in web advertising based on page ranks. Even in our tiny example, page ranks might be used to convince advertisers to pay up to four times as much to place an ad on page 0 as on page 4. Computing page ranks is mathematically sound, an interesting computer science problem, and big business, all rolled into one.



Visualizing the histogram. With `StdDraw`, it is also easy to create a visual representation that can give you a feeling for how the random surfer visit frequencies converge to the page ranks. If you enable double buffering; scale the x - and y -coordinates appropriately; add this code

```
StdDraw.clear();
for (int i = 0; i < n; i++)
    StdDraw.filledRectangle(i, freq[i]/2.0, 0.25, freq[i]/2.0);
StdDraw.show();
StdDraw.pause(10);
```

to the random move loop; and run `RandomSurfer` for a large number of trials, then you will see a drawing of the frequency histogram that eventually stabilizes to the page ranks. After you have used this tool once, you are likely to find yourself using it *every* time you want to study a new model (perhaps with some minor adjustments to handle larger models).

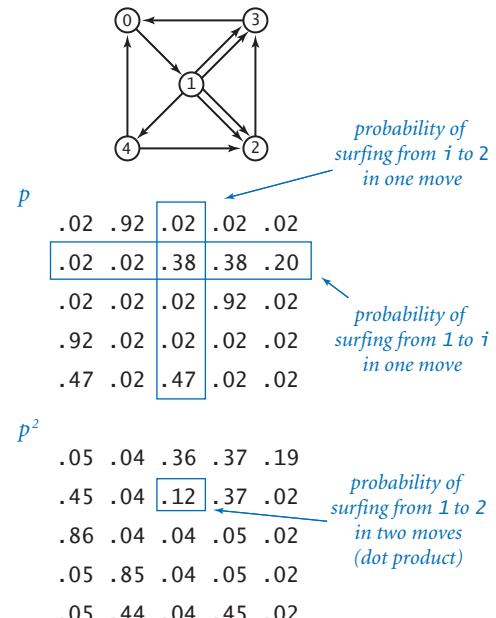
Page ranks with histogram

Studying other models. RandomSurfer and Transition are excellent examples of data-driven programs. You can easily define a graph by creating a file like `tiny.txt` that starts with an integer n and then specifies pairs of integers between 0 and $n-1$ that represent links connecting pages. You are encouraged to run it for various data models as suggested in the exercises, or to make up some graphs of your own to study. If you have ever wondered how web page ranking works, this calculation is your chance to develop better intuition about what causes one page to be ranked more highly than another. Which kind of page is likely to be rated highly? One that has many links to other pages, or one that has just a few links to other pages? The exercises in this section present many opportunities to study the behavior of the random surfer. Since RandomSurfer uses standard input, you can also write simple programs that generate large graphs, pipe their output through both Transition and RandomSurfer, and in this way study the random surfer on large graphs. Such flexibility is an important reason to use standard input and standard output.

DIRECTLY SIMULATING THE BEHAVIOR OF A random surfer to understand the structure of the web is appealing, but it has limitations. Think about the following question: could you use it to compute page ranks for a web graph with millions (or billions!) of web pages and links? The quick answer to this question is *no*, because you cannot even afford to store the transition matrix for such a large number of pages. A matrix for millions of pages would have *trillions* of elements. Do you have that much space on your computer? Could you use RandomSurfer to find page ranks for a smaller graph with, say, thousands of pages? To answer this question, you might run multiple simulations, record the results for a large number of trials, and then interpret those experimental results. We do use this approach for many scientific problems (the gambler's ruin problem is one example; SECTION 2.4 is devoted to another), but it can be very time-consuming, as a huge number of trials may be necessary to get the desired accuracy. Even for our tiny example, we saw that it takes millions of iterations to get the page ranks accurate to three or four decimal places. For larger graphs, the required number of iterations to obtain accurate estimates becomes truly huge.

Mixing a Markov chain It is important to remember that the page ranks are a property of the transition matrix, not any particular approach for computing them. That is, RandomSurfer is just *one* way to compute page ranks. Fortunately, a simple computational model based on a well-studied area of mathematics provides a far more efficient approach than simulation to the problem of computing page ranks. That model makes use of the basic arithmetic operations on two-dimensional matrices that we considered in SECTION 1.4.

Squaring a Markov chain. What is the probability that the random surfer will move from page i to page j in *two* moves? The first move goes to an intermediate page k , so we calculate the probability of moving from i to k and then from k to j for all possible k and add up the results. For our example, the probability of moving from 1 to 2 in two moves is the probability of moving from 1 to 0 to 2 (0.02×0.02), plus the probability of moving from 1 to 1 to 2 (0.02×0.38), plus the probability of moving from 1 to 2 to 2 (0.38×0.02), plus the probability of moving from 1 to 3 to 2 (0.38×0.02), plus the probability of moving from 1 to 4 to 2 (0.20×0.47), which adds up to a grand total of 0.1172. The same process works for each pair of pages. *This calculation is one that we have seen before*, in the definition of matrix multiplication: the element in row i and column j in the result is the dot product of row i and column j in the original. In other words, the result of multiplying $p[][]$ by itself is a matrix where the element in row i and column j is the probability that the random surfer moves from page i to page j in two moves. Studying the elements of the two-move transition matrix for our example is well worth your time and will help you better understand the movement of the random surfer. For instance, the largest value in the square is the one in row 2 and column 0, reflecting the fact that a surfer starting on page 2 has only one link out, to page 3, where there is also only one link out, to page 0. Therefore, by far the most likely outcome for a surfer starting on page 2 is to end up in page 0 after two moves. All of the other two-move routes involve more choices and are less probable. It is important to note that this



Squaring a Markov chain

is an exact computation (up to the limitations of Java’s floating-point precision); in contrast, `RandomSurfer` produces an estimate and needs more iterations to get a more accurate estimate.

The power method. We might then calculate the probabilities for three moves by multiplying by `p[][]` again, and for four moves by multiplying by `p[][]` yet again, and so forth. However, matrix–matrix multiplication is expensive, and we are actually interested in a *vector*–matrix multiplication. For our example, we start with the vector

[1.0 0.0 0.0 0.0 0.0]

which specifies that the random surfer starts on page 0. Multiplying this vector by the transition matrix gives the vector

[.02 .92 .02 .02 .02]

which is the probabilities that the surfer winds up on each of the pages after one step. Now, multiplying *this* vector by the transition matrix gives the vector

[.05 .04 .36 .37 .19]

which contains the probabilities that the surfer winds up on each of the pages after *two* steps. For example, the probability of moving from 0 to 2 in two moves is the probability of moving from 0 to 0 to 2 (0.02×0.02), plus the probability of moving from 0 to 1 to 2 (0.92×0.38), plus the probability of moving from 0 to 2 to 2 (0.02×0.02), plus the probability of moving from 0 to 3 to 2 (0.02×0.02), plus the probability of moving from 0 to 4 to 2 (0.02×0.47), which adds up to a grand total of 0.36. From these initial calculations, the pattern is clear: *the vector giving the probabilities that the random surfer is at each page after t steps is precisely the product of the corresponding vector for t – 1 steps and the transition matrix*. By the basic limit theorem for Markov chains, this process converges to the same vector no matter where we start; in other words, after a sufficient number of moves, the probability that the surfer ends up on any given page is independent of the starting point. `Markov` (PROGRAM 1.6.3) is an implementation that you can use to check convergence for our example. For instance, it gets the same results (the page ranks accurate to two decimal places) as `RandomSurfer`, but with just 20 matrix–vector multiplications instead of the tens of thousands of iterations needed by `RandomSurfer`. Another 20 multiplications gives the results accurate to three decimal places, as compared with millions of iterations for `RandomSurfer`, and just a few more give the results to full precision (see EXERCISE 1.6.6).

ranks[]

first move

$$[1.0 \ 0.0 \ 0.0 \ 0.0 \ 0.0 \ 0.0] * \begin{bmatrix} 0.02 & 0.92 & 0.02 & 0.02 & 0.02 \\ 0.02 & 0.02 & 0.38 & 0.38 & 0.20 \\ 0.02 & 0.02 & 0.02 & 0.92 & 0.02 \\ 0.92 & 0.02 & 0.02 & 0.02 & 0.02 \\ 0.47 & 0.02 & 0.47 & 0.02 & 0.02 \end{bmatrix} = [0.02 \ 0.92 \ 0.02 \ 0.02 \ 0.02]$$

probabilities of surfing from 0 to i in one move

second move

$$[0.02 \ 0.92 \ 0.02 \ 0.02 \ 0.02] * \begin{bmatrix} 0.02 & 0.92 & 0.02 & 0.02 & 0.02 \\ 0.02 & 0.02 & 0.38 & 0.38 & 0.20 \\ 0.02 & 0.02 & 0.02 & 0.92 & 0.02 \\ 0.92 & 0.02 & 0.02 & 0.02 & 0.02 \\ 0.47 & 0.02 & 0.47 & 0.02 & 0.02 \end{bmatrix} = [0.05 \ 0.04 \ 0.36 \ 0.37 \ 0.19]$$

probabilities of surfing from i to 2 in one move

probability of surfing from 0 to 2 in two moves (dot product)

probabilities of surfing from 0 to i in two moves

third move

$$[0.05 \ 0.04 \ 0.36 \ 0.37 \ 0.19] * \begin{bmatrix} 0.02 & 0.92 & 0.02 & 0.02 & 0.02 \\ 0.02 & 0.02 & 0.38 & 0.38 & 0.20 \\ 0.02 & 0.02 & 0.02 & 0.92 & 0.02 \\ 0.92 & 0.02 & 0.02 & 0.02 & 0.02 \\ 0.47 & 0.02 & 0.47 & 0.02 & 0.02 \end{bmatrix} = [0.44 \ 0.06 \ 0.12 \ 0.36 \ 0.03]$$

probabilities of surfing from 0 to i in three moves

▪

▪

▪

20th move

$$[0.27 \ 0.26 \ 0.15 \ 0.25 \ 0.07] * \begin{bmatrix} 0.02 & 0.92 & 0.02 & 0.02 & 0.02 \\ 0.02 & 0.02 & 0.38 & 0.38 & 0.20 \\ 0.02 & 0.02 & 0.02 & 0.92 & 0.02 \\ 0.92 & 0.02 & 0.02 & 0.02 & 0.02 \\ 0.47 & 0.02 & 0.47 & 0.02 & 0.02 \end{bmatrix} = [0.27 \ 0.26 \ 0.15 \ 0.25 \ 0.07]$$

probabilities of surfing from 0 to i in 20 moves (steady state)

The power method for computing page ranks (limit values of transition probabilities)

Program 1.6.3 Mixing a Markov chain

```

public class Markov
{ // Compute page ranks after trials moves.
    public static void main(String[] args)
    {
        int trials = Integer.parseInt(args[0]);
        int n = StdIn.readInt();
        StdIn.readInt();

        // Read transition matrix.
        double[][] p = new double[n][n];
        for (int i = 0; i < n; i++)
            for (int j = 0; j < n; j++)
                p[i][j] = StdIn.readDouble();

        // Use the power method to compute page ranks.
        double[] ranks = new double[n];
        ranks[0] = 1.0;
        for (int t = 0; t < trials; t++)
        { // Compute effect of next move on page ranks.
            double[] newRanks = new double[n];
            for (int j = 0; j < n; j++)
            { // New rank of page j is dot product
                // of old ranks and column j of p[]|.
                for (int k = 0; k < n; k++)
                    newRanks[j] += ranks[k]*p[k][j];
            }

            for (int j = 0; j < n; j++) // Update ranks[].
                ranks[j] = newRanks[j];
        }

        for (int i = 0; i < n; i++) // Print page ranks.
            StdOut.printf("%8.5f", ranks[i]);
        StdOut.println();
    }
}

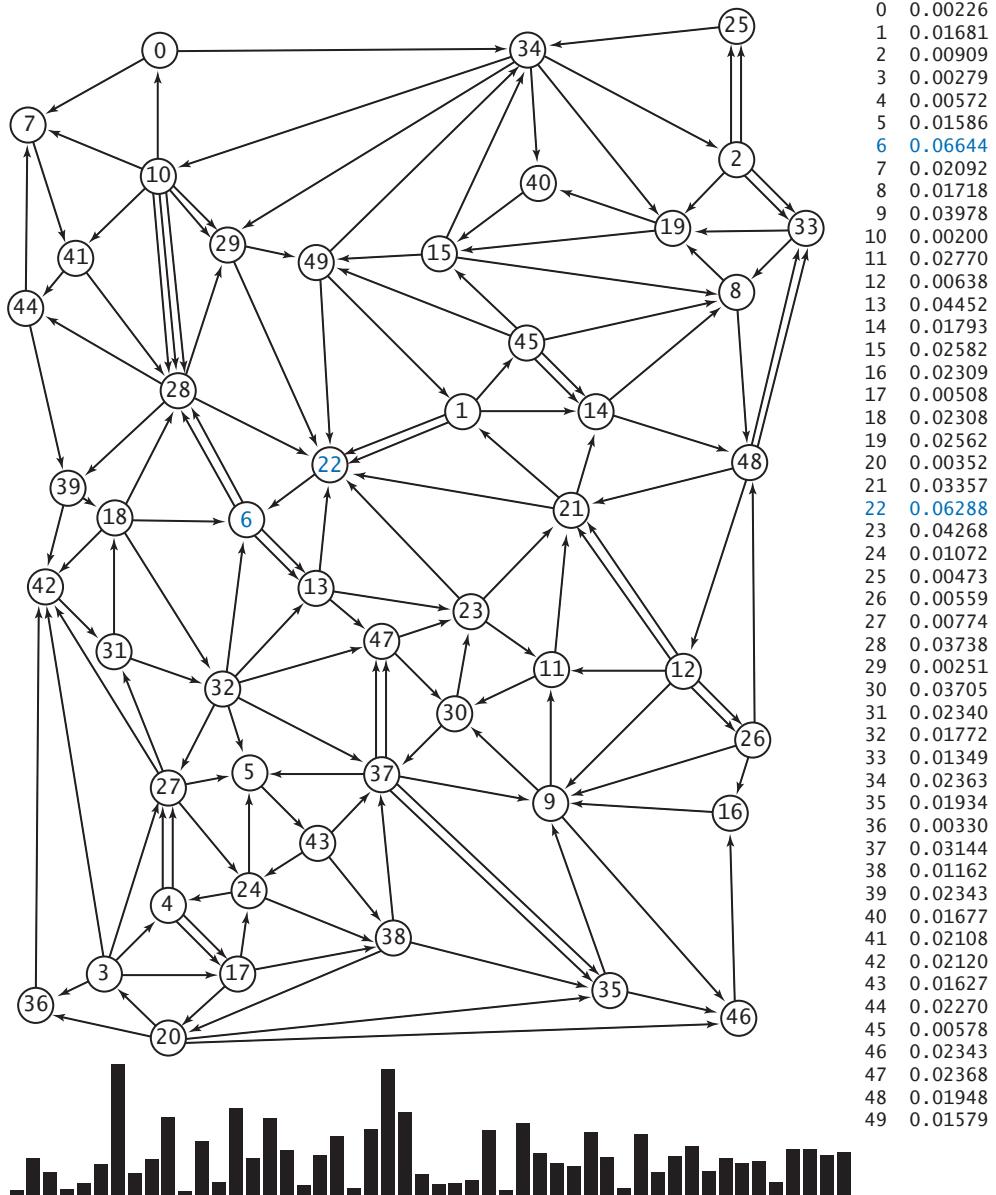
```

This program reads a transition matrix from standard input and computes the probabilities that a random surfer lands on each page (page ranks) after the number of steps specified as command-line argument.

```

% java Transition < tinyG.txt | java Markov 20
0.27245 0.26515 0.14669 0.24764 0.06806
% java Transition < tinyG.txt | java Markov 40
0.27303 0.26573 0.14618 0.24723 0.06783

```



Page ranks with histogram for a larger example

MARKOV CHAINS ARE WELL STUDIED, BUT their impact on the web was not truly felt until 1998, when two graduate students—Sergey Brin and Lawrence Page—had the audacity to build a Markov chain and compute the probabilities that a random surfer hits each page for *the whole web*. Their work revolutionized web search and is the basis for the page ranking method used by Google, the highly successful web search company that they founded. Specifically, their idea was to present to the user a list of web pages related to their search query *in decreasing order of page rank*. Page ranks (and related techniques) now predominate because they provide users with more *relevant* web pages for typical searches than earlier techniques (such as ordering pages by the number of incoming links). Computing page ranks is an enormously time-consuming task, due to the huge number of pages on the web, but the result has turned out to be enormously profitable and well worth the expense.

Lessons Developing a full understanding of the random surfer model is beyond the scope of this book. Instead, our purpose is to show you an application that involves writing a bit more code than the short programs that we have been using to teach specific concepts. Which specific lessons can we learn from this case study?

We already have a full computational model. Primitive types of data and strings, conditionals and loops, arrays, and standard input/output/drawing/audio enable you to address interesting problems of all sorts. Indeed, it is a basic precept of theoretical computer science that this model suffices to specify any computation that can be performed on any reasonable computing device. In the next two chapters, we discuss two critical ways in which the model has been extended to drastically reduce the amount of time and effort required to develop large and complex programs.

Data-driven code is prevalent. The concept of using the standard input and output streams and saving data in files is a powerful one. We write filters to convert from one kind of input to another, generators that can produce huge input files for study, and programs that can handle a wide variety of models. We can save data for archiving or later use. We can also process data derived from some other source and then save it in a file, whether it is from a scientific instrument or a distant website. The concept of data-driven code is an easy and flexible way to support this suite of activities.

Accuracy can be elusive. It is a mistake to assume that a program produces accurate answers simply because it can print numbers to many decimal places of precision. Often, the most difficult challenge that we face is ensuring that we have accurate answers.

Uniform random numbers are only a start. When we speak informally about random behavior, we often are thinking of something more complicated than the “every value equally likely” model that `Math.random()` gives us. Many of the problems that we consider involve working with random numbers from other distributions, such as `RandomSurfer`.

Efficiency matters. It is also a mistake to assume that your computer is so fast that it can do *any* computation. Some problems require much more computational effort than others. For example, the method used in `Markov` is far more efficient than directly simulating the behavior of a random surfer, but it is still too slow to compute page ranks for the huge web graphs that arise in practice. CHAPTER 4 is devoted to a thorough discussion of evaluating the performance of the programs that you write. We defer detailed consideration of such issues until then, but remember that you always need to have some general idea of the performance requirements of your programs.

PERHAPS THE MOST IMPORTANT LESSON TO learn from writing programs for complicated problems like the example in this section is that *debugging is difficult*. The polished programs in the book mask that lesson, but you can rest assured that each one is the product of a long bout of testing, fixing bugs, and running the programs on numerous inputs. Generally we avoid describing bugs and the process of fixing them in the text because that makes for a boring account and overly focuses attention on bad code, but you can find some examples and descriptions in the exercises and on the booksite.

Exercises

1.6.1 Modify `Transition` to take the leap probability as a command-line argument and use your modified version to examine the effect on page ranks of switching to an 80–20 rule or a 95–5 rule.

1.6.2 Modify `Transition` to ignore the effect of multiple links. That is, if there are multiple links from one page to another, count them as one link. Create a small example that shows how this modification can change the order of page ranks.

1.6.3 Modify `Transition` to handle pages with no outgoing links, by filling rows corresponding to such pages with the value $1/n$, where n is the number of columns.

1.6.4 The code fragment in `RandomSurfer` that generates the random move fails if the probabilities in the row `p[page]` do not add up to 1. Explain what happens in that case, and suggest a way to fix the problem.

1.6.5 Determine, to within a factor of 10, the number of iterations required by `RandomSurfer` to compute page ranks accurate to 4 decimal places and to 5 decimal places for `tiny.txt`.

1.6.6 Determine the number of iterations required by `Markov` to compute page ranks accurate to 3 decimal places, to 4 decimal places, and to ten 10 places for `tiny.txt`.

1.6.7 Download the file `medium.txt` from the booksite (which reflects the 50-page example depicted in this section) and add to it links *from* page 23 *to* every other page. Observe the effect on the page ranks, and discuss the result.

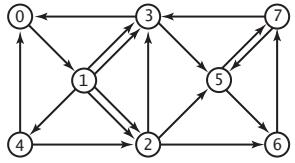
1.6.8 Add to `medium.txt` (see the previous exercise) links *to* page 23 *from* every other page, observe the effect on the page ranks, and discuss the result.

1.6.9 Suppose that your page is page 23 in `medium.txt`. Is there a link that you could add from your page to some other page that would *raise* the rank of *your* page?

1.6.10 Suppose that your page is page 23 in `medium.txt`. Is there a link that you could add from your page to some other page that would *lower* the rank of *that* page?

1.6.11 Use Transition and RandomSurfer to determine the page ranks for the eight-page graph shown below.

1.6.12 Use Transition and Markov to determine the page ranks for the eight-page graph shown below.



Eight-page example

Creative Exercises

1.6.13 *Matrix squaring.* Write a program like `Markov` that computes page ranks by repeatedly squaring the matrix, thus computing the sequence p, p^2, p^4, p^8, p^{16} , and so forth. Verify that all of the rows in the matrix converge to the same values.

1.6.14 *Random web.* Write a generator for `Transition` that takes as command-line arguments a page count n and a link count m and prints to standard output n followed by m random pairs of integers from 0 to $n-1$. (See SECTION 4.5 for a discussion of more realistic web models.)

1.6.15 *Hubs and authorities.* Add to your generator from the previous exercise a fixed number of *hubs*, which have links pointing to them from 10% of the pages, chosen at random, and *authorities*, which have links pointing from them to 10% of the pages. Compute page ranks. Which rank higher, hubs or authorities?

1.6.16 *Page ranks.* Design a graph in which the highest-ranking page has fewer links pointing to it than some other page.

1.6.17 *Hitting time.* The hitting time for a page is the expected number of moves between times the random surfer visits the page. Run experiments to estimate the hitting times for `tiny.txt`, compare hitting times with page ranks, formulate a hypothesis about the relationship, and test your hypothesis on `medium.txt`.

1.6.18 *Cover time.* Write a program that estimates the time required for the random surfer to visit every page at least once, starting from a random page.

1.6.19 *Graphical simulation.* Create a graphical simulation where the size of the dot representing each page is proportional to its page rank. To make your program data driven, design a file format that includes coordinates specifying where each page should be drawn. Test your program on `medium.txt`.

Functions and Modules

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THIS CHAPTER CENTERS ON A CONSTRUCT that has as profound an impact on control flow as do conditionals and loops: the *function*, which allows us to transfer control back and forth between different pieces of code. Functions (which are known as *static methods* in Java) are important because they allow us to clearly separate tasks within a program and because they provide a general mechanism that enables us to reuse code.

We group functions together in *modules*, which we can compile independently. We use modules to break a computational task into subtasks of a reasonable size. You will learn in this chapter how to build modules of your own and how to use them, in a style of programming known as *modular programming*.

Some modules are developed with the primary intent of providing code that can be reused later by many other programs. We refer to such modules as *libraries*. In particular, we consider in this chapter libraries for generating random numbers, analyzing data, and providing input/output for arrays. Libraries vastly extend the set of operations that we use in our programs.

We pay special attention to functions that transfer control to themselves—a process known as *recursion*. At first, recursion may seem counterintuitive, but it allows us to develop simple programs that can address complex tasks that would otherwise be much more difficult to carry out.

Whenever you can clearly separate tasks within programs, you should do so. We repeat this mantra throughout this chapter, and end the chapter with a case study showing how a complex programming task can be handled by breaking it into smaller subtasks, then independently developing modules that interact with one another to address the subtasks.

2.1 Defining Functions

THE JAVA CONSTRUCT FOR IMPLEMENTING A function is known as the *static method*. The modifier `static` distinguishes this kind of method from the kind discussed in CHAPTER 3—we will apply it consistently for now and discuss the difference then. You have actually been using static methods since the beginning of this book, from mathematical functions such as `Math.abs()` and `Math.sqrt()` to all of the methods in `StdIn`, `StdOut`, `StdDraw`, and `StdAudio`. Indeed, every Java program that you have written has a static method named `main()`. In this section, you will learn how to *define* your own static methods.

In mathematics, a *function* maps an input value of one type (the *domain*) to an output value of another type (the *range*). For example, the function $f(x) = x^2$ maps 2 to 4, 3 to 9, 4 to 16, and so forth. At first, we work with static methods that implement mathematical functions, because they are so familiar. Many standard mathematical functions are implemented in Java’s `Math` library, but scientists and engineers work with a broad variety of mathematical functions, which cannot all be included in the library. At the beginning of this section, you will learn how to implement such functions on your own.

Later, you will learn that we can do more with static methods than implement mathematical functions: static methods can have strings and other types as their range or domain, and they can produce side effects such as printing output. We also consider in this section how to use static methods to organize programs and thus to simplify complicated programming tasks.

Static methods support a key concept that will pervade your approach to programming from this point forward: *whenever you can clearly separate tasks within programs, you should do so*. We will be overemphasizing this point throughout this section and reinforcing it throughout this book. When you write an essay, you break it up into paragraphs; when you write a program, you will break it up into methods. Separating a larger task into smaller ones is much more important in programming than in writing, because it greatly facilitates *debugging*, *maintenance*, and *reuse*, which are all critical in developing good software.

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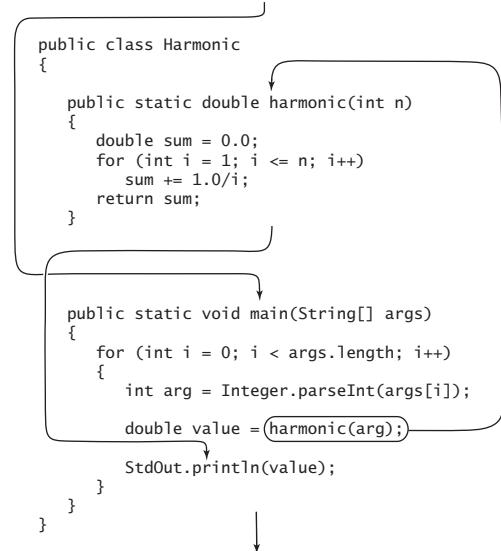
Programs in this section

Static methods As you know from using Java’s Math library, the use of static methods is easy to understand. For example, when you write `Math.abs(a-b)` in a program, the effect is as if you were to replace that code with the *return value* that is produced by Java’s `Math.abs()` method when passed the expression `a-b` as an *argument*. This usage is so intuitive that we have hardly needed to comment on it. If you think about what the system has to do to create this effect, you will see that it involves changing a program’s *control flow*. The implications of being able to change the control flow in this way are as profound as doing so for conditionals and loops.

You can *define* static methods other than `main()` in a `.java` file by specifying a method signature, followed by a sequence of statements that constitute the method. We will consider the details shortly, but we begin with a simple example—`Harmonic` (PROGRAM 2.1.1)—that illustrates how methods affect control flow. It features a static method named `harmonic()` that takes an integer argument `n` and returns the `n`th harmonic number (see PROGRAM 1.3.5).

PROGRAM 2.1.1 is superior to our original implementation for computing harmonic numbers (PROGRAM 1.3.5) because it clearly separates the two primary tasks performed by the program: calculating harmonic numbers and interacting with the user. (For purposes of illustration, PROGRAM 2.1.1 takes several command-line arguments instead of just one.) *Whenever you can clearly separate tasks within programs, you should do so.*

Control flow. While `Harmonic` appeals to our familiarity with mathematical functions, we will examine it in detail so that you can think carefully about what a static method is and how it operates. `Harmonic` comprises two static methods: `harmonic()` and `main()`. Even though `harmonic()` appears first in the code, the first statement that Java executes is, as usual, the first statement in `main()`. The next few statements operate as usual, except that the code `harmonic(arg)`, which is known as a *call* on the static method `harmonic()`, causes a *transfer of control* to the first line of code in `harmonic()`, each time that it is encountered. Moreover, Java



Flow of control for a call on a static method

Program 2.1.1 Harmonic numbers (revisited)

```

public class Harmonic
{
    public static double harmonic(int n)
    {
        double sum = 0.0;
        for (int i = 1; i <= n; i++)
            sum += 1.0/i;
        return sum;
    }

    public static void main(String[] args)
    {
        for (int i = 0; i < args.length; i++)
        {
            int arg = Integer.parseInt(args[i]);
            double value = harmonic(arg);
            StdOut.println(value);
        }
    }
}

```

sum | *cumulated sum*

arg | *argument*
value | *return value*

This program defines two static methods, one named `harmonic()` that has integer argument `n` and computes the `n`th harmonic numbers (see PROGRAM 1.3.5) and one named `main()`, which tests `harmonic()` with integer arguments specified on the command line.

```
% java Harmonic 1 2 4
1.0
1.5
2.083333333333333
```

```
% java Harmonic 10 100 1000 10000
2.9289682539682538
5.187377517639621
7.485470860550343
9.787606036044348
```

initializes the *parameter variable* `n` in `harmonic()` to the value of `arg` in `main()` at the time of the call. Then, Java executes the statements in `harmonic()` as usual, until it reaches a *return statement*, which transfers control back to the statement in `main()` containing the call on `harmonic()`. Moreover, the method call `harmonic(arg)` produces a value—the value specified by the `return` statement, which is the value of the variable `sum` in `harmonic()` at the time that the `return`

statement is executed. Java then assigns this *return value* to the variable `value`. The end result exactly matches our intuition: The first value assigned to `value` and printed is 1.0—the value computed by code in `harmonic()` when the parameter variable `n` is initialized to 1. The next value assigned to `value` and printed is 1.5—the value computed by `harmonic()` when `n` is initialized to 2. The same process is repeated for each command-line argument, transferring control back and forth between `harmonic()` and `main()`.

Function-call trace. One simple approach to following the control flow through function calls is to imagine that each function prints its name and argument value(s) when it is called and its return value just before returning, with indentation added on calls and subtracted on returns. The result enhances the process of tracing a program by printing the values of its variables, which we have been using since SECTION 1.2. The added indentation exposes the flow of the control, and helps us check that each function has the effect that we expect. Generally, adding calls on `StdOut.println()` to trace *any* program’s control flow in this way is a fine way to begin to understand what it is doing. If the return values match our expectations, we need not trace the function code in detail, saving us a substantial amount of work.

FOR THE REST OF THIS CHAPTER, your programming will center on creating and using static methods, so it is worthwhile to consider in more detail their basic properties. Following that, we will study several examples of function implementations and applications.

Terminology. It is useful to draw a distinction between abstract concepts and Java mechanisms to implement them (the Java `if` statement implements the conditional, the `while` statement implements the loop, and so forth). Several concepts are rolled up in the idea of a mathematical function, and there are Java constructs corresponding to each, as summarized in the table at the top of the next page. While these formalisms have served mathematicians well for centuries (and have served programmers well for decades), we will refrain from considering in detail all of the implications of this correspondence and focus on those that will help you learn to program.

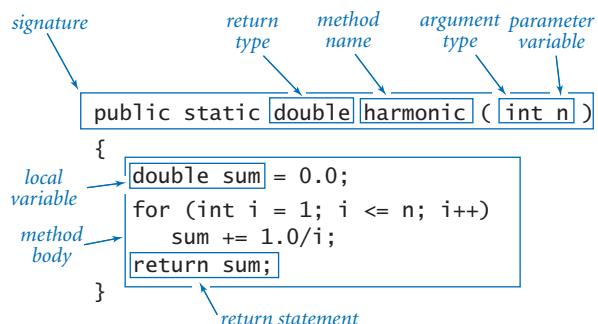
```
i = 0
arg = 1
harmonic(1)
    sum = 0.0
    sum = 1.0
    return 1.0
value = 1.0
i = 1
arg = 2
harmonic(2)
    sum = 0.0
    sum = 1.0
    sum = 1.5
    return 1.5
value = 1.5
i = 2
arg = 4
harmonic(4)
    sum = 0.0
    sum = 1.0
    sum = 1.5
    sum = 1.833333333333333
    sum = 2.083333333333333
    return 2.083333333333333
value = 2.083333333333333
```

*Function-call trace for
java Harmonic 1 2 4*

<i>concept</i>	<i>Java construct</i>	<i>description</i>
<i>function</i>	static method	mapping
<i>input value</i>	argument	input to function
<i>output value</i>	return value	output from function
<i>formula</i>	method body	function definition
<i>independent variable</i>	parameter variable	symbolic placeholder for input value

When we use a symbolic name in a formula that defines a mathematical function (such as $f(x) = 1 + x + x^2$), the symbol x is a placeholder for some input value that will be substituted into the formula to determine the output value. In Java, we use a *parameter variable* as a symbolic placeholder and we refer to a particular input value where the function is to be evaluated as an *argument*.

Static method definition. The first line of a static method definition, known as the *signature*, gives a name to the method and to each parameter variable. It also specifies the type of each parameter variable and the return type of the method. The signature consists of the keyword `public`; the keyword `static`; the *return type*; the *method name*; and a sequence of zero or more parameter variable types and names, separated by commas and enclosed in parentheses. We will discuss the meaning of the `public` keyword in the next section and the meaning of the `static` keyword in CHAPTER 3. (Technically, the signature in Java includes only the method name and parameter types, but we leave that distinction for experts.) Following the signature is the *body* of the method, enclosed in curly braces. The body consists of the kinds of statements we discussed in CHAPTER 1. It also can contain a *return statement*, which transfers control back to the point where the static method was called and returns the result of the computation or *return value*. The body may declare *local variables*, which are variables that are available only inside the method in which they are declared.



Anatomy of a static method

Function calls. As you have already seen, a static method call in Java is nothing more than the method name followed by its arguments, separated by commas and enclosed in parentheses, in precisely the same form as is customary for mathematical functions. As noted in SECTION 1.2, a method call is an expression, so you can use it to build up more complicated expressions. Similarly, an argument is an expression—Java evaluates the expression and passes the resulting value to the method. So, you can write code like `Math.exp(-x*x/2) / Math.sqrt(2*Math.PI)` and Java knows what you mean.

Multiple arguments. Like a mathematical function, a Java static method can take on more than one argument, and therefore can have more than one parameter variable. For example, the following static method computes the length of the hypotenuse of a right triangle with sides of length `a` and `b`:

```
public static double hypotenuse(double a, double b)
{   return Math.sqrt(a*a + b*b); }
```

Although the parameter variables are of the same type in this case, in general they can be of different types. The type and the name of each parameter variable are declared in the function signature, with the declarations for each variable separated by commas.

Multiple methods. You can define as many static methods as you want in a `.java` file. Each method has a body that consists of a sequence of statements enclosed in curly braces. These methods are independent and can appear in any order in the file. A static method can call any other static method in the same file or any static method in a Java library such as `Math`, as illustrated with this pair of methods:

```
public static double square(double a)
{   return a*a; }

public static double hypotenuse(double a, double b)
{   return Math.sqrt(square(a) + square(b)); }
```

Also, as we see in the next section, a static method can call static methods in other `.java` files (provided they are accessible to Java). In SECTION 2.3, we consider the ramifications of the idea that a static method can even call *itself*.

```
for (int i = 0; i < args.length; i++)
{
    arg = Integer.parseInt(args[i]);
    double value = harmonic(arg);
    StdOut.println(value);
}
```

Anatomy of a function call

Overloading. Static methods with different signatures are different static methods. For example, we often want to define the same operation for values of different numeric types, as in the following static methods for computing absolute values:

```
public static int abs(int x)
{
    if (x < 0) return -x;
    else        return x;
}

public static double abs(double x)
{
    if (x < 0.0) return -x;
    else          return x;
}
```

These are two different methods, but are sufficiently similar so as to justify using the same name (`abs`). Using the same name for two static methods whose signatures differ is known as *overloading*, and is a common practice in Java programming. For example, the Java Math library uses this approach to provide implementations of `Math.abs()`, `Math.min()`, and `Math.max()` for all primitive numeric types. Another common use of overloading is to define two different versions of a method: one that takes an argument and another that uses a default value for that argument.

Multiple return statements. You can put `return` statements in a method wherever you need them: control goes back to the calling program as soon as the first `return` statement is reached. This *primality-testing* function is an example of a function that is natural to define using multiple `return` statements:

```
public static boolean isPrime(int n)
{
    if (n < 2) return false;
    for (int i = 2; i <= n/i; i++)
        if (n % i == 0) return false;
    return true;
}
```

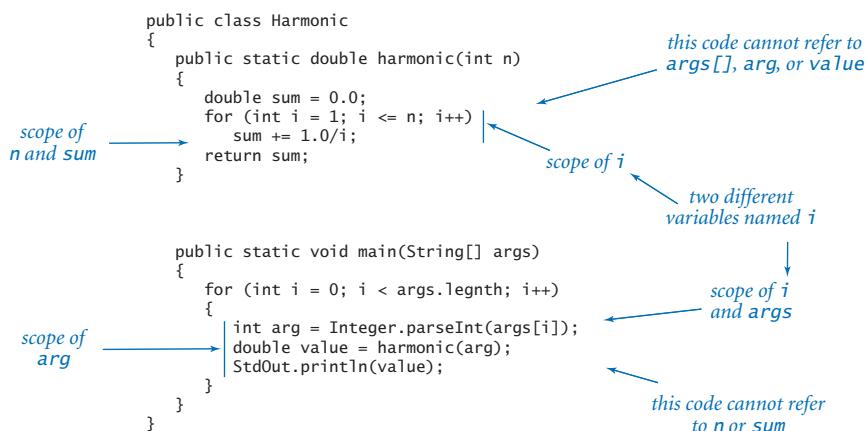
Even though there may be multiple `return` statements, any static method returns a single value each time it is invoked: the value following the first `return` statement encountered. Some programmers insist on having only one `return` per method, but we are not so strict in this book.

<i>absolute value of an int value</i>	<pre>public static int abs(int x) { if (x < 0) return -x; else return x; }</pre>
<i>absolute value of a double value</i>	<pre>public static double abs(double x) { if (x < 0.0) return -x; else return x; }</pre>
<i>primality test</i>	<pre>public static boolean isPrime(int n) { if (n < 2) return false; for (int i = 2; i <= n/i; i++) if (n % i == 0) return false; return true; }</pre>
<i>hypotenuse of a right triangle</i>	<pre>public static double hypotenuse(double a, double b) { return Math.sqrt(a*a + b*b); }</pre>
<i>harmonic number</i>	<pre>public static double harmonic(int n) { double sum = 0.0; for (int i = 1; i <= n; i++) sum += 1.0 / i; return sum; }</pre>
<i>uniform random integer in [0, n)</i>	<pre>public static int uniform(int n) { return (int) (Math.random() * n); }</pre>
<i>draw a triangle</i>	<pre>public static void drawTriangle(double x0, double y0, double x1, double y1, double x2, double y2) { StdDraw.line(x0, y0, x1, y1); StdDraw.line(x1, y1, x2, y2); StdDraw.line(x2, y2, x0, y0); }</pre>

Typical code for implementing functions (static methods)

Single return value. A Java method provides only one return value to the caller, of the type declared in the method signature. This policy is not as restrictive as it might seem because Java data types can contain more information than the value of a single primitive type. For example, you will see later in this section that you can use arrays as return values.

Scope. The *scope* of a variable is the part of the program that can refer to that variable by name. The general rule in Java is that the scope of the variables declared in a block of statements is limited to the statements in that block. In particular, the scope of a variable declared in a static method is limited to that method's body. Therefore, you cannot refer to a variable in one static method that is declared in another. If the method includes smaller blocks—for example, the body of an `if` or a `for` statement—the scope of any variables declared in one of those blocks is limited to just the statements within that block. Indeed, it is common practice to use the same variable names in independent blocks of code. When we do so, we are declaring different independent variables. For example, we have been following this practice when we use an index `i` in two different `for` loops in the same program. A guiding principle when designing software is that each variable should be declared so that its scope is as small as possible. One of the important reasons that we use static methods is that they ease debugging by limiting variable scope.



Scope of local and parameter variables

Side effects. In mathematics, a function maps one or more input values to some output value. In computer programming, many functions fit that same model: they accept one or more arguments, and their only purpose is to return a value. A *pure function* is a function that, given the same arguments, always returns the same value, without producing any observable *side effects*, such as consuming input, producing output, or otherwise changing the state of the system. The functions `harmonic()`, `abs()`, `isPrime()`, and `hypotenuse()` are examples of pure functions.

However, in computer programming it is also useful to define functions that do produce side effects. In fact, we often define functions whose *only* purpose is to produce side effects. In Java, a static method may use the keyword `void` as its return type, to indicate that it has no return value. An explicit `return` is not necessary in a `void` static method: control returns to the caller after Java executes the method's last statement.

For example, the static method `StdOut.println()` has the side effect of printing the given argument to standard output (and has no return value). Similarly, the following static method has the side effect of drawing a triangle to standard drawing (and has no specified return value):

```
public static void drawTriangle(double x0, double y0,
                                double x1, double y1,
                                double x2, double y2)
{
    StdDraw.line(x0, y0, x1, y1);
    StdDraw.line(x1, y1, x2, y2);
    StdDraw.line(x2, y2, x0, y0);
}
```

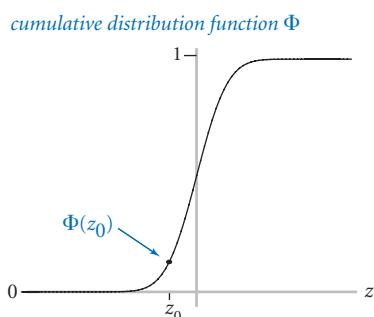
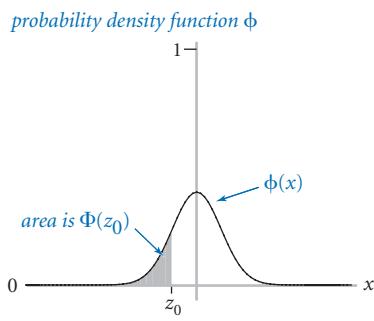
It is generally poor style to write a static method that both produces side effects and returns a value. One notable exception arises in functions that read input. For example, `StdIn.readInt()` both returns a value (an integer) and produces a side effect (consuming one integer from standard input). In this book, we use `void` static methods for two primary purposes:

- For I/O, using `StdIn`, `StdOut`, `StdDraw`, and `StdAudio`
- To manipulate the contents of arrays

You have been using `void` static methods for output since `main()` in `HelloWorld`, and we will discuss their use with arrays later in this section. It is possible in Java to write methods that have other side effects, but we will avoid doing so until CHAPTER 3, where we do so in a specific manner supported by Java.

Implementing mathematical functions Why not just use the methods that are defined within Java, such as `Math.sqrt()`? The answer to this question is that we *do* use such implementations when they are present. Unfortunately, there are an unlimited number of mathematical functions that we may wish to use and only a small set of functions in the library. When you encounter a mathematical function that is not in the library, you need to implement a corresponding static method.

As an example, we consider the kind of code required for a familiar and important application that is of interest to many high school and college students in the United States. In a recent year, more than 1 million students took a standard college entrance examination. Scores range from 400 (lowest) to 1600 (highest) on the multiple-choice parts of the test. These scores play a role in making important decisions: for example, student athletes are required to have a score of at least 820, and the minimum eligibility requirement for certain academic scholarships is 1500. What percentage of test takers are ineligible for athletics? What percentage are eligible for the scholarships?



Gaussian probability functions

Two functions from statistics enable us to compute accurate answers to these questions. The *Gaussian (normal) probability density function* is characterized by the familiar bell-shaped curve and defined by the formula $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$. The *Gaussian cumulative distribution function* $\Phi(z)$ is defined to be the area under the curve defined by $\phi(x)$ above the x -axis and to the left of the vertical line $x=z$. These functions play an important role in science, engineering, and finance because they arise as accurate models throughout the natural world and because they are essential in understanding experimental error.

In particular, these functions are known to accurately describe the distribution of test scores in our example, as a function of the mean (average value of the scores) and the standard deviation (square root of the average of the sum of the squares of the differences between each score and the mean), which are published each year. Given the mean μ and the standard deviation σ of the test scores, the percentage of students with scores less than a given value z is closely approximated by the function $\Phi((z-\mu)/\sigma)$. Static methods to calculate ϕ and Φ are not available in Java's `Math` library, so we need to develop our own implementations.

Program 2.1.2 Gaussian functions

```

public class Gaussian
{ // Implement Gaussian (normal) distribution functions.
    public static double pdf(double x)
    {
        return Math.exp(-x*x/2) / Math.sqrt(2*Math.PI);
    }

    public static double cdf(double z)
    {
        if (z < -8.0) return 0.0;
        if (z > 8.0) return 1.0;
        double sum = 0.0;
        double term = z;
        for (int i = 3; sum != sum + term; i += 2)
        {
            sum = sum + term;
            term = term * z * z / i;
        }
        return 0.5 + pdf(z) * sum;
    }

    public static void main(String[] args)
    {
        double z      = Double.parseDouble(args[0]);
        double mu     = Double.parseDouble(args[1]);
        double sigma = Double.parseDouble(args[2]);
        StdOut.printf("%.3f\n", cdf((z - mu) / sigma));
    }
}

```

sum | *cumulated sum*
term | *current term*

This code implements the Gaussian probability density function (`pdf`) and Gaussian cumulative distribution function (`cdf`), which are not implemented in Java's `Math` library. The `pdf()` implementation follows directly from its definition, and the `cdf()` implementation uses a Taylor series and also calls `pdf()` (see accompanying text and EXERCISE 1.3.38).

```

% java Gaussian 820 1019 209
0.171
% java Gaussian 1500 1019 209
0.989
% java Gaussian 1500 1025 231
0.980

```

Closed form. In the simplest situation, we have a closed-form mathematical formula defining our function in terms of functions that are implemented in the library. This situation is the case for ϕ —the Java Math library includes methods to compute the exponential and the square root functions (and a constant value for π), so a static method `pdf()` corresponding to the mathematical definition is easy to implement (see PROGRAM 2.1.2).

No closed form. Otherwise, we may need a more complicated algorithm to compute function values. This situation is the case for Φ —no closed-form expression exists for this function. Such algorithms sometimes follow immediately from Taylor series approximations, but developing reliably accurate implementations of mathematical functions is an art that needs to be addressed carefully, taking advantage of the knowledge built up in mathematics over the past several centuries. Many different approaches have been studied for evaluating Φ . For example, a Taylor series approximation to the ratio of Φ and ϕ turns out to be an effective basis for evaluating the function:

$$\Phi(z) = 1/2 + \phi(z) (z + z^3/3 + z^5/(3\cdot 5) + z^7/(3\cdot 5\cdot 7) + \dots)$$

This formula readily translates to the Java code for the static method `cdf()` in PROGRAM 2.1.2. For small (respectively large) z , the value is extremely close to 0 (respectively 1), so the code directly returns 0 (respectively 1); otherwise, it uses the Taylor series to add terms until the sum converges.

Running `Gaussian` with the appropriate arguments on the command line tells us that about 17% of the test takers were ineligible for athletics and that only about 1% qualified for the scholarship. In a year when the mean was 1025 and the standard deviation 231, about 2% qualified for the scholarship.

COMPUTING WITH MATHEMATICAL FUNCTIONS OF ALL kinds has always played a central role in science and engineering. In a great many applications, the functions that you need are expressed in terms of the functions in Java's Math library, as we have just seen with `pdf()`, or in terms of Taylor series approximations that are easy to compute, as we have just seen with `cdf()`. Indeed, support for such computations has played a central role throughout the evolution of computing systems and programming languages. You will find many examples on the booksite and throughout this book.

Using static methods to organize code Beyond evaluating mathematical functions, the process of calculating an output value on the basis of an input value is important as a general technique for organizing control flow in *any* computation. Doing so is a simple example of an extremely important principle that is a prime guiding force for any good programmer: *whenever you can clearly separate tasks within programs, you should do so.*

Functions are natural and universal for expressing computational tasks. Indeed, the “bird’s-eye view” of a Java program that we began with in SECTION 1.1 was equivalent to a function: we began by thinking of a Java program as a function that transforms command-line arguments into an output string. This view expresses itself at many different levels of computation. In particular, it is generally the case that a long program is more naturally expressed in terms of functions instead of as a sequence of Java assignment, conditional, and loop statements. With the ability to define functions, we can better organize our programs by defining functions within them when appropriate.

For example, `Coupon` (PROGRAM 2.1.3) is a version of `CouponCollector` (PROGRAM 1.4.2) that better separates the individual components of the computation. If you study PROGRAM 1.4.2, you will identify three separate tasks:

- Given n , compute a random coupon value.
- Given n , do the coupon collection experiment.
- Get n from the command line, and then compute and print the result.

`Coupon` rearranges the code in `CouponCollector` to reflect the reality that these three functions underlie the computation. With this organization, we could change `getCoupon()` (for example, we might want to draw the random numbers from a different distribution) or `main()` (for example, we might want to take multiple inputs or run multiple experiments) without worrying about the effect of any changes in `collectCoupons()`.

Using static methods isolates the implementation of each component of the collection experiment from others, or *encapsulates* them. Typically, programs have many independent components, which magnifies the benefits of separating them into different static methods. We will discuss these benefits in further detail after we have seen several other examples, but you certainly can appreciate that it is better to express a computation in a program by breaking it up into functions, just as it is better to express an idea in an essay by breaking it up into paragraphs. *Whenever you can clearly separate tasks within programs, you should do so.*

Program 2.1.3 Coupon collector (revisited)

```

public class Coupon
{
    public static int getCoupon(int n)
    { // Return a random integer between 0 and n-1.
        return (int) (Math.random() * n);
    }

    public static int collectCoupons(int n)
    { // Collect coupons until getting one of each value
        // and return the number of coupons collected.
        boolean[] isCollected = new boolean[n];
        int count = 0, distinct = 0;
        while (distinct < n)
        {
            int r = getcoupon(n);           n          # coupon values (0 to n-1)
            count++;                      isCollected[i] has coupon i been collected?
            if (!isCollected[r])          count      # coupons collected
                distinct++;               distinct   # distinct coupons collected
            isCollected[r] = true;         r          random coupon
        }
        return count;
    }

    public static void main(String[] args)
    { // Collect n different coupons.
        int n = Integer.parseInt(args[0]);
        int count = collectCoupons(n);
        StdOut.println(count);
    }
}

```

This version of PROGRAM 1.4.2 illustrates the style of encapsulating computations in static methods. This code has the same effect as *CouponCollector*, but better separates the code into its three constituent pieces: generating a random integer between 0 and $n-1$, running a coupon collection experiment, and managing the I/O.

```
% java Coupon 1000
6522
% java Coupon 1000
6481
```

```
% java Coupon 10000
105798
% java Coupon 1000000
12783771
```

Passing arguments and returning values Next, we examine the specifics of Java’s mechanisms for passing arguments to and returning values from functions. These mechanisms are conceptually very simple, but it is worthwhile to take the time to understand them fully, as the effects are actually profound. Understanding argument-passing and return-value mechanisms is key to learning any new programming language.

Pass by value. You can use parameter variables anywhere in the code in the body of the function in the same way you use local variables. The only difference between a parameter variable and a local variable is that Java evaluates the argument provided by the calling code and initializes the parameter variable with the resulting value. This approach is known as *pass by value*. The method works with the *value* of its arguments, not the arguments themselves. One consequence of this approach is that changing the value of a parameter variable within a static method has no effect on the calling code. (For clarity, we do not change parameter variables in the code in this book.) An alternative approach known as *pass by reference*, where the method works directly with the calling code’s arguments, is favored in some programming environments.

A STATIC METHOD CAN TAKE AN array as an argument or return an array to the caller. This capability is a special case of Java’s object orientation, which is the subject of CHAPTER 3. We consider it in the present context because the basic mechanisms are easy to understand and to use, leading us to compact solutions to a number of problems that naturally arise when we use arrays to help us process large amounts of data.

Arrays as arguments. When a static method takes an array as an argument, it implements a function that operates on an arbitrary number of values of the same type. For example, the following static method computes the mean (average) of an array of double values:

```
public static double mean(double[] a)
{
    double sum = 0.0;
    for (int i = 0; i < a.length; i++)
        sum += a[i];
    return sum / a.length;
}
```

We have been using arrays as arguments since our first program. The code

```
public static void main(String[] args)
```

defines `main()` as a static method that takes an array of strings as an argument and returns nothing. By convention, the Java system collects the strings that you type after the program name in the `java` command into an array and calls `main()` with that array as argument. (Most programmers use the name `args` for the parameter variable, even though any name at all would do.) Within `main()`, we can manipulate that array just like any other array.

Side effects with arrays. It is often the case that the purpose of a static method that takes an array as argument is to produce a side effect (change values of array elements). A prototypical example of such a method is one that exchanges the values at two given indices in a given array. We can adapt the code that we examined at the beginning of SECTION 1.4:

```
public static void exchange(String[] a, int i, int j)
{
    String temp = a[i];
    a[i] = a[j];
    a[j] = temp;
}
```

This implementation stems naturally from the Java array representation. The parameter variable in `exchange()` is a reference to the array, not a copy of the array values: when you pass an array as an argument to a method, the method has an opportunity to reassign values to the elements in that array. A second prototypical example of a static method that takes an array argument and produces side effects is one that randomly shuffles the values in the array, using this version of the algorithm that we examined in SECTION 1.4 (and the `exchange()` and `uniform()` methods considered earlier in this section):

```
public static void shuffle(String[] a)
{
    int n = a.length;
    for (int i = 0; i < n; i++)
        exchange(a, i, i + uniform(n-i));
}
```

<i>find the maximum of the array values</i>	<pre>public static double max(double[] a) { double max = Double.NEGATIVE_INFINITY; for (int i = 0; i < a.length; i++) if (a[i] > max) max = a[i]; return max; }</pre>
<i>dot product</i>	<pre>public static double dot(double[] a, double[] b) { double sum = 0.0; for (int i = 0; i < a.length; i++) sum += a[i] * b[i]; return sum; }</pre>
<i>exchange the values of two elements in an array</i>	<pre>public static void exchange(String[] a, int i, int j) { String temp = a[i]; a[i] = a[j]; a[j] = temp; }</pre>
<i>print a one-dimensional array (and its length)</i>	<pre>public static void print(double[] a) { StdOut.println(a.length); for (int i = 0; i < a.length; i++) StdOut.println(a[i]); }</pre>
<i>read a 2D array of double values (with dimensions) in row-major order</i>	<pre>public static double[][] readDouble2D() { int m = StdIn.readInt(); int n = StdIn.readInt(); double[][] a = new double[m][n]; for (int i = 0; i < m; i++) for (int j = 0; j < n; j++) a[i][j] = StdIn.readDouble(); return a; }</pre>

Typical code for implementing functions with array arguments or return values

Similarly, we will consider in SECTION 4.2 methods that *sort* an array (rearrange its values so that they are in order). All of these examples highlight the basic fact that the mechanism for passing arrays in Java is *call by value* with respect to the array reference but *call by reference* with respect to the array elements. Unlike primitive-type arguments, the changes that a method makes to the elements of an array *are* reflected in the client program. A method that takes an array as its argument cannot change the array itself—the memory location, length, and type of the array are the same as they were when the array was created—but a method can assign different values to the elements in the array.

Arrays as return values. A method that sorts, shuffles, or otherwise modifies an array taken as an argument does not have to return a reference to that array, because it is changing the elements of a client array, not a copy. But there are many situations where it is useful for a static method to provide an array as a return value. Chief among these are static methods that create arrays for the purpose of returning multiple values of the same type to a client. For example, the following static method creates and returns an array of the kind used by `StdAudio` (see PROGRAM 1.5.7): it contains values sampled from a sine wave of a given frequency (in hertz) and duration (in seconds), sampled at the standard 44,100 samples per second.

```
public static double[] tone(double hz, double t)
{
    int SAMPLING_RATE = 44100;
    int n = (int) (SAMPLING_RATE * t);
    double[] a = new double[n+1];
    for (int i = 0; i <= n; i++)
        a[i] = Math.sin(2 * Math.PI * i * hz / SAMPLING_RATE);
    return a;
}
```

In this code, the length of the array returned depends on the duration: if the given duration is t , the length of the array is about $44100 \cdot t$. With static methods like this one, we can write code that treats a sound wave as a single entity (an array containing sampled values), as we will see next in PROGRAM 2.1.4.

Example: superposition of sound waves As discussed in SECTION 1.5, the simple audio model that we studied there needs to be embellished to create sound that resembles the sound produced by a musical instrument. Many different embellishments are possible; with static methods we can systematically apply them to produce sound waves that are far more complicated than the simple sine waves that we produced in SECTION 1.5. As an illustration of the effective use of static methods to solve an interesting computational problem, we consider a program that has essentially the same functionality as `PlayThatTune` (PROGRAM 1.5.7), but adds harmonic tones one octave above and one octave below each note to produce a more realistic sound.

Chords and harmonics. Notes like concert A have a pure sound that is not very musical, because the sounds that you are accustomed to hearing have many other components. The sound from the guitar string echoes off the wooden part of the

A		440.00
C♯		554.37
E		659.26

A major chord



A		440.00
A		220.00
A		880.00

concert A with harmonics



Superposing waves to make composite sounds

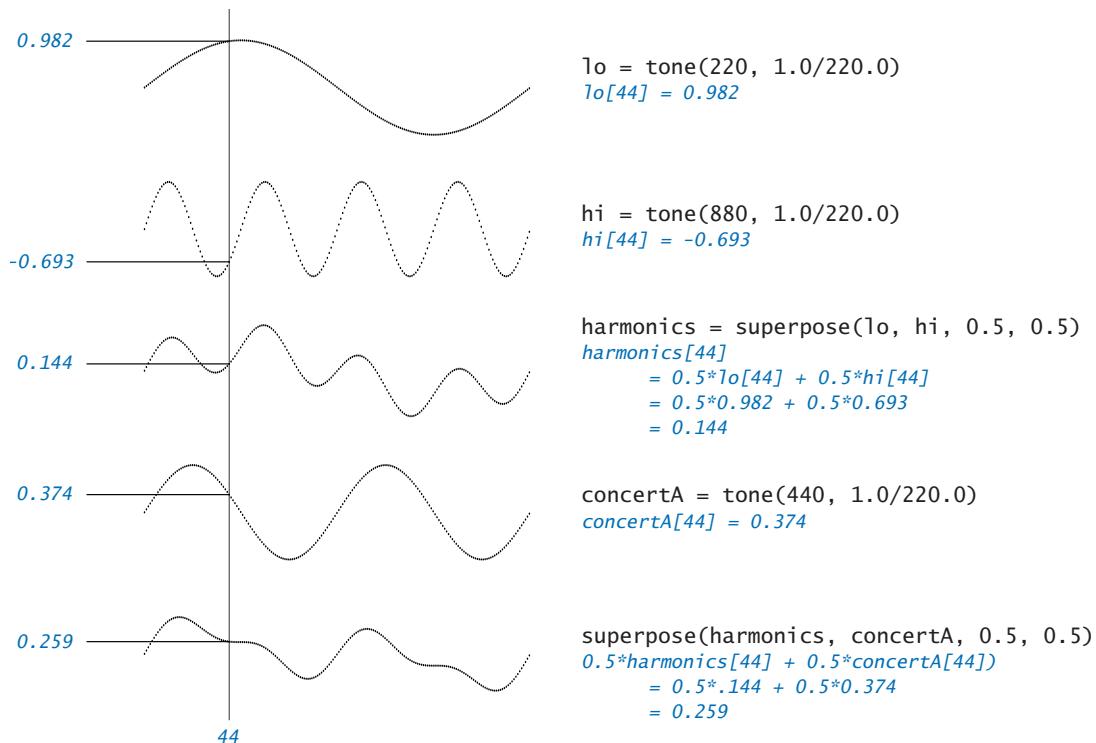
complicated waves. Indeed, one of the triumphs of 19th-century mathematics was the development of the idea that any smooth periodic function can be expressed as a sum of sine and cosine waves, known as a *Fourier series*. This mathematical idea corresponds to the notion that we can create a large range of sounds with musical instruments or our vocal cords and that all sound consists of a composition of various oscillating curves. Any sound corresponds to a curve and any curve corresponds to a sound, and we can create arbitrarily complex curves with superposition.

instrument, the walls of the room that you are in, and so forth. You may think of such effects as modifying the basic sine wave. For example, most musical instruments produce *harmonics* (the same note in different octaves and not as loud), or you might play chords (multiple notes at the same time). To combine multiple sounds, we use *superposition*: simply add the waves together and rescale to make sure that all values stay between -1 and $+1$. As it turns out, when we superpose sine waves of different frequencies in this way, we can get arbitrarily

Weighted superposition. Since we represent sound waves by arrays of numbers that represent their values at the same sample points, superposition is simple to implement: we add together the values at each sample point to produce the combined result and then rescale. For greater control, we specify a relative weight for each of the two waves to be added, with the property that the weights are positive and sum to 1. For example, if we want the first sound to have three times the effect of the second, we would assign the first a weight of 0.75 and the second a weight of 0.25. Now, if one wave is in an array `a[]` with relative weight `awt` and the other is in an array `b[]` with relative weight `bwt`, we compute their weighted sum with the following code:

```
double[] c = new double[a.length];
for (int i = 0; i < a.length; i++)
    c[i] = a[i]*awt + b[i]*bwt;
```

The conditions that the weights are positive and sum to 1 ensure that this operation preserves our convention of keeping the values of all of our waves between -1 and $+1$.



Adding harmonics to concert A (1/220 second at 44,100 samples/second)

Program 2.1.4 Play that tune (revisited)

```

public class PlayThatTuneDeluxe
{
    public static double[] superpose(double[] a, double[] b,
                                     double awt, double bwt)
    { // Weighted superposition of a and b.
        double[] c = new double[a.length];
        for (int i = 0; i < a.length; i++)
            c[i] = a[i]*awt + b[i]*bwt;
        return c;
    }

    public static double[] tone(double hz, double t)
    { /* see text */ }

    public static double[] note(int pitch, double t)
    { // Play note of given pitch, with harmonics.
        double hz = 440.0 * Math.pow(2, pitch / 12.0);
        double[] a = tone(hz, t);
        double[] hi = tone(2*hz, t);
        double[] lo = tone(hz/2, t);
        double[] h = superpose(hi, lo, 0.5, 0.5);
        return superpose(a, h, 0.5, 0.5);
    }

    public static void main(String[] args)
    { // Read and play a tune, with harmonics.
        while (!StdIn.isEmpty())
        { // Read and play a note, with harmonics.
            int pitch = StdIn.readInt();
            double duration = StdIn.readDouble();
            double[] a = note(pitch, duration);
            StdAudio.play(a);
        }
    }
}

```

hz	frequency
a[]	pure tone
hi[]	upper harmonic
lo[]	lower harmonic
h[]	tone with harmonics

This code embellishes the sounds produced by PROGRAM 1.5.7 by using static methods to create harmonics, which results in a more realistic sound than the pure tone.

```
% more elise.txt
7 0.25
6 0.25
7 0.25
6 0.25
...

```

```
% java PlayThatTuneDeluxe < elise.txt
```

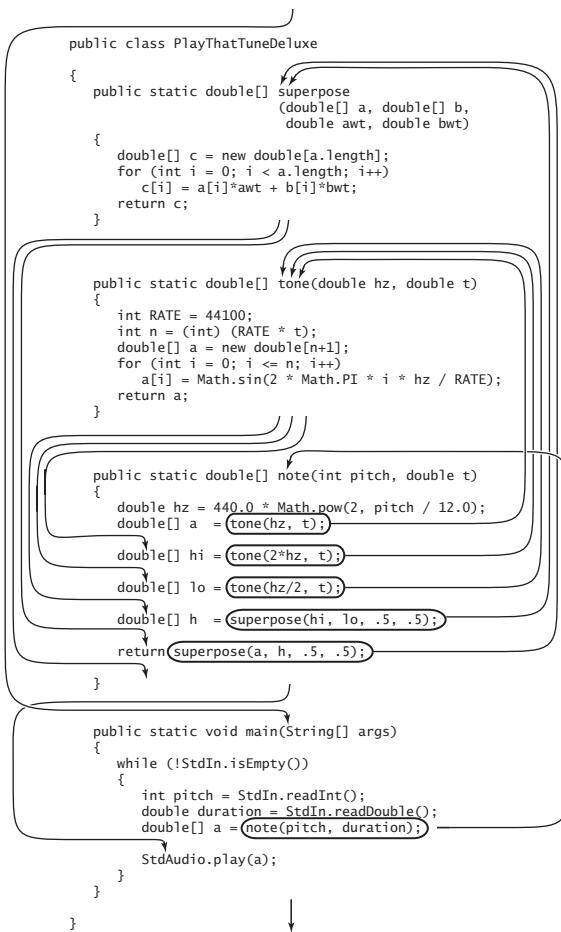


PROGRAM 2.1.4 IS AN IMPLEMENTATION THAT applies these concepts to produce a more realistic sound than that produced by PROGRAM 1.5.7. To do so, it makes use of functions to divide the computation into four parts:

- Given a frequency and duration, create a pure tone.
- Given two sound waves and relative weights, superpose them.
- Given a pitch and duration, create a note with harmonics.
- Read and play a sequence of pitch/duration pairs from standard input.

These tasks are each amenable to implementation as a function, with all of the functions then depending on one another. Each function is well defined and straightforward to implement. All of them (and StdAudio) represent sound as a sequence of floating-point numbers kept in an array, corresponding to sampling a sound wave at 44,100 samples per second.

Up to this point, the use of functions has been somewhat of a notational convenience. For example, the control flow in PROGRAM 2.1.1–2.1.3 is simple—each function is called in just one place in the code. By contrast, PlayThatTuneDeluxe (PROGRAM 2.1.4) is a convincing example of the effectiveness of defining functions to organize a computation because the functions are each called multiple times. For example, the function note() calls the function tone() three times and the function sum() twice. Without functions methods, we would need multiple copies of the code in



Flow of control among several static methods

`tone()` and `sum()`; with functions, we can deal directly with concepts close to the application. Like loops, functions have a simple but profound effect: one sequence of statements (those in the method definition) is executed multiple times during the execution of our program—once for each time the function is called in the control flow in `main()`.

FUNCTIONS (STATIC METHODS) ARE IMPORTANT BECAUSE they give us the ability to *extend* the Java language within a program. Having implemented and debugged functions such as `harmonic()`, `pdf()`, `cdf()`, `mean()`, `abs()`, `exchange()`, `shuffle()`, `isPrime()`, `uniform()`, `superpose()`, `note()`, and `tone()`, we can use them almost as if they were built into Java. The flexibility to do so opens up a whole new world of programming. Before, you were safe in thinking about a Java program as a sequence of statements. Now you need to think of a Java program as a *set of static methods* that can call one another. The statement-to-statement control flow to which you have been accustomed is still present within static methods, but programs have a higher-level control flow defined by static method calls and returns. This ability enables you to think in terms of operations called for by the application, not just the simple arithmetic operations on primitive types that are built into Java.

Whenever you can clearly separate tasks within programs, you should do so. The examples in this section (and the programs throughout the rest of the book) clearly illustrate the benefits of adhering to this maxim. With static methods, we can

- Divide a long sequence of statements into independent parts.
- Reuse code without having to copy it.
- Work with higher-level concepts (such as sound waves).

This produces code that is easier to understand, maintain, and debug than a long program composed solely of Java assignment, conditional, and loop statements. In the next section, we discuss the idea of using static methods defined in *other* programs, which again takes us to another level of programming.

Q&A

Q. What happens if I leave out the keyword `static` when defining a static method?

A. As usual, the best way to answer a question like this is to try it yourself and see what happens. Here is the result of omitting the `static` modifier from `harmonic()` in `Harmonic`:

```
Harmonic.java:15: error: non-static method harmonic(int)
    cannot be referenced from a static context
        double value = harmonic(arg);
                           ^
1 error
```

Non-static methods are different from static methods. You will learn about the former in CHAPTER 3.

Q. What happens if I write code after a `return` statement?

A. Once a `return` statement is reached, control immediately returns to the caller, so any code after a `return` statement is useless. Java identifies this situation as a compile-time error, reporting `unreachable code`.

Q. What happens if I do not include a `return` statement?

A. There is no problem, if the return type is `void`. In this case, control will return to the caller after the last statement. When the return type is not `void`, Java will report a `missing return statement` compile-time error if there is *any* path through the code that does not end in a `return` statement.

Q. Why do I need to use the return type `void`? Why not just omit the return type?

A. Java requires it; we have to include it. Second-guessing a decision made by a programming-language designer is the first step on the road to becoming one.

Q. Can I return from a `void` function by using `return`? If so, which return value should I use?

A. Yes. Use the statement `return;` with no return value.

Q. This issue with side effects and arrays passed as arguments is confusing. Is it really all that important?

A. Yes. Properly controlling side effects is one of a programmer’s most important tasks in large systems. Taking the time to be sure that you understand the difference between passing a value (when arguments are of a primitive type) and passing a reference (when arguments are arrays) will certainly be worthwhile. The very same mechanism is used for all other types of data, as you will learn in CHAPTER 3.

Q. So why not just eliminate the possibility of side effects by making all arguments pass by value, including arrays?

A. Think of a huge array with, say, millions of elements. Does it make sense to copy all of those values for a static method that is going to exchange just two of them? For this reason, most programming languages support passing an array to a function without creating a copy of the array elements—Matlab is a notable exception.

Q. In which order does Java evaluate method calls?

A. Regardless of operator precedence or associativity, Java evaluates subexpressions (including method calls) and argument lists from left to right. For example, when evaluating the expression

$$f1() + f2() * f3(f4(), f5())$$

Java calls the methods in the order `f1()`, `f2()`, `f4()`, `f5()`, and `f3()`. This is most relevant for methods that produce side effects. As a matter of style, we avoid writing code that depends on the order of evaluation.

Exercises

2.1.1 Write a static method `max3()` that takes three `int` arguments and returns the value of the largest one. Add an overloaded function that does the same thing with three `double` values.

2.1.2 Write a static method `odd()` that takes three `boolean` arguments and returns `true` if an odd number of the argument values are `true`, and `false` otherwise.

2.1.3 Write a static method `majority()` that takes three `boolean` arguments and returns `true` if at least two of the argument values are `true`, and `false` otherwise. Do not use an `if` statement.

2.1.4 Write a static method `eq()` that takes two `int` arrays as arguments and returns `true` if the arrays have the same length and all corresponding pairs of elements are equal, and `false` otherwise.

2.1.5 Write a static method `areTriangular()` that takes three `double` arguments and returns `true` if they could be the sides of a triangle (none of them is greater than or equal to the sum of the other two). See EXERCISE 1.2.15.

2.1.6 Write a static method `sigmoid()` that takes a `double` argument x and returns the `double` value obtained from the formula $1 / (1 + e^{-x})$.

2.1.7 Write a static method `sqrt()` that takes a `double` argument and returns the square root of that number. Use Newton's method (see PROGRAM 1.3.6) to compute the result.

2.1.8 Give the function-call trace for `java Harmonic 3 5`

2.1.9 Write a static method `lg()` that takes a `double` argument n and returns the base-2 logarithm of n . You may use Java's Math library.

2.1.10 Write a static method `lg()` that takes an `int` argument n and returns the largest integer not larger than the base-2 logarithm of n . Do *not* use the Math library.

2.1.11 Write a static method `signum()` that takes an `int` argument n and returns -1 if n is less than 0, 0 if n is equal to 0, and $+1$ if n is greater than 0.

2.1.12 Consider the static method `duplicate()` below.

```
public static String duplicate(String s)
{
    String t = s + s;
    return t;
}
```

What does the following code fragment do?

```
String s = "Hello";
s = duplicate(s);
String t = "Bye";
t = duplicate(duplicate(duplicate(t)));
StdOut.println(s + t);
```

2.1.13 Consider the static method `cube()` below.

```
public static void cube(int i)
{
    i = i * i * i;
}
```

How many times is the following `for` loop iterated?

```
for (int i = 0; i < 1000; i++)
    cube(i);
```

Answer: Just 1,000 times. A call to `cube()` has no effect on the client code. It changes the value of its local parameter variable `i`, but that change has no effect on the `i` in the `for` loop, which is a different variable. If you replace the call to `cube(i)` with the statement `i = i * i * i;` (maybe that was what you were thinking), then the loop is iterated five times, with `i` taking on the values 0, 1, 2, 9, and 730 at the beginning of the five iterations.

2.1.14 The following *checksum* formula is widely used by banks and credit card companies to validate legal account numbers:

$$d_0 + f(d_1) + d_2 + f(d_3) + d_4 + f(d_5) + \dots = 0 \pmod{10}$$

The d_i are the decimal digits of the account number and $f(d)$ is the sum of the decimal digits of $2d$ (for example, $f(7) = 5$ because $2 \times 7 = 14$ and $1 + 4 = 5$). For example, 17,327 is valid because $1 + 5 + 3 + 4 + 7 = 20$, which is a multiple of 10. Implement the function f and write a program to take a 10-digit integer as a command-line argument and print a valid 11-digit number with the given integer as its first 10 digits and the checksum as the last digit.

2.1.15 Given two stars with angles of *declination* and *right ascension* (d_1, a_1) and (d_2, a_2), the angle they subtend is given by the formula

$$2 \arcsin((\sin^2(d/2) + \cos(d_1)\cos(d_2)\sin^2(a/2))^{1/2})$$

where a_1 and a_2 are angles between -180 and 180 degrees, d_1 and d_2 are angles between -90 and 90 degrees, $a = a_2 - a_1$, and $d = d_2 - d_1$. Write a program to take the declination and right ascension of two stars as command-line arguments and print the angle they subtend. *Hint:* Be careful about converting from degrees to radians.

2.1.16 Write a static method `scale()` that takes a `double` array as its argument and has the side effect of scaling the array so that each element is between 0 and 1 (by subtracting the minimum value from each element and then dividing each element by the difference between the minimum and maximum values). Use the `max()` method defined in the table in the text, and write and use a matching `min()` method.

2.1.17 Write a static method `reverse()` that takes an array of strings as its argument and returns a new array with the strings in reverse order. (Do not change the order of the strings in the argument array.) Write a static method `reverseInplace()` that takes an array of strings as its argument and produces the side effect of reversing the order of the strings in the argument array.

2.1.18 Write a static method `readBoolean2D()` that reads a two-dimensional boolean matrix (with dimensions) from standard input and returns the resulting two-dimensional array.

2.1.19 Write a static method `histogram()` that takes an `int` array `a[]` and an integer `m` as arguments and returns an array of length `m` whose `i`th element is the number of times the integer `i` appeared in `a[]`. Assuming the values in `a[]` are all between 0 and `m-1`, the sum of the values in the returned array should equal `a.length`.

2.1.20 Assemble code fragments in this section and in SECTION 1.4 to develop a program that takes an integer command-line argument `n` and prints `n` five-card hands, separated by blank lines, drawn from a randomly shuffled card deck, one card per line using card names like Ace of Clubs.

2.1.21 Write a static method `multiply()` that takes two square matrices of the same dimension as arguments and produces their product (another square matrix of that same dimension). *Extra credit:* Make your program work whenever the number of columns in the first matrix is equal to the number of rows in the second matrix.

2.1.22 Write a static method `any()` that takes a `boolean` array as its argument and returns `true` if any of the elements in the array is `true`, and `false` otherwise. Write a static method `all()` that takes an array of `boolean` values as its argument and returns `true` if all of the elements in the array are `true`, and `false` otherwise.

2.1.23 Develop a version of `getCoupon()` that better models the situation when one of the coupons is rare: choose one of the `n` values at random, return that value with probability $1/(1,000n)$, and return all other values with equal probability. *Extra credit:* How does this change affect the expected number of coupons that need to be collected in the coupon collector problem?

2.1.24 Modify `PlayThatTune` to add harmonics two octaves away from each note, with half the weight of the one-octave harmonics.

Creative Exercises

2.1.25 *Birthday problem.* Develop a class with appropriate static methods for studying the birthday problem (see EXERCISE 1.4.38).

2.1.26 *Euler's totient function.* Euler's totient function is an important function in number theory: $\varphi(n)$ is defined as the number of positive integers less than or equal to n that are relatively prime with n (no factors in common with n other than 1). Write a class with a static method that takes an integer argument n and returns $\varphi(n)$, and a `main()` that takes an integer command-line argument, calls the method with that argument, and prints the resulting value.

2.1.27 *Harmonic numbers.* Write a program `Harmonic` that contains three static methods `harmonic()`, `harmonicSmall()`, and `harmonicLarge()` for computing the harmonic numbers. The `harmonicSmall()` method should just compute the sum (as in PROGRAM 1.3.5), the `harmonicLarge()` method should use the approximation $H_n = \log_e(n) + \gamma + 1/(2n) - 1/(12n^2) + 1/(120n^4)$ (the number $\gamma = 0.577215664901532\dots$ is known as *Euler's constant*), and the `harmonic()` method should call `harmonicSmall()` for $n < 100$ and `harmonicLarge()` otherwise.

2.1.28 *Black–Scholes option valuation.* The Black–Scholes formula supplies the theoretical value of a European call option on a stock that pays no dividends, given the current stock price s , the exercise price x , the continuously compounded risk-free interest rate r , the volatility σ , and the time (in years) to maturity t . The Black–Scholes value is given by the formula $s \Phi(a) - x e^{-rt} \Phi(b)$, where $\Phi(z)$ is the Gaussian cumulative distribution function, $a = (\ln(s/x) + (r + \sigma^2/2)t) / (\sigma\sqrt{t})$, and $b = a - \sigma\sqrt{t}$. Write a program that takes s , r , σ , and t from the command line and prints the Black–Scholes value.

2.1.29 *Fourier spikes.* Write a program that takes a command-line argument n and plots the function

$$(\cos(t) + \cos(2t) + \cos(3t) + \dots + \cos(nt)) / n$$

for 500 equally spaced samples of t from -10 to 10 (in radians). Run your program for $n = 5$ and $n = 500$. Note: You will observe that the sum converges to a spike (0 everywhere except a single value). This property is the basis for a proof that *any* smooth function can be expressed as a sum of sinusoids.

2.1.30 *Calendar.* Write a program `Calendar` that takes two integer command-line arguments `m` and `y` and prints the monthly calendar for month `m` of year `y`, as in this example:

```
% java Calendar 2 2009
February 2009
S M Tu W Th F S
1 2 3 4 5 6 7
8 9 10 11 12 13 14
15 16 17 18 19 20 21
22 23 24 25 26 27 28
```

Hint: See `LeapYear` (PROGRAM 1.2.4) and EXERCISE 1.2.29.

2.1.31 *Horner's method.* Write a class `Horner` with a method `evaluate()` that takes a floating-point number `x` and array `p[]` as arguments and returns the result of evaluating the polynomial whose coefficients are the elements in `p[]` at `x`:

$$p(x) = p_0 + p_1x^1 + p_2x^2 + \dots + p_{n-2}x^{n-2} + p_{n-1}x^{n-1}$$

Use *Horner's method*, an efficient way to perform the computations that is suggested by the following parenthesization:

$$p(x) = p_0 + x(p_1 + x(p_2 + \dots + x(p_{n-2} + xp_{n-1})) \dots)$$

Write a test client with a static method `exp()` that uses `evaluate()` to compute an approximation to e^x , using the first n terms of the Taylor series expansion $e^x = 1 + x + x^2/2! + x^3/3! + \dots$. Your client should take a command-line argument `x` and compare your result against that computed by `Math.exp(x)`.

2.1.32 *Chords.* Develop a version of `PlayThatTune` that can handle songs with chords (including harmonics). Develop an input format that allows you to specify different durations for each chord and different amplitude weights for each note within a chord. Create test files that exercise your program with various chords and harmonics, and create a version of *Für Elise* that uses them.

2.1.33 Benford's law. The American astronomer Simon Newcomb observed a quirk in a book that compiled logarithm tables: the beginning pages were much grubbier than the ending pages. He suspected that scientists performed more computations with numbers starting with 1 than with 8 or 9, and postulated that, under general circumstances, the leading digit is much more likely to be 1 (roughly 30%) than the digit 9 (less than 4%). This phenomenon is known as *Benford's law* and is now often used as a statistical test. For example, IRS forensic accountants rely on it to discover tax fraud. Write a program that reads in a sequence of integers from standard input and tabulates the number of times each of the digits 1–9 is the leading digit, breaking the computation into a set of appropriate static methods. Use your program to test the law on some tables of information from your computer or from the web. Then, write a program to foil the IRS by generating random amounts from \$1.00 to \$1,000.00 with the same distribution that you observed.

2.1.34 Binomial distribution. Write a function

```
public static double binomial(int n, int k, double p)
```

to compute the probability of obtaining exactly k heads in n biased coin flips (heads with probability p) using the formula

$$f(n, k, p) = p^k(1-p)^{n-k} n! / (k!(n-k)!)$$

Hint: To stave off overflow, compute $x = \ln f(n, k, p)$ and then return e^x . In `main()`, take n and p from the command line and check that the sum over all values of k between 0 and n is (approximately) 1. Also, compare every value computed with the normal approximation

$$f(n, k, p) \approx \phi(np, np(1-p))$$

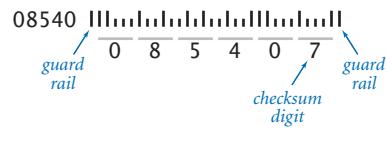
(see EXERCISE 2.2.1).

2.1.35 Coupon collecting from a binomial distribution. Develop a version of `getCoupon()` that uses `binomial()` from the previous exercise to return coupon values according to the binomial distribution with $p = 1/2$. *Hint:* Generate a uniformly random number x between 0 and 1, then return the smallest value of k for which the sum of $f(n, j, p)$ for all $j < k$ exceeds x . *Extra credit:* Develop a hypothesis for describing the behavior of the coupon collector function under this assumption.

2.1.36 Postal bar codes. The barcode used by the U.S. Postal System to route mail is defined as follows: Each decimal digit in the ZIP code is encoded using a sequence of three half-height and two full-height bars. The barcode starts and ends with a full-height bar (the guard rail) and includes a checksum digit (after the five-digit ZIP code or ZIP+4), computed by summing up the original digits modulo 10. Implement the following functions

- Draw a half-height or full-height bar on `StdDraw`.
- Given a digit, draw its sequence of bars.
- Compute the checksum digit.

Also implement a test client that reads in a five- (or nine-) digit ZIP code as the command-line argument and draws the corresponding postal bar code.



2.2 Libraries and Clients

EACH PROGRAM THAT YOU HAVE WRITTEN so far consists of Java code that resides in a single .java file. For large programs, keeping all the code in a single file in this way is restrictive and unnecessary. Fortunately, it is very easy in Java to refer to a method in one file that is defined in another. This ability has two important consequences on our style of programming.

First, it enables *code reuse*. One program can make use of code that is already written and debugged, not by copying the code, but just by referring to it. This ability to define code that can be reused is an essential part of modern programming. It amounts to extending Java—you can define and use your own operations on data.

Second, it enables *modular programming*. You can not only divide a program up into static methods, as just described in SECTION 2.1, but also keep those methods in different files, grouped together according to the needs of the application. Modular programming is important because it allows us to *independently* develop, compile, and debug parts of big programs one piece at a time, leaving each finished piece in its own file for later use without having to worry about its details again. We develop libraries of static methods for use by any other program, keeping each library in its own file and using its methods in any other program. Java’s Math library and our Std* libraries for input/output are examples that you have already used. More importantly, you will soon see that it is very easy to define libraries of your own. The ability to define libraries and then to use them in multiple programs is a critical aspect of our ability to build programs to address complex tasks.

Having just moved in SECTION 2.1 from thinking of a Java program as a sequence of statements to thinking of a Java program as a class comprising a set of static methods (one of which is `main()`), you will be ready after this section to think of a Java program as a set of *classes*, each of which is an independent module consisting of a set of methods. Since each method can call a method in another class, all of your code can interact as a network of methods that call one another, grouped together in classes. With this capability, you can start to think about managing complexity when programming by breaking up programming tasks into classes that can be implemented and tested independently.

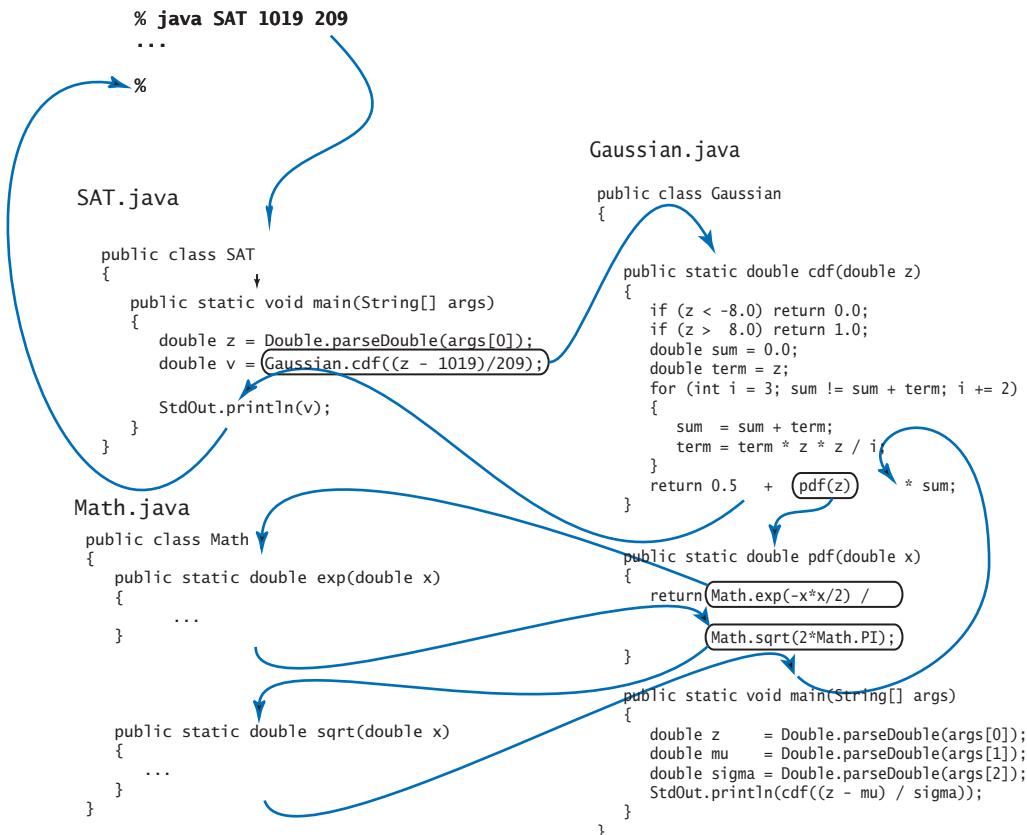
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Programs in this section

Using static methods in other programs To refer to a static method in one class that is defined in another, we use the same mechanism that we have been using to invoke methods such as `Math.sqrt()` and `StdOut.println()`:

- Make both classes accessible to Java (for example, by putting them both in the same directory in your computer).
- To call a method, prepend its class name and a period separator.

For example, we might wish to write a simple client `SAT.java` that takes an SAT score z from the command line and prints the percentage of students scoring less than z in a given year (in which the mean score was 1,019 and its standard deviation was 209). To get the job done, `SAT.java` needs to compute $\Phi((z - 1,019)/209)$, a



Flow of control in a modular program

task perfectly suited for the `cdf()` method in `Gaussian.java` (PROGRAM 2.1.2). All that we need to do is to keep `Gaussian.java` in the same directory as `SAT.java` and prepend the class name when calling `cdf()`. Moreover, any other class in that directory can make use of the static methods defined in `Gaussian`, by calling `Gaussian.pdf()` or `Gaussian.cdf()`. The `Math` library is always accessible in Java, so any class can call `Math.sqrt()` and `Math.exp()`, as usual. The files `Gaussian.java`, `SAT.java`, and `Math.java` implement Java classes that interact with one another: `SAT` calls a method in `Gaussian`, which calls another method in `Gaussian`, which then calls two methods in `Math`.

The potential effect of programming by defining multiple files, each an independent class with multiple methods, is another profound change in our programming style. Generally, we refer to this approach as *modular programming*. We independently develop and debug methods for an application and then utilize them at any later time. In this section, we will consider numerous illustrative examples to help you get used to the idea. However, there are several details about the process that we need to discuss before considering more examples.

The public keyword. We have been identifying every static method as `public` since `HelloWorld`. This modifier identifies the method as available for use by any other program with access to the file. You can also identify methods as `private` (and there are a few other categories), but you have no reason to do so at this point. We will discuss various options in SECTION 3.3.

Each module is a class. We use the term *module* to refer to all the code that we keep in a single file. In Java, by convention, each module is a Java `class` that is kept in a file with the same name of the class but has a `.java` extension. In this chapter, each `class` is merely a set of static methods (one of which is `main()`). You will learn much more about the general structure of the Java `class` in CHAPTER 3.

The .class file. When you compile the program (by typing `javac` followed by the class name), the Java compiler makes a file with the class name followed by a `.class` extension that has the code of your program in a language more suited to your computer. If you have a `.class` file, you can use the module's methods in another program even without having the source code in the corresponding `.java` file (but you are on your own if you discover a bug!).

Compile when necessary. When you compile a program, Java typically compiles everything that needs to be compiled in order to run that program. If you call `Gaussian.cdf()` in `SAT`, then, when you type `javac SAT.java`, the compiler will also check whether you modified `Gaussian.java` since the last time it was compiled (by checking the time it was last changed against the time `Gaussian.class` was created). If so, it will also compile `Gaussian.java`! If you think about this approach, you will agree that it is actually quite helpful. After all, if you find a bug in `Gaussian.java` (and fix it), you want all the classes that call methods in `Gaussian` to use the new version.

Multiple `main()` methods. Another subtle point is to note that more than one class might have a `main()` method. In our example, both `SAT` and `Gaussian` have their own `main()` method. If you recall the rule for executing a program, you will see that there is no confusion: when you type `java` followed by a class name, Java transfers control to the machine code corresponding to the `main()` method defined in that class. Typically, we include a `main()` method in every class, to test and debug its methods. When we want to run `SAT`, we type `java SAT`; when we want to debug `Gaussian`, we type `java Gaussian` (with appropriate command-line arguments).

IF YOU THINK OF EACH PROGRAM that you write as something that you might want to make use of later, you will soon find yourself with all sorts of useful tools. Modular programming allows us to view every solution to a computational problem that we may develop as adding value to our computational environment.

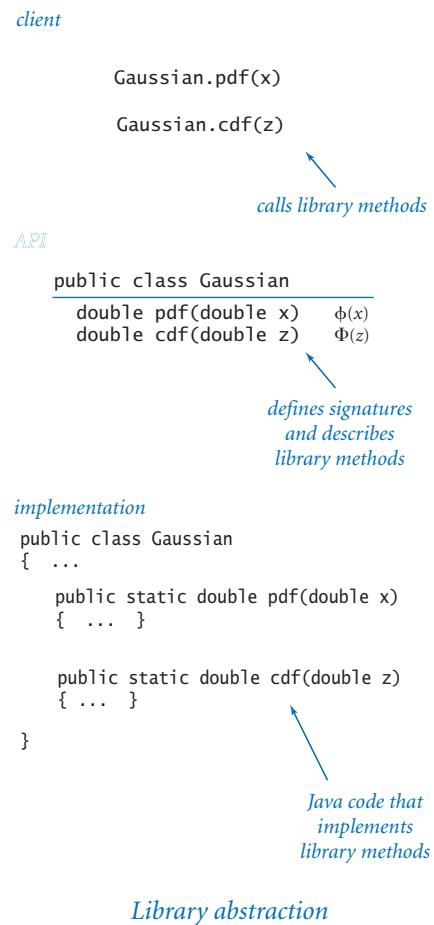
For example, suppose that you need to evaluate Φ for some future application. Why not just cut and paste the code that implements `cdf()` from `Gaussian`? That would work, but would leave you with two copies of the code, making it more difficult to maintain. If you later want to fix or improve this code, you would need to do so in both copies. Instead, you can just call `Gaussian.cdf()`. Our implementations and uses of our methods are soon going to proliferate, so having just one copy of each is a worthy goal.

From this point forward, you should write *every* program by identifying a reasonable way to divide the computation into separate parts of a manageable size and implementing each part as if someone will want to use it later. Most frequently, that someone will be you, and you will have yourself to thank for saving the effort of rewriting and re-debugging code.

Libraries We refer to a module whose methods are primarily intended for use by many other programs as a *library*. One of the most important characteristics of programming in Java is that thousands of libraries have been predefined for your use. We reveal information about those that might be of interest to you throughout the book, but we will postpone a detailed discussion of the scope of Java libraries, because many of them are designed for use by experienced programmers. Instead, we focus in this chapter on the even more important idea that we can build *user-defined libraries*, which are nothing more than classes that contain a set of related methods for use by other programs. No Java library can contain all the methods that we might need for a given computation, so this ability to create our own library of methods is a crucial step in addressing complex programming applications.

Clients. We use the term *client* to refer to a program that calls a given library method. When a class contains a method that is a client of a method in another class, we say that the first class is a client of the second class. In our example, SAT is a client of Gaussian. A given class might have multiple clients. For example, all of the programs that you have written that call `Math.sqrt()` or `Math.random()` are clients of Math.

APIs. Programmers normally think in terms of a *contract* between the client and the implementation that is a clear specification of what the method is to do. When you are writing both clients and implementations, you are making contracts with yourself, which by itself is helpful because it provides extra help in debugging. More important, this approach enables code reuse. You have been able to write programs that are clients of `Std*` and `Math` and other built-in Java classes because of an informal contract (an English-



language description of what they are supposed to do) along with a precise specification of the signatures of the methods that are available for use. Collectively, this information is known as an *application programming interface* (API). This same mechanism is effective for user-defined libraries. The API allows any client to use the library without having to examine the code in the implementation, as you have been doing for Math and Std*. The guiding principle in API design is to *provide to clients the methods they need and no others*. An API with a huge number of methods may be a burden to implement; an API that is lacking important methods may be unnecessarily inconvenient for clients.

Implementations. We use the term *implementation* to describe the Java code that implements the methods in an API, kept by convention in a file with the library name and a .java extension. Every Java program is an implementation of some API, and no API is of any use without some implementation. Our goal when developing an implementation is to honor the terms of the contract. Often, there are many ways to do so, and separating client code from implementation code gives us the freedom to substitute new and improved implementations.

FOR EXAMPLE, CONSIDER THE GAUSSIAN DISTRIBUTION functions. These do not appear in Java's Math library but are important in applications, so it is worthwhile for us to put them in a library where they can be accessed by future client programs and to articulate this API:

```
public class Gaussian
```

double pdf(double x)	$\phi(x)$
double pdf(double x, double mu, double sigma)	$\phi(x, \mu, \sigma)$
double cdf(double z)	$\Phi(z)$
double cdf(double z, double mu, double sigma)	$\Phi(z, \mu, \sigma)$

API for our library of static methods for Gaussian distribution functions

The API includes not only the one-argument Gaussian distribution functions that we have previously considered (see PROGRAM 2.1.2) but also three-argument versions (in which the client specifies the mean and standard deviation of the distribution) that arise in many statistical applications. Implementing the three-argument Gaussian distribution functions is straightforward (see EXERCISE 2.2.1).

How much information should an API contain? This is a gray area and a hotly debated issue among programmers and computer-science educators. We might try to put as much information as possible in the API, but (as with any contract!) there are limits to the amount of information that we can productively include. In this book, we stick to a principle that parallels our guiding design principle: *provide to client programmers the information they need and no more*. Doing so gives us vastly more flexibility than the alternative of providing detailed information about implementations. Indeed, any extra information amounts to implicitly extending the contract, which is undesirable. Many programmers fall into the bad habit of checking implementation code to try to understand what it does. Doing so might lead to client code that depends on behavior not specified in the API, which would not work with a new implementation. Implementations change more often than you might think. For example, each new release of Java contains many new implementations of library functions.

Often, the implementation comes first. You might have a working module that you later decide would be useful for some task, and you can just start using its methods in other programs. In such a situation, it is wise to carefully articulate the API at some point. The methods may not have been designed for reuse, so it is worthwhile to use an API to do such a design (as we did for *Gaussian*).

The remainder of this section is devoted to several examples of libraries and clients. Our purpose in considering these libraries is twofold. First, they provide a richer programming environment for your use as you develop increasingly sophisticated client programs of your own. Second, they serve as examples for you to study as you begin to develop libraries for your own use.

Random numbers We have written several programs that use `Math.random()`, but our code often uses particular idioms that convert the random `double` values between 0 and 1 that `Math.random()` provides to the type of random numbers that we want to use (random `boolean` values or random `int` values in a specified range, for example). To effectively reuse our code that implements these idioms, we will, from now on, use the `StdRandom` library in PROGRAM 2.2.1. `StdRandom` uses overloading to generate random numbers from various distributions. You can use any of them in the same way that you use our standard I/O libraries (see the first Q&A at the end of SECTION 2.1). As usual, we summarize the methods in our `StdRandom` library with an API:

```

public class StdRandom
    void setSeed(long seed)           set the seed for reproducible results
    int uniform(int n)               integer between 0 and n-1
    double uniform(double lo, double hi) floating-point number between lo and hi
    boolean bernoulli(double p)      true with probability p, false otherwise
    double gaussian()                Gaussian, mean 0, standard deviation 1
    double gaussian(double mu, double sigma) Gaussian, mean mu, standard deviation sigma
    int discrete(double[] p)         i with probability p[i]
    void shuffle(double[] a)          randomly shuffle the array a[]

```

API for our library of static methods for random numbers

These methods are sufficiently familiar that the short descriptions in the API suffice to specify what they do. By collecting all of these methods that use `Math.random()` to generate random numbers of various types in one file (`StdRandom.java`), we concentrate our attention on generating random numbers to this one file (and reuse the code in that file) instead of spreading them through every program that uses these methods. Moreover, each program that uses one of these methods is clearer than code that calls `Math.random()` directly, because its purpose for using `Math.random()` is clearly articulated by the choice of method from `StdRandom`.

API design. We make certain assumptions about the values passed to each method in `StdRandom`. For example, we assume that clients will call `uniform(n)` only for positive integers `n`, `bernoulli(p)` only for `p` between 0 and 1, and `discrete()` only for an array whose elements are between 0 and 1 and sum to 1. All of these assumptions are part of the contract between the client and the implementation. We strive to design libraries such that the contract is clear and unambiguous and to avoid getting bogged down with details. As with many tasks in programming, a good API design is often the result of several iterations of trying and living with various possibilities. We always take special care in designing APIs, because when we change an API we might have to change all clients and all implementations. Our goal is to articulate what clients can expect *separate from the code* in the API. This practice frees us to change the code, and perhaps to use an implementation that achieves the desired effect more efficiently or with more accuracy.

Program 2.2.1 Random number library

```
public class StdRandom
{
    public static int uniform(int n)
    {   return (int) (Math.random() * n);   }

    public static double uniform(double lo, double hi)
    {   return lo + Math.random() * (hi - lo);   }

    public static boolean bernoulli(double p)
    {   return Math.random() < p;   }

    public static double gaussian()
    {   /* See Exercise 2.2.17. */   }

    public static double gaussian(double mu, double sigma)
    {   return mu + sigma * gaussian();   }

    public static int discrete(double[] probabilities)
    {   /* See Program 1.6.2. */   }

    public static void shuffle(double[] a)
    {   /* See Exercise 2.2.4. */   }

    public static void main(String[] args)
    {   /* See text. */   }
}
```

The methods in this library compute various types of random numbers: random nonnegative integer less than a given value, uniformly distributed in a given range, random bit (Bernoulli), standard Gaussian, Gaussian with given mean and standard deviation, and distributed according to a given discrete distribution.

```
% java StdRandom 5
90 26.36076 false 8.79269 0
13 18.02210 false 9.03992 1
58 56.41176 true 8.80501 0
29 16.68454 false 8.90827 0
85 86.24712 true 8.95228 0
```

Unit testing. Even though we implement StdRandom without reference to any particular client, it is good programming practice to include a *test client* `main()` that, although not used when a client class uses the library, is helpful when debugging and testing the methods in the library. *Whenever you create a library, you should include a `main()` method for unit testing and debugging.* Proper unit testing can be a significant programming challenge in itself (for example, the best way of testing whether the methods in StdRandom produce numbers that have the same characteristics as truly random numbers is still debated by experts). At a minimum, you should always include a `main()` method that

- Exercises all the code
- Provides some assurance that the code is working
- Takes an argument from the command line to allow more testing

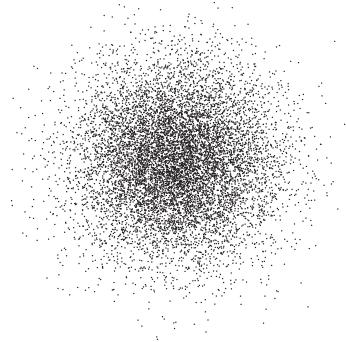
Then, you should refine that `main()` method to do more exhaustive testing as you use the library more extensively. For example, we might start with the following code for StdRandom (leaving the testing of `shuffle()` for an exercise):

```
public static void main(String[] args)
{
    int n = Integer.parseInt(args[0]);
    double[] probabilities = { 0.5, 0.3, 0.1, 0.1 };
    for (int i = 0; i < n; i++)
    {
        StdOut.printf(" %2d " , uniform(100));
        StdOut.printf("%8.5f " , uniform(10.0, 99.0));
        StdOut.printf("%5b " , bernoulli(0.5));
        StdOut.printf("%7.5f " , gaussian(9.0, 0.2));
        StdOut.printf("%2d " , discrete(probabilities));
        StdOut.println();
    }
}
```

When we include this code in `StdRandom.java` and invoke this method as illustrated in PROGRAM 2.2.1, the output includes no surprises: the integers in the first column might be equally likely to be any value from 0 to 99; the numbers in the second column might be uniformly spread between 10.0 and 99.0; about half of the values in the third column are `true`; the numbers in the fourth column seem to average about 9.0, and seem unlikely to be too far from 9.0; and the last column seems to be not far from 50% 0s, 30% 1s, 10% 2s, and 10% 3s. If something seems

amiss in one of the columns, we can type `java StdRandom 10` or `100` to see many more results. In this particular case, we can (and should) do far more extensive testing in a separate client to check that the numbers have many of the same properties as truly random numbers drawn from the cited distributions (see EXERCISE 2.2.3). One effective approach is to write test clients that use `StdDraw`, as data visualization can be a quick indication that a program is behaving as intended. For example, a plot of a large number of points whose x - and y -coordinates are

```
public class RandomPoints
{
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        for (int i = 0; i < n; i++)
        {
            double x = StdRandom.gaussian(.5, .2);
            double y = StdRandom.gaussian(.5, .2);
            StdDraw.point(x, y);
        }
    }
}
```



A `StdRandom` test client

both drawn from various distributions often produces a pattern that gives direct insight into the important properties of the distribution. More important, a bug in the random number generation code is likely to show up immediately in such a plot.

Stress testing. An extensively used library such as `StdRandom` should also be subjected to *stress testing*, where we make sure that it does not crash when the client does not follow the contract or makes some assumption that is not explicitly covered. Java libraries have already been subjected to such stress testing, which requires carefully examining each line of code and questioning whether some condition might cause a problem. What should `discrete()` do if the array elements do not sum to exactly 1? What if the argument is an array of length 0? What should the two-argument

`uniform()` do if one or both of its arguments is `NaN`? `Infinity`? Any question that you can think of is fair game. Such cases are sometimes referred to as *corner cases*. You are certain to encounter a teacher or a supervisor who is a stickler about corner cases. With experience, most programmers learn to address them early, to avoid an unpleasant bout of debugging later. Again, a reasonable approach is to implement a stress test as a separate client.

Input and output for arrays We have seen—and will continue to see—many examples where we wish to keep data in arrays for processing. Accordingly, it is useful to build a library that complements StdIn and StdOut by providing static methods for reading arrays of primitive types from standard input and printing them to standard output. The following API provides these methods:

public class StdArrayIO	
double[] readDouble1D()	<i>read a one-dimensional array of double values</i>
double[][] readDouble2D()	<i>read a two-dimensional array of double values</i>
void print(double[] a)	<i>print a one-dimensional array of double values</i>
void print(double[][] a)	<i>print a two-dimensional array of double values</i>

Note 1. 1D format is an integer n followed by n values.

Note 2. 2D format is two integers m and n followed by $m \times n$ values in row-major order.

Note 3. Methods for int and boolean are also included.

API for our library of static methods for array input and output

The first two notes at the bottom of the table reflect the idea that we need to settle on a *file format*. For simplicity and harmony, we adopt the convention that all values appearing in standard input include the dimension(s) and appear in the order indicated. The `read*`() methods expect input in this format; the `print()` methods produce output in this format. The third note at the bottom of the table indicates that `StdArrayIO` actually contains 12 methods—four each for `int`, `double`, and `boolean`. The `print()` methods are overloaded (they all have the same name `print()` but different types of arguments), but the `read*`() methods need different names, formed by adding the type name (capitalized, as in `StdIn`) followed by `1D` or `2D`.

Implementing these methods is straightforward from the array-processing code that we have considered in SECTION 1.4 and in SECTION 2.1, as shown in `StdArrayIO` (PROGRAM 2.2.2). Packaging up all of these static methods into one file—`StdArrayIO.java`—allows us to easily reuse the code and saves us from having to worry about the details of reading and printing arrays when writing client programs later on.

Program 2.2.2 Array I/O library

```

public class StdArrayIO
{
    public static double[] readDouble1D()
    { /* See Exercise 2.2.11. */ }

    public static double[][] readDouble2D()
    {
        int m = StdIn.readInt();
        int n = StdIn.readInt();
        double[][] a = new double[m][n];
        for (int i = 0; i < m; i++)
            for (int j = 0; j < n; j++)
                a[i][j] = StdIn.readDouble();
        return a;
    }

    public static void print(double[] a)
    { /* See Exercise 2.2.11. */ }

    public static void print(double[][] a)
    {
        int m = a.length;
        int n = a[0].length;
        System.out.println(m + " " + n);
        for (int i = 0; i < m; i++)
        {
            for (int j = 0; j < n; j++)
                StdOut.printf("%9.5f ", a[i][j]);
            StdOut.println();
        }
        StdOut.println();
    }

    // Methods for other types are similar (see booksite).

    public static void main(String[] args)
    { print(readDouble2D()); }
}

```

```

% more tiny2D.txt
4 3
0.000  0.270  0.000
0.246  0.224 -0.036
0.222  0.176  0.0893
-0.032 0.739  0.270

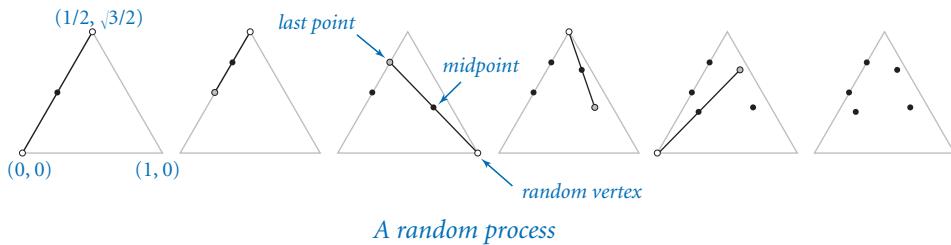
% java StdArrayIO < tiny2D.txt
4 3
0.00000  0.27000  0.00000
0.24600  0.22400 -0.03600
0.22200  0.17600  0.08930
-0.03200 0.73900  0.27000

```

This library of static methods facilitates reading one-dimensional and two-dimensional arrays from standard input and printing them to standard output. The file format includes the dimensions (see accompanying text). Numbers in the output in the example are truncated.

Iterated function systems Scientists have discovered that complex visual images can arise unexpectedly from simple computational processes. With StdRandom, StdDraw, and StdArrayIO, we can study the behavior of such systems.

Sierpinski triangle. As a first example, consider the following simple process: Start by plotting a point at one of the vertices of a given equilateral triangle. Then pick one of the three vertices at random and plot a new point halfway between the point just plotted and that vertex. Continue performing this same operation. Each time, we are pick a random vertex from the triangle to establish the line whose midpoint will be the next point plotted. Since we make random choices, the set of points should have some of the characteristics of random points, and that does seem to be the case after the first few iterations:

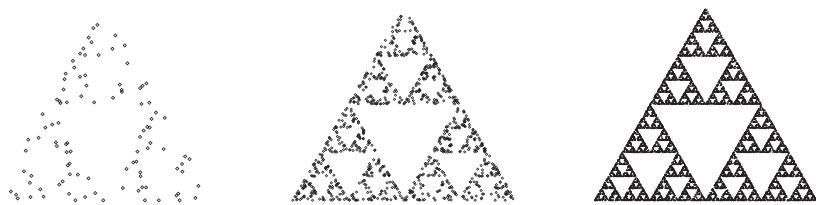


We can study the process for a large number of iterations by writing a program to plot `trials` points according to the rules:

```
double[] cx = { 0.000, 1.000, 0.500 };
double[] cy = { 0.000, 0.000, 0.866 };
double x = 0.0, y = 0.0;
for (int t = 0; t < trials; t++)
{
    int r = StdRandom.uniform(3);
    x = (x + cx[r]) / 2.0;
    y = (y + cy[r]) / 2.0;
    StdDraw.point(x, y);
}
```

We keep the x - and y -coordinates of the triangle vertices in the arrays `cx[]` and `cy[]`, respectively. We use `StdRandom.uniform()` to choose a random index `r` into

these arrays—the coordinates of the chosen vertex are $(cx[r], cy[r])$. The x -coordinate of the midpoint of the line from (x, y) to that vertex is given by the expression $(x + cx[r])/2.0$, and a similar calculation gives the y -coordinate. Adding a call to `StdDraw.point()` and putting this code in a loop completes the implementation. Remarkably, despite the randomness, the same figure always emerges after a large number of iterations! This figure is known as the *Sierpinski triangle* (see EXERCISE 2.3.27). Understanding why such a regular figure should arise from such a random process is a fascinating question.



A random process?

Barnsley fern. To add to the mystery, we can produce pictures of remarkable diversity by playing the same game with different rules. One striking example is known as the *Barnsley fern*. To generate it, we use the same process, but this time driven by the following table of formulas. At each step, we choose the formulas to use to update x and y with the indicated probability (1% of the time we use the first pair of formulas, 85% of the time we use the second pair of formulas, and so forth).

<i>probability</i>	<i>x-update</i>		<i>y-update</i>	
1%	$x =$	0.500	$y =$	$0.16y$
85%	$x =$	$0.85x + 0.04y + 0.075$	$y =$	$-0.04x + 0.85y + 0.180$
7%	$x =$	$0.20x - 0.26y + 0.400$	$y =$	$0.23x + 0.22y + 0.045$
7%	$x =$	$-0.15x + 0.28y + 0.575$	$y =$	$0.26x + 0.24y - 0.086$

Program 2.2.3 Iterated function systems

```

public class IFS
{
    public static void main(String[] args)
    { // Plot trials iterations of IFS on StdIn.
        int trials = Integer.parseInt(args[0]);
        double[] dist   = StdArrayIO.readDouble1D();
        double[][] cx   = StdArrayIO.readDouble2D();
        double[][] cy   = StdArrayIO.readDouble2D();
        double x = 0.0, y = 0.0;
        for (int t = 0; t < trials; t++)
        { // Plot 1 iteration.
            int r = StdRandom.discrete(dist);
            double x0 = cx[r][0]*x + cx[r][1]*y + cx[r][2];
            double y0 = cy[r][0]*x + cy[r][1]*y + cy[r][2];
            x = x0;
            y = y0;
            StdDraw.point(x, y);
        }
    }
}

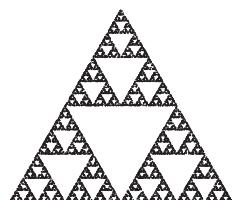
```

trials *iterations*
dist[] *probabilities*
cx[][] *x coefficients*
cy[][] *y coefficients*
x, y *current point*

This data-driven client of `StdArrayIO`, `StdRandom`, and `StdDraw` iterates the function system defined by a 1-by- m vector (*probabilities*) and two m -by-3 matrices (*coefficients* for updating *x* and *y*, respectively) on standard input, plotting the result as a set of points on standard drawing. Curiously, this code does not need to know the value of m , as it uses separate methods to create and process the matrices.

```
% more sierpinski.txt
3
.33 .33 .34
3 3
.50 .00 .00
.50 .00 .50
.50 .00 .25
3 3
.00 .50 .00
.00 .50 .00
.00 .50 .433
```

```
% java IFS 10000 < sierpinski.txt
```



```
% more barnsley.txt          % java IFS 20000 < barnsley.txt
```

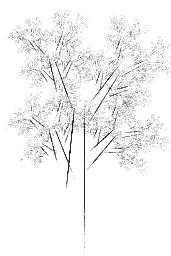
```
4
 0.01 0.85 0.07 0.07
4 3
 0.00 0.00 0.500
 0.85 0.04 0.075
 0.20 -0.26 0.400
 -0.15 0.28 0.575
4 3
 0.00 0.16 0.000
 -0.04 0.85 0.180
 0.23 0.22 0.045
 0.26 0.24 -0.086
```



```
% more tree.txt
```

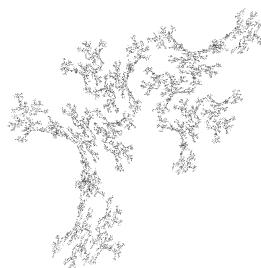
```
6
 0.1 0.1 0.2 0.2 0.2 0.2
6 3
 0.00 0.00 0.550
 -0.05 0.00 0.525
 0.46 -0.15 0.270
 0.47 -0.15 0.265
 0.43 0.28 0.285
 0.42 0.26 0.290
6 3
 0.00 0.60 0.000
 -0.50 0.00 0.750
 0.39 0.38 0.105
 0.17 0.42 0.465
 -0.25 0.45 0.625
 -0.35 0.31 0.525
```

```
% java IFS 20000 < tree.txt
```



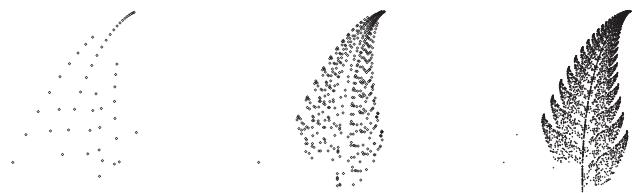
```
% java IFS 20000 < coral.txt
```

```
% more coral.txt
3
 0.40 0.15 0.45
3 3
 0.3077 -0.5315 0.8863
 0.3077 -0.0769 0.2166
 0.0000 0.5455 0.0106
3 3
 -0.4615 -0.2937 1.0962
 0.1538 -0.4476 0.3384
 0.6923 -0.1958 0.3808
```



Examples of iterated function systems

We could write code just like the code we just wrote for the Sierpinski triangle to iterate these rules, but matrix processing provides a uniform way to generalize that code to handle any set of rules. We have m different transformations, chosen from a 1-by- m vector with `StdRandom.discrete()`. For each transformation, we have an equation for updating x and an equation for updating y , so we use two m -by-3 matrices for the equation coefficients, one for x and one for y . IFS (PROGRAM 2.2.3) implements this data-driven version of the computation. This program enables limitless exploration: it performs the iteration for any input containing a vector that defines the probability distribution and the two matrices that define the coefficients, one for updating x and the other for updating y . For the coefficients just given, again, even though we choose a random equation at each step, the same figure emerges every time that we do this computation: an image that looks remarkably similar to a fern that you might see in the woods, not something generated by a random process on a computer.



Generating a Barnsley fern

That the same short program that takes a few numbers from standard input and plots points on standard drawing can (given different data) produce both the Sierpinski triangle and the Barnsley fern (and many, many other images) is truly remarkable. Because of its simplicity and the appeal of the results, this sort of calculation is useful in making synthetic images that have a realistic appearance in computer-generated movies and games.

Perhaps more significantly, the ability to produce such realistic diagrams so easily suggests intriguing scientific questions: What does computation tell us about nature? What does nature tell us about computation?

Statistics Next, we consider a library for a set of mathematical calculations and basic visualization tools that arise in all sorts of applications in science and engineering and are not all implemented in standard Java libraries. These calculations relate to the task of understanding the statistical properties of a set of numbers. Such a library is useful, for example, when we perform a series of scientific experiments that yield measurements of a quantity. One of the most important challenges facing modern scientists is proper analysis of such data, and computation is playing an increasingly important role in such analysis. These basic data analysis methods that we will consider are summarized in the following API:

<code>public class StdStats</code>	
<code>double max(double[] a)</code>	<i>largest value</i>
<code>double min(double[] a)</code>	<i>smallest value</i>
<code>double mean(double[] a)</code>	<i>average</i>
<code>double var(double[] a)</code>	<i>sample variance</i>
<code>double stddev(double[] a)</code>	<i>sample standard deviation</i>
<code>double median(double[] a)</code>	<i>median</i>
<code>void plotPoints(double[] a)</code>	<i>plot points at $(i, a[i])$</i>
<code>void plotLines(double[] a)</code>	<i>plot lines connecting points at $(i, a[i])$</i>
<code>void plotBars(double[] a)</code>	<i>plot bars to points at $(i, a[i])$</i>

Note: Overloaded implementations are included for other numeric types.

API for our library of static methods for data analysis

Basic statistics. Suppose that we have n measurements x_0, x_1, \dots, x_{n-1} . The average value of those measurements, otherwise known as the *mean*, is given by the formula $\mu = (x_0 + x_1 + \dots + x_{n-1}) / n$ and is an estimate of the value of the quantity. The minimum and maximum values are also of interest, as is the median (the value that is smaller than and larger than half the values). Also of interest is the *sample variance*, which is given by the formula

$$\sigma^2 = ((x_0 - \mu)^2 + (x_1 - \mu)^2 + \dots + (x_{n-1} - \mu)^2) / (n-1)$$

Program 2.2.4 Data analysis library

```

public class StdStats
{
    public static double max(double[] a)
    { // Compute maximum value in a[].
        double max = Double.NEGATIVE_INFINITY;
        for (int i = 0; i < a.length; i++)
            if (a[i] > max) max = a[i];
        return max;
    }

    public static double mean(double[] a)
    { // Compute the average of the values in a[].
        double sum = 0.0;
        for (int i = 0; i < a.length; i++)
            sum = sum + a[i];
        return sum / a.length;
    }

    public static double var(double[] a)
    { // Compute the sample variance of the values in a[].
        double avg = mean(a);
        double sum = 0.0;
        for (int i = 0; i < a.length; i++)
            sum += (a[i] - avg) * (a[i] - avg);
        return sum / (a.length - 1);
    }

    public static double stddev(double[] a)
    { return Math.sqrt(var(a)); }

    // See Program 2.2.5 for plotting methods.

    public static void main(String[] args)
    { /* See text. */ }
}

```

This code implements methods to compute the maximum, mean, variance, and standard deviation of numbers in a client array. The method for computing the minimum is omitted; plotting methods are in PROGRAM 2.2.5; see EXERCISE 4.2.20 for median().

```
% more tiny1D.txt
5
3.0 1.0 2.0 5.0 4.0
```

```
% java StdStats < tiny1D.txt
min   1.000
mean   3.000
max   5.000
std dev 1.581
```

and the *sample standard deviation*, the square root of the sample variance. StdStats (PROGRAM 2.2.4) shows implementations of static methods for computing these basic statistics (the median is more difficult to compute than the others—we will consider the implementation of `median()` in SECTION 4.2). The `main()` test client for StdStats reads numbers from standard input into an array and calls each of the methods to print the minimum, mean, maximum, and standard deviation, as follows:

```
public static void main(String[] args)
{
    double[] a = StdArrayIO.readDouble1D();
    StdOut.printf("      min %7.3f\n", min(a));
    StdOut.printf("      mean %7.3f\n", mean(a));
    StdOut.printf("      max %7.3f\n", max(a));
    StdOut.printf(" std dev %7.3f\n", stddev(a));
}
```

As with `StdRandom`, a more extensive test of the calculations is called for (see EXERCISE 2.2.3). Typically, as we debug or test new methods in the library, we adjust the unit testing code accordingly, testing the methods one at a time. A mature and widely used library like `StdStats` also deserves a stress-testing client for extensively testing everything after any change. If you are interested in seeing what such a client might look like, you can find one for `StdStats` on the booksite. Most experienced programmers will advise you that any time spent doing unit testing and stress testing will more than pay for itself later.

Plotting. One important use of `StdDraw` is to help us visualize data rather than relying on tables of numbers. In a typical situation, we perform experiments, save the experimental data in an array, and then compare the results against a model, perhaps a mathematical function that describes the data. To expedite this process for the typical case where values of one variable are equally spaced, our `StdStats` library contains static methods that you can use for plotting data in an array. PROGRAM 2.2.5 is an implementation of the `plotPoints()`, `plotLines()`, and `plotBars()` methods for `StdStats`. These methods display the values in the argument array at evenly spaced intervals in the drawing window, either connected together by line segments (`lines`), filled circles at each value (`points`), or bars from the x -axis to the value (`bars`). They all plot the points with x -coordinate i and y -coordinate $a[i]$ using filled circles, lines through the points, and bars, respectively. In addition,

Program 2.2.5 Plotting data values in an array

```

public static void plotPoints(double[] a)
{ // Plot points at (i, a[i]).
    int n = a.length;
    StdDraw.setXscale(-1, n);
    StdDraw.setPenRadius(1/(3.0*n));
    for (int i = 0; i < n; i++)
        StdDraw.point(i, a[i]);
}

public static void plotLines(double[] a)
{ // Plot lines through points at (i, a[i]).
    int n = a.length;
    StdDraw.setXscale(-1, n);
    StdDraw.setPenRadius();
    for (int i = 1; i < n; i++)
        StdDraw.line(i-1, a[i-1], i, a[i]);
}

public static void plotBars(double[] a)
{ // Plot bars from (0, a[i]) to (i, a[i]).
    int n = a.length;
    StdDraw.setXscale(-1, n);
    for (int i = 0; i < n; i++)
        StdDraw.filledRectangle(i, a[i]/2, 0.25, a[i]/2);
}

```

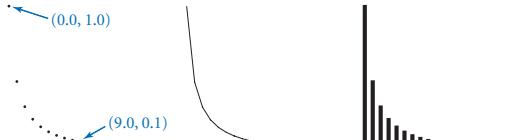
This code implements three methods in `StdStats` (PROGRAM 2.2.4) for plotting data. They plot the points $(i, a[i])$ with filled circles, connecting line segments, and bars, respectively.

```

plotPoints(a);   plotLines(a);   plotBars(a);

int n = 20;
double[] a = new double[n];
for (int i = 0; i < n; i++)
    a[i] = 1.0/(i+1);

```



The diagram illustrates three distinct data visualizations produced by the code:

- Scatter Plot:** A series of blue dots representing individual data points. The first point is labeled $(0.0, 1.0)$ and the last point is labeled $(9.0, 0.1)$.
- Line Plot:** A continuous black line connecting the points, showing a downward-curving trend.
- Bar Chart:** A series of vertical black bars representing the data as discrete segments. The height of each bar corresponds to the value of $a[i]$ at index i .

they all rescale x to fill the drawing window (so that the points are evenly spaced along the x -coordinate) and leave to the client scaling of the y -coordinates.

These methods are not intended to be a general-purpose plotting package, but you can certainly think of all sorts of things that you might want to add: different types of spots, labeled axes, color, and many other artifacts are commonly found in modern systems that can plot data. Some situations might call for more complicated methods than these.

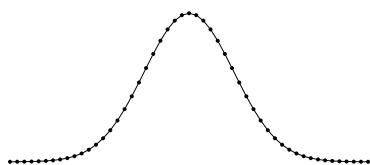
Our intent with StdStats is to introduce you to data analysis while showing you how easy it is to define a library to take care of useful tasks. Indeed, this library has already proved useful—we use these plotting methods to produce the figures in this book that depict function graphs, sound waves, and experimental results. Next, we consider several examples of their use.

Plotting function graphs. You can use the `StdStats.plot*` methods to draw a plot of the function graph for any function at all: choose an x -interval where you want to plot the function, compute function values evenly spaced through that interval and store them in an array, determine and set the y -scale, and then call `StdStats.plotLines()` or another `plot*` method. For example, to plot a sine function, rescale the y -axis to cover values between -1 and $+1$. Scaling the x -axis is automatically handled by the StdStats methods. If you do not know the range, you can handle the situation by calling:

```
StdDraw.setScale(StdStats.min(a), StdStats.max(a));
```

The smoothness of the curve is determined by properties of the function and by the number of points plotted. As we discussed when first considering `StdDraw`, you have to be careful to sample enough points to catch fluctuations in the function. We will consider another approach to plotting functions based on sampling values that are not equally spaced in SECTION 2.4.

```
int n = 50;
double[] a = new double[n+1];
for (int i = 0; i <= n; i++)
    a[i] = Gaussian.pdf(-4.0 + 8.0*i/n);
StdStats.plotPoints(a);
StdStats.plotLines(a);
```



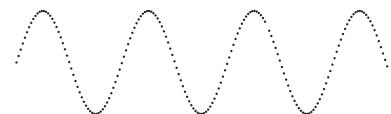
Plotting a function graph

Plotting sound waves. Both the `StdAudio` library and the `StdStats` plot methods work with arrays that contain sampled values at regular intervals. The diagrams of sound waves in SECTION 1.5 and at the beginning of this section were all produced by first scaling the y -axis with `StdDraw.setScale(-1, 1)`, then plotting the points with `StdStats.plotPoints()`. As you have seen, such plots give direct insight into processing audio. You can also produce interesting effects by plotting sound waves as you play them with `StdAudio`, although this task is a bit challenging because of the huge amount of data involved (see EXERCISE 1.5.23).

Plotting experimental results. You can put multiple plots on the same drawing. One typical reason to do so is to compare experimental results with a theoretical model. For example, `Bernoulli` (PROGRAM 2.2.6) counts the number of heads found when a fair coin is flipped n times and compares the result with the predicted Gaussian probability density function. A famous result from probability theory is that the distribution of this quantity is the *binomial distribution*, which is extremely well approximated by the Gaussian distribution with mean $n/2$ and standard deviation $\sqrt{n}/2$. The more trials we perform, the more accurate the approximation. The drawing produced by `Bernoulli` is a succinct summary of the results of the experiment and a convincing validation of the theory. This example is prototypical of a scientific approach to applications programming that we use often throughout this book and that you should use whenever you run an experiment. If a theoretical model that can explain your results is available, a visual plot comparing the experiment to the theory can validate both.

THESE FEW EXAMPLES ARE INTENDED TO suggest what is possible with a well-designed library of static methods for data analysis. Several extensions and other ideas are explored in the exercises. You will find `StdStats` to be useful for basic plots, and you are encouraged to experiment with these implementations and to modify them or to add methods to make your own library that can draw plots of your own design. As you continue to address an ever-widening circle of programming tasks, you will naturally be drawn to the idea of developing tools like these for your own use.

```
StdDraw.setScale(-1.0, 1.0);
double[] hi;
hi = PlayThatTune.tone(880, 0.01);
StdStats.plotPoints(hi);
```



Plotting a sound wave

Program 2.2.6 Bernoulli trials

```

public class Bernoulli
{
    public static int binomial(int n)
    { // Simulate flipping a coin n times; return # heads.
        int heads = 0;
        for (int i = 0; i < n; i++)
            if (StdRandom.bernoulli(0.5)) heads++;
        return heads;
    }
    public static void main(String[] args)
    { // Perform Bernoulli trials, plot results and model.
        int n = Integer.parseInt(args[0]);
        int trials = Integer.parseInt(args[1]);
        int[] freq = new int[n+1];
        for (int t = 0; t < trials; t++)
            freq[binomial(n)]++;
        double[] norm = new double[n+1];
        for (int i = 0; i <= n; i++)
            norm[i] = (double) freq[i] / trials;
        StdStats.plotBars(norm);

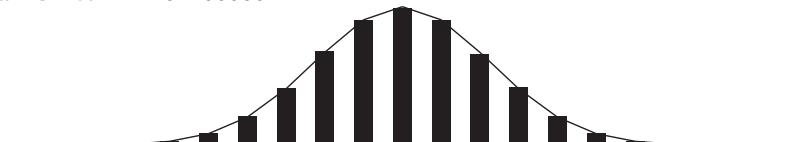
        double mean   = n / 2.0;
        double stddev = Math.sqrt(n) / 2.0;
        double[] phi  = new double[n+1];
        for (int i = 0; i <= n; i++)
            phi[i] = Gaussian.pdf(i, mean, stddev);
        StdStats.plotLines(phi);
    }
}

```

n	<i>number of flips per trial</i>
trials	<i>number of trials</i>
freq[]	<i>experimental results</i>
norm[]	<i>normalized results</i>
phi[]	<i>Gaussian model</i>

This `StdStats`, `StdRandom`, and `Gaussian` client provides visual evidence that the number of heads observed when a fair coin is flipped n times obeys a Gaussian distribution.

```
% java Bernoulli 20 100000
```



Modular programming The library implementations that we have developed illustrate a programming style known as *modular programming*. Instead of writing a new program that is self-contained within its own file to address a new problem, we break up each task into smaller, more manageable subtasks, then implement and independently debug code that addresses each subtask. Good libraries facilitate modular programming by allowing us to define and provide solutions for important subtasks for future clients. *Whenever you can clearly separate tasks within a program, you should do so.* Java supports such separation by allowing us to independently debug and later use classes in separate files. Traditionally, programmers use the term *module* to refer to code that can be compiled and run independently; in Java, each *class* is a module.

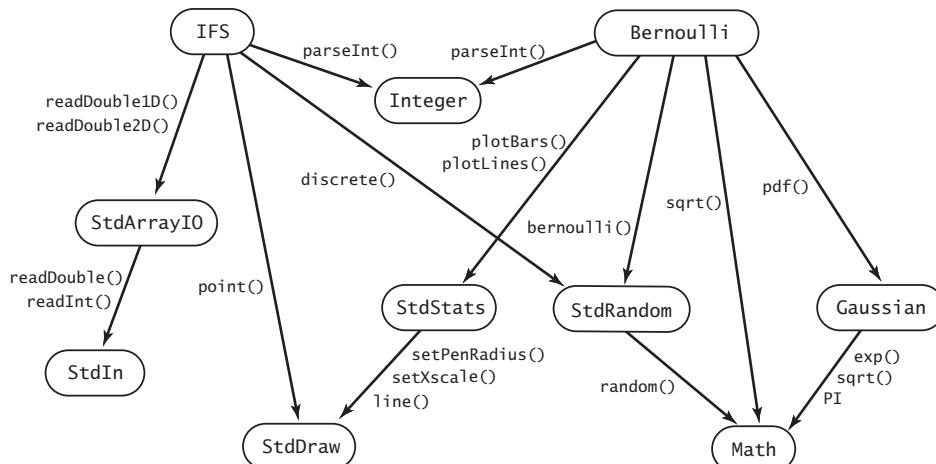
`IFS` (PROGRAM 2.2.3) exemplifies modular programming. This relatively sophisticated computation is implemented with several relatively small modules, developed independently. It uses `StdRandom` and `StdArrayIO`, as well as the methods from `Integer` and `StdDraw` that we are accustomed to using. If we were to put all of the code required for `IFS` in a single file, we would have a large amount of code on our hands to maintain and debug; with modular programming, we can study iterated function systems with some confidence that the arrays are read properly and that the random number generator will produce properly distributed values, because we already implemented and tested the code for these tasks in separate modules.

Similarly, `Bernoulli` (PROGRAM 2.2.6) exemplifies modular programming. It is a client of `Gaussian`, `Integer`, `Math`, `StdRandom`, and `StdStats`. Again, we can have some confidence that the methods in these modules produce the expected results because they are system libraries or libraries that we have tested, debugged, and used before.

API	description
<code>Gaussian</code>	Gaussian distribution functions
<code>StdRandom</code>	random numbers
<code>StdArrayIO</code>	input and output for arrays
<code>IFS</code>	client for iterated function systems
<code>StdStats</code>	functions for data analysis
<code>Bernoulli</code>	client for Bernoulli trials

Summary of classes in this section

To describe the relationships among modules in a modular program, we often draw a *dependency graph*, where we connect two class names with an arrow labeled with the name of a method if the first class contains a method call and the second class contains the definition of the method. Such diagrams play an important role because understanding the relationships among modules is necessary for proper development and maintenance.



Dependency graph (partial) for the modules in this section

We emphasize modular programming throughout this book because it has many important advantages that have come to be accepted as essential in modern programming, including the following:

- We can have programs of a reasonable size, even in large systems.
- Debugging is restricted to small pieces of code.
- We can reuse code without having to re-implement it.
- Maintaining (and improving) code is much simpler.

The importance of these advantages is difficult to overstate, so we will expand upon each of them.

Programs of a reasonable size. No large task is so complex that it cannot be divided into smaller subtasks. If you find yourself with a program that stretches to more than a few pages of code, you must ask yourself the following questions: Are there subtasks that could be implemented separately? Could some of these subtasks be

logically grouped together in a separate library? Could other clients use this code in the future? At the other end of the range, if you find yourself with a huge number of tiny modules, you must ask yourself questions such as these: Is there some group of subtasks that logically belong in the same module? Is each module likely to be used by multiple clients? There is no hard-and-fast rule on module size: one implementation of a critically important abstraction might properly be a few lines of code, whereas another library with a large number of overloaded methods might properly stretch to hundreds of lines of code.

Debugging. Tracing a program rapidly becomes more difficult as the number of statements and interacting variables increases. Tracing a program with hundreds of variables requires keeping track of hundreds of values, as any statement might affect or be affected by any variable. To do so for hundreds or thousands of statements or more is untenable. With modular programming and our guiding principle of keeping the scope of variables local to the extent possible, we severely restrict the number of possibilities that we have to consider when debugging. Equally important is the idea of a contract between client and implementation. Once we are satisfied that an implementation is meeting its end of the bargain, we can debug all its clients under that assumption.

Code reuse. Once we have implemented libraries such as `StdStats` and `StdRandom`, we do not have to worry about writing code to compute averages or standard deviations or to generate random numbers again—we can simply reuse the code that we have written. Moreover, we do not need to make copies of the code: any module can just refer to any public method in any other module.

Maintenance. Like a good piece of writing, a good program can always be improved, and modular programming facilitates the process of continually improving your Java programs because improving a module improves all of its clients. For example, it is normally the case that there are several different approaches to solving a particular problem. With modular programming, you can implement more than one and try them independently. More importantly, suppose that while developing a new client, you find a bug in some module. With modular programming, fixing that bug essentially fixes bugs in all of the module's clients.

IF YOU ENOUNTER AN OLD PROGRAM (or a new program written by an old programmer!), you are likely to find one huge module—a long sequence of statements, stretching to several pages or more, where any statement can refer to any variable in the program. Old programs of this kind are found in critical parts of our computational infrastructure (for example, some nuclear power plants and some banks) precisely because the programmers charged with maintaining them cannot even understand them well enough to rewrite them in a modern language! With support for modular programming, modern languages like Java help us avoid such situations by separately developing libraries of methods in independent classes.

The ability to share static methods among different files fundamentally extends our programming model in two different ways. First, it allows us to reuse code without having to maintain multiple copies of it. Second, by allowing us to organize a program into files of manageable size that can be independently debugged and compiled, it strongly supports our basic message: *whenever you can clearly separate tasks within a program, you should do so.*

In this section, we have supplemented the `Std*` libraries of SECTION 1.5 with several other libraries that you can use: `Gaussian`, `StdArrayIO`, `StdRandom`, and `StdStats`. Furthermore, we have illustrated their use with several client programs. These tools are centered on basic mathematical concepts that arise in any scientific project or engineering task. Our intent is not just to provide tools, but also to illustrate that it is easy to create your own tools. The first question that most modern programmers ask when addressing a complex task is “Which tools do I need?” When the needed tools are not conveniently available, the second question is “How difficult would it be to implement them?” To be a good programmer, you need to have the confidence to build a software tool when you need it and the wisdom to know when it might be better to seek a solution in a library.

After libraries and modular programming, you have one more step to learn a complete modern programming model: *object-oriented programming*, the topic of CHAPTER 3. With object-oriented programming, you can build libraries of functions that use side effects (in a tightly controlled manner) to vastly extend the Java programming model. Before moving to object-oriented programming, we consider in this chapter the profound ramifications of the idea that any method can call itself (in SECTION 2.3) and a more extensive case study (in SECTION 2.4) of modular programming than the small clients in this section.

Q&A

Q. I tried to use `StdRandom`, but got the error message `Exception in thread "main" java.lang.NoClassDefFoundError: StdRandom`. What's wrong?

A. You need to make `StdRandom` accessible to Java. See the first Q&A at the end of SECTION 1.5.

Q. Is there a keyword that identifies a class as a library?

A. No, any set of public methods will do. There is a bit of a conceptual leap in this viewpoint because it is one thing to sit down to create a `.java` file that you will compile and run, quite another thing to create a `.java` file that you will rely on much later in the future, and still another thing to create a `.java` file for *someone else* to use in the future. You need to develop some libraries for your own use before engaging in this sort of activity, which is the province of experienced systems programmers.

Q. How do I develop a new version of a library that I have been using for a while?

A. With care. Any change to the API might break any client program, so it is best to work in a separate directory. When you use this approach, you are working with a copy of the code. If you are changing a library that has a lot of clients, you can appreciate the problems faced by companies putting out new versions of their software. If you just want to add a few methods to a library, go ahead: that is usually not too dangerous, though you should realize that you might find yourself in a situation where you have to support that library for years!

Q. How do I know that an implementation behaves properly? Why not automatically check that it satisfies the API?

A. We use informal specifications because writing a detailed specification is not much different from writing a program. Moreover, a fundamental tenet of theoretical computer science says that doing so does not even solve the basic problem, because generally there is no way to check that two different programs perform the same computation.

Exercises

- 2.2.1** Add to Gaussian (PROGRAM 2.1.2) an implementation of the three-argument static method `pdf(x, mu, sigma)` specified in the API that computes the Gaussian probability density function with a given mean μ and standard deviation σ , based on the formula $\phi(x, \mu, \sigma) = \phi((x - \mu) / \sigma) / \sigma$. Also add an implementation of the associated cumulative distribution function `cdf(z, mu, sigma)`, based on the formula $\Phi(z, \mu, \sigma) = \Phi((z - \mu) / \sigma)$.
- 2.2.2** Write a library of static methods that implements the *hyperbolic* functions based on the definitions $\sinh(x) = (e^x - e^{-x}) / 2$ and $\cosh(x) = (e^x + e^{-x}) / 2$, with $\tanh(x)$, $\coth(x)$, $\text{sech}(x)$, and $\text{csch}(x)$ defined in a manner analogous to standard trigonometric functions.
- 2.2.3** Write a test client for both `StdStats` and `StdRandom` that checks that the methods in both libraries operate as expected. Take a command-line argument n , generate n random numbers using each of the methods in `StdRandom`, and print their statistics. *Extra credit:* Defend the results that you get by comparing them to those that are to be expected from analysis.
- 2.2.4** Add to `StdRandom` a method `shuffle()` that takes an array of `double` values as argument and rearranges them in random order. Implement a test client that checks that each permutation of the array is produced about the same number of times. Add overloaded methods that take arrays of integers and strings.
- 2.2.5** Develop a client that does stress testing for `StdRandom`. Pay particular attention to `discrete()`. For example, do the probabilities sum to 1?
- 2.2.6** Write a static method that takes `double` values `ymin` and `ymax` (with `ymin` strictly less than `ymax`), and a `double` array `a[]` as arguments and uses the `StdStats` library to linearly scale the values in `a[]` so that they are all between `ymin` and `ymax`.
- 2.2.7** Write a `Gaussian` and `StdStats` client that explores the effects of changing the mean and standard deviation for the Gaussian probability density function. Create one plot with the Gaussian distributions having a fixed mean and various standard deviations and another with Gaussian distributions having a fixed standard deviation and various means.

2.2.8 Add a method `exp()` to `StdRandom` that takes an argument λ and returns a random number drawn from the *exponential distribution* with rate λ . *Hint:* If x is a random number uniformly distributed between 0 and 1, then $-\ln x / \lambda$ is a random number from the exponential distribution with rate λ .

2.2.9 Add to `StdRandom` a static method `maxwellBoltzmann()` that returns a random value drawn from a *Maxwell–Boltzmann distribution* with parameter σ . To produce such a value, return the square root of the sum of the squares of three random numbers drawn from the Gaussian distribution with mean 0 and standard deviation σ . The speeds of molecules in an ideal gas obey a Maxwell–Boltzmann distribution.

2.2.10 Modify `Bernoulli` (PROGRAM 2.2.6) to animate the bar graph, replotting it after each experiment, so that you can watch it converge to the Gaussian distribution. Then add a command-line argument and an overloaded `binomial()` implementation to allow you to specify the probability p that a biased coin comes up heads, and run experiments to get a feeling for the distribution corresponding to a biased coin. Be sure to try values of p that are close to 0 and close to 1.

2.2.11 Develop a full implementation of `StdArrayIO` (implement all 12 methods indicated in the API).

2.2.12 Write a library `Matrix` that implements the following API:

<code>public class Matrix</code>	
<code>double dot(double[] a, double[] b)</code>	<i>vector dot product</i>
<code>double[][] multiply(double[][] a, double[][] b)</code>	<i>matrix–matrix product</i>
<code>double[][] transpose(double[][] a)</code>	<i>transpose</i>
<code>double[] multiply(double[][] a, double[] x)</code>	<i>matrix–vector product</i>
<code>double[] multiply(double[] x, double[][] a)</code>	<i>vector–matrix product</i>

(See SECTION 1.4.) As a test client, use the following code, which performs the same calculation as `Markov` (PROGRAM 1.6.3):

```

public static void main(String[] args)
{
    int trials = Integer.parseInt(args[0]);
    double[][] p = StdArrayIO.readDouble2D();
    double[] ranks = new double[p.length];
    rank[0] = 1.0;
    for (int t = 0; t < trials; t++)
        ranks = Matrix.multiply(ranks, p);
    StdArrayIO.print(ranks);
}

```

Mathematicians and scientists use mature libraries or special-purpose matrix-processing languages for such tasks. See the booksite for details on using such libraries.

2.2.13 Write a `Matrix` client that implements the version of Markov described in SECTION 1.6 but is based on squaring the matrix, instead of iterating the vector-matrix multiplication.

2.2.14 Rewrite `RandomSurfer` (PROGRAM 1.6.2) using the `StdArrayIO` and `StdRandom` libraries.

Partial solution.

```

...
double[][] p = StdArrayIO.readDouble2D();
int page = 0; // Start at page 0.
int[] freq = new int[n];
for (int t = 0; t < trials; t++)
{
    page = StdRandom.discrete(p[page]);
    freq[page]++;
}
...

```

Creative Exercises

2.2.15 Sicherman dice. Suppose that you have two six-sided dice, one with faces labeled 1, 3, 4, 5, 6, and 8 and the other with faces labeled 1, 2, 2, 3, 3, and 4. Compare the probabilities of occurrence of each of the values of the sum of the dice with those for a standard pair of dice. Use `StdRandom` and `StdStats`.

2.2.16 Craps. The following are the rules for a *pass bet* in the game of *craps*. Roll two six-sided dice, and let x be their sum.

- If x is 7 or 11, you win.
- If x is 2, 3, or 12, you lose.

Otherwise, repeatedly roll the two dice until their sum is either x or 7.

- If their sum is x , you win.
- If their sum is 7, you lose.

Write a modular program to estimate the probability of winning a pass bet. Modify your program to handle loaded dice, where the probability of a die landing on 1 is taken from the command line, the probability of landing on 6 is $1/6$ minus that probability, and 2–5 are assumed equally likely. Hint: Use `StdRandom.discrete()`.

2.2.17 Gaussian random values. Implement the no-argument `gaussian()` function in `StdRandom` (PROGRAM 2.2.1) using the Box–Muller formula (see EXERCISE 1.2.27). Next, consider an alternative approach, known as *Marsaglia’s method*, which is based on generating a random point in the unit circle and using a form of the Box–Muller formula (see the discussion of `do-while` at the end of SECTION 1.3).

```
public static double gaussian()
{
    double r, x, y;
    do
    {
        x = uniform(-1.0, 1.0);
        y = uniform(-1.0, 1.0);
        r = x*x + y*y;
    } while (r >= 1 || r == 0);
    return x * Math.sqrt(-2 * Math.log(r) / r);
}
```

For each approach, generate 10 million random values from the Gaussian distribution, and measure which is faster.

2.2.18 *Dynamic histogram.* Suppose that the standard input stream is a sequence of double values. Write a program that takes an integer n and two double values lo and hi from the command line and uses `StdStats` to plot a histogram of the count of the numbers in the standard input stream that fall in each of the n intervals defined by dividing (lo, hi) into n equal-sized intervals. Use your program to add code to your solution to EXERCISE 2.2.3 to plot a histogram of the distribution of the numbers produced by each method, taking n from the command line.

2.2.19 *Stress test.* Develop a client that does stress testing for `StdStats`. Work with a classmate, with one person writing code and the other testing it.

2.2.20 *Gambler's ruin.* Develop a `StdRandom` client to study the gambler's ruin problem (see PROGRAM 1.3.8 and EXERCISE 1.3.24–25). *Note:* Defining a static method for the experiment is more difficult than for `Bernoulli` because you cannot return two values.

2.2.21 *IFS.* Experiment with various inputs to `IFS` to create patterns of your own design like the Sierpinski triangle, the Barnsley fern, or the other examples in the table in the text. You might begin by experimenting with minor modifications to the given inputs.

2.2.22 *IFS matrix implementation.* Write a version of `IFS` that uses the static method `multiply()` from `Matrix` (see EXERCISE 2.2.12) instead of the equations that compute the new values of x_0 and y_0 .

2.2.23 *Library for properties of integers.* Develop a library based on the functions that we have considered in this book for computing properties of integers. Include functions for determining whether a given integer is prime; determining whether two integers are relatively prime; computing all the factors of a given integer; computing the greatest common divisor and least common multiple of two integers; Euler's totient function (EXERCISE 2.1.26); and any other functions that you think might be useful. Include overloaded implementations for `long` values. Create an API, a client that performs stress testing, and clients that solve several of the exercises earlier in this book.

2.2.24 *Music library.* Develop a library based on the functions in `PlayThatTune` (PROGRAM 2.1.4) that you can use to write client programs to create and manipulate songs.

2.2.25 *Voting machines.* Develop a `StdRandom` client (with appropriate static methods of its own) to study the following problem: Suppose that in a population of 100 million voters, 51% vote for candidate *A* and 49% vote for candidate *B*. However, the voting machines are prone to make mistakes, and 5% of the time they produce the wrong answer. Assuming the errors are made independently and at random, is a 5% error rate enough to invalidate the results of a close election? What error rate can be tolerated?

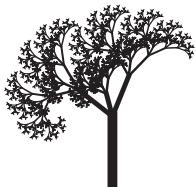
2.2.26 *Poker analysis.* Write a `StdRandom` and `StdStats` client (with appropriate static methods of its own) to estimate the probabilities of getting one pair, two pair, three of a kind, a full house, and a flush in a five-card poker hand via simulation. Divide your program into appropriate static methods and defend your design decisions. *Extra credit:* Add straight and straight flush to the list of possibilities.

2.2.27 *Animated plots.* Write a program that takes a command-line argument *m* and produces a bar graph of the *m* most recent `double` values on standard input. Use the same animation technique that we used for `BouncingBall` (PROGRAM 1.5.6): `erase`, `redraw`, `show`, and `wait briefly`. Each time your program reads a new number, it should redraw the whole bar graph. Since most of the picture does not change as it is redrawn slightly to the left, your program will produce the effect of a fixed-size window dynamically sliding over the input values. Use your program to plot a huge time-variant data file, such as stock prices.

2.2.28 *Array plot library.* Develop your own plot methods that improve upon those in `StdStats`. Be creative! Try to make a plotting library that you think will be useful for some application in the future.

2.3 Recursion

THE IDEA OF CALLING ONE FUNCTION from another immediately suggests the possibility of a function calling *itself*. The function-call mechanism in Java and most modern programming languages supports this possibility, which is known as *recursion*. In this section, we will study examples of elegant and efficient recursive solutions to a variety of problems. Recursion is a powerful programming technique that we use often in this book. Recursive programs are often more compact and easier to understand than their nonrecursive counterparts. Few programmers become sufficiently comfortable with recursion to use it in everyday code, but solving a problem with an elegantly crafted recursive program is a satisfying experience that is certainly accessible to every programmer (even you!).



*A recursive model
of the natural world*

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Programs in this section

Recursion is much more than a programming technique. In many settings, it is a useful way to describe the natural world. For example, the recursive tree (to the left) resembles a real tree, and has a natural recursive description. Many, many phenomena are well explained by recursive models. In particular, recursion plays a central role in computer science. It provides a simple computational model that embraces everything that can be computed with any computer; it helps us to organize and to analyze programs; and it is the key to numerous critically important computational applications, ranging from combinatorial search to tree data structures that support information processing to the fast Fourier transform for signal processing.

One important reason to embrace recursion is that it provides a straightforward way to build simple mathematical models that we can use to prove important facts about our programs. The proof technique that we use to do so is known as *mathematical induction*. Generally, we avoid going into the details of mathematical proofs in this book, but you will see in this section that it is worthwhile to understand that point of view and make the effort to convince yourself that recursive programs have the intended effect.

Your first recursive program The “Hello, World” for recursion is the *factorial* function, defined for positive integers n by the equation

$$n! = n \times (n-1) \times (n-2) \times \dots \times 2 \times 1$$

In other words, $n!$ is the product of the positive integers less than or equal to n . Now, $n!$ is easy to compute with a for loop, but an even easier method is to use the following recursive function:

```
public static long factorial(int n)
{
    if (n == 1) return 1;
    return n * factorial(n-1);
}
```

This function calls itself. The implementation clearly produces the desired effect. You can persuade yourself that it does so by noting that `factorial()` returns $1 = 1!$ when n is 1 and that if it properly computes the value

$$(n-1)! = (n-1) \times (n-2) \times \dots \times 2 \times 1$$

then it properly computes the value

$$\begin{aligned} n! &= n \times (n-1)! \\ &= n \times (n-1) \times (n-2) \times \dots \times 2 \times 1 \end{aligned}$$

To compute `factorial(5)`, the recursive function multiplies 5 by `factorial(4)`; to compute `factorial(4)`, it multiplies 4 by `factorial(3)`; and so forth. This process is repeated until calling `factorial(1)`, which directly returns the value 1. We can trace this computation in precisely the same way that we trace any sequence of function calls. Since we treat all of the calls as being independent copies of the code, the fact that they are recursive is immaterial.

Our `factorial()` implementation exhibits the two main components that are required for every recursive function. First, the *base case* returns a value without making any subsequent recursive calls. It does this for one or more special input values for which the function can be evaluated without recursion. For `factorial()`, the base case is $n = 1$. Second, the *reduction step* is the central

```
factorial(5)
factorial(4)
factorial(3)
factorial(2)
factorial(1)
    return 1
    return 2*1 = 2
    return 3*2 = 6
    return 4*6 = 24
return 5*24 = 120
```

Function-call trace for `factorial(5)`

part of a recursive function. It relates the value of the function at one (or more) arguments to the value of function at one (or more) other arguments. For `factorial()`, the reduction step is `n * factorial(n-1)`. All recursive functions must have these two components. Furthermore, the sequence of argument values must converge to the base case. For `factorial()`, the value of n decreases by 1 for each call, so the sequence of argument values converges to the base case $n = 1$.

Tiny programs such as `factorial()` perhaps become slightly clearer if we put the reduction step in an `else` clause. However, adopting this convention for every recursive program would unnecessarily complicate larger programs because it would involve putting most of the code (for the reduction step) within curly braces after the `else`. Instead, we adopt the convention of always putting the base case as the first statement, ending with a `return`, and then devoting the rest of the code to the reduction step.

The `factorial()` implementation itself is not particularly useful in practice because $n!$ grows so quickly that the multiplication will overflow a `long` and produce incorrect answers for $n > 20$. But the same technique is effective for computing all sorts of functions. For example, the recursive function

```
public static double harmonic(int n)
{
    if (n == 1) return 1.0;
    return harmonic(n-1) + 1.0/n;
}
```

computes the n th harmonic numbers (see PROGRAM 1.3.5) when n is small, based on the following equations:

$$\begin{aligned} H_n &= 1 + 1/2 + \dots + 1/n \\ &= (1 + 1/2 + \dots + 1/(n-1)) + 1/n \\ &= H_{n-1} + 1/n \end{aligned}$$

Indeed, this same approach is effective for computing, with only a few lines of code, the value of *any* finite sum (or product) for which you have a compact formula. Recursive functions like these are just loops in disguise, but recursion can help us better understand the underlying computation.

1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3628800
11	39916800
12	479001600
13	6227020800
14	87178291200
15	1307674368000
16	20922789888000
17	355687428096000
18	6402373705728000
19	121645100408832000
20	2432902008176640000

Values of $n!$ in long

Mathematical induction Recursive programming is directly related to *mathematical induction*, a technique that is widely used for proving facts about the natural numbers.

Proving that a statement involving an integer n is true for infinitely many values of n by mathematical induction involves the following two steps:

- The *base case*: prove the statement true for some specific value or values of n (usually 0 or 1).
- The *induction step* (the central part of the proof): assume the statement to be true for all positive integers less than n , then use that fact to prove it true for n .

Such a proof suffices to show that the statement is true for *infinitely* many values of n : we can start at the base case, and use our proof to establish that the statement is true for each larger value of n , one by one.

Everyone's first induction proof is to demonstrate that the sum of the positive integers less than or equal to n is given by the formula $n(n + 1) / 2$. That is, we wish to prove that the following equation is valid for all $n \geq 1$:

$$1 + 2 + 3 \dots + (n-1) + n = n(n+1)/2$$

The equation is certainly true for $n = 1$ (base case) because $1 = 1(1 + 1) / 2$. If we assume it to be true for all positive integers less than n , then, in particular, it is true for $n - 1$, so

$$1 + 2 + 3 \dots + (n-1) = (n-1)n/2$$

and we can add n to both sides of this equation and simplify to get the desired equation (induction step).

Every time we write a recursive program, we need mathematical induction to be convinced that the program has the desired effect. The correspondence between induction and recursion is self-evident. The difference in nomenclature indicates a difference in outlook: in a recursive program, our outlook is to get a computation done by reducing to a smaller problem, so we use the term *reduction step*; in an induction proof, our outlook is to establish the truth of the statement for larger problems, so we use the term *induction step*.

When we write recursive programs we usually do not write down a full formal proof that they produce the desired result, but we are always dependent upon the existence of such a proof. We often appeal to an informal induction proof to convince ourselves that a recursive program operates as expected. For example, we just discussed an informal proof to become convinced that `factorial()` computes the product of the positive integers less than or equal to n .

Program 2.3.1 Euclid's algorithm

```

public class Euclid
{
    public static int gcd(int p, int q)
    {
        if (q == 0) return p;
        return gcd(q, p % q);
    }
    public static void main(String[] args)
    {
        int p = Integer.parseInt(args[0]);
        int q = Integer.parseInt(args[1]);
        int divisor = gcd(p, q);
        StdOut.println(divisor);
    }
}

```

p, q | *arguments*
 divisor | *greatest common divisor*

% java Euclid 1440 408
24
% java Euclid 314159 271828
1

This program prints the greatest common divisor of its two command-line arguments, using a recursive implementation of Euclid's algorithm.

Euclid's algorithm The *greatest common divisor* (gcd) of two positive integers is the largest integer that divides evenly into both of them. For example, the greatest common divisor of 102 and 68 is 34 since both 102 and 68 are multiples of 34, but no integer larger than 34 divides evenly into 102 and 68. You may recall learning about the greatest common divisor when you learned to reduce fractions. For example, we can simplify $68/102$ to $2/3$ by dividing both numerator and denominator by 34, their gcd. Finding the gcd of huge numbers is an important problem that arises in many commercial applications, including the famous RSA cryptosystem.

We can efficiently compute the gcd using the following property, which holds for positive integers p and q :

If $p > q$, the gcd of p and q is the same as the gcd of q and $p \% q$.

To convince yourself of this fact, first note that the gcd of p and q is the same as the gcd of q and $p-q$, because a number divides both p and q if and only if it divides both q and $p-q$. By the same argument, q and $p-2q$, q and $p-3q$, and so forth have the same gcd, and one way to compute $p \% q$ is to subtract q from p until getting a number less than q .

The static method `gcd()` in `Euclid` (PROGRAM 2.3.1) is a compact recursive function whose reduction step is based on this property. The base case is when q is 0, with $\text{gcd}(p, 0) = p$. To see that the reduction step converges to the base case, observe that the second argument value strictly decreases in each recursive call since $p \% q < q$. If $p < q$, the first recursive call effectively switches the order of the two arguments. In fact, the second argument value decreases by at least a factor of 2 for every second recursive call, so the sequence of argument values quickly converges to the base case (see EXERCISE 2.3.11). This recursive solution to the problem of computing the greatest common divisor is known as *Euclid’s algorithm* and is one of the oldest known algorithms—it is more than 2,000 years old.

```
gcd(1440, 408)
gcd(408, 216)
gcd(216, 192)
gcd(192, 24)
gcd(24, 0)
    return 24
return 24
return 24
return 24
return 24
return 24
```

Function-call trace for gcd()

Towers of Hanoi No discussion of recursion would be complete without the ancient *towers of Hanoi* problem. In this problem, we have three poles and n discs that fit onto the poles. The discs differ in size and are initially stacked on one of the poles, in order from largest (disc n) at the bottom to smallest (disc 1) at the top. The task is to move all n discs to another pole, while obeying the following rules:

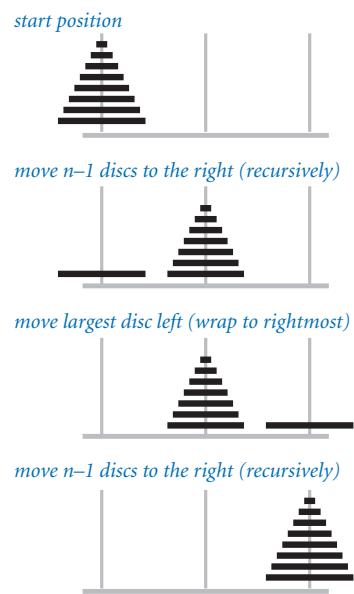
- Move only one disc at a time.
- Never place a larger disc on a smaller one.

One legend says that the world will end when a certain group of monks accomplishes this task in a temple with 64 golden discs on three diamond needles. But how can the monks accomplish the task at all, playing by the rules?

To solve the problem, our goal is to issue a sequence of instructions for moving the discs. We assume that the poles are arranged in a row, and that each instruction to move a disc specifies its number and whether to move it left or right. If a disc is on the left pole, an instruction to move left means to wrap to the right pole; if a disc is on the right pole, an instruction to move right means to wrap to the left pole. When the discs are all on one pole, there are two possible moves (move the smallest disc left or right); otherwise, there are three possible moves

(move the smallest disc left or right, or make the one legal move involving the other two poles). Choosing among these possibilities on each move to achieve the goal is a challenge that requires a plan. Recursion provides just the plan that we need, based on the following idea: first we move the top $n-1$ discs to an empty pole, then we move the largest disc to the other empty pole (where it does not interfere with the smaller ones), and then we complete the job by moving the $n-1$ discs onto the largest disc.

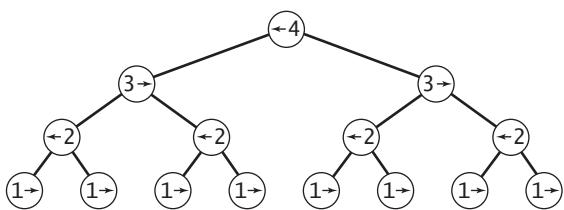
TowersOfHanoi (PROGRAM 2.3.2) is a direct implementation of this recursive strategy. It takes a command-line argument n and prints the solution to the towers of Hanoi problem on n discs. The recursive function moves() prints the sequence of moves to move the stack of discs to the left (if the argument left is true) or to the right (if left is false). It does so exactly according to the plan just described.



Recursive plan for towers of Hanoi

Function-call trees To better understand the behavior of modular programs that have multiple recursive calls (such as TowersOfHanoi), we use a visual representation known as a *function-call tree*. Specifically, we represent each method call as a *tree node*, depicted as a circle labeled with the values of the arguments for that call. Below each tree node, we draw the tree nodes corresponding to each call in that use of the method (in order from left to right) and lines connecting to them. This diagram contains all the information we need to understand the behavior of the program. It contains a tree node for each function call.

We can use function-call trees to understand the behavior of any modular program, but they are particularly useful in exposing the behavior of recursive programs. For example, the tree corresponding to a call to move() in TowersOfHanoi is easy to construct. Start by drawing a tree node labeled with the values of the command-line arguments. The first argument is the number



Function-call tree for moves(4, true) in TowersOfHanoi

Program 2.3.2 Towers of Hanoi

```
public class TowersOfHanoi
{
    public static void moves(int n, boolean left)
    {
        if (n == 0) return;
        moves(n-1, !left);
        if (left) StdOut.println(n + " left");
        else      StdOut.println(n + " right");
        moves(n-1, !left);
    }
    public static void main(String[] args)
    { // Read n, print moves to move n discs left.
        int n = Integer.parseInt(args[0]);
        moves(n, true);
    }
}
```

n *number of discs*
left *direction to move pile*

The recursive method `moves()` prints the moves needed to move `n` discs to the left (if `left` is true) or to the right (if `left` is false).

<pre>% java TowersOfHanoi 1 1 left</pre> <pre>% java TowersOfHanoi 2 1 right 2 left 1 right</pre> <pre>% java TowersOfHanoi 3 1 left 2 right 1 left 3 left 1 left 2 right 1 left</pre>	<pre>% java TowersOfHanoi 4 1 right 2 left 1 right 3 right 1 right 2 left 1 right 3 right 1 right 2 left 1 right</pre>
--	--

of discs in the pile to be moved (and the label of the disc to actually be moved); the second is the direction to move the disc. For clarity, we depict the direction (a boolean value) as an arrow that points left or right, since that is our interpretation of the value—the direction to move the piece. Then draw two tree nodes below with the number of discs decremented by 1 and the direction switched, and continue doing so until only nodes with labels corresponding to a first argument value 1 have no nodes below them. These nodes correspond to calls on `moves()` that do not lead to further recursive calls.

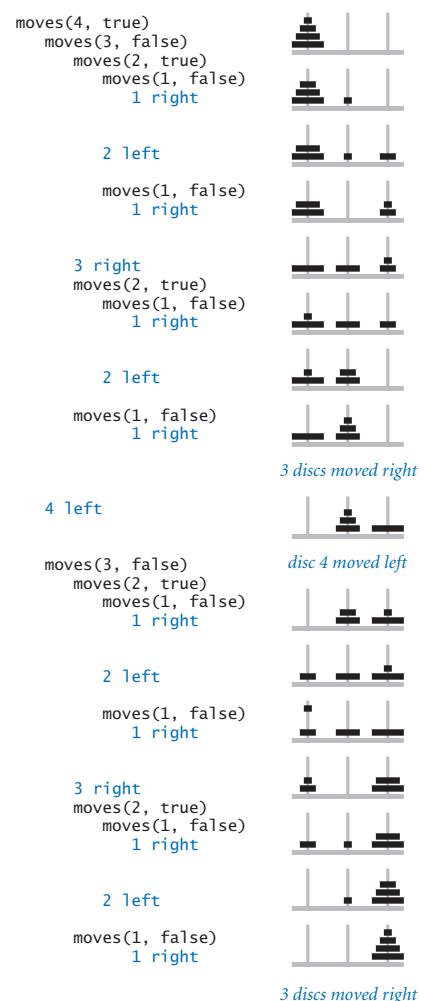
Take a moment to study the function-call tree depicted earlier in this section and to compare it with the corresponding function-call trace depicted at right. When you do so, you will see that the recursion tree is just a compact representation of the trace. In particular, reading the node labels from left to right gives the moves needed to solve the problem.

Moreover, when you study the tree, you probably notice several patterns, including the following two:

- Alternate moves involve the smallest disc.
- That disc always moves in the same direction.

These observations are relevant because they give a solution to the problem that does not require recursion (or even a computer): every other move involves the smallest disc (including the first and last), and each intervening move is the only legal move at the time not involving the smallest disc. We can *prove* that this approach produces the same outcome as the recursive program, using induction. Having started centuries ago without the benefit of a computer, perhaps our monks are using this approach.

Trees are relevant and important in understanding recursion because the tree is a quintessential recursive object. As an abstract mathematical model, trees play an essential role in many applications, and in CHAPTER 4, we will consider the use of trees as a computational model to structure data for efficient processing.



Function-call trace for `moves(4, true)`

Exponential time One advantage of using recursion is that often we can develop mathematical models that allow us to prove important facts about the behavior of recursive programs. For the towers of Hanoi problem, we can estimate the amount of time until the end of the world (assuming that the legend is true). This exercise is important not just because it tells us that the end of the world is quite far off (even if the legend is true), but also because it provides insight that can help us avoid writing programs that will not finish until then.

The mathematical model for the towers of Hanoi problem is simple: if we define the function $T(n)$ to be the number of discs moved by `TowersOfHanoi` to solve an n -disc problem, then the recursive code implies that $T(n)$ must satisfy the following equation:

$$T(n) = 2 T(n-1) + 1 \text{ for } n > 1, \text{ with } T(1) = 1$$

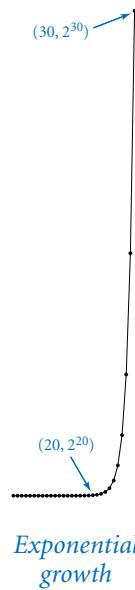
Such an equation is known in discrete mathematics as a *recurrence relation*. Recurrence relations naturally arise in the study of recursive programs. We can often use them to derive a closed-form expression for the quantity of interest. For $T(n)$, you may have already guessed from the initial values $T(1) = 1$, $T(2) = 3$, $T(3) = 7$, and $T(4) = 15$ that $T(n) = 2^n - 1$. The recurrence relation provides a way to *prove* this to be true, by mathematical induction:

- *Base case:* $T(1) = 2^1 - 1 = 1$
- *Induction step:* if $T(n-1) = 2^{n-1} - 1$, $T(n) = 2(2^{n-1} - 1) + 1 = 2^n - 1$

Therefore, by induction, $T(n) = 2^n - 1$ for all $n > 0$. The minimum possible number of moves also satisfies the same recurrence (see EXERCISE 2.3.11).

Knowing the value of $T(n)$, we can estimate the amount of time required to perform all the moves. If the monks move discs at the rate of one per second, it would take more than one week for them to finish a 20-disc problem, more than 34 years to finish a 30-disc problem, and more than 348 centuries for them to finish a 40-disc problem (assuming that they do not make a mistake). The 64-disc problem would take more than 5.8 billion centuries. The end of the world is likely to be even further off than that because those monks presumably never have had the benefit of using PROGRAM 2.3.2, and might not be able to move the discs so rapidly or to figure out so quickly which disc to move next.

Even computers are no match for exponential growth. A computer that can do a billion operations per second will still take centuries to do 2^{64} operations, and no computer will ever do $2^{1,000}$ operations, say. The lesson is profound: with recursion, you can easily write simple short programs



Exponential growth

that take exponential time, but they simply will not run to completion when you try to run them for large n . Novices are often skeptical of this basic fact, so it is worth your while to pause now to think about it. To convince yourself that it is true, take the print statements out of `TowersOfHanoi` and run it for increasing values of n starting at 20. You can easily verify that each time you increase the value of n by 1, the running time doubles, and you will quickly lose patience waiting for it to finish. If you wait for an hour for some value of n , you will wait more than a day for $n + 5$, more than a month for $n + 10$, and more than a century for $n + 20$ (no one has that much patience). Your computer is just not fast enough to run every short Java program that you write, no matter how simple the program might seem! *Beware of programs that might require exponential time.*

We are often interested in predicting the running time of our programs. In SECTION 4.1, we will discuss the use of the same process that we just used to help estimate the running time of other programs.

Gray codes The towers of Hanoi problem is no toy. It is intimately related to basic algorithms for manipulating numbers and discrete objects. As an example, we consider *Gray codes*, a mathematical abstraction with numerous applications.

The playwright Samuel Beckett, perhaps best known for *Waiting for Godot*, wrote a play called *Quad* that had the following property: starting with an empty stage, characters enter and exit one at a time so that each subset of characters on the stage appears exactly once. How did Beckett generate the stage directions for this play?

One way to represent a subset of n discrete objects is to use a string of n bits. For Beckett's problem, we use a 4-bit string, with bits numbered from right to left and a bit value of 1 indicating the character onstage. For example, the string 0 1 0 1 corresponds to the scene with characters 3 and 1 onstage. This representation gives a quick proof of a basic fact: *the number of different subsets of n objects is exactly 2^n .* *Quad* has four characters, so there are $2^4 = 16$ different scenes. Our task is to generate the stage directions.

An n -bit *Gray code* is a list of the 2^n different n -bit binary numbers such that each element in the list differs in precisely one bit from its predecessor. Gray codes directly apply to Beckett's problem because changing the value of a bit from 0 to 1

code	subset	move
0 0 0 0	empty	
0 0 0 1	1	enter 1
0 0 1 1	2 1	enter 2
0 0 1 0	2	exit 1
0 1 1 0	3 2	enter 3
0 1 1 1	3 2 1	enter 1
0 1 0 1	3 1	exit 2
0 1 0 0	3	exit 1
1 1 0 0	4 3	enter 4
1 1 0 1	4 3 1	enter 1
1 1 1 1	4 3 2 1	enter 2
1 1 1 0	4 3 2	exit 1
1 0 1 0	4 2	exit 3
1 0 1 1	4 2 1	enter 1
1 0 0 1	4 1	exit 2
1 0 0 0	4	exit 1

Gray code representations

corresponds to a character entering the subset onstage; changing a bit from 1 to 0 corresponds to a character exiting the subset.

How do we generate a Gray code? A recursive plan that is very similar to the one that we used for the towers of Hanoi problem is effective. The n -bit *binary-reflected Gray code* is defined recursively as follows:

- The $(n-1)$ bit code, with 0 prepended to each word, followed by
- The $(n-1)$ bit code *in reverse order*, with 1 prepended to each word

The 0-bit code is defined to be empty, so the 1-bit code is 0 followed by 1. From this recursive definition, we can verify by induction that the n -bit binary reflected Gray code has the required property: adjacent codewords differ in one bit position. It is true by the inductive hypothesis, except possibly for the last codeword in the first half and the first codeword in the second half: this pair differs only in their first bit.

The recursive definition leads, after some careful thought, to the implementation in Beckett (PROGRAM 2.3.3) for printing Beckett's stage directions. This program is remarkably similar to TowersOfHanoi. Indeed, except for nomenclature, the only difference is in the values of the second arguments in the recursive calls!

As with the directions in TowersOfHanoi, the enter and exit directions are redundant in Beckett, since `exit` is issued only when an actor is onstage, and `enter` is issued only when an actor is not onstage. Indeed, both Beckett and TowersOfHanoi directly involve the ruler function that we considered in one of our first programs (PROGRAM 1.2.1). Without the printing instructions, they both implement a simple recursive function that could allow Ruler to print the values of the ruler function for any value given as a command-line argument.

Gray codes have many applications, ranging from analog-to-digital converters to experimental design. They have been used in pulse code communication, the minimization of logic circuits, and hypercube architectures, and were even proposed to organize books on library shelves.

		1-bit code	3-bit code
	2-bit	0 0 0 1 1 1 1 0	0 0 0 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 1 1 0 1 0 1 0 1 0 0
2-bit		1-bit code (reversed)	4-bit
3-bit	2-bit	2-bit code ↓	3-bit code ↓
	3-bit	0 0 0 0 0 1 0 1 1 0 1 0 1 1 0 1 1 1 1 0 1 1 0 0	1 1 0 0 1 1 0 1 1 1 1 1 1 1 1 0 1 0 1 0 1 0 1 1 1 0 0 1 1 0 0 0
		2-bit code (reversed)	3-bit code (reversed)

2-, 3-, and 4-bit Gray codes

Program 2.3.3 Gray code

```

public class Beckett
{
    public static void moves(int n, boolean enter)
    {
        if (n == 0) return;
        moves(n-1, true);
        if (enter) StdOut.println("enter " + n);
        else       StdOut.println("exit   " + n);
        moves(n-1, false);
    }

    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        moves(n, true);
    }
}

```

n number of actors
enter stage direction

This recursive program gives Beckett's stage instructions (the bit positions that change in a binary-reflected Gray code). The bit position that changes is precisely described by the ruler function, and (of course) each actor alternately enters and exits.

```
% java Beckett 1
enter 1
% java Beckett 2
enter 1
enter 2
exit  1
% java Beckett 3
enter 1
enter 2
exit  1
enter 3
enter 1
exit  2
exit  1
exit  2
exit  1
```

```
% java Beckett 4
enter 1
enter 2
exit  1
enter 3
enter 1
exit  2
exit  1
enter 4
enter 1
enter 2
exit  1
exit  3
enter 1
exit  2
exit  1
```

Recursive graphics Simple recursive drawing schemes can lead to pictures that are remarkably intricate. Recursive drawings not only relate to numerous applications, but also provide an appealing platform for developing a better understanding of properties of recursive functions, because we can watch the process of a recursive figure taking shape.

As a first simple example, consider `Htree` (PROGRAM 2.3.4), which, given a command-line argument n , draws an *H-tree* of *order* n , defined as follows: The base case is to draw nothing for $n = 0$. The reduction step is to draw, within the unit square

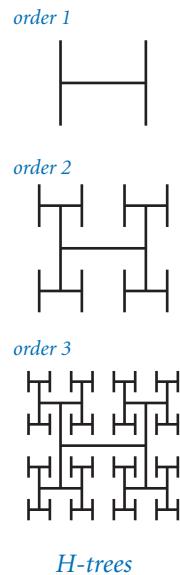
- three lines in the shape of the letter H
- four H-trees of order $n - 1$, one centered at each tip of the H

with the additional proviso that the H-trees of order $n - 1$ are halved in size.

Drawings like these have many practical applications. For example, consider a cable company that needs to run cable to all of the homes distributed throughout its region. A reasonable strategy is to use an H-tree to get the signal to a suitable number of centers distributed throughout the region, then run cables connecting each home to the nearest center. The same problem is faced by computer designers who want to distribute power or signal throughout an integrated circuit chip.

Though every drawing is in a fixed-size window, H-trees certainly exhibit exponential growth. An H-tree of order n connects 4^n centers, so you would be trying to plot more than a million lines with $n = 10$, and more than a billion with $n = 15$. The program will certainly not finish the drawing with $n = 30$.

If you take a moment to run `Htree` on your computer for a drawing that takes a minute or so to complete, you will, just by watching the drawing progress, have the opportunity to gain substantial insight into the nature of recursive programs, because you can see the *order* in which the H figures appear and how they form into H-trees. An even more instructive exercise, which derives from the fact that the same drawing results no matter in which order the recursive `draw()` calls and the `StdDraw.line()` calls appear, is to observe the effect of rearranging the order of these calls on the order in which the lines appear in the emerging drawing (see EXERCISE 2.3.14).

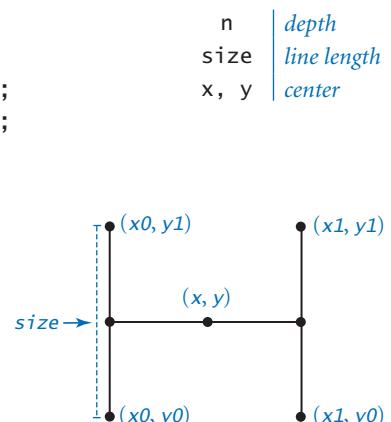


Program 2.3.4 Recursive graphics

```

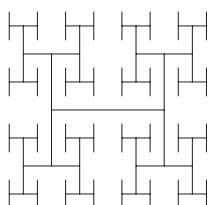
public class Htree
{
    public static void draw(int n, double size, double x, double y)
    {   // Draw an H-tree centered at x, y
        // of depth n and given size.
        if (n == 0) return;
        double x0 = x - size/2, x1 = x + size/2;
        double y0 = y - size/2, y1 = y + size/2;
        StdDraw.line(x0, y, x1, y);
        StdDraw.line(x0, y0, x0, y1);
        StdDraw.line(x1, y0, x1, y1);
        draw(n-1, size/2, x0, y0);
        draw(n-1, size/2, x0, y1);
        draw(n-1, size/2, x1, y0);
        draw(n-1, size/2, x1, y1);
    }
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        draw(n, 0.5, 0.5, 0.5);
    }
}

```

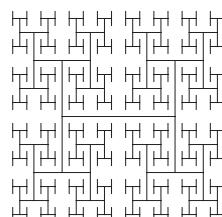


The function `draw()` draws three lines, each of length `size`, in the shape of the letter H, centered at (x, y) . Then, it calls itself recursively for each of the four tips, halving the `size` argument in each call and using an integer argument `n` to control the depth of the recursion.

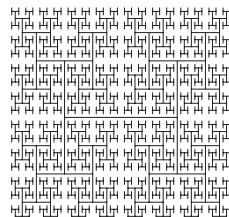
% java Htree 3



% java Htree 4



% java Htree 5



Brownian bridge An H-tree is a simple example of a *fractal*: a geometric shape that can be divided into parts, each of which is (approximately) a reduced-size copy of the original. Fractals are easy to produce with recursive programs, although scientists, mathematicians, and programmers study them from many different points of view. We have already encountered fractals several times in this book—for example, IFS (PROGRAM 2.2.3).

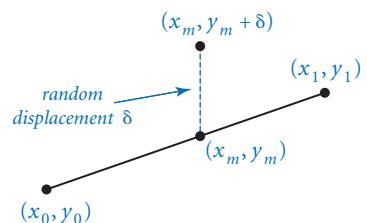
The study of fractals plays an important and lasting role in artistic expression, economic analysis, and scientific discovery. Artists and scientists use fractals to build compact models of complex shapes that arise in nature and resist description using conventional geometry, such as clouds, plants, mountains, riverbeds, human skin, and many others. Economists use fractals to model function graphs of economic indicators.

Fractional Brownian motion is a mathematical model for creating realistic fractal models for many naturally rugged shapes. It is used in computational finance and in the study of many natural phenomena, including ocean flows and nerve membranes. Computing the exact fractals specified by the model can be a difficult challenge, but it is not difficult to compute approximations with recursive programs.

Brownian (PROGRAM 2.3.5) produces a function graph that approximates a simple example of fractional Brownian motion known as a *Brownian bridge* and closely related functions. You can think of this graph as a random walk that connects the two points (x_0, y_0) and (x_1, y_1) , controlled by a few parameters. The implementation is based on the *midpoint displacement method*, which is a recursive plan for drawing the plot within the x -interval $[x_0, x_1]$. The base case (when the length of the interval is smaller than a given tolerance) is to draw a straight line connecting the two endpoints. The reduction case is to divide the interval into two halves, proceeding as follows:

- Compute the midpoint (x_m, y_m) of the interval.
- Add to the y -coordinate y_m of the midpoint a random value δ , drawn from the Gaussian distribution with mean 0 and a given variance.
- Recur on the subintervals, dividing the variance by a given scaling factor s .

The shape of the curve is controlled by two parameters: the *volatility* (initial value of the variance) controls the distance the function graph strays from the straight



Brownian bridge calculation

Program 2.3.5 Brownian bridge

```

public class Brownian
{
    public static void curve(double x0, double y0,
                            double x1, double y1,
                            double var, double s)
    {
        if (x1 - x0 < 0.01)
        {
            StdDraw.line(x0, y0, x1, y1);
            return;
        }
        double xm = (x0 + x1) / 2;
        double ym = (y0 + y1) / 2;
        double delta = StdRandom.gaussian(0, Math.sqrt(var));
        curve(x0, y0, xm, ym + delta, var/s, s);
        curve(xm, ym+delta, x1, y1, var/s, s);
    }
    public static void main(String[] args)
    {
        double hurst = Double.parseDouble(args[0]);
        double s = Math.pow(2, 2*hurst);
        curve(0, 0.5, 1.0, 0.5, 0.01, s);
    }
}

```

x0, y0	<i>left endpoint</i>
x1, y1	<i>right endpoint</i>
xm, ym	<i>middle</i>
delta	<i>displacement</i>
var	<i>variance</i>
hurst	<i>Hurst exponent</i>

By adding a small, random Gaussian to a recursive program that would otherwise plot a straight line, we get fractal curves. The command-line argument *hurst*, known as the Hurst exponent, controls the smoothness of the curves.

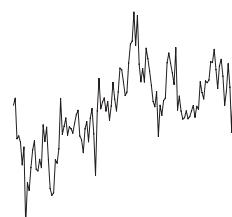
% java Brownian 1



% java Brownian 0.5



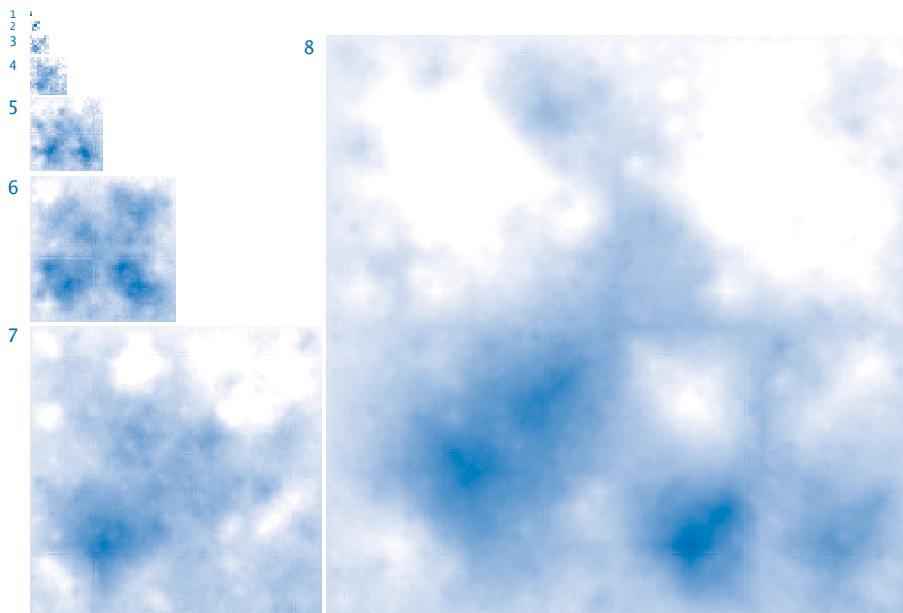
% java Brownian 0.05



line connecting the points, and *the Hurst exponent* controls the smoothness of the curve. We denote the Hurst exponent by H and divide the variance by 2^{2H} at each recursive level. When H is $1/2$ (halved at each level), the curve is a Brownian bridge—a continuous version of the gambler’s ruin problem (see PROGRAM 1.3.8). When $0 < H < 1/2$, the displacements tend to increase, resulting in a rougher curve. Finally, when $2 > H > 1/2$, the displacements tend to decrease, resulting in a smoother curve. The value $2 - H$ is known as the *fractal dimension* of the curve.

The volatility and initial endpoints of the interval have to do with scale and positioning. The `main()` test client in `Brownian` allows you to experiment with the Hurst exponent. With values larger than $1/2$, you get plots that look something like the horizon in a mountainous landscape; with values smaller than $1/2$, you get plots similar to those you might see for the value of a stock index.

Extending the midpoint displacement method to two dimensions yields fractals known as *plasma clouds*. To draw a rectangular plasma cloud, we use a recursive plan where the base case is to draw a rectangle of a given color and the reduction step is to draw a plasma cloud in each of the four quadrants with colors that are perturbed from the average with a random Gaussian. Using the same volatility and smoothness controls as in `Brownian`, we can produce synthetic clouds that are remarkably realistic. We can use the same code to produce synthetic terrain, by interpreting the color value as the altitude. Variants of this scheme are widely used in the entertainment industry to generate background scenery for movies and games.



Plasma clouds

Pitfalls of recursion By now, you are perhaps persuaded that recursion can help you to write compact and elegant programs. As you begin to craft your own recursive programs, you need to be aware of several common pitfalls that can arise. We have already discussed one of them in some detail (the running time of your program might grow exponentially). Once identified, these problems are generally not difficult to overcome, but you will learn to be very careful to avoid them when writing recursive programs.

Missing base case. Consider the following recursive function, which is supposed to compute harmonic numbers, but is missing a base case:

```
public static double harmonic(int n)
{
    return harmonic(n-1) + 1.0/n;
}
```

If you run a client that calls this function, it will repeatedly call itself and never return, so your program will never terminate. You probably already have encountered infinite loops, where you invoke your program and nothing happens (or perhaps you get an unending sequence of printed output). With infinite recursion, however, the result is different because the system keeps track of each recursive call (using a mechanism that we will discuss in SECTION 4.3, based on a data structure known as a *stack*) and eventually runs out of memory trying to do so. Eventually, Java reports a `StackOverflowError` at run time. When you write a recursive program, you should always try to convince yourself that it has the desired effect by an informal argument based on mathematical induction. Doing so might uncover a missing base case.

No guarantee of convergence. Another common problem is to include within a recursive function a recursive call to solve a subproblem that is not smaller than the original problem. For example, the following method goes into an infinite recursive loop for any value of its argument (except 1) because the sequence of argument values does not converge to the base case:

```
public static double harmonic(int n)
{
    if (n == 1) return 1.0;
    return harmonic(n) + 1.0/n;
}
```

Bugs like this one are easy to spot, but subtle versions of the same problem can be harder to identify. You may find several examples in the exercises at the end of this section.

Excessive memory requirements. If a function calls itself recursively an excessive number of times before returning, the memory required by Java to keep track of the recursive calls may be prohibitive, resulting in a `StackOverflowError`. To get an idea of how much memory is involved, run a small set of experiments using our recursive function for computing the harmonic numbers for increasing values of n :

```
public static double harmonic(int n)
{
    if (n == 1) return 1.0;
    return harmonic(n-1) + 1.0/n;
}
```

The point at which you get `StackOverflowError` will give you some idea of how much memory Java uses to implement recursion. By contrast, you can run PROGRAM 1.3.5 to compute H_n for huge n using only a tiny bit of memory.

Excessive recomputation. The temptation to write a simple recursive function to solve a problem must always be tempered by the understanding that a function might take exponential time (unnecessarily) due to excessive recomputation. This effect is possible even in the simplest recursive functions, and you certainly need to learn to avoid it. For example, the *Fibonacci sequence*

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

is defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has many interesting properties and arise in numerous applications. A novice programmer might implement this recursive function to compute numbers in the Fibonacci sequence:

```
// Warning: this function is spectacularly inefficient.
public static long fibonacci(int n)
{
    if (n == 0) return 0;
    if (n == 1) return 1;
    return fibonacci(n-1) + fibonacci(n-2);
}
```

```
fibonacci(8)
  fibonacci(7)
    fibonacci(6)
      fibonacci(5)
        fibonacci(4)
          fibonacci(3)
            fibonacci(2)
              fibonacci(1)
                return 1
              fibonacci(0)
                return 0
                return 1
              fibonacci(1)
                return 1
                return 2
            fibonacci(2)
              fibonacci(1)
                return 1
              fibonacci(0)
                return 0
                return 1
              return 3
            fibonacci(3)
              fibonacci(2)
                fibonacci(1)
                  return 1
                fibonacci(0)
                  return 0
                  return 1
              fibonacci(1)
                return 1
                return 2
              return 5
            fibonacci(4)
              fibonacci(3)
                fibonacci(2)
                :
                :
```

However, this function is spectacularly inefficient! Novice programmers often refuse to believe this fact, and run code like this expecting that the computer is certainly fast enough to crank out an answer. Go ahead; see if your computer is fast enough to use this function to compute `fibonacci(50)`. To see why it is futile to do so, consider what the function does to compute $\text{fibonacci}(8) = 21$. It first computes $\text{fibonacci}(7) = 13$ and $\text{fibonacci}(6) = 8$. To compute $\text{fibonacci}(7)$, it recursively computes $\text{fibonacci}(6) = 8$ again and $\text{fibonacci}(5) = 5$. Things rapidly get worse because both times it computes $\text{fibonacci}(6)$, it ignores the fact that it already computed $\text{fibonacci}(5)$, and so forth. In fact, the number of times this program computes $\text{fibonacci}(1)$ when computing $\text{fibonacci}(n)$ is precisely F_n (see EXERCISE 2.3.12). The mistake of recomputation is compounded exponentially. As an example, $\text{fibonacci}(200)$ makes $F_{200} > 10^{43}$ recursive calls to $\text{fibonacci}(1)$! No imaginable computer will ever be able to do this many calculations. *Beware of programs that might require exponential time.* Many calculations that arise and find natural expression as recursive functions fall into this category. Do not fall into the trap of implementing and trying to run them.

NEXT, WE CONSIDER A SYSTEMATIC TECHNIQUE known as *dynamic programming*, an elegant technique for avoiding such problems. The idea is to avoid the excessive recomputation inherent in some recursive functions by saving away the previously computed values for later reuse, instead of constantly recomputing them.

Wrong way to compute Fibonacci numbers

Dynamic programming A general approach to implementing recursive programs, known as *dynamic programming*, provides effective and elegant solutions to a wide class of problems. The basic idea is to recursively divide a complex problem into a number of simpler subproblems; store the answer to each of these subproblems; and, ultimately, use the stored answers to solve the original problem. By solving each subproblem only once (instead of over and over), this technique avoids a potential exponential blow-up in the running time.

For example, if our original problem is to compute the n th Fibonacci number, then it is natural to define $n + 1$ subproblems, where subproblem i is to compute the i th Fibonacci number for each $0 \leq i \leq n$. We can solve subproblem i easily if we already know the solutions to smaller subproblems—specifically, subproblems $i - 1$ and $i - 2$. Moreover, the solution to our original problem is simply the solution to one of the subproblems—subproblem n .

Top-down dynamic programming. In *top-down* dynamic programming, we store or *cache* the result of each subproblem that we solve, so that the next time we need to solve the same subproblem, we can use the cached values instead of solving the subproblem from scratch. For our Fibonacci example, we use an array $f[]$ to store the Fibonacci numbers that have already been computed. We accomplish this in Java by using a `static` variable, also known as a *class variable* or *global variable*, that is declared outside of any method. This allows us to save information from one function call to the next.

```
public class TopDownFibonacci
{
    static variable
    (declared outside
     of any method)
    private static long[] f = new long[92];
    public static long fibonacci(int n)
    {
        if (n == 0) return 0;
        if (n == 1) return 1;
        if (f[n] > 0) return f[n];
        f[n] = fibonacci(n-1) + fibonacci(n-2);
        return f[n];
    }
}
```

Top-down dynamic programming approach for computing Fibonacci numbers

Top-down dynamic programming is also known as *memoization* because it avoids duplicating work by *remembering* the results of function calls.

Bottom-up dynamic programming. In *bottom-up* dynamic programming, we compute solutions to *all* of the subproblems, starting with the “simplest” subproblems and gradually building up solutions to more and more complicated subproblems. To apply bottom-up dynamic programming, we must *order* the subproblems so that each subsequent subproblem can be solved by combining solutions to subproblems earlier in the order (which have already been solved). For our Fibonacci example, this is easy: solve the subproblems in the order 0, 1, and 2, and so forth. By the time we need to solve subproblem i , we have already solved all smaller subproblems—in particular, subproblems $i-1$ and $i-2$.

```
public static long fibonacci(int n)
{
    long[] f = new int[n+1];
    f[0] = 0;
    f[1] = 1;
    for (int i = 2; i <= n; i++)
        f[i] = f[i-1] + f[i-2];
    return f[n];
}
```

When the ordering of the subproblems is clear, and space is available to store all the solutions, bottom-up dynamic programming is a very effective approach.

NEXT, WE CONSIDER A MORE SOPHISTICATED application of dynamic programming, where the order of solving the subproblems is not so clear (until you see it). Unlike the problem of computing Fibonacci numbers, this problem would be much more difficult to solve without thinking recursively and also applying a bottom-up dynamic programming approach.

Longest common subsequence problem. We consider a fundamental string-processing problem that arises in computational biology and other domains. Given two strings x and y , we wish to determine how *similar* they are. Some examples include comparing two DNA sequences for homology, two English words for spelling, or two Java files for repeated code. One measure of similarity is the length of the *longest common subsequence* (LCS). If we delete some characters from x and some characters from y , and the resulting two strings are equal, we call the resulting string a *common subsequence*. The LCS problem is to find a common subsequence of two strings that is as long as possible. For example, the LCS of GGCACCCACG and ACGGCGGATACG is GGCAACG, a string of length 7.

Algorithms to compute the LCS are used in data comparison programs like the `diff` command in Unix, which has been used for decades by programmers wanting to understand differences and similarities in their text files. Similar algorithms play important roles in scientific applications, such as the Smith–Waterman algorithm in computational biology and the Viterbi algorithm in digital communications theory.

Longest common subsequence recurrence. Now we describe a recursive formulation that enables us to find the LCS of two given strings s and t . Let m and n be the lengths of s and t , respectively. We use the notation $s[i..m]$ to denote the suffix of s starting at index i , and $t[j..n]$ to denote the suffix of t starting at index j . On the one hand, if s and t begin with the same character, then the LCS of x and y contains that first character. Thus, our problem reduces to finding the LCS of the suffixes $s[1..m]$ and $t[1..n]$. On the other hand, if s and t begin with different characters, both characters cannot be part of a common subsequence, so we can safely discard one or the other. In either case, the problem reduces to finding the LCS of two strings—either $s[0..m]$ and $t[1..n]$ or $s[1..m]$ and $t[0..n]$ —one of which is strictly shorter. In general, if we let $\text{opt}[i][j]$ denote the length of the LCS of the suffixes $s[i..m]$ and $t[j..n]$, then the following recurrence expresses $\text{opt}[i][j]$ in terms of the length of the LCS for shorter suffixes.

$$\text{opt}[i][j] = \begin{cases} 0 & \text{if } i = m \text{ or } j = n \\ \text{opt}[i+1, j+1] + 1 & \text{if } s[i] = t[j] \\ \max(\text{opt}[i, j+1], \text{opt}[i+1, j]) & \text{otherwise} \end{cases}$$

Dynamic programming solution. `LongestCommonSubsequence` (PROGRAM 2.3.6) begins with a bottom-up dynamic programming approach to solving this recurrence. We maintain a two-dimensional array $\text{opt}[i][j]$ that stores the length of the LCS of the suffixes $s[i..m]$ and $t[j..n]$. Initially, the bottom row (the values for $i = m$) and the right column (the values for $j = n$) are 0. These are the initial values. From the recurrence, the order of the rest of the computation is clear: we start with $\text{opt}[m][n]$. Then, as long as we decrease either i or j or both, we know that we will have computed what we need to compute $\text{opt}[i][j]$, since the two options involve an $\text{opt}[\cdot][\cdot]$ entry with a larger value of i or j or both. The method `lcs()` in PROGRAM 2.3.6 computes the elements in $\text{opt}[\cdot][\cdot]$ by filling in values in rows from bottom to top ($i = m-1$ to 0) and from right to left in each row ($j = n-1$ to 0). The alternative choice of filling in values in columns from right to left and

Program 2.3.6 Longest common subsequence

```

public class LongestCommonSubsequence
{
    public static String lcs(String s, String t)
    { // Compute length of LCS for all subproblems.
        int m = s.length(), n = t.length();
        int[][] opt = new int[m+1][n+1];
        for (int i = m-1; i >= 0; i--)
            for (int j = n-1; j >= 0; j--)
                if (s.charAt(i) == t.charAt(j))
                    opt[i][j] = opt[i+1][j+1] + 1;
                else
                    opt[i][j] = Math.max(opt[i+1][j], opt[i][j+1]);
        // Recover LCS itself.
        String lcs = "";
        int i = 0, j = 0;                                s, t      | two strings
        while(i < m && j < n)                         m, n      | lengths of two strings
        {                                                 opt[i][j] | length of LCS of
            if (s.charAt(i) == t.charAt(j))           lcs       | x[i..m] and y[j..n]
            {
                lcs += s.charAt(i);                  lcs       | longest common subsequence
                i++;
                j++;
            }
            else if (opt[i+1][j] >= opt[i][j+1])   i++;     |
            else                                     j++;     |
        }
        return lcs;
    }

    public static void main(String[] args)
    { StdOut.println(lcs(args[0], args[1]));  }
}

```

The function `lcs()` computes and returns the LCS of two strings `s` and `t` using bottom-up dynamic programming. The method call `s.charAt(i)` returns character `i` of string `s`.

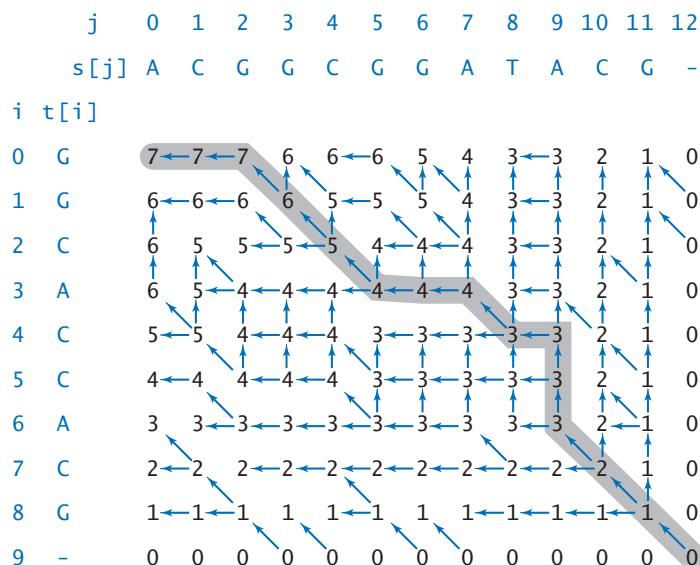
```
% java LongestCommonSubsequence GGCACCACG ACGGGGGATACG
GGCAACG
```

from bottom to top in each row would work as well. The diagram at the bottom of this page has a blue arrow pointing to each entry that indicates which value was used to compute it. (When there is a tie in computing the maximum, both options are shown.)

The final challenge is to recover the longest common subsequence itself, not just its length. The key idea is to retrace the steps of the dynamic programming algorithm *backward*, rediscovering the path of choices (highlighted in gray in the diagram) from $\text{opt}[0][0]$ to $\text{opt}[m][n]$. To determine the choice that led to $\text{opt}[i][j]$, we consider the three possibilities:

- The character $s[i]$ equals $t[j]$. In this case, we must have $\text{opt}[i][j] = \text{opt}[i+1][j+1] + 1$, and the next character in the LCS is $s[i]$ (or $t[j]$), so we include the character $s[i]$ (or $t[j]$) in the LCS and continue tracing back from $\text{opt}[i+1][j+1]$.
- The LCS does not contain $s[i]$. In this case, $\text{opt}[i][j] = \text{opt}[i+1][j]$ and we continue tracing back from $\text{opt}[i+1][j]$.
- The LCS does not contain $t[j]$. In this case, $\text{opt}[i][j] = \text{opt}[i][j+1]$ and we continue tracing back from $\text{opt}[i][j+1]$.

We begin tracing back at $\text{opt}[0][0]$ and continue until we reach $\text{opt}[m][n]$. At each step in the traceback either i increases or j increases (or both), so the process terminates after at most $m + n$ iterations of the while loop.



Longest common subsequence of GGCACCACCG and ACCGGGGATACG

DYNAMIC PROGRAMMING IS A FUNDAMENTAL ALGORITHM design paradigm, intimately linked to recursion. If you take later courses in algorithms or operations research, you are sure to learn more about it. The idea of recursion is fundamental in computation, and the idea of avoiding recomputation of values that have been computed before is certainly a natural one. Not all problems immediately lend themselves to a recursive formulation, and not all recursive formulations admit an order of computation that easily avoids recomputation—arranging for both can seem a bit miraculous when one first encounters it, as you have just seen for the LCS problem.

Perspective Programmers who do not use recursion are missing two opportunities. First recursion leads to compact solutions to complex problems. Second, recursive solutions embody an argument that the program operates as anticipated. In the early days of computing, the overhead associated with recursive programs was prohibitive in some systems, and many people avoided recursion. In modern systems like Java, recursion is often the method of choice.

Recursive functions truly illustrate the power of a carefully articulated abstraction. While the concept of a function having the ability to call itself seems absurd to many people at first, the many examples that we have considered are certainly evidence that mastering recursion is essential to understanding and exploiting computation and in understanding the role of computational models in studying natural phenomena.

Recursion has reinforced for us the idea of proving that a program operates as intended. The natural connection between recursion and mathematical induction is essential. For everyday programming, our interest in correctness is to save time and energy tracking down bugs. In modern applications, security and privacy concerns make correctness an *essential* part of programming. If the programmer cannot be convinced that an application works as intended, how can a user who wants to keep personal data private and secure be so convinced?

Recursion is the last piece in a programming model that served to build much of the computational infrastructure that was developed as computers emerged to take a central role in daily life in the latter part of the 20th century. Programs built from libraries of functions consisting of statements that operate on primitive types of data, conditionals, loops, and function calls (including recursive ones) can solve important problems of all sorts. In the next section, we emphasize this point and review these concepts in the context of a large application. In CHAPTER 3 and in CHAPTER 4, we will examine extensions to these basic ideas that embrace the more expansive style of programming that now dominates the computing landscape.

Q&A

Q. Are there situations when iteration is the only option available to address a problem?

A. No, any loop can be replaced by a recursive function, though the recursive version might require excessive memory.

Q. Are there situations when recursion is the only option available to address a problem?

A. No, any recursive function can be replaced by an iterative counterpart. In SECTION 4.3, we will see how compilers produce code for function calls by using a data structure called a *stack*.

Q. Which should I prefer, recursion or iteration?

A. Whichever leads to the simpler, more easily understood, or more efficient code.

Q. I get the concern about excessive space and excessive recomputation in recursive code. Anything else to be concerned about?

A. Be extremely wary of creating arrays in recursive code. The amount of space used can pile up very quickly, as can the amount of time required for memory management.

Exercises

2.3.1 What happens if you call `factorial()` with a negative value of n ? With a large value of, say, 35?

2.3.2 Write a recursive function that takes an integer n as its argument and returns $\ln(n!)$.

2.3.3 Give the sequence of integers printed by a call to `ex233(6)`:

```
public static void ex233(int n)
{
    if (n <= 0) return;
    StdOut.println(n);
    ex233(n-2);
    ex233(n-3);
    StdOut.println(n);
}
```

2.3.4 Give the value of `ex234(6)`:

```
public static String ex234(int n)
{
    if (n <= 0) return "";
    return ex234(n-3) + n + ex234(n-2) + n;
}
```

2.3.5 Criticize the following recursive function:

```
public static String ex235(int n)
{
    String s = ex235(n-3) + n + ex235(n-2) + n;
    if (n <= 0) return "";
    return s;
}
```

Answer: The base case will never be reached because the base case appears after the reduction step. A call to `ex235(3)` will result in calls to `ex235(0)`, `ex235(-3)`, `ex235(-6)`, and so forth until a `StackOverflowError`.

2.3.6 Given four positive integers a , b , c , and d , explain what value is computed by $\text{gcd}(\text{gcd}(a, b), \text{gcd}(c, d))$.

2.3.7 Explain in terms of integers and divisors the effect of the following Euclid-like function:

```
public static boolean gcdlike(int p, int q)
{
    if (q == 0) return (p == 1);
    return gcdlike(q, p % q);
}
```

2.3.8 Consider the following recursive function:

```
public static int mystery(int a, int b)
{
    if (b == 0) return 0;
    if (b % 2 == 0) return mystery(a+a, b/2);
    return mystery(a+a, b/2) + a;
}
```

What are the values of `mystery(2, 25)` and `mystery(3, 11)`? Given positive integers a and b , describe what value `mystery(a, b)` computes. Then answer the same question, but replace `+` with `*` and `return 0` with `return 1`.

2.3.9 Write a recursive program `Ruler` to plot the subdivisions of a ruler using `StdDraw`, as in PROGRAM 1.2.1.

2.3.10 Solve the following recurrence relations, all with $T(1) = 1$. Assume n is a power of 2.

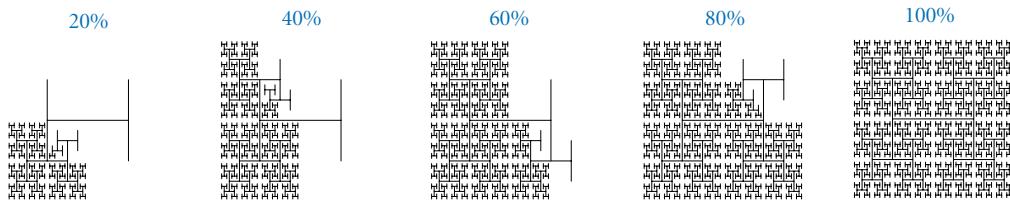
- $T(n) = T(n/2) + 1$
- $T(n) = 2T(n/2) + 1$
- $T(n) = 2T(n/2) + n$
- $T(n) = 4T(n/2) + 3$

2.3.11 Prove by induction that the minimum possible number of moves needed to solve the towers of Hanoi satisfies the same recurrence as the number of moves used by our recursive solution.

2.3.12 Prove by induction that the recursive program given in the text makes exactly F_n recursive calls to `fibonacci(1)` when computing `fibonacci(n)`.

2.3.13 Prove that the second argument to `gcd()` decreases by at least a factor of 2 for every second recursive call, and then prove that `gcd(p, q)` uses at most $2 \log_2 n + 1$ recursive calls where n is the larger of p and q .

2.3.14 Modify `Htree` (PROGRAM 2.3.4) to animate the drawing of the H-tree. Next, rearrange the order of the recursive calls (and the base case), view the resulting animation, and explain each outcome.



Creative Exercises

2.3.15 *Binary representation.* Write a program that takes a positive integer n (in decimal) as a command-line argument and prints its binary representation. Recall, in PROGRAM 1.3.7, that we used the method of subtracting out powers of 2. Now, use the following simpler method: repeatedly divide 2 into n and read the remainders backward. First, write a `while` loop to carry out this computation and print the bits in the wrong order. Then, use recursion to print the bits in the correct order.

2.3.16 *A4 paper.* The width-to-height ratio of paper in the ISO format is the square root of 2 to 1. Format A0 has an area of 1 square meter. Format A1 is A0 cut with a vertical line into two equal halves, A2 is A1 cut with a horizontal line into two halves, and so on. Write a program that takes an integer command-line argument n and uses `StdDraw` to show how to cut a sheet of A0 paper into 2^n pieces.

2.3.17 *Permutations.* Write a program `Permutations` that takes an integer command-line argument n and prints all $n!$ permutations of the n letters starting at a (assume that n is no greater than 26). A permutation of n elements is one of the $n!$ possible orderings of the elements. As an example, when $n = 3$, you should get the following output (but do not worry about the order in which you enumerate them):

```
bca cba cab acb bac abc
```

2.3.18 *Permutations of size k .* Modify `Permutations` from the previous exercise so that it takes two command-line arguments n and k , and prints all $P(n, k) = n! / (n-k)!$ permutations that contain exactly k of the n elements. Below is the desired output when $k = 2$ and $n = 4$ (again, do not worry about the order):

```
ab ac ad ba bc bd ca cb cd da db dc
```

2.3.19 *Combinations.* Write a program `Combinations` that takes an integer command-line argument n and prints all 2^n combinations of any size. A combination is a subset of the n elements, independent of order. As an example, when $n = 3$, you should get the following output:

```
a ab abc ac b bc c
```

Note that your program needs to print the empty string (subset of size 0).

2.3.20 *Combinations of size k.* Modify Combinations from the previous exercise so that it takes two integer command-line arguments n and k , and prints all $C(n, k) = n! / (k!(n-k)!)$ combinations of size k . For example, when $n = 5$ and $k = 3$, you should get the following output:

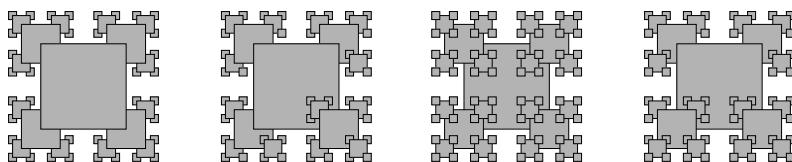
```
abc abd abe acd ace ade bcd bce bde cde
```

2.3.21 *Hamming distance.* The Hamming distance between two bit strings of length n is equal to the number of bits in which the two strings differ. Write a program that reads in an integer k and a bit string s from the command line, and prints all bit strings that have Hamming distance at most k from s . For example, if k is 2 and s is 0000, then your program should print

```
0011 0101 0110 1001 1010 1100
```

Hint: Choose k of the bits in s to flip.

2.3.22 *Recursive squares.* Write a program to produce each of the following recursive patterns. The ratio of the sizes of the squares is 2.2:1. To draw a shaded square, draw a filled gray square, then an unfilled black square.



2.3.23 *Pancake flipping.* You have a stack of n pancakes of varying sizes on a griddle. Your goal is to rearrange the stack in order so that the largest pancake is on the bottom and the smallest one is on top. You are only permitted to flip the top k pancakes, thereby reversing their order. Devise a recursive scheme to arrange the pancakes in the proper order that uses at most $2n - 3$ flips.

2.3.24 *Gray code.* Modify Beckett (PROGRAM 2.3.3) to print the Gray code (not just the sequence of bit positions that change).

2.3.25 *Towers of Hanoi variant.* Consider the following variant of the towers of Hanoi problem. There are $2n$ discs of increasing size stored on three poles. Initially all of the discs with odd size ($1, 3, \dots, 2n-1$) are piled on the left pole from top to bottom in increasing order of size; all of the discs with even size ($2, 4, \dots, 2n$) are piled on the right pole. Write a program to provide instructions for moving the odd discs to the right pole and the even discs to the left pole, obeying the same rules as for towers of Hanoi.

2.3.26 *Animated towers of Hanoi.* Use StdDraw to animate a solution to the towers of Hanoi problem, moving the discs at a rate of approximately 1 per second.

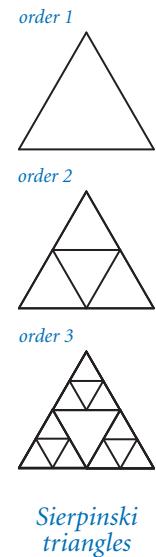
2.3.27 *Sierpinski triangles.* Write a recursive program to draw Sierpinski triangles (see PROGRAM 2.2.3). As with Htree, use a command-line argument to control the depth of the recursion.

2.3.28 *Binomial distribution.* Estimate the number of recursive calls that would be used by the code

```
public static double binomial(int n, int k)
{
    if ((n == 0) && (k == 0)) return 1.0;
    if ((n < 0) || (k < 0)) return 0.0;
    return (binomial(n-1, k) + binomial(n-1, k-1))/2.0;
}
```

to compute `binomial(100, 50)`. Develop a better implementation that is based on dynamic programming. *Hint:* See EXERCISE 1.4.41.

2.3.29 *Collatz function.* Consider the following recursive function, which is related to a famous unsolved problem in number theory, known as the *Collatz problem*, or the *3n+1 problem*:



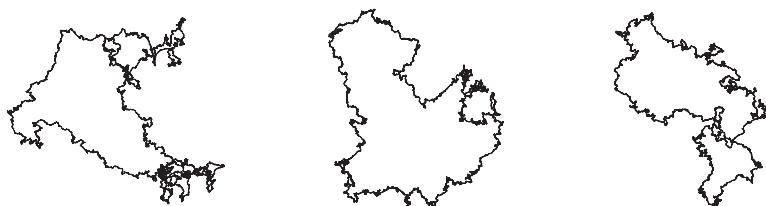
```
public static void collatz(int n)
{
    StdOut.print(n + " ");
    if (n == 1) return;
    if (n % 2 == 0) collatz(n / 2);
    else            collatz(3*n + 1);
}
```

For example, a call to `collatz(7)` prints the sequence

```
7 22 11 34 17 52 26 13 40 20 10 5 16 8 4 2 1
```

as a consequence of 17 recursive calls. Write a program that takes a command-line argument n and returns the value of $i < n$ for which the number of recursive calls for `collatz(i)` is maximized. The unsolved problem is that no one knows whether the function terminates for all integers (mathematical induction is no help, because one of the recursive calls is for a larger value of the argument).

2.3.30 *Brownian island.* B. Mandelbrot asked the famous question *How long is the coast of Britain?* Modify `Brownian` to get a program `BrownianIsland` that plots Brownian islands, whose coastlines resemble that of Great Britain. The modifications are simple: first, change `curve()` to add a random Gaussian to the x -coordinate as well as to the y -coordinate; second, change `main()` to draw a curve from the point at the center of the canvas back to itself. Experiment with various values of the parameters to get your program to produce islands with a realistic look.



Brownian islands with Hurst exponent of 0.76

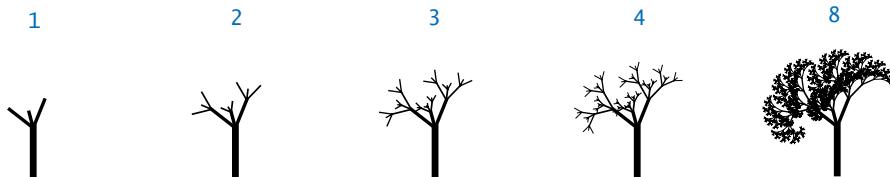
2.3.31 *Plasma clouds.* Write a recursive program to draw plasma clouds, using the method suggested in the text.

2.3.32 *A strange function.* Consider McCarthy's 91 function:

```
public static int McCarthy(int n)
{
    if (n > 100) return n - 10;
    return McCarthy(McCarthy(n+11));
}
```

Determine the value of `McCarthy(50)` without using a computer. Give the number of recursive calls used by `McCarthy()` to compute this result. Prove that the base case is reached for all positive integers n or find a value of n for which this function goes into an infinite recursive loop.

2.3.33 *Recursive tree.* Write a program `Tree` that takes a command-line argument n and produces the following recursive patterns for n equal to 1, 2, 3, 4, and 8.



2.3.34 *Longest palindromic subsequence.* Write a program `LongestPalindromicSubsequence` that takes a string as a command-line argument and determines the longest subsequence of the string that is a palindrome (the same when read forward or backward). Hint: Compute the longest common subsequence of the string and its reverse.

2.3.35 *Longest common subsequence of three strings.* Given three strings, write a program that computes the longest common subsequence of the three strings.

2.3.36 *Longest strictly increasing subsequence.* Given an integer array, find the longest subsequence that is strictly increasing. *Hint:* Compute the longest common subsequence of the original array and a sorted version of the array, where any duplicate values are removed.

2.3.37 *Longest common strictly increasing subsequence.* Given two integer arrays, find the longest increasing subsequence that is common to both arrays.

2.3.38 *Binomial coefficients.* The binomial coefficient $C(n, k)$ is the number of ways of choosing a subset of k elements from a set of n elements. *Pascal's identity* expresses the binomial coefficient $C(n, k)$ in terms of smaller binomial coefficients: $C(n, k) = C(n-1, k-1) + C(n-1, k)$, with $C(n, 0) = 1$ for each integer n . Write a recursive function (do not use dynamic programming) to computer $C(n, k)$. How long does it take to computer $C(100, 15)$? Repeat the question, first using top-down dynamic programming, then using bottom-up dynamic programming.

2.3.39 *Painting houses.* Your job is to paint a row of n houses red, green, or blue so as to minimize total cost, where $\text{cost}(i, \text{color})$ = cost to pain house i the specified *color*. You may not paint two adjacent houses the same color. Write a program to determine an optimal solution to the problem. *Hint:* Use bottom-up dynamic programming and solve the following subproblems for each $i = 1, 2, \dots, n$:

- $\text{red}(i)$ = min cost to paint houses 1, 2, ..., i so that the house i is red
- $\text{green}(i)$ = min cost to paint houses 1, 2, ..., i so that the house i is green
- $\text{blue}(i)$ = min cost to paint houses 1, 2, ..., i so that the house i is blue

2.4 Case Study: Percolation

THE PROGRAMMING TOOLS THAT WE HAVE considered to this point allow us to attack all manner of important problems. We conclude our study of functions and modules by considering a case study of developing a program to solve an interesting scientific problem. Our purpose in doing so is to review the basic elements that we have covered, in the context of the various challenges that you might face in solving a specific problem, and to illustrate a programming style that you can apply broadly.

Our example applies a widely applicable computational technique known as *Monte Carlo simulation* to study a natural model known as *percolation*. The term “Monte Carlo simulation” is broadly used to encompass any computational technique that employs randomness to estimate an unknown quantity by performing multiple trials (known as *simulations*). We have used it in several other contexts already—for example, in the gambler’s ruin and coupon collector problems. Rather than develop a complete mathematical model or measure all possible outcomes of an experiment, we rely on the laws of probability.

In this case study we will learn quite a bit about percolation, a model which underlies many natural phenomena. Our focus, however, is on the process of developing modular programs to address computational tasks. We identify subtasks that can be independently addressed, striving to identify the key underlying abstractions and asking ourselves questions such as the following: Is there some specific subtask that would help solve this problem? What are the essential characteristics of this specific subtask? Might a solution that addresses these essential characteristics be useful in solving other problems? Asking such questions pays significant dividend, because they lead us to develop software that is easier to create, debug, and reuse, so that we can more quickly address the main problem of interest.

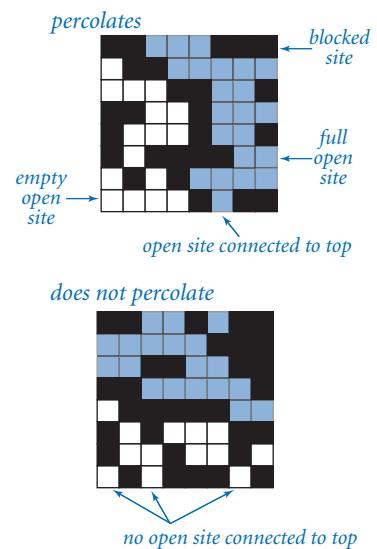
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Programs in this section

Percolation It is not unusual for local interactions in a system to imply global properties. For example, an electrical engineer might be interested in composite systems consisting of randomly distributed insulating and metallic materials: which fraction of the materials need to be metallic so that the composite system is an electrical conductor? As another example, a geologist might be interested in a porous landscape with water on the surface (or oil below). Under which conditions will the water be able to drain through to the bottom (or the oil to gush through to the surface)? Scientists have defined an abstract process known as *percolation* to model such situations. It has been studied widely, and shown to be an accurate model in a dizzying variety of applications, beyond insulating materials and porous substances to the spread of forest fires and disease epidemics to evolution to the study of the Internet.

For simplicity, we begin by working in two dimensions and model the system as an n -by- n grid of *sites*. Each site is either *blocked* or *open*; open sites are initially *empty*. A *full* site is an open site that can be connected to an open site in the top row via a chain of neighboring (left, right, up, down) open sites. If there is a full site in the bottom row, then we say that the system *percolates*. In other words, a system percolates if we fill all open sites connected to the top row and that process fills some open site on the bottom row. For the insulating/metallic materials example, the open sites correspond to metallic materials, so that a system that percolates has a metallic path from top to bottom, with full sites conducting. For the porous substance example, the open sites correspond to empty space through which water might flow, so that a system that percolates lets water fill open sites, flowing from top to bottom.

In a famous scientific problem that has been heavily studied for decades, scientists are interested in the following question: if sites are independently set to be open with *site vacancy probability* p (and therefore blocked with probability $1-p$), what is the probability that the system percolates? No mathematical solution to this problem has yet been derived. Our task is to write computer programs to help study the problem.

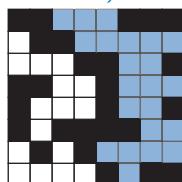


Percolation examples

Basic scaffolding To address percolation with a Java program, we face numerous decisions and challenges, and we certainly will end up with much more code than in the short programs that we have considered so far in this book. Our goal is to illustrate an incremental style of programming where we independently develop modules that address parts of the problem, building confidence with a small computational infrastructure of our own design and construction as we proceed.

The first step is to pick a representation of the data. This decision can have substantial impact on the kind of code that we write later, so it is not to be taken lightly. Indeed, it is often the case that we learn something while working with a chosen representation that causes us to scrap it and start all over using a new one.

percolation system



blocked sites

```
1 1 0 0 0 1 1 1
0 1 1 0 0 0 0 0
0 0 0 1 1 0 0 1
1 1 0 0 1 0 0 0
1 0 0 0 1 0 0 1
1 0 1 1 1 1 0 0
0 1 0 1 0 0 0 0
0 0 0 0 1 0 1 1
```

open sites

```
0 0 1 1 1 0 0 0
1 0 0 1 1 1 1 1
1 1 1 0 0 1 1 0
0 0 1 1 0 1 1 1
0 1 1 1 0 1 1 0
0 1 0 0 0 0 1 1
1 0 1 0 1 1 1 1
1 1 1 1 0 1 0 0
```

full sites

```
0 0 1 1 1 0 0 0
0 0 0 1 1 1 1 1
0 0 0 0 0 1 1 0
0 0 0 0 0 1 1 1
0 0 0 0 0 0 1 1 0
0 0 0 0 0 0 0 1 1
0 0 0 0 1 1 1 1
0 0 0 0 0 1 0 0 0
```

Percolation representations

For percolation, the path to an effective representation is clear: use an n -by- n array. Which type of data should we use for each element? One possibility is to use integers, with the convention that 0 indicates an empty site, 1 indicates a blocked site, and 2 indicates a full site. Alternatively, note that we typically describe sites in terms of questions: Is the site open or blocked? Is the site full or empty? This characteristic of the elements suggests that we might use n -by- n arrays in which element is either true or false. We refer to such two-dimensional arrays as *boolean* matrices. Using boolean matrices leads to code that is easier to understand than the alternative.

Boolean matrices are fundamental mathematical objects with many applications. Java does not provide direct support for operations on boolean matrices, but we can use the methods in `StdArrayIO` (see PROGRAM 2.2.2) to read and write them. This choice illustrates a basic principle that often comes up in programming: *the effort required to build a more general tool usually pays dividends*.

Eventually, we will want to work with random data, but we also want to be able to read and write to files because debugging programs with random inputs can be counterproductive. With random data, you get different input each time that you run the program; after fixing a bug, what you want to see is the *same* input that you just used, to check that the fix was effective. Accordingly, it is best to start with some specific cases that we understand, kept in files formatted compatible with `StdArrayIO` (dimensions followed by 0 and 1 values in row-major order).

When you start working on a new problem that involves several files, it is usually worthwhile to create a new folder (directory) to isolate those files from others that you may be working on. For example, we might create a folder named `percolation` to store all of the files for this case study. To get started, we can implement and debug the basic code for reading and writing percolation systems, create test files, check that the files are compatible with the code, and so forth, before worrying about percolation at all. This type of code, sometimes called *scaffolding*, is straightforward to implement, but making sure that it is solid at the outset will save us from distraction when approaching the main problem.

Now we can turn to the code for testing whether a boolean matrix represents a system that percolates. Referring to the helpful interpretation in which we can think of the task as simulating what would happen if the top were flooded with water (does it flow to the bottom or not?), our first design decision is that we will want to have a `flow()` method that takes as an argument a boolean matrix `isOpen[][]` that specifies which sites are open and returns another boolean matrix `isFull[][]` that specifies which sites are full. For the moment, we will not worry at all about how to implement this method; we are just deciding how to organize the computation. It is also clear that we will want client code to be able to use a `percolates()` method that checks whether the array returned by `flow()` has any full sites on the bottom.

`Percolation` (PROGRAM 2.4.1) summarizes these decisions. It does not perform any interesting computation, but after running and debugging this code we can start thinking about actually solving the problem. A method that performs no computation, such as `flow()`, is sometimes called a *stub*. Having this stub allows us to test and debug `percolates()` and `main()` in the context in which we will need them. We refer to code like PROGRAM 2.4.1 as *scaffolding*. As with scaffolding that construction workers use when erecting a building, this kind of code provides the support that we need to develop a program. By fully implementing and debugging this code (much, if not all, of which we need, anyway) at the outset, we provide a sound basis for building code to solve the problem at hand. Often, we carry the analogy one step further and remove the scaffolding (or replace it with something better) after the implementation is complete.

Program 2.4.1 Percolation scaffolding

```

public class Percolation
{
    public static boolean[][] flow(boolean[][] isOpen)
    {
        int n = isOpen.length;
        boolean[][] isFull = new boolean[n][n];
        // The isFull[][] matrix computation goes here.
        return isFull;
    }

    public static boolean percolates(boolean[][] isOpen)
    {
        boolean[][] isFull = flow(isOpen);
        int n = isOpen.length;
        for (int j = 0; j < n; j++)
            if (isFull[n-1][j]) return true;
        return false;
    }

    public static void main(String[] args)
    {
        boolean[][] isOpen = StdArrayIO.readBoolean2D();
        StdArrayIO.print(flow(isOpen));
        StdOut.println(percolates(isOpen));
    }
}

```

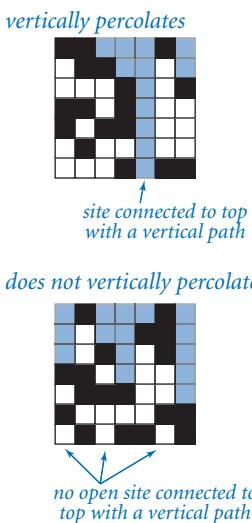
n | system size (n-by-n)
 $\text{isFull}[][]$ | full sites
 $\text{isOpen}[][]$ | open sites

To get started with percolation, we implement and debug this code, which handles all the straightforward tasks surrounding the computation. The primary function `flow()` returns a boolean matrix giving the full sites (none, in the placeholder code here). The helper function `percolates()` checks the bottom row of the returned matrix to decide whether the system percolates. The test client `main()` reads a boolean matrix from standard input and prints the result of calling `flow()` and `percolates()` for that matrix.

```
% more test5.txt
5 5
0 1 1 0 1
0 0 1 1 1
1 1 0 1 1
1 0 0 0 1
0 1 1 1 1
```

```
% java Percolation < test5.txt
5 5
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
false
```

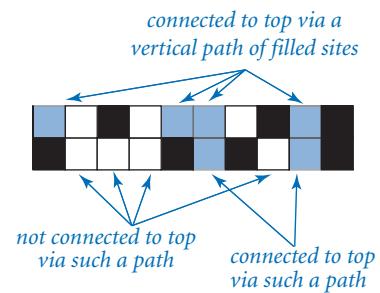
Vertical percolation Given a boolean matrix that represents the open sites, how do we figure out whether it represents a system that percolates? As we will see later in this section, this computation turns out to be directly related to a fundamental question in computer science. For the moment, we will consider a much simpler version of the problem that we call *vertical percolation*.



The simplification is to restrict attention to vertical connection paths. If such a path connects top to bottom in a system, we say that the system *vertically percolates* along the path (and that the system itself vertically percolates). This restriction is perhaps intuitive if we are talking about sand traveling through cement, but not if we are talking about water traveling through cement or about electrical conductivity. Simple as it is, vertical percolation is a problem that is interesting in its own right because it suggests various mathematical questions. Does the restriction make a significant difference? How many vertical percolation paths do we expect?

Determining the sites that are filled by some path that is connected vertically to the top is a simple calculation. We initialize the top row of our result array from the top row of the percolation system, with full sites corresponding to open ones. Then, moving from top to bottom, we fill in each row of the array by checking the corresponding row of the percolation system. Proceeding from top to bottom, we fill in the rows of `isFull[][]` to mark as `true` all elements that correspond to sites in `isOpen[][]` that are vertically connected to a full site on the previous row. PROGRAM 2.4.2 is an implementation of `flow()` for `Percolation` that returns a boolean matrix of full sites (`true` if connected to the top via a vertical path, `false` otherwise).

Testing After we become convinced that our code is behaving as planned, we want to run it on a broader variety of test cases and address some of our scientific questions. At this point, our initial scaffolding becomes less useful, as representing large boolean matrices with 0s and 1s on standard input and standard output and maintaining large numbers of test cases quickly becomes unwieldy. Instead,



Vertical percolation calculation

Program 2.4.2 Vertical percolation detection

```

public static boolean[][] flow(boolean[][] isOpen)
{ // Compute full sites for vertical percolation.
    int n = isOpen.length;
    boolean[][] isFull = new boolean[n][n];
    for (int j = 0; j < n; j++)
        isFull[0][j] = isOpen[0][j];
    for (int i = 1; i < n; i++)
        for (int j = 0; j < n; j++)
            isFull[i][j] = isOpen[i][j] && isFull[i-1][j];
    return isFull;
}

```

n *system size (n-by-n)*
isFull[][] *full sites*
isOpen[][] *open sites*

Substituting this method for the stub in PROGRAM 2.4.1 gives a solution to the vertical-only percolation problem that solves our test case as expected (see text).

<pre>% more test5.txt 5 5 0 1 1 0 1 0 0 1 1 1 1 1 0 1 1 1 0 0 0 1 0 1 1 1 1</pre>	<pre>% java Percolation < test5.txt 5 5 0 1 1 0 1 0 0 1 0 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1 true</pre>
---	---

we want to automatically generate test cases and observe the operation of our code on them, to be sure that it is operating as we expect. Specifically, to gain confidence in our code and to develop a better understanding of percolation, our next goals are to:

- Test our code for large random boolean matrices.
- Estimate the probability that a system percolates for a given p .

To accomplish these goals, we need new clients that are slightly more sophisticated than the scaffolding we used to get the program up and running. Our modular programming style is to develop such clients in independent classes *without modifying our percolation code at all*.

Data visualization. We can work with much bigger problem instances if we use StdDraw for output. The following static method for Percolation allows us to visualize the contents of boolean matrices as a subdivision of the StdDraw canvas into squares, one for each site:

```
public static void show(boolean[][] a, boolean which)
{
    int n = a.length;
    StdDraw.setXscale(-1, n);
    StdDraw.setYscale(-1, n);
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++)
            if (a[i][j] == which)
                StdDraw.filledSquare(j, n-i-1, 0.5);
}
```

The second argument `which` specifies which squares we want to fill—those corresponding to `true` elements or those corresponding to `false` elements. This method is a bit of a diversion from the calculation, but pays dividends in its ability to help us visualize large problem instances. Using `show()` to draw our boolean matrices representing blocked and full sites in different colors gives a compelling visual representation of percolation.

Monte Carlo simulation. We want our code to work properly for *any* boolean matrix. Moreover, the scientific question of interest involves random boolean matrices. To this end, we add another static method to `Percolation`:

```
public static boolean[][] random(int n, double p)
{
    boolean[][] a = new boolean[n][n];
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++)
            a[i][j] = StdRandom.bernoulli(p);
    return a;
}
```

This method generates a random n -by- n boolean matrix of any given size n , each element `true` with probability p .

Having debugged our code on a few specific test cases, we are ready to test it on random systems. It is possible that such cases may uncover a few more bugs, so some care is in order to check results. However, having debugged our code for a small system, we can proceed with some confidence. It is easier to focus on new bugs after eliminating the obvious bugs.

WITH THESE TOOLS, A CLIENT FOR testing our percolation code on a much larger set of trials is straightforward. `PercolationVisualizer` (PROGRAM 2.4.3) consists of just a `main()` method that takes `n` and `p` from the command line and displays the result of the percolation flow calculation.

This kind of client is typical. Our eventual goal is to compute an accurate estimate of percolation probabilities, perhaps by running a large number of trials, but this simple tool gives us the opportunity to gain more familiarity with the problem by studying some large cases (while at the same time gaining confidence that our code is working properly). Before reading further, you are encouraged to download and run this code from the booksite to study the percolation process. When you run `PercolationVisualizer` for moderate-size n (50 to 100, say) and various p , you will immediately be drawn into using this program to try to answer some questions about percolation. Clearly, the system never percolates when p is low and always percolates when p is very high. How does it behave for intermediate values of p ? How does the behavior change as n increases?

Estimating probabilities The next step in our program development process is to write code to estimate the probability that a random system (of size n with site vacancy probability p) percolates. We refer to this quantity as the *percolation probability*. To estimate its value, we simply run a number of trials. The situation is no different from our study of coin flipping (see PROGRAM 2.2.6), but instead of flipping a coin, we generate a random system and check whether it percolates.

`PercolationProbability` (PROGRAM 2.4.4) encapsulates this computation in a method `estimate()`, which takes three arguments `n`, `p`, and `trials` and returns an estimate of the probability that an n -by- n system with site vacancy probability p percolates, obtained by generating `trials` random systems and calculating the fraction of them that percolate.

How many trials do we need to obtain an accurate estimate? This question is addressed by basic methods in probability and statistics, which are beyond the scope of this book, but we can get a feeling for the problem with computational experience. With just a few runs of `PercolationProbability`, you can learn that if the site vacancy probability is close to either 0 or 1, then we do not need many trials, but that there are values for which we need as many as 10,000 trials to be able to estimate it within two decimal places. To study the situation in more detail, we might modify `PercolationProbability` to produce output like `Bernoulli` (PROGRAM 2.2.6), plotting a histogram of the data points so that we can see the distribution of values (see EXERCISE 2.4.9).

Program 2.4.3 Visualization client

```

public class PercolationVisualizer
{
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        double p = Double.parseDouble(args[1]);
        StdDraw.enableDoubleBuffering();

        // Draw blocked sites in black.
        boolean[][] isOpen = Percolation.random(n, p);
        StdDraw.setPenColor(StdDraw.BLACK);
        Percolation.show(isOpen, false);

        // Draw full sites in blue.
        StdDraw.setPenColor(StdDraw.BOOK_BLUE);
        boolean[][] isFull = Percolation.flow(isOpen);
        Percolation.show(isFull, true);

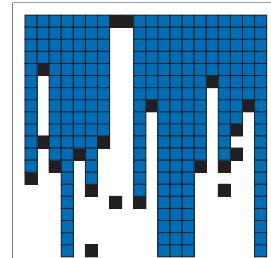
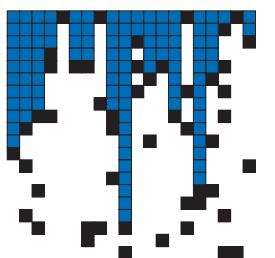
        StdDraw.show();
    }
}

```

<code>n</code> <code>p</code> <code>isOpen[][]</code> <code>isFull[][]</code>	<i>system size (n-by-n)</i> <i>site vacancy probability</i> <i>open sites</i> <i>full sites</i>
--	--

This client takes two command-line argument n and p , generates an n -by- n random system with site vacancy probability p , determines which sites are full, and draws the result on standard drawing. The diagrams below show the results for vertical percolation.

% java PercolationVisualizer 20 0.9 % java PercolationVisualizer 20 0.95



Using `PercolationProbability.estimate()` represents a giant leap in the amount of computation that we are doing. All of a sudden, it makes sense to run thousands of trials. It would be unwise to try to do so without first having thoroughly debugged our percolation methods. Also, we need to begin to take the time required to complete the computation into account. The basic methodology for doing so is the topic of SECTION 4.1, but the structure of these programs is sufficiently simple that we can do a quick calculation, which we can verify by running the program. If we perform T trials, each of which involves n^2 sites, then the total running time of `PercolationProbability.estimate()` is proportional to n^2T . If we increase T by a factor of 10 (to gain more precision), the running time increases by about a factor of 10. If we increase n by a factor of 10 (to study percolation for larger systems), the running time increases by about a factor of 100.

Can we run this program to determine percolation probabilities for a system with billions of sites with several digits of precision? No computer is fast enough to use `PercolationProbability.estimate()` for this purpose. Moreover, in a scientific experiment on percolation, the value of n is likely to be much higher. We can hope to formulate a hypothesis from our simulation that can be tested experimentally on a much larger system, but not to precisely simulate a system that corresponds atom-for-atom with the real world. Simplification of this sort is essential in science.

You are encouraged to download `PercolationProbability` from the book site to get a feel for both the percolation probabilities and the amount of time required to compute them. When you do so, you are not just learning more about percolation, but are also testing the hypothesis that the models we have just described apply to the running times of our simulations of the percolation process.

What is the probability that a system with site vacancy probability p vertically percolates? Vertical percolation is sufficiently simple that elementary probabilistic models can yield an exact formula for this quantity, which we can validate experimentally with `PercolationProbability`. Since our only reason for studying vertical percolation was an easy starting point around which we could develop supporting software for studying percolation methods, we leave further study of vertical percolation for an exercise (see EXERCISE 2.4.11) and turn to the main problem.

Program 2.4.4 Percolation probability estimate

```

public class PercolationProbability
{
    public static double estimate(int n, double p, int trials)
    { // Generate trials random n-by-n systems; return empirical
      // percolation probability estimate.
        int count = 0;
        for (int t = 0; t < trials; t++)
        { // Generate one random n-by-n boolean matrix.
          boolean[][] isOpen = Percolation.random(n, p);
          if (Percolation.percolates(isOpen)) count++;
        }
        return (double) count / trials;
    }
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);           isOpen[][]      n
        double p = Double.parseDouble(args[1]);       q             p
        int trials = Integer.parseInt(args[2]);       trials        number of trials
        double q = estimate(n, p, trials);           isopen[][]      open sites
        StdOut.println(q);                         q             percolation probability
    }
}

```

The method `estimate()` generates `trials` random n -by- n systems with site vacancy probability p and computes the fraction of them that percolate. This is a Bernoulli process, like coin flipping (see PROGRAM 2.2.6). Increasing the number of `trials` increases the accuracy of the estimate. If p is close to 0 or to 1, not many trials are needed to achieve an accurate estimate. The results below are for vertical percolation.

```

% java PercolationProbability 20 0.05 10
0.0
% java PercolationProbability 20 0.95 10
1.0
% java PercolationProbability 20 0.85 10
0.7
% java PercolationProbability 20 0.85 1000
0.564
% java PercolationProbability 40 0.85 100
0.1

```

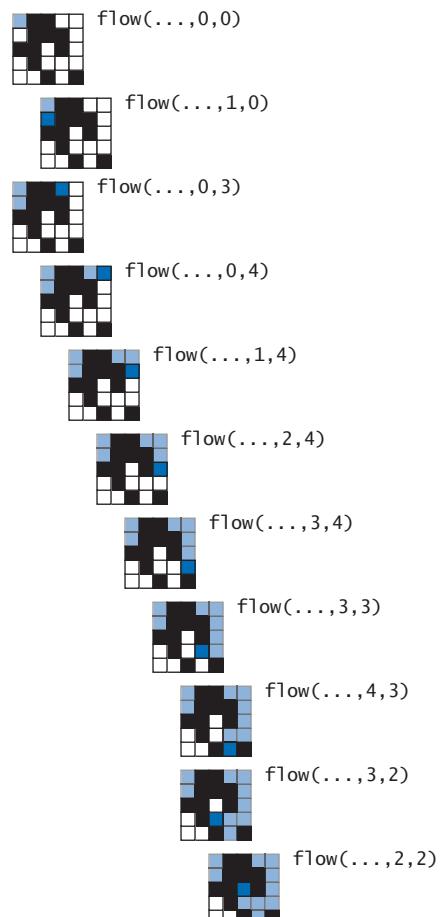
Recursive solution for percolation How do we test whether a system percolates in the general case when *any* path starting at the top and ending at the bottom (not just a vertical one) will do the job?

Remarkably, we can solve this problem with a compact program, based on a classic recursive scheme known as *depth-first search*. PROGRAM 2.4.5 is an implementation of `flow()` that computes the matrix `isFull[][]`, based on a recursive four-argument version of `flow()` that takes as arguments the site vacancy matrix `isOpen[][]`, the current matrix `isFull[][]`, and a site position specified by a row index i and a column index j . The base case is a recursive call that just returns (we refer to such a call as a *null call*), for one of the following reasons:

- Either i or j is outside the array bounds.
- The site is blocked (`isOpen[i][j]` is `false`).
- We have already marked the site as full (`isFull[i][j]` is `true`).

The reduction step is to mark the site as filled and issue recursive calls for the site's four neighbors: `isOpen[i+1][j]`, `isOpen[i][j+1]`, `isOpen[i][j-1]`, and `isOpen[i-1][j]`. The one-argument `flow()` calls the recursive method for every site on the top row. The recursion always terminates because each recursive call either is null or marks a new site as full. We can show by an induction-based argument (as usual for recursive programs) that a site is marked as full if and only if it is connected to one of the sites on the top row.

Tracing the operation of `flow()` on a tiny test case is an instructive exercise. You will see that it calls `flow()` for every site that can be reached via a path of open sites from the top row. This example illustrates that simple recursive programs can mask computations that otherwise are quite sophisticated. This method is a special case of the depth-first search algorithm, which has many important applications.



Recursive percolation (null calls omitted)

Program 2.4.5 Percolation detection

```

public static boolean[][] flow(boolean[][] isOpen)
{ // Fill every site reachable from the top row.
    int n = isOpen.length;
    boolean[][] isFull = new boolean[n][n];           n
    for (int j = 0; j < n; j++)           isOpen[][] | system size (n-by-n)
        flow(isOpen, isFull, 0, j);           isFull[][] | open sites
    return isFull;           i, j | full sites
}                           current site row, column
public static void flow(boolean[][] isOpen,
                        boolean[][] isFull, int i, int j)
{ // Fill every site reachable from (i, j).
    int n = isFull.length;
    if (i < 0 || i >= n) return;
    if (j < 0 || j >= n) return;
    if (!isOpen[i][j]) return;
    if (isFull[i][j]) return;
    isFull[i][j] = true;
    flow(isOpen, isFull, i+1, j); // Down.
    flow(isOpen, isFull, i, j+1); // Right.
    flow(isOpen, isFull, i, j-1); // Left.
    flow(isOpen, isFull, i-1, j); // Up.
}

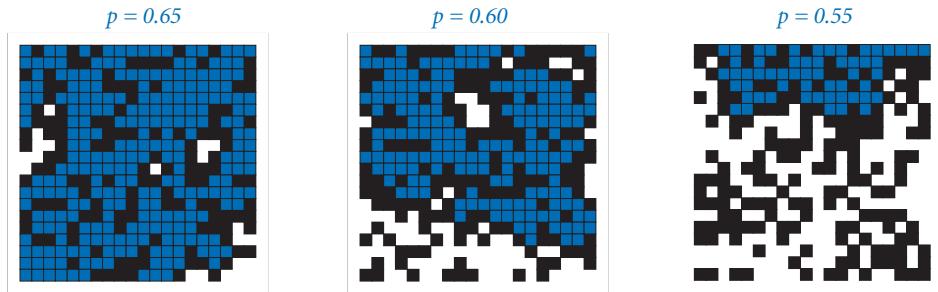
```

Substituting these methods for the stub in PROGRAM 2.4.1 gives a depth-first-search-based solution to the percolation problem. The recursive `flow()` sets to `true` the element in `isFull[][]` corresponding to any site that can be reached from `isOpen[i][j]` via a chain of neighboring open sites. The one-argument `flow()` calls the recursive method for every site on the top row.

```
% more test8.txt
8 8
0 0 1 1 1 0 0 0
1 0 0 1 1 1 1 1
1 1 1 0 0 1 1 0
0 0 1 1 0 1 1 1
0 1 1 1 0 1 1 0
0 1 0 0 0 0 1 1
1 0 1 0 1 1 1 1
1 1 1 1 0 1 0 0
```

```
% java Percolation < test8.txt
8 8
0 0 1 1 1 0 0 0
0 0 0 1 1 1 1 1
0 0 0 0 0 1 1 0
0 0 0 0 0 1 1 1
0 0 0 0 0 0 1 1
0 0 0 0 0 0 1 1
0 0 0 0 1 1 1 1
0 0 0 0 0 1 0 0
true
```

To avoid conflict with our solution for vertical percolation (PROGRAM 2.4.2), we might rename that class `PercolationVertical`, making another copy of `Percolation` (PROGRAM 2.4.1) and substituting the two `flow()` methods in PROGRAM 2.4.5 for the placeholder `flow()`. Then, we can visualize and perform experiments with this algorithm with the `PercolationVisualizer` and `PercolationProbability` tools that we have developed. If you do so, and try various values for n and p , you will quickly get a feeling for the situation: the systems always percolate when the site vacancy probability p is high and never percolate when p is low, and (particularly as n increases) there is a value of p above which the systems (almost) always percolate and below which they (almost) never percolate.



Percolation is less probable as the site vacancy probability p decreases

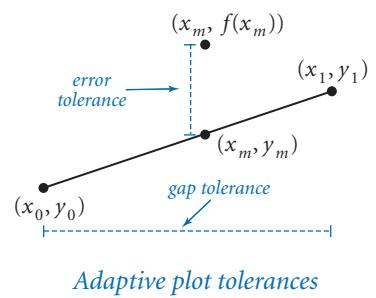
Having debugged `PercolationVisualizer` and `PercolationProbability` on the simple vertical percolation process, we can use them with more confidence to study percolation, and turn quickly to study the scientific problem of interest. Note that if we want to experiment with vertical percolation again, we would need to edit `PercolationVisualizer` and `PercolationProbability` to refer to `PercolationVertical` instead of `Percolation`, or write other clients of both `PercolationVertical` and `Percolation` that run methods in both classes to compare them.

Adaptive plot To gain more insight into percolation, the next step in program development is to write a program that plots the percolation probability as a function of the site vacancy probability p for a given value of n . Perhaps the best way to produce such a plot is to first derive a mathematical equation for the function, and then use that equation to make the plot. For percolation, however, no one has been able to derive such an equation, so the next option is to use the Monte Carlo method: run simulations and plot the results.

Immediately, we are faced with numerous decisions. For how many values of p should we compute an estimate of the percolation probability? Which values of p should we choose? How much precision should we aim for in these calculations? These decisions constitute an experimental design problem. Much as we might like to instantly produce an accurate rendition of the curve for any given n , the computation cost can be prohibitive. For example, the first thing that comes to mind is to plot, say, 100 to 1,000 equally spaced points, using `StdStats` (PROGRAM 2.2.5). But, as you learned from using `PercolationProbability`, computing a sufficiently precise value of the percolation probability for each point might take several seconds or longer, so the whole plot might take minutes or hours or even longer. Moreover, it is clear that a lot of this computation time is completely wasted, because we know that values for small p are 0 and values for large p are 1. We might prefer to spend that time on more precise computations for intermediate p . How should we proceed?

`PercolationPlot` (PROGRAM 2.4.6) implements a recursive approach with the same structure as `Brownian` (PROGRAM 2.3.5) that is widely applicable to similar problems. The basic idea is simple: we choose the maximum distance that we wish to allow between values of the x -coordinate (which we refer to as the *gap tolerance*), the maximum known error that we wish to tolerate in the y -coordinate (which we refer to as the *error tolerance*), and the number of trials T per point that we wish to perform. The recursive method draws the plot within a given interval $[x_0, x_1]$, from (x_0, y_0) to (x_1, y_1) . For our problem, the plot is from $(0, 0)$ to $(1, 1)$. The base case (if the distance between x_0 and x_1 is less than the gap tolerance, or the distance between the line connecting the two endpoints and the value of the function at the midpoint is less than the error tolerance) is to simply draw a line from (x_0, y_0) to (x_1, y_1) . The reduction step is to (recursively) plot the two halves of the curve, from (x_0, y_0) to $(x_m, f(x_m))$ and from $(x_m, f(x_m))$ to (x_1, y_1) .

The code in `PercolationPlot` is relatively simple and produces a good-looking curve at relatively low cost. We can use it to study the shape of the curve for various values of n or choose smaller tolerances to be more confident that the curve is close to the actual values. Precise mathematical statements about quality of approximation can, in principle, be derived, but it is perhaps not appropriate to go into too much detail while exploring and experimenting, since our goal is simply to develop a hypothesis about percolation that can be tested by scientific experimentation.



Adaptive plot tolerances

Program 2.4.6 Adaptive plot client

```

public class PercolationPlot
{
    public static void curve(int n,
                            double x0, double y0,
                            double x1, double y1) {
        // Perform experiments and plot results.
        double gap = 0.01;
        double err = 0.0025;
        int trials = 10000;
        double xm = (x0 + x1)/2;
        double ym = (y0 + y1)/2;
        double fxm = PercolationProbability.estimate(n, xm, trials);
        if (x1 - x0 < gap || Math.abs(ym - fxm) < err)
        {
            StdDraw.line(x0, y0, x1, y1);
            return;
        }
        curve(n, x0, y0, xm, f xm);
        StdDraw.filledCircle(xm, f xm, 0.005);
        curve(n, xm, f xm, x1, y1);
    }

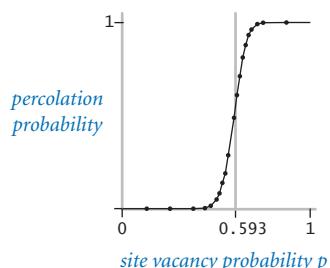
    public static void main(String[] args)
    {
        // Plot experimental curve for n-by-n percolation system.
        int n = Integer.parseInt(args[0]);
        curve(n, 0.0, 0.0, 1.0, 1.0);
    }
}

```

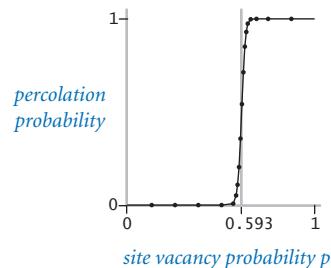
n system size
 x_0, y_0 left endpoint
 x_1, y_1 right endpoint
 x_m, y_m midpoint
 f_{xm} value at midpoint
 gap gap tolerance
 err error tolerance
 $trials$ number of trials

This recursive program draws a plot of the percolation probability (experimental observations) against the site vacancy probability p (control variable) for random n -by- n systems.

% java PercolationPlot 20



% java PercolationPlot 100



Indeed, the curves produced by `PercolationPlot` immediately confirm the hypothesis that there is a *threshold* value (about 0.593): if p is greater than the threshold, then the system almost certainly percolates; if p is less than the threshold, then the system almost certainly does not percolate. As n increases, the curve approaches a step function that changes value from 0 to 1 at the threshold. This phenomenon, known as a *phase transition*, is found in many physical systems.

The simple form of the output of PROGRAM 2.4.6 masks the huge amount of computation behind it. For example, the curve drawn for $n = 100$ has 18 points, each the result of 10,000 trials, with each trial involving n^2 sites. Generating and testing each site involves a few lines of code, so this plot comes at the cost of executing *billions* of statements. There are two lessons to be learned from this observation. First, we need to have confidence in any line of code that might be executed billions of times, so our care in developing and debugging code incrementally is justified. Second, although we might be interested in systems that are much larger, we need further study in computer science to be able to handle larger cases—that is, to develop faster algorithms and a framework for knowing their performance characteristics.

With this reuse of all of our software, we can study all sorts of variants on the percolation problem, just by implementing different `flow()` methods. For example, if you leave out the last recursive call in the recursive `flow()` method in PROGRAM 2.4.5, it tests for a type of percolation known as *directed percolation*, where paths that go up are not considered. This model might be important for a situation like a liquid percolating through porous rock, where gravity might play a role, but not for a situation like electrical connectivity. If you run `PercolationPlot` for both methods, will you be able to discern the difference (see EXERCISE 2.4.10)?

```

PercolationPlot.curve()
PercolationProbability.estimate()
Percolation.random()
StdRandom.bernoulli()
  : n2 times
StdRandom.bernoulli()
return
Percolation.percolates()
  flow()
  return
return
: T times
Percolation.random()
StdRandom.bernoulli()
  : n2 times
StdRandom.bernoulli()
return
Percolation.percolates()
  flow()
  return
return
once for each point
PercolationProbability.estimate()
Percolation.random()
StdRandom.bernoulli()
  : n2 times
StdRandom.bernoulli()
return
Percolation.percolates()
  flow()
  return
return
: T times
Percolation.random()
StdRandom.bernoulli()
  : n2 times
StdRandom.bernoulli()
return
Percolation.percolates()
  flow()
  return
return
return
return

```

Function-call trace for `PercolationPlot`

To model physical situations such as water flowing through porous substances, we need to use three-dimensional arrays. Is there a similar threshold in the three-dimensional problem? If so, what is its value? Depth-first search is effective for studying this question, though the addition of another dimension requires that we pay even more attention to the computational cost of determining whether a system percolates (see EXERCISE 2.4.18). Scientists also study more complex lattice structures that are not well modeled by multidimensional arrays—we will see how to model such structures in SECTION 4.5.

Percolation is interesting to study via *in silico* experimentation because no one has been able to derive the threshold value mathematically for several natural models. The only way that scientists know the value is by using simulations like `Percolation`. A scientist needs to do experiments to see whether the percolation model reflects what is observed in nature, perhaps through refining the model (for example, using a different lattice structure). Percolation is an example of an increasing number of problems where computer science of the kind described here is an essential part of the scientific process.

Lessons We might have approached the problem of studying percolation by sitting down to design and implement a single program, which probably would run to hundreds of lines, to produce the kind of plots that are drawn by PROGRAM 2.4.6. In the early days of computing, programmers had little choice but to work with such programs, and would spend enormous amounts of time isolating bugs and correcting design decisions. With modern programming tools like Java, we can do better, using the incremental modular style of programming presented in this chapter and keeping in mind some of the lessons that we have learned.

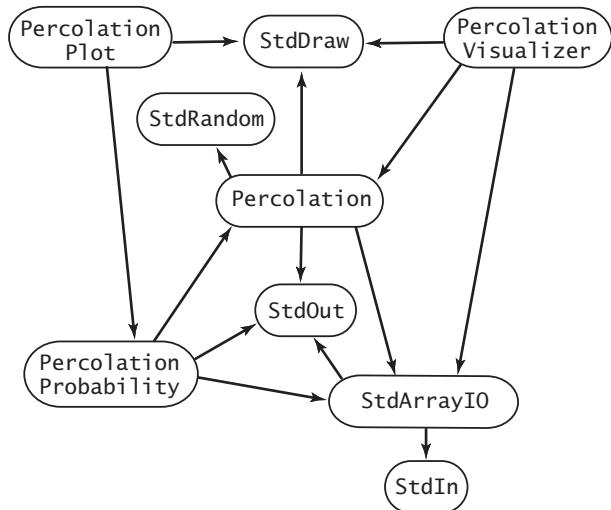
Expect bugs. Every interesting piece of code that you write is going to have at least one or two bugs, if not many more. By running small pieces of code on small test cases that you understand, you can more easily isolate any bugs and then more easily fix them when you find them. Once debugged, you can depend on using a library as a building block for any client.

Keep modules small. You can focus attention on at most a few dozen lines of code at a time, so you may as well break your code into small modules as you write it. Some classes that contain libraries of related methods may eventually grow to contain hundreds of lines of code; otherwise, we work with small files.

Limit interactions. In a well-designed modular program, most modules should depend on just a few others. In particular, a module that *calls* a large number of other modules needs to be divided into smaller pieces. Modules that *are called by* a large number of other modules (you should have only a few) need special attention, because if you do need to make changes in a module's API, you have to reflect those changes in all its clients.

Develop code incrementally. You should run and debug each small module as you implement it. That way, you are never working with more than a few dozen lines of unreliable code at any given time. If you put all your code in one big module, it is difficult to be confident that *any* of it is free from bugs. Running code early also forces you to think sooner rather than later about I/O formats, the nature of problem instances, and other issues. Experience gained when thinking about such issues and debugging related code makes the code that you develop later in the process more effective.

Solve an easier problem. Some working solution is better than no solution, so it is typical to begin by putting together the simplest code that you can craft that solves a given problem, as we did with vertical percolation. This implementation is the first step in a process of continual refinements and improvements as we develop a more complete understanding of the problem by examining a broader variety of test cases and developing support software such as our `PercolationVisualizer` and `PercolationProbability` classes.



Case study dependency graph (not including system calls)

Consider a recursive solution. Recursion is an indispensable tool in modern programming that you should learn to trust. If you are not already convinced of this fact by the simplicity and elegance of `Percolation` and `PercolationPlot`, you might wish to try to develop a nonrecursive program for testing whether a system percolates and then reconsider the issue.

Build tools when appropriate. Our visualization method `show()` and random boolean matrix generation method `random()` are certainly useful for many other applications, as is the adaptive plotting method of `PercolationPlot`. Incorporating these methods into appropriate libraries would be simple. It is no more difficult (indeed, perhaps easier) to implement general-purpose methods like these than it would be to implement special-purpose methods for percolation.

Reuse software when possible. Our `StdIn`, `StdRandom`, and `StdDraw` libraries all simplified the process of developing the code in this section, and we were also immediately able to reuse programs such as `PercolationVisualizer`, `PercolationProbability`, and `PercolationPlot` for percolation after developing them for vertical percolation. After you have written a few programs of this kind, you might find yourself developing versions of these programs that you can reuse for other Monte Carlo simulations or other experimental data analysis problems.

THE PRIMARY PURPOSE OF THIS CASE study is to convince you that modular programming will take you much further than you could get without it. Although no approach to programming is a panacea, the tools and approach that we have discussed in this section will allow you to attack complex programming tasks that might otherwise be far beyond your reach.

The success of modular programming is only a start. Modern programming systems have a vastly more flexible programming model than the class-as-a-library-of-static-methods model that we have been considering. In the next two chapters, we develop this model, along with many examples that illustrate its utility.

Q&A

Q. Editing `PercolationVisualizer` and `PercolationProbability` to rename `Percolation` to `PercolationVertical` or whatever method we want to study seems to be a bother. Is there a way to avoid doing so?

A. Yes, this is a key issue to be revisited in CHAPTER 3. In the meantime, you can keep the implementations in separate subdirectories, but that can get confusing. Advanced Java mechanisms (such as the *classpath*) are also helpful, but they also have their own problems.

Q. That recursive `flow()` method makes me nervous. How can I better understand what it's doing?

A. Run it for small examples of your own making, instrumented with instructions to print a function-call trace. After a few runs, you will gain confidence that it always marks as full the sites connected to the start site via a chain of neighboring open sites.

Q. Is there a simple nonrecursive approach to identifying the full sites?

A. There are several methods that perform the same basic computation. We will revisit the problem in SECTION 4.5, where we consider *breadth-first search*. In the meantime, working on developing a nonrecursive implementation of `flow()` is certain to be an instructive exercise, if you are interested.

Q. `PercolationPlot` (PROGRAM 2.4.6) seems to involve a huge amount of computation to produce a simple function graph. Is there some better way?

A. Well, the best would be a simple mathematical formula describing the function, but that has eluded scientists for decades. Until scientists discover such a formula, they must resort to computational experiments like the ones in this section.

Exercises

2.4.1 Write a program that takes a command-line argument n and creates an n -by- n boolean matrix with the element in row i and column j set to `true` if i and j are relatively prime, then shows the matrix on the standard drawing (see EXERCISE 1.4.16). Then, write a similar program to draw the Hadamard matrix of order n (see EXERCISE 1.4.29). Finally, write a program to draw the boolean matrix such that the element in row n and column j is set to `true` if the coefficient of x^j in $(1 + x)^i$ (binomial coefficient) is odd (see EXERCISE 1.4.41). You may be surprised at the pattern formed by the third example.

2.4.2 Implement a `print()` method for `Percolation` that prints 1 for blocked sites, 0 for open sites, and * for full sites.

2.4.3 Give the recursive calls for `flow()` in PROGRAM 2.4.5 given the following input:

```
3 3
1 0 1
0 0 0
1 1 0
```

2.4.4 Write a client of `Percolation` like `PercolationVisualizer` that does a series of experiments for a value of n taken from the command line where the site vacancy probability p increases from 0 to 1 by a given increment (also taken from the command line).

2.4.5 Describe the order in which the sites are marked when `Percolation` is used on a system with no blocked sites. Which is the last site marked? What is the depth of the recursion?

2.4.6 Experiment with using `PercolationPlot` to plot various mathematical functions (by replacing the call `PercolationProbability.estimate()` with a different expression that evaluates a mathematical function). Try the function $f(x) = \sin x + \cos 10x$ to see how the plot adapts to an oscillating curve, and come up with interesting plots for three or four functions of your own choosing.

2.4.7 Modify `Percolation` to animate the flow computation, showing the sites filling one by one. Check your answer to the previous exercise.

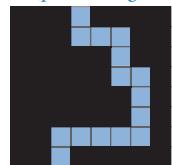
2.4.8 Modify `Percolation` to compute that maximum depth of the recursion used in the flow calculation. Plot the expected value of that quantity as a function of the site vacancy probability p . How does your answer change if the order of the recursive calls is reversed?

2.4.9 Modify `PercolationProbability` to produce output like that produced by `Bernoulli` (PROGRAM 2.2.6). *Extra credit:* Use your program to validate the hypothesis that the data obeys a Gaussian distribution.

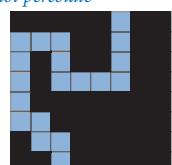
2.4.10 Create a program `PercolationDirected` that tests for *directed* percolation (by leaving off the last recursive call in the recursive `flow()` method in PROGRAM 2.4.5, as described in the text), then use `PercolationPlot` to draw a plot of the directed percolation probability as a function of the site vacancy probability p .

2.4.11 Write a client of `Percolation` and `PercolationDirected` that takes a site vacancy probability p from the command line and prints an estimate of the probability that a system percolates but does not percolate down. Use enough experiments to get an estimate that is accurate to three decimal places.

percolates (path never goes up)



does not percolate



Directed percolation

Creative Exercises

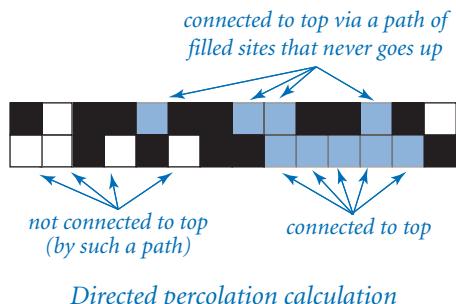
2.4.12 *Vertical percolation.* Show that a system with site vacancy probability p vertically percolates with probability $1 - (1 - p^n)^n$, and use `PercolationProbability` to validate your analysis for various values of n .

2.4.13 *Rectangular percolation systems.* Modify the code in this section to allow you to study percolation in rectangular systems. Compare the percolation probability plots of systems whose ratio of width to height is 2 to 1 with those whose ratio is 1 to 2.

2.4.14 *Adaptive plotting.* Modify `PercolationPlot` to take its control parameters (gap tolerance, error tolerance, and number of trials) as command-line arguments. Experiment with various values of the parameters to learn their effect on the quality of the curve and the cost of computing it. Briefly describe your findings.

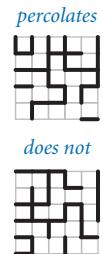
2.4.15 *Nonrecursive directed percolation.* Write a nonrecursive program that tests for directed percolation by moving from top to bottom as in our vertical percolation code. Base your solution on the following computation: if any site in a contiguous subrow of open sites in the current row is connected to some full site on the previous row, then all of the sites in the subrow become full.

2.4.16 *Fast percolation test.* Modify the recursive `flow()` method in PROGRAM 2.4.5 so that it returns as soon as it finds a site on the bottom row (and fills no more sites). Hint: Use an argument `done` that is `true` if the bottom has been hit, `false` otherwise. Give a rough estimate of the performance improvement factor for this change when running `PercolationPlot`. Use values of n for which the programs run at least a few seconds but not more than a few minutes. Note that the improvement is ineffective unless the first recursive call in `flow()` is for the site *below* the current site.



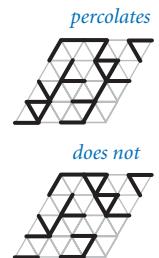
Directed percolation calculation

2.4.17 Bond percolation. Write a modular program for studying percolation under the assumption that the edges of the grid provide connectivity. That is, an edge can be either empty or full, and a system percolates if there is a path consisting of full edges that goes from top to bottom. *Note:* This problem has been solved analytically, so your simulations should validate the hypothesis that the bond percolation threshold approaches $1/2$ as n gets large.



2.4.18 Percolation in three dimensions. Implement a class `Percolation3D` and a class `BooleanMatrix3D` (for I/O and random generation) to study percolation in three-dimensional cubes, generalizing the two-dimensional case studied in this section. A percolation system is an n -by- n -by- n cube of sites that are unit cubes, each open with probability p and blocked with probability $1-p$. Paths can connect an open cube with any open cube that shares a common face (one of six neighbors, except on the boundary). The system percolates if there exists a path connecting any open site on the bottom plane to any open site on the top plane. Use a recursive version of `flow()` like PROGRAM 2.4.5, but with six recursive calls instead of four. Plot the percolation probability versus site vacancy probability p for as large a value of n as you can. Be sure to develop your solution incrementally, as emphasized throughout this section.

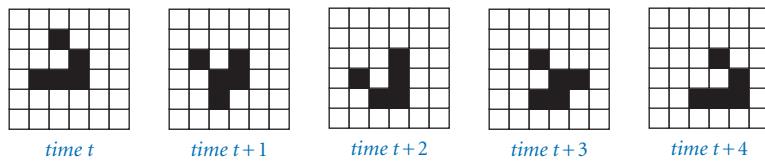
2.4.19 Bond percolation on a triangular grid. Write a modular program for studying bond percolation on a triangular grid, where the system is composed of $2n^2$ equilateral triangles packed together in an n -by- n grid of rhombus shapes. Each interior point has six bonds; each point on the edge has four; and each corner point has two.



2.4.20 *Game of Life.* Implement a class `GameOfLife` that simulates Conway's *Game of Life*. Consider a boolean matrix corresponding to a system of cells that we refer to as being either live or dead. The game consists of checking and perhaps updating the value of each cell, depending on the values of its neighbors (the adjacent cells in every direction, including diagonals). Live cells remain live and dead cells remain dead, with the following exceptions:

- A dead cell with exactly three live neighbors becomes live.
- A live cell with exactly one live neighbor becomes dead.
- A live cell with more than three live neighbors becomes dead.

Initialize with a random boolean matrix, or use one of the starting patterns on the booksite. This game has been heavily studied, and relates to foundations of computer science (see the booksite for more information).



Five generations of a glider

Object-Oriented Programming

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YOUR NEXT STEP IN PROGRAMMING EFFECTIVELY is conceptually simple. Now that you know how to use primitive types of data, you will learn in this chapter how to *use, create, and design* higher-level data types.

An *abstraction* is a simplified description of something that captures its essential elements while suppressing all other details. In science, engineering, and programming, we are always striving to understand complex systems through abstraction. In Java programming, we do so with *object-oriented programming*, where we break a large and potentially complex program into a set of interacting elements, or *objects*. The idea originates from modeling (in software) real-world entities such as electrons, people, buildings, or solar systems and readily extends to modeling abstract entities such as bits, numbers, colors, images, or programs.

A data type is a set of values and a set of operations defined on those values. The values and operations for primitive types such as `int` and `double` are predefined by Java. In object-oriented programming, we write Java code to define new data types. An *object* is an entity that holds a data-type value; you can manipulate this data-type value by applying one of the object's data-type operations.

This ability to define new data types and to manipulate objects holding data-type values is also known as *data abstraction*, and leads us to a style of modular programming that naturally extends the *procedural programming* style for primitive types that was the basis for CHAPTER 2. A data type allows us to isolate *data* as well as functions. Our mantra for this chapter is this: *whenever you can clearly separate data and associated tasks within a computation, you should do so*.

3.1 Using Data Types

ORGANIZING DATA FOR PROCESSING IS AN essential step in the development of a computer program. Programming in Java is largely based on doing so with data types known as *reference types* that are designed to support object-oriented programming, a style of programming that facilitates organizing and processing data.

The eight primitive data types (`boolean`, `byte`, `char`, `double`, `float`, `int`, `long`, and `short`) that you have been using are supplemented in Java by extensive libraries of reference types that are tailored for a large variety of applications. The `String` data type is one such example that you have already used. You will learn more about the `String` data type in this section, as well as how to use several other reference types for image processing and input/output. Some of them are built into Java (`String` and `Color`), and some were developed for this book (`In`, `Out`, `Draw`, and `Picture`) and are useful as general resources.

You certainly noticed in the first two chapters of this book that our programs were largely confined to operations on numbers. Of course, the reason is that Java's primitive types represent numbers. The one exception has been strings, a reference type that is built into Java. With reference types you can write programs that operate not just on strings, but on images, sounds, or any of hundreds of other abstractions that are available in Java's libraries or on our booksite.

In this section, we focus on client programs that *use* existing data types, to give you some concrete reference points for understanding these new concepts and to illustrate their broad reach. We will consider programs that manipulate strings, colors, images, files, and web pages—quite a leap from the primitive types of CHAPTER 1.

In the next section, you will take another leap, by learning how to *define* your own data types to implement any abstraction whatsoever, taking you to a whole new level of programming. Writing programs that operate on your own types of data is an extremely powerful and useful style of programming that has dominated the landscape for many years.

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Programs in this section

Basic definitions A *data type* is a set of values and a set of operations defined on those values. This statement is one of several mantras that we repeat often because of its importance. In CHAPTER 1, we discussed in detail Java’s *primitive* data types. For example, the values of the primitive data type `int` are integers between -2^{31} and $2^{31} - 1$; the operations defined for the `int` data type include those for basic arithmetic and comparisons, such as `+`, `*`, `%`, `<`, and `>`.

You also have been using a data type that is not primitive—the `String` data type. You know that values of the `String` data type are sequences of characters and that you can perform the operation of concatenating two `String` values to produce a `String` result. You will learn in this section that there are dozens of other operations available for processing strings, such as finding a string’s length, extracting individual characters from the string, and comparing two strings.

Every data type is defined by its set of values and the operations defined on them, but when we *use* the data type, we focus on the *operations*, not the values. When you write programs that use `int` or `double` values, you are not concerning yourself with *how* they are represented (we never did spell out the details), and the same holds true when you write programs that use reference types, such as `String`, `Color`, or `Picture`. In other words, *you do not need to know how a data type is implemented to be able to use it* (yet another mantra)

The `String` data type. As a running example, we will revisit Java’s `String` data type in the context of object-oriented programming. We do so for two reasons. First, you have been using the `String` data type since your first program, so it is a familiar example. Second, string processing is critical to many computational applications. Strings lie at the heart of our ability to compile and run Java programs and to perform many other core computations; they are the basis of the information-processing systems that are critical to most business systems; people use them every day when typing into email, blog, or chat applications or preparing documents for publication; and they have proved to be critical ingredients in scientific progress in several fields, particularly molecular biology.

We will write programs that declare, create, and manipulate values of type `String`. We begin by describing the `String` API, which documents the available operations. Then, we consider Java language mechanisms for *declaring variables*, *creating objects* to hold data-type values, and *invoking instance methods* to apply data-type operations. These mechanisms differ from the corresponding ones for primitive types, though you will notice many similarities.

API. The Java *class* provides a mechanism for defining data types. In a class, we specify the data-type values and implement the data-type operations. To fulfill our promise that *you do not need to know how a data type is implemented to be able to use it*, we specify the behavior of classes for clients by listing their instance methods in an *API (application programming interface)*, in the same manner as we have been doing for libraries of static methods. The purpose of an API is to provide the information that you need to write a client program that uses the data type.

The following table summarizes the instance methods from Java's *String* API that we use most often; the full API has more than 60 methods! Several of the methods use integers to refer to a character's index within a string; as with arrays, these indices start at 0.

`public class String (Java string data type)`

<code>String(String s)</code>	<i>create a string with the same value as s</i>
<code>String(char[] a)</code>	<i>create a string that represents the same sequence of characters as in a[]</i>
<code>int length()</code>	<i>number of characters</i>
<code>char charAt(int i)</code>	<i>the character at index i</i>
<code>String substring(int i, int j)</code>	<i>characters at indices i through (j-1)</i>
<code>boolean contains(String substring)</code>	<i>does this string contain substring?</i>
<code>boolean startsWith(String pre)</code>	<i>does this string start with pre?</i>
<code>boolean endsWith(String post)</code>	<i>does this string end with post?</i>
<code>int indexOf(String pattern)</code>	<i>index of first occurrence of pattern</i>
<code>int indexOf(String pattern, int i)</code>	<i>index of first occurrence of pattern after i</i>
<code>String concat(String t)</code>	<i>this string with t appended</i>
<code>int compareTo(String t)</code>	<i>string comparison</i>
<code>String toLowerCase()</code>	<i>this string, with lowercase letters</i>
<code>String toUpperCase()</code>	<i>this string, with uppercase letters</i>
<code>String replaceAll(String a, String b)</code>	<i>this string, with a replaced by b</i>
<code>String[] split(String delimiter)</code>	<i>strings between occurrences of delimiter</i>
<code>boolean equals(Object t)</code>	<i>is this string's value the same as t's?</i>
<code>int hashCode()</code>	<i>an integer hash code</i>

See the online documentation and booksite for many other available methods.

Excerpts from the API for Java's String data type

The first entry, with the same name as the class and no return type, defines a special method known as a *constructor*. The other entries define *instance methods* that can take arguments and return values in the same manner as the static methods that we have been using, but they are *not* static methods: they implement operations for the data type. For example, the instance method `length()` returns the number of characters in the string and `charAt()` returns the character at a specified index.

Declaring variables. You declare variables of a reference type in precisely the same way that you declare variables of a primitive type, using a declaration statement consisting of the data type name followed by a variable name. For example, the statement

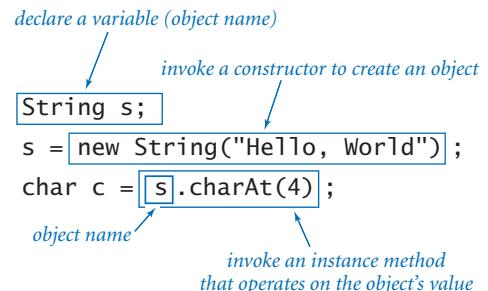
```
String s;
```

declares a variable `s` of type `String`. This statement does not *create* anything; it just says that we will use the variable name `s` to refer to a `String` object. By convention, reference types begin with uppercase letters and primitive types begin with lowercase letters.

Creating objects. In Java, each data-type value is stored in an *object*. When a client invokes a constructor, the Java system creates (or *instantiates*) an individual object (or *instance*). To invoke a constructor, use the keyword `new`; followed by the class name; followed by the constructor's arguments, enclosed in parentheses and separated by commas, in the same manner as a static method call. For example, `new String("Hello, World")` creates a new `String` object corresponding to the sequence of characters `Hello, World`. Typically, client code invokes a constructor to create an object and assigns it to a variable in the same line of code as the declaration:

```
String s = new String("Hello, World");
```

You can create any number of objects from the same class; each object has its own identity and may or may not store the same value as another object of the same type. For example, the code



```
String s1 = new String("Cat");
String s2 = new String("Dog");
String s3 = new String("Cat");
```

creates three different `String` objects. In particular, `s1` and `s3` refer to different objects, even though the two objects represent the same sequence of characters.

Invoking instance methods. The most important difference between a variable of a reference type and a variable of a primitive type is that you can use reference-type variables to invoke the methods that implement data-type operations (in contrast to the built-in syntax involving operators such as `+` and `*` that we used with primitive types).

Such methods are known as *instance methods*. Invoking (or *calling*) an instance method is similar to calling a static method in another class, except that an instance method is associated not just with a class, but also with an individual object. Accordingly, we typically use an *object* name (variable of the given type) instead of the *class* name to identify the method.

For example, if `s1` and `s2` are variables of type `String` as defined earlier, then `s1.length()` returns the integer 3, `s1.charAt(1)` returns the character 'a', and `s1.concat(s2)` returns a new string `CatDog`.

String shortcuts. As you already know, Java provides special language support for the `String` data type. You can create a `String` object using a string literal instead of an explicit constructor call. Also, you can concatenate two strings using the string concatenation operator (`+`) instead of making an explicit call to the `concat()` method. We introduced the longhand version here solely to demonstrate the syntax you need for other data types; these two shortcuts are unique to the `String` data type.

<i>shorthand</i> <code>String s = "abc";</code> <i>longhand</i> <code>String s = new String("abc");</code>	<code>String t = r + s;</code> <code>String t = r.concat(s);</code>
---	--

```
String a = new String("now is");
String b = new String("the time");
String c = new String(" the");
```

<i>instance method call</i>	<i>return type</i>	<i>return value</i>
<code>a.length()</code>	<code>int</code>	6
<code>a.charAt(4)</code>	<code>char</code>	'i'
<code>a.substring(2, 5)</code>	<code>String</code>	"w i"
<code>b.startsWith("the")</code>	<code>boolean</code>	true
<code>a.indexOf("is")</code>	<code>int</code>	4
<code>a.concat(c)</code>	<code>String</code>	"now is the"
<code>b.replace("t", "T")</code>	<code>String</code>	"The Time"
<code>a.split(" ")</code>	<code>String[]</code>	{ "now", "is" }
<code>b.equals(c)</code>	<code>boolean</code>	false

Examples of String data-type operations

The following code fragments illustrate the use of various string-processing methods. This code clearly exhibits the idea of developing an abstract model and separating the code that implements the abstraction from the code that uses it. This ability characterizes object-oriented programming and is a turning point in this book: we have not yet seen any code of this nature, but virtually all of the code that we write from this point forward will be based on defining and invoking methods that implement data-type operations.

<i>extract file name and extension from a command-line argument</i>	<pre>String s = args[0]; int dot = s.indexOf("."); String base = s.substring(0, dot); String extension = s.substring(dot + 1, s.length());</pre>
<i>print all lines on standard input that contain a string specified as a command-line argument</i>	<pre>String query = args[0]; while (StdIn.hasNextLine()) { String line = StdIn.readLine(); if (line.contains(query)) StdOut.println(line); }</pre>
<i>is the string a palindrome?</i>	<pre>public static boolean isPalindrome(String s) { int n = s.length(); for (int i = 0; i < n/2; i++) if (s.charAt(i) != s.charAt(n-1-i)) return false; return true; }</pre>
<i>translate from DNA to mRNA (replace 'T' with 'U')</i>	<pre>public static String translate(String dna) { dna = dna.toUpperCase(); String rna = dna.replaceAll("T", "U"); return rna; }</pre>

Typical string-processing code

String-processing application: genomics To give you more experience with string processing, we will give a very brief overview of the field of *genomics* and consider a program that a bioinformatician might use to identify potential genes. Biologists use a simple model to represent the building blocks of life, in which the letters A, C, G, and T represent the four bases in the DNA of living organisms. In each living organism, these basic building blocks appear in a set of long sequences (one for each chromosome) known as a *genome*. Understanding properties of genomes is a key to understanding the processes that manifest themselves in living organisms. The genomic sequences for many living things are known, including the human genome, which is a sequence of about 3 billion bases. Since the sequences have been identified, scientists have begun composing computer programs to study their structure. String processing is now one of the most important methodologies—experimental or computational—in molecular biology.

Gene prediction. A *gene* is a substring of a genome that represents a functional unit of critical importance in understanding life processes. A gene consists of a sequence of *codons*, each of which is a sequence of three bases that represents one amino acid. The *start codon* ATG marks the beginning of a gene, and any of the *stop codons* TAG, TAA, or TGA marks the end of a gene (and no other occurrences of any of these stop codons can appear within the gene). One of the first steps in analyzing a genome is to identify its potential genes, which is a string-processing problem that Java’s `String` data type equips us to solve.

PotentialGene (PROGRAM 3.1.1) is a program that serves as a first step. The `isPotentialGene()` function takes a DNA string as an argument and determines whether it corresponds to a potential gene based on the following criteria: length is a multiple of 3, starts with the start codon, ends with a stop codon, and has no intervening stop codons. To make the determination, the program uses a variety of string instance methods: `length()`, `charAt()`, `startsWith()`, `endsWith()`, `substring()`, and `equals()`.

Although the rules that define genes are a bit more complicated than those we have sketched here, PotentialGene exemplifies how a basic knowledge of programming can enable a scientist to study genomic sequences more effectively.

IN THE PRESENT CONTEXT, OUR INTEREST in the `String` data type is that it illustrates what a data type can be—a well-developed encapsulation of an important abstraction that is useful to clients. Before proceeding to other examples, we consider a few basic properties of reference types and objects in Java.

Program 3.1.1 Identifying a potential gene

```

public class PotentialGene
{
    public static boolean isPotentialGene(String dna)
    {
        // Length is a multiple of 3.
        if (dna.length() % 3 != 0) return false;

        // Starts with start codon.
        if (!dna.startsWith("ATG")) return false;

        // No intervening stop codons.
        for (int i = 3; i < dna.length() - 3; i++)
        {
            if (i % 3 == 0)
            {
                String codon = dna.substring(i, i+3);
                dna | string to analyze
                codon | 3 consecutive bases
                if (codon.equals("TAA")) return false;
                if (codon.equals("TAG")) return false;
                if (codon.equals("TGA")) return false;
            }
        }

        // Ends with a stop codon.
        if (dna.endsWith("TAA")) return true;
        if (dna.endsWith("TAG")) return true;
        if (dna.endsWith("TGA")) return true;

        return false;
    }
}

```

The `isPotentialGene()` function takes a DNA string as an argument and determines whether it corresponds to a potential gene: length is a multiple of 3, starts with the start codon (ATG), ends with a stop codon (TAA or TAG or TGA), and has no intervening stop codons. See EXERCISE 3.1.19 for the test client.

```

% java PotentialGene ATGCGCCTGCGTCTGTACTAG
true

% java PotentialGene ATGCGCTGCCCTGTACTAG
false

```

Object references. A constructor creates an object and returns to the client a *reference* to that object, not the object itself (hence the name *reference type*). What is an object reference? Nothing more than a mechanism for accessing an object. There are several different ways for Java to implement references, but we do not need to know the details to use them. Still, it is worthwhile to have a mental model of one common implementation. One approach is for new to assign memory space to hold the object's current data-type value and return a *pointer* (memory address) to that space. We refer to the memory address associated with the object as the object's *identity*.

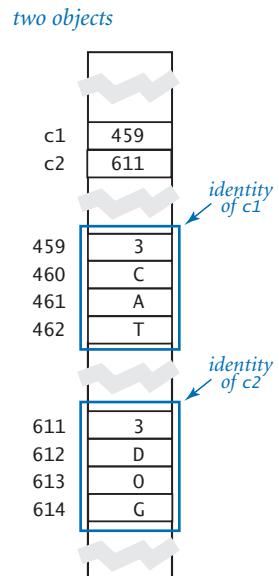
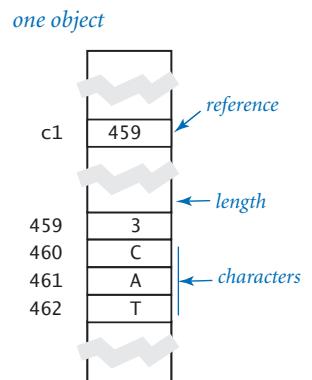
Why not just process the object itself? For small objects, it might make sense to do so, but for large objects, cost becomes an issue: data-type values can consume large amounts of memory. It does not make sense to copy or move all of its data every time that we pass an object as an argument to a method. If this reasoning seems familiar to you, it is because we have used precisely the same reasoning before, when talking about passing arrays as arguments to static methods in SECTION 2.1. Indeed, arrays *are* objects, as we will see later in this section. By contrast, primitive types have values that are natural to represent directly in memory, so that it does not make sense to use a reference to access each value.

We will discuss properties of object references in more detail after you have seen several examples of client code that use reference types.

Using objects. A variable declaration gives us a variable name for an object that we can use in code in much the same way as we use a variable name for an `int` or `double`:

- As an argument or return value for a method
- In an assignment statement
- In an array

We have been using `String` objects in this way ever since `HelloWorld`: most of our programs call `StdOut.println()` with a `String` argument, and all of our programs have a `main()` method that takes an argument that is a `String`



Object representation

array. As we have already seen, there is one critically important addition to this list for variables that refer to objects:

- To invoke an instance method defined on it

This usage is not available for variables of a primitive type, where operations are built into the language and invoked only via operators such as +, -, *, and /.

Uninitialized variables. When you declare a variable of a reference type but do not assign a value to it, the variable is *uninitialized*, which leads to the same behavior as for primitive types when you try to use the variable. For example, the code

```
String bad;
boolean value = bad.startsWith("Hello");
```

leads to the compile-time error variable `bad` might not have been initialized because it is trying to use an uninitialized variable.

Type conversion. If you want to convert an object from one type to another, you have to write code to do it. Often, there is no issue, because values for different data types are so different that no conversion is contemplated. For instance, what would it mean to convert a `String` object to a `Color` object? But there is one important case where conversion is very often worthwhile: all Java reference types have a special instance method `toString()` that returns a `String` object. The nature of the conversion is completely up to the implementation, but usually the string encodes the object's value. Programmers typically call the `toString()` method to print traces when debugging code. Java automatically calls the `toString()` method in certain situations, including with string concatenation and `StdOut.println()`. For example, for any object reference `x`, Java automatically converts the expression `"x = " + x` to `"x = " + x.toString()` and the expression `StdOut.println(x)` to `StdOut.println(x.toString())`. We will examine the Java language mechanism that enables this feature in SECTION 3.3.

Accessing a reference data type. As with libraries of static methods, the code that implements each class resides in a file that has the same name as the class but carries a `.java` extension. To write a client program that uses a data type, you need to make the class available to Java. The `String` data type is part of the Java language, so it is always available. You can make a user-defined data type available either by placing a copy of the `.java` file in the same directory as the client or by using Java's classpath mechanism (described on the booksite). With this understood, you will next learn how to use a data type in your own client code.

Distinction between instance methods and static methods. Finally, you are ready to appreciate the meaning of the modifier `static` that we have been using since PROGRAM 1.1.1—one of the last mysterious details in the Java programs that you have been writing. The primary purpose of static methods is to implement functions; the primary purpose of instance (non-static) methods is to implement data-type operations. You can distinguish between the uses of the two types of methods in our client code, because a static method call typically starts with a *class name* (uppercase, by convention) and an instance method call typically starts with an *object name* (lowercase, by convention). These differences are summarized in the following table, but after you have written some client code yourself, you will be able to quickly recognize the difference.

	<i>instance method</i>	<i>static method</i>
<i>sample call</i>	<code>s.startsWith("Hello")</code>	<code>Math.sqrt(2.0)</code>
<i>invoked with</i>	object name (or object reference)	class name
<i>parameters</i>	reference to invoking object and argument(s)	argument(s)
<i>primary purpose</i>	manipulate object's value	compute return value

Instance methods versus static methods

THE BASIC CONCEPTS THAT WE HAVE just covered are the starting point for object-oriented programming, so it is worthwhile to briefly summarize them here. A *data type* is a set of values and a set of operations defined on those values. We implement data types in independent modules and write client programs that use them. An *object* is an *instance* of a data type. Objects are characterized by three essential properties: *state*, *behavior*, and *identity*. The *state* of an object is a value from its data type. The *behavior* of an object is defined by the data type's operations. The *identity* of an object is the location in memory where it is stored. In object-oriented programming, we invoke *constructors* to create *objects* and then modify their state by invoking their *instance methods*. In Java, we manipulate objects via *object references*.

To demonstrate the power of object orientation, we next consider several more examples. First, we consider the familiar world of image processing, where we process `Color` and `Picture` objects. Then, we revisit our input/output libraries in the context of object-oriented programming, enabling us to access information from files and the web.

Color. *Color* is a sensation in the eye from electromagnetic radiation. Since we want to view and manipulate color images on our computers, color is a widely used abstraction in computer graphics, and Java provides a *Color* data type. In professional publishing, in print, and on the web, working with color is a complex task. For example, the appearance of a color image depends in a significant way on the medium used to present it. The *Color* data type separates the creative designer's problem of specifying a desired color from the system's problem of faithfully reproducing it.

Java has hundreds of data types in its libraries, so we need to explicitly list which Java libraries we are using in our program to avoid naming conflicts. Specifically, we include the statement

```
import java.awt.Color;
```

at the beginning of any program that uses *Color*. (Until now, we have been using standard Java libraries or our own, so there has been no need to import them.)

To represent color values, *Color* uses the *RGB color model* where a color is defined by three integers (each between 0 and 255) that represent the intensity of the red, green, and blue (respectively) components of the color. Other color values are obtained by mixing the red, green, and blue components. That is, the data-type values of *Color* are three 8-bit integers. We do not need to know whether the implementation uses *int*, *short*, or *char* values to represent these integers. With this convention, Java is using 24 bits to represent each color and can represent $256^3 = 2^{24} \approx 16.7$ million possible colors. Scientists estimate that the human eye can distinguish only about 10 million distinct colors.

The *Color* data type has a constructor that takes three integer arguments. For example, you can write

```
Color red      = new Color(255, 0, 0);
Color bookBlue = new Color(9, 90, 166);
```

to create objects whose values represent pure red and the blue used to print this book, respectively. We have been using colors in *StdDraw* since SECTION 1.5, but have been limited to a set of predefined colors, such as *StdDraw.BLACK*, *StdDraw.RED*, and *StdDraw.PINK*. Now you have millions of colors available for your use. *AlbersSquares* (PROGRAM 3.1.2) is a *StdDraw* client that allows you to experiment with them.

Some color values

red	green	blue	
255	0	0	red
0	255	0	green
0	0	255	blue
0	0	0	black
100	100	100	dark gray
255	255	255	white
255	255	0	yellow
255	0	255	magenta
9	90	166	this color

Program 3.1.2 Albers squares

```

import java.awt.Color;
public class AlbersSquares
{
    public static void main(String[] args)
    {
        int r1 = Integer.parseInt(args[0]);
        int g1 = Integer.parseInt(args[1]);
        int b1 = Integer.parseInt(args[2]);
        Color c1 = new Color(r1, g1, b1);
        int r2 = Integer.parseInt(args[3]);
        int g2 = Integer.parseInt(args[4]);
        int b2 = Integer.parseInt(args[5]);
        Color c2 = new Color(r2, g2, b2);

        StdDraw.setPenColor(c1);
        StdDraw.filledSquare(.25, 0.5, 0.2);
        StdDraw.setPenColor(c2);
        StdDraw.filledSquare(.25, 0.5, 0.1);
        StdDraw.setPenColor(c2);
        StdDraw.filledSquare(.75, 0.5, 0.2);
        StdDraw.setPenColor(c1);
        StdDraw.filledSquare(.75, 0.5, 0.1);
    }
}

```

r1, g1, b1 *RGB values*
c1 *first color*
r2, g2, b2 *RGB values*
c2 *second color*

This program displays the two colors entered in RGB representation on the command line in the familiar format developed in the 1960s by the color theorist Josef Albers, which revolutionized the way that people think about color.

% java AlbersSquares 9 90 166 100 100 100



As usual, when we address a new abstraction, we are introducing you to `Color` by describing the essential elements of Java's color model, not all of the details. The API for `Color` contains several constructors and more than 20 methods; the ones that we will use are briefly summarized next.

```
public class java.awt.Color
    Color(int r, int g, int b)
    int getRed()           red intensity
    int getGreen()          green intensity
    int getBlue()           blue intensity
    Color brighter()        brighter version of this color
    Color darker()          darker version of this color
    String toString()       string representation of this color
    String equals(Object c) is this color's value the same as c ?
```

See the online documentation and booksite for other available methods.

Excerpts from the API for Java's `Color` data type

Our primary purpose is to use `Color` as an example to illustrate object-oriented programming, while at the same time developing a few useful tools that we can use to write programs that process colors. Accordingly, we choose one color property as an example to convince you that writing object-oriented code to process abstract concepts like color is a convenient and useful approach.

Luminance. The quality of the images on modern displays such as LCD monitors, plasma TVs, and cellphone screens depends on an understanding of a color property known as *monochrome luminance*, or effective brightness. A standard formula for luminance is derived from the eye's sensitivity to red, green, and blue. It is a linear combination of the three intensities: if a color's red, green, and blue values are r , g , and b , respectively, then its monochrome luminance Y is defined by this equation:

$$Y = 0.299r + 0.587g + 0.114b$$

Since the coefficients are positive and sum to 1, and the intensities are all integers between 0 and 255, the luminance is a real number between 0 and 255.

Grayscale. The RGB color model has the property that when all three color intensities are the same, the resulting color is on a grayscale that ranges from black (all 0s) to white (all 255s). To print a color photograph in a black-and-white newspaper (or a book), we need a function to convert from color to grayscale. A simple way to convert a color to grayscale is to replace the color with a new one whose red, green, and blue values equal its monochrome luminance.

Color compatibility. The monochrome luminance is also crucial in determining whether two colors are compatible, in the sense that printing text in one of the colors on a background in the other color will be readable. A widely used rule of thumb is that the difference between the luminance of the foreground and background colors should be at least 128. For example, black text on a white background has a luminance difference of 255, but black text on a (book) blue background has a luminance difference of only 74. This rule is important in the design of advertising, road signs, websites, and many other applications. Luminance (PROGRAM 3.1.3) is a library of static methods that we can use to convert a color to grayscale and to test whether two colors are compatible. The static methods in Luminance illustrate the utility of using data types to organize information. Using `Color` objects as arguments and return values substantially simplifies the implementation: the alternative of passing around three intensity values is cumbersome and returning multiple values is not possible without reference types.

HAVING AN ABSTRACTION FOR COLOR IS important not just for direct use, but also in building higher-level data types that have `Color` values. Next, we illustrate this point by building on the color abstraction to develop a data type that allows us to write programs to process digital images.

red	green	blue	
9	90	166	<i>this color</i>
74	74	74	grayscale version
0	0	0	black

$$0.299 * 9 + 0.587 * 90 + 0.114 * 166 = 74.445$$

Grayscale example

<i>luminance</i>		<i>difference</i>
0		232
74		158
232		74

Compatibility example

Program 3.1.3 Luminance library

```

import java.awt.Color;
public class Luminance
{
    public static double intensity(Color color)
    { // Monochrome luminance of color.
        int r = color.getRed();                                r, g, b | RGB values
        int g = color.getGreen();
        int b = color.getBlue();
        return 0.299*r + 0.587*g + 0.114*b;
    }

    public static Color toGray(Color color)
    { // Use luminance to convert to grayscale.
        int y = (int) Math.round(intensity(color));
        Color gray = new Color(y, y, y);                      y | luminance of color
        return gray;
    }

    public static boolean areCompatible(Color a, Color b)
    { // True if colors are compatible, false otherwise.
        return Math.abs(intensity(a) - intensity(b)) >= 128.0;
    }

    public static void main(String[] args)
    { // Are the two specified RGB colors compatible?
        int[] a = new int[6];
        for (int i = 0; i < 6; i++)
            a[i] = Integer.parseInt(args[i]);                  a[] | int values of args[]
        Color c1 = new Color(a[0], a[1], a[2]);              c1 | first color
        Color c2 = new Color(a[3], a[4], a[5]);              c2 | second color
        StdOut.println(areCompatible(c1, c2));
    }
}

```

This library comprises three important functions for manipulating color: monochrome luminance, conversion to grayscale, and background/foreground compatibility.

```

% java Luminance 232 232 232    0   0   0
true
% java Luminance   9  90 166  232 232 232
true
% java Luminance   9  90 166    0   0   0
false

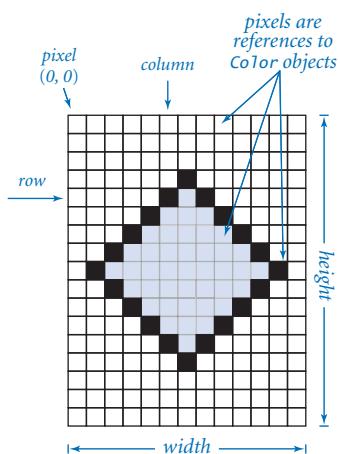
```

Digital image processing

You are familiar with the concept of a *photograph*. Technically, we might define a photograph as a two-dimensional image created by collecting and focusing visible wavelengths of electromagnetic radiation that constitutes a representation of a scene at a point in time. That technical definition is beyond our scope, except to note that the history of photography is a history of technological development. During the last century, photography was based on chemical processes, but its future is now based in computation. Your camera and your cellphone are computers with lenses and light-sensitive devices capable of capturing images in digital form, and your computer has photo-editing software that allows you to process those images. You can crop them, enlarge and reduce them, adjust the contrast, brighten or darken them, remove redeye, or perform scores of other operations. Many such operations are remarkably easy to implement, given a simple basic data type that captures the idea of a digital image, as you will now see.

Digital images. Which set of values do we need to process digital images, and which operations do we need to perform on those values? The basic abstraction for computer displays is the same one that is used for digital photographs and is very simple: a *digital image* is a rectangular grid of *pixels* (picture elements), where the color of each pixel is individually defined. Digital images are sometimes referred to as *raster* or *bitmapped* images. In contrast, the types of images that we produce with StdDraw (which involve geometric objects such as points, lines, circles, and squares) are referred to as *vector* images.

Our class `Picture` is a data type for digital images whose definition follows immediately from the digital image abstraction. The set of values is nothing more than a two-dimensional matrix of `Color` values, and the operations are what you might expect: create a blank image with a given width and height, load an image from a file, set the value of a pixel to a given color, return the color of a given pixel, return the width or the height, show the image in a window on your computer screen, and save the image to a file. In this description, we intentionally use the word *matrix* instead of *array* to emphasize that we are referring to an abstraction (a matrix of pixels), not a specific implementation (a Java two-dimensional array of `Color` objects). *You do not need to know how a data type is*



Anatomy of a digital image

implemented to be able to use it. Indeed, typical images have so many pixels that implementations are likely to use a more efficient representation than an array of `Color` objects. In any case, to write client programs that manipulate images, you just need to know this API:

<code>public class Picture</code>	
<code>Picture(String filename)</code>	<i>create a picture from a file</i>
<code>Picture(int w, int h)</code>	<i>create a blank w-by-h picture</i>
<code>int width()</code>	<i>return the width of the picture</i>
<code>int height()</code>	<i>return the height of the picture</i>
<code>Color get(int col, int row)</code>	<i>return the color of pixel (col, row)</i>
<code>void set(int col, int row, Color c)</code>	<i>set the color of pixel (col, row) to c</i>
<code>void show()</code>	<i>display the picture in a window</i>
<code>void save(String filename)</code>	<i>save the picture to a file</i>

API for our data type for image processing

By convention, $(0, 0)$ is the upper-leftmost pixel, so the image is laid as in the customary order for two-dimensional arrays (by contrast, the convention for `StdDraw` is to have the point $(0,0)$ at the lower-left corner, so that drawings are oriented as in the customary manner for Cartesian coordinates). Most image-processing programs are filters that scan through all of the pixels in a source image and then perform some computation to determine the color of each pixel in a target image. The supported file formats for the first constructor and the `save()` method are the widely used PNG and JPEG formats, so that you can write programs to process your own digital photos and add the results to an album or a website. The `show()` window also has an interactive option for saving to a file. These methods, together with Java's `Color` data type, open the door to image processing.

Grayscale. You will find many examples of color images on the booksite, and all of the methods that we describe are effective for full-color images, but all our example images in this book will be grayscale. Accordingly, our first task is to write a program that converts images from color to grayscale. This task is a prototypical image-processing task: for each pixel in the source, we set a pixel in the target to a

Program 3.1.4 Converting color to grayscale

```

import java.awt.Color;
public class Grayscale
{
    public static void main(String[] args)
    { // Show image in grayscale.
        Picture picture = new Picture(args[0]);
        for (int col = 0; col < picture.width(); col++)
        {
            for (int row = 0; row < picture.height(); row++)
            {
                Color color = picture.get(col, row);
                Color gray = Luminance.toGray(color);
                picture.set(col, row, gray);
            }
        }
        picture.show();
    }
}

```

picture	<i>image from file</i>
col, row	<i>pixel coordinates</i>
color	<i>pixel color</i>
gray	<i>pixel grayscale</i>

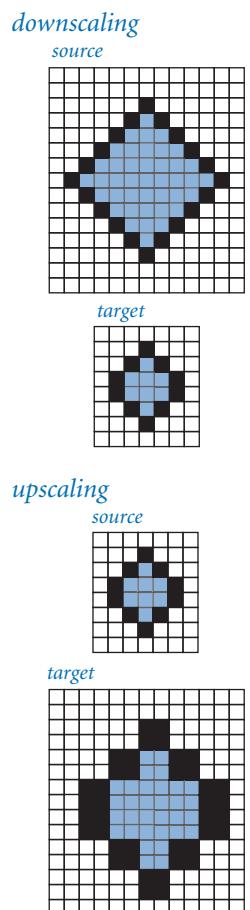
This program illustrates a simple image-processing client. First, it creates a *Picture* object initialized with an image file named by the command-line argument. Then it converts each pixel in the picture to grayscale by creating a grayscale version of each pixel's color and resetting the pixel to that color. Finally, it shows the picture. You can perceive individual pixels in the picture on the right, which was upscaled from a low-resolution picture (see “Scaling” on the next page).

different color. `Grayscale` (PROGRAM 3.1.4) is a filter that takes a file name from the command line and produces a grayscale version of that image. It creates a new `Picture` object initialized with the color image, then sets the color of each pixel to a new `Color` having a grayscale value computed by applying the `toGray()` method in `Luminance` (PROGRAM 3.1.3) to the color of the corresponding pixel in the source.

Scaling. One of the most common image-processing tasks is to make an image smaller or larger. Examples of this basic operation, known as *scaling*, include making small thumbnail photos for use in a chat room or a cellphone, changing the size of a high-resolution photo to make it fit into a specific space in a printed publication or on a web page, and zooming in on a satellite photograph or an image produced by a microscope. In optical systems, we can just move a lens to achieve a desired scale, but in digital imagery, we have to do more work.

In some cases, the strategy is clear. For example, if the target image is to be half the size (in each dimension) of the source image, we simply choose half the pixels, say, by deleting half the rows and half the columns. This technique is known as *sampling*. If the target image is to be double the size (in each dimension) of the source image, we can replace each source pixel by four target pixels of the same color. Note that we can lose information when we downscale, so halving an image and then doubling it generally does not give back the same image.

A single strategy is effective for both downscaling and upscaling. Our goal is to produce the target image, so we proceed through the pixels in the target, one by one, scaling each pixel's coordinates to identify a pixel in the source whose color can be assigned to the target. If the width and height of the source are w_s and h_s (respectively) and the width and height of the target are w_t and h_t (respectively), then we scale the column index by w_s/w_t and the row index by h_s/h_t . That is, we get the color of the pixel in column c and row r and of the target from column $c \times w_s/w_t$ and row $r \times h_s/h_t$ in the source. For example, if we are halving the size of an image, the scale factors are 2, so the pixel in column 3 and row 2 of the target gets the color of the pixel in column 6 and row 4 of the source; if we are doubling the size of the image, the scale factors are 1/2, so the pixel



Scaling a digital image

Program 3.1.5 Image scaling

```

public class Scale
{
    public static void main(String[] args)
    {
        int w = Integer.parseInt(args[1]);
        int h = Integer.parseInt(args[2]);
        Picture source = new Picture(args[0]);
        Picture target = new Picture(w, h);
        for (int colT = 0; colT < w; colT++)
        {
            for (int rowT = 0; rowT < h; rowT++)
            {
                int colS = colT * source.width() / w;
                int rowS = rowT * source.height() / h;
                target.set(colT, rowT, source.get(colS, rowS));
            }
        }
        source.show();
        target.show();
    }
}

```

w, h	<i>target dimensions</i>
source	<i>source image</i>
target	<i>target image</i>
colT, rowT	<i>target pixel coords</i>
colS, rowS	<i>source pixel coords</i>

This program takes the name of an image file and two integers (width *w* and height *h*) as command-line arguments, scales the picture to *w*-by-*h*, and displays both images.

in column 4 and row 6 of the target gets the color of the pixel in column 2 and row 3 of the source. `Scale` (PROGRAM 3.1.5) is an implementation of this strategy. More sophisticated strategies can be effective for low-resolution images of the sort that you might find on old web pages or from old cameras. For example, we might downscale to half size by averaging the values of four pixels in the source to make one pixel in the target. For the high-resolution images that are common in most applications today, the simple approach used in `Scale` is effective.

The same basic idea of computing the color value of each target pixel as a function of the color values of specific source pixels is effective for all sorts of image-processing tasks. Next, we consider one more example, and you will find numerous other examples in the exercises and on the booksite.

Fade effect. Our final image-processing example is an entertaining computation where we transform one image into another in a series of discrete steps. Such a transformation is sometimes known as a *fade effect*. `Fade` (PROGRAM 3.1.6) is a `Picture` and `Color` client that uses a *linear interpolation* strategy to implement this effect. It computes $n - 1$ intermediate pictures, with each pixel in picture i being a weighted average of the corresponding pixels in the source and target. The static method `blend()` implements the interpolation: the source color is weighted by a factor of $1 - i / n$ and the target color by a factor of i / n (when i is 0, we have the source color, and when i is n , we have the target color). This simple computation can produce striking results. When you run `Fade` on your computer, the change appears to happen dynamically. Try running it on some images from your photo library. Note that `Fade` assumes that the images have the same width and height; if you have images for which this is not the case, you can use `Scale` to create a scaled version of one or both of them for `Fade`.

Program 3.1.6 Fade effect

```

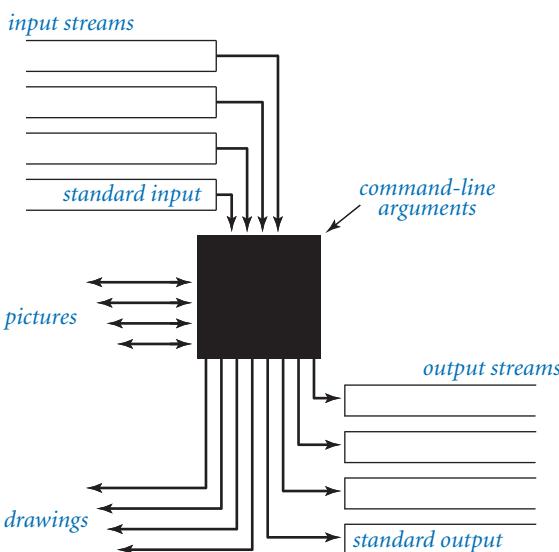
import java.awt.Color;
public class Fade
{
    public static Color blend(Color c1, Color c2, double alpha)
    { // Compute blend of colors c1 and c2, weighted by alpha.
        double r = (1-alpha)*c1.getRed() + alpha*c2.getRed();
        double g = (1-alpha)*c1.getGreen() + alpha*c2.getGreen();
        double b = (1-alpha)*c1.getBlue() + alpha*c2.getBlue();
        return new Color((int) r, (int) g, (int) b);
    }
    public static void main(String[] args)
    { // Show m-image fade sequence from source to target.
        Picture source = new Picture(args[0]);
        Picture target = new Picture(args[1]);
        int n = Integer.parseInt(args[2]);
        int width = source.width();
        int height = source.height();
        Picture picture = new Picture(width, height);
        for (int i = 0; i <= n; i++)
        {
            for (int col = 0; col < width; col++)
            {
                for (int row = 0; row < height; row++)
                {
                    Color c1 = source.get(col, row);
                    Color c2 = target.get(col, row);
                    double alpha = (double) i / n;
                    Color color = blend(c1, c2, alpha);
                    picture.set(col, row, color);
                }
            }
            picture.show();
        }
    }
}

```

n i $c1$ $c2$ $color$	<i>number of pictures</i> <i>current picture</i> <i>picture counter</i> <i>source color</i> <i>target color</i> <i>blended color</i>
---------------------------------------	---

To fade from one picture into another in n steps, we set each pixel in picture i to a weighted average of the corresponding pixel in the source and destination pictures, with the source getting weight $1 - i / n$ and the destination getting weight i / n . An example transformation is shown on the facing page.

Input and output revisited In SECTION 1.5 you learned how to read and write numbers and text using `StdIn` and `StdOut` and to make drawings with `StdDraw`. You have certainly come to appreciate the utility of these mechanism in getting information into and out of your programs. One reason that they are convenient is that the “standard” conventions make them accessible from anywhere within a program. One disadvantage of these conventions is that they leave us dependent upon the operating system’s piping and redirection mechanism for access to files, and they restrict us to working with just one input file, one output file, and one drawing for any given program. With object-oriented programming, we can define mechanisms that are similar to those in `StdIn`, `StdOut`, and `StdDraw` but allow us to work with *multiple* input streams, output streams, and drawings within one program.



A bird’s-eye view of a Java program (revisited again)

Specifically, we define in this section the data types `In`, `Out`, and `Draw` for input streams, output streams, and drawings, respectively. As usual, you must make these classes accessible to Java (see the Q&A at the end of SECTION 1.5).

These data types give us the flexibility that we need to address many common data-processing tasks within our Java programs. Rather than being restricted to just one input stream, one output stream, and one drawing, we can easily define multiple objects of each type, connecting the streams to various sources and destinations. We also get the flexibility to assign such objects to variables, pass them as arguments or return values from

methods, and create arrays of them, manipulating them just as we manipulate objects of any type. We will consider several examples of their use after we have presented the APIs.

Input stream data type. Our In data type is a more general version of StdIn that supports reading numbers and text from files and websites as well as the standard input stream. It implements the *input stream* data type, with the API at the bottom of this page. Instead of being restricted to one abstract input stream (standard input), this data type gives you the ability to directly specify the source of an input stream. Moreover, that source can be either a *file* or a *website*. When you call the constructor with a string argument, the constructor first tries to find a file in the current directory of your local computer with that name. If it cannot do so, it assumes the argument is a website name and tries to connect to that website. (If no such website exists, it generates a run-time exception.) In either case, the specified file or website becomes the source of the input for the input stream object thus created, and the `read*()` methods will read input from that stream.

```
public class In
```

<code>In()</code>	<i>create an input stream from standard input</i>
-------------------	---

<code>In(String name)</code>	<i>create an input stream from a file or website</i>
------------------------------	--

instance methods that read individual tokens from the input stream

<code>boolean isEmpty()</code>	<i>is standard input empty (or only whitespace)?</i>
--------------------------------	--

<code>int readInt()</code>	<i>read a token, convert it to an int, and return it</i>
----------------------------	--

<code>double readDouble()</code>	<i>read a token, convert it to a double, and return it</i>
----------------------------------	--

...

instance methods that read characters from the input stream

<code>boolean hasNextChar()</code>	<i>does standard input have any remaining characters?</i>
------------------------------------	---

<code>char readChar()</code>	<i>read a character from standard input and return it</i>
------------------------------	---

instance methods that read lines from the input stream

<code>boolean hasNextLine()</code>	<i>does standard input have a next line?</i>
------------------------------------	--

<code>String readLine()</code>	<i>read the rest of the line and return it as a String</i>
--------------------------------	--

instance methods that read the rest of the input stream

<code>int[] readAllInts()</code>	<i>read all remaining tokens; return as array of integers</i>
----------------------------------	---

<code>double[] readAllDoubles()</code>	<i>read all remaining tokens; return as array of doubles</i>
--	--

...

Note: All operations supported by StdIn are also supported for In objects.

API for our data type for input streams

This arrangement makes it possible to process multiple files within the same program. Moreover, the ability to directly access the web opens up the whole web as potential input for your programs. For example, it allows you to process data that is provided and maintained by someone else. You can find such files all over the web. Scientists now regularly post data files with measurements or results of experiments, ranging from genome and protein sequences to satellite photographs to astronomical observations; financial services companies, such as stock exchanges, regularly publish on the web detailed information about the performance of stock and other financial instruments; governments publish election results; and so forth. Now you can write Java programs that read these kinds of files directly. The In data type gives you a great deal of flexibility to take advantage of the multitude of data sources that are now available.

Output stream data type. Similarly, our Out data type is a more general version of StdOut that supports printing text to a variety of output streams, including standard output and files. Again, the API specifies the same methods as its StdOut counterpart. You specify the file that you want to use for output by using the one-argument constructor with the file's name as the argument. Out interprets this string as the name of a new file on your local computer, and sends its output there. If you use the no-argument constructor, then you obtain the standard output stream.

<code>public class Out</code>	
<code> Out()</code>	<i>create an output stream to standard output</i>
<code> Out(String name)</code>	<i>create an output stream to a file</i>
<code> void print(String s)</code>	<i>print s to the output stream</i>
<code> void println(String s)</code>	<i>print s and a newline to the output stream</i>
<code> void println()</code>	<i>print a newline to the output stream</i>
<code> void printf(String format, ...)</code>	<i>print the arguments to the output stream, as specified by the format string format</i>

API for our data type for output streams

Program 3.1.7 Concatenating files

```
public class Cat
{
    public static void main(String[] args)
    {
        Out out = new Out(args[args.length-1]);
        for (int i = 0; i < args.length - 1; i++)
        {
            In in = new In(args[i]);
            String s = in.readAll();
            out.println(s);
        }
    }
}
```

out	<i>output stream</i>
i	<i>argument index</i>
in	<i>current input stream</i>
s	<i>contents of in</i>

This program creates an output file whose name is given by the last command-line argument and whose contents are the concatenation of the input files whose names are given as the other command-line arguments.

% more in1.txt This is	% java Cat in1.txt in2.txt out.txt % more out.txt This is
% more in2.txt a tiny test.	a tiny test.

File concatenation and filtering. PROGRAM 3.1.7 is a sample client of In and Out that uses multiple input streams to concatenate several input files into a single output file. Some operating systems have a command known as cat that implements this function. However, a Java program that does the same thing is perhaps more useful, because we can tailor it to *filter* the input files in various ways: we might wish to ignore irrelevant information, change the format, or select only some of the data, to name just a few examples. We now consider one example of such processing, and you will find several others in the exercises.

Screen scraping. The combination of the `In` data type (which allows us to create an input stream from any page on the web) and the `String` data type (which provides powerful tools for processing text strings) opens up the entire web to direct access by our Java programs, without any direct dependence on the operating system or browser. One paradigm is known as *screen scraping*: the goal is to extract some information from a web page with a program, rather than having to browse to find it. To do so, we take advantage of the fact that many web pages are defined with text files in a highly structured format (because they are created by computer programs!). Your browser has a mechanism that allows you to examine the source code that produces the web page that you are viewing, and by examining that source you can often figure out what to do.

Suppose that we want to take a stock trading symbol as a command-line argument and print that stock's current trading price. Such information is published on the web by financial service companies and Internet service providers. For example, you can find the stock price of a company whose symbol is `goog` by browsing to `http://finance.yahoo.com/q?s=goog`. Like many web pages, the name encodes an argument (`goog`), and we could substitute any other ticker symbol to get a web page with financial information for any other company. Also, like many other files on the web, the referenced file is a text file, written in a formatting language known as HTML. From the point of view of a Java program, it is just a `String` value accessible through an `In` object. You can use your browser to download the source of that file, or you could use

```
% java Cat "http://finance.yahoo.com/q?s=goog" goog.html
```

to put the source into a file `goog.html` on your local computer (though there is no real need to do so). Now, suppose that `goog` is trading at \$1,100.62 at the moment. If you search for the string "`1,100.62`" in the source of that page, you will find the stock price buried within some HTML code. Without having to know details of HTML, you can figure out something about the context in which the price appears. In this case, you can see that the stock price is enclosed between the substrings `` and ``.

```
...
(GOOG)</h2> <span class="rtq_exch"><span class="rtq_dash">-</span>
NMS </span><span class="wl_sign">
</span></div></div>
<div class="yfi_rt_quote_summary_rt_top_sigfig_promo_1"><div>
<span class="time_rtq_ticker">
<span id="yfs_184goog">1,100.62</span>
</span> <span class="down_r time_rtq_content"><span id="yfs_c63_goog">
...

```

HTML code from the web

With the `String` data type's `indexOf()` and `substring()` methods, you easily can grab this information, as illustrated in `StockQuote` (PROGRAM 3.1.8). This program depends on the web page format used by `http://finance.yahoo.com`; if this format changes, `StockQuote` will not work. Indeed, by the time you read this page, the format may have changed. Even so, making appropriate changes is not likely to be difficult. You can entertain yourself by embellishing `StockQuote` in all kinds of interesting ways. For example, you could grab the stock price on a periodic basis and plot it, compute a moving average, or save the results to a file for later analysis. Of course, the same technique works for sources of data found all over the web, as you can see in examples in the exercises at the end of this section and on the booksite.

Extracting data. The ability to maintain multiple input and output streams gives us a great deal of flexibility in meeting the challenges of processing large amounts of data coming from a variety of sources. We consider one more example: Suppose that a scientist or a financial analyst has a large amount of data within a spreadsheet program. Typically such spreadsheets are tables with a relatively large number of rows and a relatively small number of columns. You are not likely to be interested in all the data in the spreadsheet, but you may be interested in a few of the columns. You can do some calculations within the spreadsheet program (this is its purpose, after all), but you certainly do not have the flexibility that you have with Java programming. One way to address this situation is to have the spreadsheet *export* the data to a text file, using some special character to delimit the columns, and then write a Java program that reads that file from an input stream. One standard practice is to use commas as delimiters: print one line per row, with commas separating column entries. Such files are known as *comma-separated-value* or `.csv` files. With the `split()` method in Java's `String` data type, we can read the file line-by-line and isolate the data that we want. We will see several examples of this approach later in the book. `Split` (PROGRAM 3.1.9) is an `In` and `Out` client that goes one step further: it creates multiple output streams and makes one file for each column.

THESE EXAMPLES ARE CONVINCING ILLUSTRATIONS OF the utility of working with text files, with multiple input and output streams, and with direct access to web pages. Web pages are written in HTML precisely so that they are accessible to any program that can read strings. People use text formats such as `.csv` files rather than data formats that are beholden to particular applications precisely to allow as many people as possible to access the data with simple programs like `Split`.

Program 3.1.8 Screen scraping for stock quotes

```

public class StockQuote
{
    private static String readHTML(String symbol)
    { // Return HTML corresponding to stock symbol.
        In page = new In("http://finance.yahoo.com/q?s=" + symbol);
        return page.readAll();
    }

    public static double priceOf(String symbol)
    { // Return current stock price for symbol.
        String html = readHTML(symbol);
        int p      = html.indexOf("yfs_184", 0);
        int from   = html.indexOf(">", p);
        int to     = html.indexOf("</span>", from);
        String price = html.substring(from + 1, to);
        return Double.parseDouble(price.replaceAll(",",""));
    }

    public static void main(String[] args)
    { // Print price of stock specified by symbol.
        String symbol = args[0];
        double price = priceOf(symbol);
        StdOut.println(price);
    }
}

```

symbol *stock symbol*
page *input stream*
html *contents of page*
p *yfs_184 index*
from *> index*
to * index*
price *current price*

This program accepts a stock ticker symbol as a command-line argument and prints to standard output the current stock price for that stock, as reported by the website <http://finance.yahoo.com>. It uses the `indexOf()`, `substring()`, and `replaceAll()` methods from `String`.

```

% java StockQuote goog
1100.62
% java StockQuote adbe
70.51

```

Program 3.1.9 Splitting a file

```

public class Split
{
    public static void main(String[] args)
    { // Split file by column into n files.
        String name = args[0];
        int n = Integer.parseInt(args[1]);
        String delimiter = ",";
        // Create output streams.
        Out[] out = new Out[n];
        for (int i = 0; i < n; i++)
            out[i] = new Out(name + i + ".txt");
        In in = new In(name + ".csv");
        while (in.hasNextLine())
        { // Read a line and write fields to output streams.
            String line = in.readLine();
            String[] fields = line.split(delimiter);
            for (int i = 0; i < n; i++)
                out[i].println(fields[i]);
        }
    }
}

```

name n delimiter in out[] line fields[]	base file name number of fields delimiter (comma) input stream output streams current line values in current line
---	---

This program uses multiple output streams to split a .csv file into separate files, one for each comma-delimited field. The name of the output file corresponding to the *i*th field is formed by concatenating *i* and then .txt to the end of the original file name.

```
% more DJIA.csv
...
31-Oct-29,264.97,7150000,273.51
30-Oct-29,230.98,10730000,258.47
29-Oct-29,252.38,16410000,230.07
28-Oct-29,295.18,9210000,260.64
25-Oct-29,299.47,5920000,301.22
24-Oct-29,305.85,12900000,299.47
23-Oct-29,326.51,6370000,305.85
22-Oct-29,322.03,4130000,326.51
21-Oct-29,323.87,6090000,320.91
...
```

```
% java Split DJIA 4
% more DJIA2.txt
...
7150000
10730000
16410000
9210000
5920000
12900000
6370000
4130000
6090000
...
```

Drawing data type. When using the Picture data type that we considered earlier in this section, we could write programs that manipulated multiple pictures, arrays of pictures, and so forth, precisely because the data type provides us with the capability for computing with Picture objects. Naturally, we would like the same capability for computing with the kinds of geometric objects that we create with StdDraw. Accordingly, we have a Draw data type with the following API:

```
public class Draw
    Draw()
    drawing commands
        void line(double x0, double y0, double x1, double y1)
        void point(double x, double y)
        void circle(double x, double y, double radius)
        void filledCircle(double x, double y, double radius)
        ...
    control commands
        void setXscale(double x0, double x1)
        void setYscale(double y0, double y1)
        void setPenRadius(double radius)
        ...
    
```

Note: All operations supported by StdDraw are also supported for Draw objects.

As for any data type, you can create a new drawing by using new to create a Draw object, assign it to a variable, and use that variable name to call the methods that create the graphics. For example, the code

```
Draw draw = new Draw();
draw.circle(0.5, 0.5, 0.2);
```

draws a circle in the center of a window on your screen. As with Picture, each drawing has its own window, so that you can address applications that call for displaying multiple different drawings at the same time.

Properties of reference types Now that you have seen several examples of reference types (Charge, Color, Picture, String, In, Out, and Draw) and client programs that use them, we discuss in more detail some of their essential properties. To a large extent, Java protects novice programmers from having to know these details. Experienced programmers, however, know that a firm understanding of these properties is helpful in writing correct, effective, and efficient object-oriented programs.

A reference captures the distinction between a thing and its name. This distinction is a familiar one, as illustrated in these examples:

<i>type</i>	<i>typical object</i>	<i>typical name</i>
website	our booksite	<code>http://introcs.cs.princeton.edu</code>
person	father of computer science	Alan Turing
planet	third rock from the sun	Earth
building	our office	35 Olden Street
ship	superliner that sank in 1912	<i>RMS Titanic</i>
number	circumference/diameter of a circle	π
Picture	<code>new Picture("mandrill.jpg")</code>	picture

A given object may have multiple names, but each object has its own identity. We can create a new name for an object without changing the object's value (via an assignment statement), but when we change an object's value (by invoking an instance method), all of the object's names refer to the changed object.

The following analogy may help you keep this crucial distinction clear in your mind. Suppose that you want to have your house painted, so you write the street address of your house in pencil on a piece of paper and give it to a few house painters. Now, if you hire one of the painters to paint the house, it becomes a different color. No changes have been made to any of the pieces of paper, but the house that they all refer to has changed. One of the painters might erase what you've written and write the address of another house, but changing what is written on one piece of paper does not change what is written on another piece of paper. Java references are like the pieces of paper: they hold names of objects. Changing a reference does not change the object, but changing an object makes the change apparent to everyone having a reference to it.

The famous Belgian artist René Magritte captured this same concept in a painting where he created an image of a pipe along with the caption *ceci n'est pas une pipe* (*this is not a pipe*) below it. We might interpret the caption as saying that the image is not actually a pipe, just an image of a pipe. Or perhaps Magritte meant that the caption is neither a pipe nor an image of a pipe, just a caption! In the present context, this image reinforces the idea that a reference to an object is nothing more than a reference; it is not the object itself.

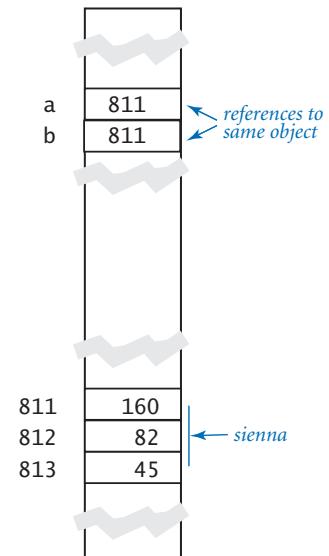
Aliasing. An assignment statement with a reference type creates a second copy of the reference. The assignment statement *does not create a new object*, just another reference to an existing object. This situation is known as *aliasing*: both variables refer to the same object. Aliasing also arises when passing an object reference to a method: The parameter variable becomes another reference to the corresponding object. The effect of aliasing is a bit unexpected, because it is different from that for variables holding values of a primitive type. *Be sure that you understand the difference.* If *x* and *y* are variables of a primitive type, then the assignment statement *x = y* copies the value of *y* to *x*. For reference types, the reference is copied (not the value).

Aliasing is a common source of bugs in Java programs, as illustrated by the following example:

```
Picture a = new Picture("mandrill.jpg");
Picture b = a;
a.set(col, row, color1); // a updated
b.set(col, row, color2); // a updated again
```

After the second assignment statement, variables *a* and *b* both refer to the same *Picture* object. Changing the state of an object impacts *all* code involving aliased variables referencing that object. We are used to thinking of two different variables of primitive types as being independent, but that intuition does not carry over to reference objects. For example, if the preceding code assumes that *a* and *b* refer to different *Picture* objects, then it will produce the wrong result. Such *aliasing bugs* are common in programs written by people without much experience in using reference objects (that's you, so pay attention here!).

```
Color a;
a = new Color(160, 82, 45);
Color b = a;
```



Aliasing

Immutable types. For this very reason, it is common to define data types whose values cannot change. An object from a data type is *immutable* if its data-type value cannot change once created. An *immutable data type* is one in which all objects of that type are immutable. For example, `String` is an immutable data type because there are no operations available to clients that change a string's characters. In contrast, a *mutable data type* is one in which objects of that type have values that are designed to change. For example, `Picture` is mutable data type because we can change pixel colors. We will consider immutability in more detail in SECTION 3.3.

Comparing objects. When applied to reference types, the `==` operator checks whether the two *object references* are equal (that is, whether they point to the same object). That is not the same as checking whether the *objects* have the same value. For example, consider the following code:

```
Color a = new Color(160, 82, 45);
Color b = new Color(160, 82, 45);
Color c = b;
```

Now `(a == b)` is `false` and `(b == c)` is `true`, but when you are thinking about equality testing for `Color`, you probably are thinking that you want to test whether their *values* are the same—you might want all three of these to test as equal. Java does not have an automatic mechanism for testing the equality of object values, which leaves programmers with the opportunity (and responsibility) to define it for themselves by defining for any class a customized method named `equals()`, as described in SECTION 3.3. For example, `Color` has such a method, and `a.equals(c)` is `true` in our example. `String` also contains an implementation of `equals()` because we often want to test whether two `String` objects have the same value (the same sequence of characters).

Pass by value. When you call a method with arguments, the effect in Java is as if each argument were to appear on the right-hand side of an assignment statement with the corresponding argument name on the left-hand side. That is, Java passes a *copy* of the argument value from the caller to the method. If the argument value is a primitive type, Java passes a copy of that value; if the argument value is an object reference, Java passes a copy of the object reference. This arrangement is known as *pass by value*.

One important consequence of this arrangement is that a method cannot directly change the value of a caller's variable. For primitive types, this policy is what

we expect (the two variables are independent), but each time that we use a reference type as a method argument, we create an alias, so we must be cautious. For example, if we pass an object reference of type `Picture` to a method, the method cannot change the caller's object reference (for example, make it refer to a different `Picture`), but it *can* change the value of the object, such as by invoking the `set()` method to change a pixel's color.

Arrays are objects. In Java, every value of any nonprimitive type is an object. In particular, arrays are objects. As with strings, special language support is provided for certain operations on arrays: declarations, initialization, and indexing. As with any other object, when we pass an array to a method or use an array variable on the right-hand side of an assignment statement, we are making a copy of the array reference, not a copy of the array. Arrays are mutable objects—a convention that is appropriate for the typical case where we expect the method to be able to modify the array by rearranging the values of its elements, as in, for example, the `exchange()` and `shuffle()` methods that we considered in SECTION 2.1.

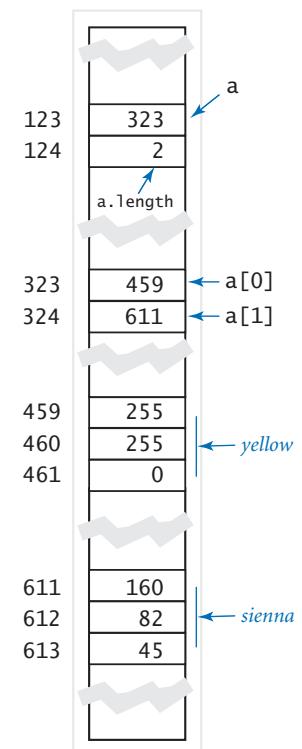
Arrays of objects. Array elements can be of any type, as we have already seen on several occasions, from `args[]` (an array of strings) in our `main()` implementations, to the array of `Out` objects in PROGRAM 3.1.9. When we create an array of objects, we do so in two steps:

- Create the array by using `new` and the square bracket syntax for array creation
- Create each object in the array, by using `new` to call a constructor

For example, we would use the following code to create an array of two `Color` objects:

```
Color[] a = new Color[2];
a[0] = new Color(255, 255, 0);
a[1] = new Color(160, 82, 45);
```

Naturally, an array of objects in Java is an array of *object references*, not the objects themselves. If the objects are large, then we gain efficiency by not having to move them around, just their references. If they are small, we lose efficiency by having to follow a reference each time we need to get to some information.

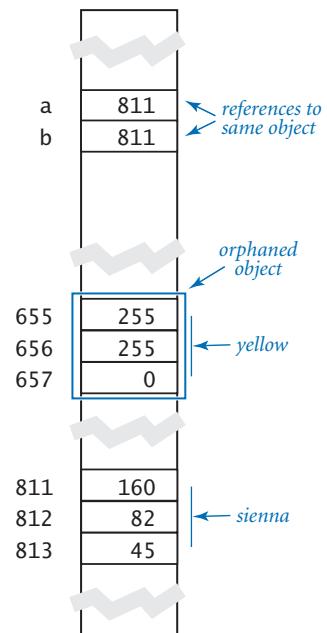


An array of objects

Safe pointers. To provide the capability to manipulate memory addresses that refer to data, many programming languages include the *pointer* (which is like the Java reference) as a primitive data type. Programming with pointers is notoriously error prone, so operations provided for pointers need to be carefully designed to help programmers avoid errors. Java takes this point of view to an extreme (one that is favored by many modern programming-language designers). In Java, there is only *one* way to create a reference (with new) and only *one* way to manipulate that reference (with an assignment statement). That is, the only things that a programmer can do with references is to create them and copy them. In programming-language jargon, Java references are known as *safe pointers*, because Java can guarantee that each reference points to an object of the specified type (and not to an arbitrary memory address). Programmers used to writing code that directly manipulates pointers think of Java as having no pointers at all, but people still debate whether it is desirable to have unsafe pointers. In short, when you program in Java, you will not be directly manipulating memory addresses, but if you find yourself doing so in some other language in the future, be careful!

Orphaned objects. The ability to assign different objects to a reference variable creates the possibility that a program may have created an object that it can no longer reference. For example, consider the three assignment statements in the figure at right. After the third assignment statement, not only do a and b refer to the same Color object (the one whose RGB values are 160, 82, and 45), but also there is no longer a reference to the Color object that was created and used to initialize b. The only reference to that object was in the variable b, and this reference was overwritten by the assignment, so there is no way to refer to the object again. Such an object is said to be *orphaned*. Objects are also orphaned when they go out of scope. Java programmers pay little attention to orphaned objects because the system automatically reuses the memory that they occupy, as we discuss next.

```
Color a, b;
a = new Color(160, 82, 45);
b = new Color(255, 255, 0);
b = a;
```



An orphaned object

Memory management. Programs tend to create huge numbers of objects but have a need for only a small number of them at any given point in time. Accordingly, programming languages and systems need mechanisms to *allocate* memory for data-type values during the time they are needed and to *free* the memory when they are no longer needed (for an object, sometime after it is orphaned). Memory management is easier for primitive types because all of the information needed for memory allocation is known at compile time. Java (and most other systems) reserves memory for variables when they are declared and frees that memory when they go out of scope. Memory management for objects is more complicated: Java knows to allocate memory for an object when it is created (with `new`), but cannot know precisely when to free the memory associated with that object because the dynamics of a program in execution determine when the object is orphaned.

Memory leaks. In many languages (such as C and C++), the programmer is responsible for both allocating and freeing memory. Doing so is tedious and notoriously error prone. For example, suppose that a program deallocates the memory for an object, but then continues to refer to it (perhaps much later in the program). In the meantime, the system may have reallocated the same memory for another use, so all kinds of havoc can result. Another insidious problem occurs when a programmer neglects to ensure that the memory for an orphaned object is deallocated. This bug is known as a *memory leak* because it can result in a steadily increasing amount of memory devoted to orphaned objects (and therefore not available for use). The effect is that performance degrades, as if memory were leaking out of your computer. Have you ever had to reboot your computer because it was gradually getting less and less responsive? A common cause of such behavior is a memory leak in one of your applications.

Garbage collection. One of Java's most significant features is its ability to automatically manage memory. The idea is to free the programmer from the responsibility of managing memory by keeping track of orphaned objects and returning the memory they use to a pool of free memory. Reclaiming memory in this way is known as *garbage collection*, and Java's safe pointer policy enables it to do this efficiently and automatically. Programmers still debate whether the overhead of automatic garbage collection justifies the convenience of not having to worry about memory management. The same conclusion that we drew for pointers holds: when you program in Java, you will not be writing code to allocate and free memory, but if you find yourself doing so in some other language in the future, be careful!

FOR REFERENCE, WE SUMMARIZE THE EXAMPLES that we have considered in this section in the table below. These examples are chosen to help you understand the essential properties of data types and object-oriented programming.

A *data type* is a set of values and a set of operations defined on those values. With primitive data types, we worked with a small and simple set of values. Strings, colors, pictures, and I/O streams are high-level data types that indicate the breadth of applicability of data abstraction. You do not need to know how a data type is implemented to be able to use it. Each data type (there are hundreds in the Java libraries, and you will soon learn to create your own) is characterized by an API (application programming interface) that provides the information that you need to use it. A client program creates objects that hold data-type values and invokes instance methods to manipulate those values. We write client programs with the basic statements and control constructs that you learned in CHAPTERS 1 and 2, but now have the capability to work with a vast variety of data types, not just the primitive ones to which you have grown accustomed. With greater experience, you will find that this ability opens up new horizons in programming.

When properly designed, data types lead to client programs that are clearer, easier to develop, and easier to maintain than equivalent programs that do not take advantage of data abstraction. The client programs in this section are testimony to this claim. Moreover, as you will see in the next section, implementing a data type is a straightforward application of the basic programming skills that you have already learned. In particular, addressing a large and complex application becomes a process of understanding its data and the operations to be performed on it, then writing programs that directly reflect this understanding. Once you have learned to do so, you might wonder how programmers ever developed large programs without using data abstraction.

API	description
Color	colors
Picture	digital images
String	character strings
In	input streams
Out	output streams
Draw	drawings

Summary of data types in this section

Q&A

Q. Why the distinction between primitive and reference types?

A. Performance. Java provides the *wrapper* reference types `Integer`, `Double`, and so forth that correspond to primitive types and can be used by programmers who prefer to ignore the distinction (for details, see SECTION 3.3). Primitive types are closer to the types of data that are supported by computer hardware, so programs that use them usually run faster and consume less memory than programs that use the corresponding reference types.

Q. What happens if I forget to use `new` when creating an object?

A. To Java, it looks as though you want to call a static method with a return value of the object type. Since you have not defined such a method, the error message is the same as when you refer to an undefined symbol. If you compile the code

```
Color sienna = Color(160, 82, 45);
```

you get this error message:

```
cannot find symbol
symbol  : method Color(int,int,int)
```

Constructors do not provide return values (their signature has no return type)—they can only follow `new`. You get the same kind of error message if you provide the wrong number of arguments to a constructor or method.

Q. Why can we print an object `x` with the function call `StdOut.println(x)`, as opposed to `StdOut.println(x.toString())`?

A. Good question. That latter code works fine, but Java saves us some typing by automatically invoking the `toString()` method in such situations. In SECTION 3.3, we will discuss Java’s mechanism for ensuring that this is the case.

Q. What is the difference between `=`, `==`, and `equals()`?

A. The single equals sign (`=`) is the basis of the assignment statement—you certainly are familiar with that. The double equals sign (`==`) is a binary operator for checking whether its two operands are identical. If the operands are of a primitive type, the result is `true` if they have the same value, and `false` otherwise. If the op-

erands are object references, the result is `true` if they refer to the same object, and `false` otherwise. That is, we use `==` to test object identity equality. The data-type method `equals()` is included in every Java type so that the implementation can provide the capability for clients to test whether two objects have the same *value*. Note that `(a == b)` implies `a.equals(b)`, but not the other way around.

Q. How can I arrange to pass an array as an argument to a function in such a way that the function cannot change the values of the elements in the array?

A. There is no direct way to do so—arrays are mutable. In SECTION 3.3, you will see how to achieve the same effect by building a wrapper data type and passing an object reference of that type instead (see `Vector`, in PROGRAM 3.3.3).

Q. What happens if I forget to use `new` when creating an array of objects?

A. You need to use `new` for each object that you create, so when you create an array of n objects, you need to use `new` $n + 1$ times: once for the array and once for each of the n objects. If you forget to create the array:

```
Color[] colors;
colors[0] = new Color(255, 0, 0);
```

you get the same error message that you would get when trying to assign a value to any uninitialized variable:

```
variable colors might not have been initialized
colors[0] = new Color(255, 0, 0);
^
```

In contrast, if you forget to use `new` when creating an object within the array and then try to use it to invoke a method:

```
Color[] colors = new Color[2];
int red = colors[0].getRed();
```

you get a `NullPointerException`. As usual, the best way to answer such questions is to write and compile such code yourself, then try to interpret Java's error message. Doing so might help you more quickly recognize mistakes later.

Q. Where can I find more details on how Java implements references and garbage collection?

A. One Java system might differ completely from another. For example, one natural scheme is to use a pointer (machine address); another is to use a *handle* (a pointer to a pointer). The former gives faster access to data; the latter facilitates garbage collection.

Q. Why red, green, and blue instead of red, *yellow*, and blue?

A. In theory, any three colors that contain some amount of each primary would work, but two different color models have evolved: one (RGB) that has proven to produce good colors on television screens, computer monitors, and digital cameras, and the other (CMYK) that is typically used for the printed page (see EXERCISE 1.2.32). CMYK does include yellow (cyan, magenta, yellow, and black). Two different color models are appropriate because printed inks *absorb* color; thus, where there are two different inks, there are more colors absorbed and *fewer* reflected. Conversely, video displays *emit* color, so where there are two different-colored pixels, there are *more* colors emitted.

Q. What exactly is the purpose of an `import` statement?

A. Not much: it just saves some typing. For example, in PROGRAM 3.1.2, it enables you to abbreviate `java.awt.Color` with `Color` everywhere in your code.

Q. Is there anything wrong with allocating and deallocating thousands of `Color` objects, as in `Grayscale` (PROGRAM 3.1.4)?

A. All programming-language constructs come at some cost. In this case the cost is reasonable, since the time to allocate `Color` objects is tiny compared to the time to draw the image.

Q. Why does the `String` method call `s.substring(i, j)` return the substring of `s` starting at index `i` and ending at `j-1` (and not `j`)?

A. Why do the indices of an array `a[]` go from 0 to `a.length-1` instead of from 1 to `length`? Programming-language designers make choices; we live with them. One nice consequence of this convention is that the length of the extracted substring is `j-i`.

Q. What is the difference between *pass by value* and *pass by reference*?

Q. With *pass by value*, when you call a method with arguments, each argument is evaluated and a *copy* of the resulting value is passed to the method. This means that if a method directly modifies an argument variable, that modification is not visible to the caller. With *pass by reference*, the *memory address* of each argument is passed to the method. This means that if a method modifies an argument variable, that modification is visible to the caller. Technically, Java is a purely *pass-by-value* language, in which the value is either a primitive-type value or an object reference. As a result, when you pass a primitive-type value to a method, the method cannot modify the corresponding value in the caller; when you pass an object reference to a method, the method cannot modify the object reference (say, to refer to a different object), but it can change the underlying object (by using the object reference to invoke one of the object's methods). For this reason, some Java programmers use the term *pass by object reference* to refer to Java's argument-passing conventions for reference types.

Q. I noticed that the argument to the `equals()` method in `String` and `Color` is of type `Object`. Shouldn't the argument be of type `String` and `Color`, respectively?

A. No. In Java, the `equals()` method is a special and its argument type should always be `Object`. This is an artifact of the inheritance mechanism that Java uses to support the `equals()` method, which we consider on page 454. For now, you can safely ignore the distinction.

Q. Why is the image-processing data type named `Picture` instead of `Image`?

A. There is already a built-in Java library named `Image`.

Exercises

3.1.1 Write a static method `reverse()` that takes a string as an argument and returns a string that contains the same sequence of characters as the argument string but in reverse order.

3.1.2 Write a program that takes from the command line three integers between 0 and 255 that represent red, green, and blue values of a color and then creates and shows a 256-by-256 `Picture` in which each pixel has that color.

3.1.3 Modify `AlbersSquares` (PROGRAM 3.1.2) to take *nine* command-line arguments that specify *three* colors and then draws the six squares showing all the Albers squares with the large square in each color and the small square in each different color.

3.1.4 Write a program that takes the name of a grayscale image file as a command-line argument and uses `StdDraw` to plot a histogram of the frequency of occurrence of each of the 256 grayscale intensities.

3.1.5 Write a program that takes the name of an image file as a command-line argument and flips the image horizontally.

3.1.6 Write a program that takes the name of an image file as a command-line argument, and creates and shows three `Picture` objects, one that contains only the red components, one for green, and one for blue.

3.1.7 Write a program that takes the name of an image file as a command-line argument and prints the pixel coordinates of the lower-left corner and the upper-right corner of the smallest bounding box (rectangle parallel to the *x*- and *y*-axes) that contains all of the non-white pixels.

3.1.8 Write a program that takes as command-line arguments the name of an image file and the pixel coordinates of a rectangle within the image; reads from standard input a list of `Color` values (represented as triples of `int` values); and serves as a filter, printing those color values for which all pixels in the rectangle are background/foreground compatible. (Such a filter can be used to pick a color for text to label an image.)

3.1.9 Write a static method `isValidDNA()` that takes a string as its argument and returns `true` if and only if it is composed entirely of the characters A, T, C, and G.

3.1.10 Write a function `complementWatsonCrick()` that takes a DNA string as its argument and returns its *Watson–Crick complement*: replace A with T, C with G, and vice versa.

3.1.11 Write a function `isWatsonCrickPalindrome()` that takes a DNA string as its input and returns `true` if the string is a Watson–Crick complemented palindrome, and `false` otherwise. A *Watson–Crick complemented palindrome* is a DNA string that is equal to the reverse of its Watson–Crick complement.

3.1.12 Write a program to check whether an ISBN number is valid (see EXERCISE 1.3.35), taking into account that an ISBN number can have hyphens inserted at arbitrary places.

3.1.13 What does the following code fragment print?

```
String string1 = "hello";
String string2 = string1;
string1 = "world";
StdOut.println(string1);
StdOut.println(string2);
```

3.1.14 What does the following code fragment print?

```
String s = "Hello World";
s.toUpperCase();
s.substring(6, 11);
StdOut.println(s);
```

Answer: "Hello World". String objects are immutable—string methods each return a new String object with the appropriate value (but they do not change the value of the object that was used to invoke them). This code ignores the objects returned and just prints the original string. To print "WORLD", replace the second and third statements with `s = s.toUpperCase()` and `s = s.substring(6, 11)`.

3.1.15 A string s is a *circular shift* of a string t if it matches when the characters of one string are circularly shifted by some number of positions. For example, ACT-GACG is a circular shift of TGACGAC, and vice versa. Detecting this condition is important in the study of genomic sequences. Write a function `isCircularShift()` that checks whether two given strings s and t are circular shifts of one another. *Hint:* The solution is a one-liner with `indexOf()` and string concatenation.

3.1.16 Given a string that represents a domain name, write a code fragment to determine its *top-level domain*. For example, the top-level domain of the string `cs.princeton.edu` is `edu`.

3.1.17 Write a static method that takes a domain name as its argument and returns the reverse domain name (reverse the order of the strings between periods). For example, the reverse domain name of `cs.princeton.edu` is `edu.princeton.cs`. This computation is useful for web log analysis. (See EXERCISE 4.2.36.)

3.1.18 What does the following recursive function return?

```
public static String mystery(String s)
{
    int n = s.length();
    if (n <= 1) return s;
    String a = s.substring(0, n/2);
    String b = s.substring(n/2, n);
    return mystery(b) + mystery(a);
}
```

3.1.19 Write a test client for `PotentialGene` (PROGRAM 3.1.1) that takes a string as a command-line argument and reports whether it is a potential gene.

3.1.20 Write a version of `PotentialGene` (PROGRAM 3.1.1) that finds *all* potential genes contained as substrings within a long DNA string. Add a command-line argument to allow the user to specify the minimum length of a potential gene.

3.1.21 Write a filter that reads text from an input stream and prints it to an output stream, removing any lines that consist only of whitespace.

3.1.22 Write a program that takes a start string and a stop string as command-line arguments and prints all substrings of a given string that start with the first, end with the second, and otherwise contain neither. *Note:* Be especially careful of overlaps!

3.1.23 Modify StockQuote (PROGRAM 3.1.8) to take multiple symbols on the command line.

3.1.24 The example file DJIA.csv used for Split (PROGRAM 3.1.9) lists the date, high price, volume, and low price of the Dow Jones stock market average for every day since records have been kept. Download this file from the booksite and write a program that creates two Draw objects, one for the prices and one for the volumes, and plots them at a rate taken from the command line.

3.1.25 Write a program Merge that takes a delimiter string followed by an arbitrary number of file names as command-line arguments; concatenates the corresponding lines of each file, separated by the delimiter; and then prints the result to standard output, thus performing the opposite operation of Split (PROGRAM 3.1.9).

3.1.26 Find a website that publishes the current temperature in your area, and write a screen-scraper program Weather so that typing java Weather followed by your ZIP code will give you a weather forecast.

3.1.27 Suppose that `a[]` and `b[]` are both integer arrays consisting of millions of integers. What does the following code do, and how long does it take?

```
int[] temp = a; a = b; b = temp;
```

Solution. It swaps the arrays, but it does so by copying object references, so that it is not necessary to copy millions of values.

3.1.28 Describe the effect of the following function.

```
public void swap(Color a, Color b)
{
    Color temp = a;
    a = b;
    b = temp;
}
```

Creative Exercises

3.1.29 *Picture file format.* Write a library of static methods `RawPicture` with `read()` and `write()` methods for saving and reading pictures from a file. The `write()` method takes a `Picture` and the name of a file as arguments and writes the picture to the specified file, using the following format: if the picture is w -by- h , write w , then h , then $w \times h$ triples of integers representing the pixel color values, in row-major order. The `read()` method takes the name of a picture file as an argument and returns a `Picture`, which it creates by reading a picture from the specified file, in the format just described. *Note:* Be aware that this will use up much more disk space than necessary—the standard formats *compress* this information so that it will not take up so much space.

3.1.30 *Sound visualization.* Write a program that uses `StdAudio` and `Picture` to create an interesting two-dimensional color visualization of a sound file while it is playing. Be creative!

3.1.31 *Kamasutra cipher.* Write a filter `KamasutraCipher` that takes two strings as command-line argument (the *key* strings), then reads strings (separated by whitespace) from standard input, substitutes for each letter as specified by the key strings, and prints the result to standard output. This operation is the basis for one of the earliest known cryptographic systems. The condition on the key strings is that they must be of equal length and that any letter in standard input must appear in exactly one of them. For example, if the two keys are THEQUICKBROWN and FXJMPSPVLAZYDG, then we make the table

T	H	E	Q	U	I	C	K	B	R	O	W	N
F	X	J	M	P	S	V	L	A	Z	Y	D	G

which tells us that we should substitute F for T, T for F, H for X, X for H, and so forth when filtering standard input to standard output. The message is encoded by replacing each letter with its pair. For example, the message MEET AT ELEVEN is encoded as QJJF BF JKJCJG. The person receiving the message can use the same keys to get the message back.

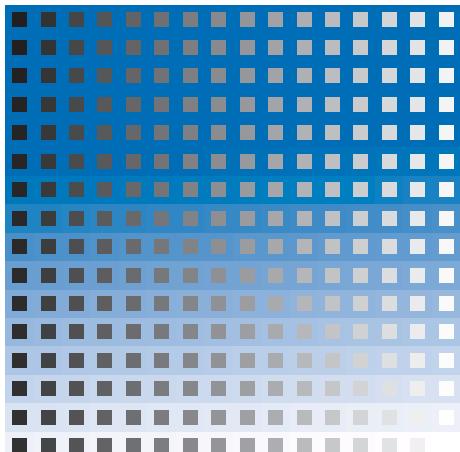
3.1.32 Safe password verification. Write a static method that takes a string as an argument and returns `true` if it meets the following conditions, `false` otherwise:

- At least eight characters long
- Contains at least one digit (0–9)
- Contains at least one uppercase letter
- Contains at least one lowercase letter
- Contains at least one character that is neither a letter nor a number

Such checks are commonly used for passwords on the web.

3.1.33 Color study. Write a program that displays the color study shown at right, which gives Albers squares corresponding to each of the 256 levels of blue (blue-to-white in row-major order) and gray (black-to-white in column-major order) that were used to print this book.

3.1.34 Entropy. The *Shannon entropy* measures the information content of an input string and plays a cornerstone role in information theory and data compression. Given a string of n characters, let f_c be the frequency of occurrence of character c . The quantity $p_c = f_c / n$ is an estimate of the probability that c would be in the string if it were a random string, and the entropy is defined to be the sum of the quantity $-p_c \log_2 p_c$, over all characters that appear in the string. The entropy is said to measure the *information content* of a string; if each character appears the same number times, the entropy is at its minimum value among strings of a given length. Write a program that takes the name of a file as a command-line argument and prints the entropy of the text in that file. Run your program on a web page that you read regularly, a recent paper that you wrote, and the fruit fly genome found on the website.



A color study

3.1.35 *Tile.* Write a program that takes the name of an image file and two integers m and n as command-line arguments and creates an m -by- n tiling of the image.

3.1.36 *Rotation filter.* Write a program that takes two command-line arguments (the name of an image file and a real number θ) and rotates the image θ degrees counterclockwise. To rotate, copy the color of each pixel (s_i, s_j) in the source image to a target pixel (t_i, t_j) whose coordinates are given by the following formulas:

$$t_i = (s_i - c_i)\cos \theta - (s_j - c_j)\sin \theta + c_i$$

$$t_j = (s_i - c_i)\sin \theta + (s_j - c_j)\cos \theta + c_j$$

where (c_i, c_j) is the center of the image.

3.1.37 *Swirl filter.* Creating a swirl effect is similar to rotation, except that the angle changes as a function of distance to the center of the image. Use the same formulas as in the previous exercise, but compute θ as a function of (s_i, s_j) , specifically $\pi/256$ times the distance to the center.

3.1.38 *Wave filter.* Write a filter like those in the previous two exercises that creates a wave effect, by copying the color of each pixel (s_i, s_j) in the source image to a target pixel (t_i, t_j) , where $t_i = s_i$ and $t_j = s_j + 20 \sin(2\pi s_j / 64)$. Add code to take the amplitude (20 in the accompanying figure) and the frequency (64 in the accompanying figure) as command-line arguments. Experiment with various values of these parameters.

3.1.39 *Glass filter.* Write a program that takes the name of an image file as a command-line argument and applies a *glass filter*: set each pixel p to the color of a random neighboring pixel (whose pixel coordinates both differ from p 's coordinates by at most 5).

3.1.40 *Slide show.* Write a program that takes the names of several image files as command-line arguments and displays them in a slide show (one every two seconds), using a fade effect to black and a fade from black between images.

3.1.41 *Morph.* The example images in the text for `Fade` do not quite line up in the vertical direction (the mandrill's mouth is much lower than Darwin's). Modify `Fade` to add a transformation in the vertical dimension that makes a smoother transition.

3.1.42 *Digital zoom.* Write a program `Zoom` that takes the name of an image file and three numbers s , x , and y as command-line arguments, and shows an output image that zooms in on a portion of the input image. The numbers are all between 0 and 1, with s to be interpreted as a scale factor and (x, y) as the relative coordinates of the point that is to be at the center of the output image. Use this program to zoom in on a relative or pet in some digital photo on your computer. (If your photo came from an old cell phone or camera, you may not be able to zoom in too close without having visible artifacts from scaling.)

3.2 Creating Data Types

IN PRINCIPLE, WE COULD WRITE ALL of our programs using only the eight built-in primitive types. However, as we saw in the last section, it is much more convenient to write programs at a higher level of abstraction. Thus, a variety of data types are built into the Java language and libraries. Still, we certainly cannot expect Java to contain every conceivable data type that we might ever wish to use, so we need to be able to *define* our own. This section explains how to build data types with the familiar Java `class`.

Implementing a data type as a Java class is not very different from implementing a library of static methods. The primary difference is that we associate *data* with the method implementations. The API specifies the constructors and instance methods that we need to implement, but we are free to choose any convenient representation. To cement the basic concepts, we begin by considering an implementation of a data type for charged particles. Next, we illustrate the process of creating data types by considering a range of examples, from complex numbers to stock accounts, including a number of software tools that we will use later in the book. Useful client code is testimony to the value of any data type, so we also consider a number of clients, including one that depicts the famous and fascinating *Mandelbrot set*.

The process of defining a data type is known as *data abstraction*. We focus on the data and implement operations on that data. *Whenever you can clearly separate data and associated operations within a program, you should do so.* Modeling physical objects or familiar mathematical abstractions is straightforward and extremely useful, but the true power of data abstraction is that it allows us to model *anything* that we can precisely specify. Once you gain experience with this style of programming, you will see that it helps us address programming challenges of arbitrary complexity.

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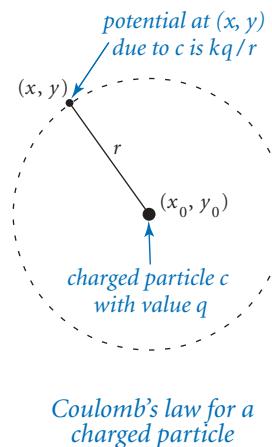
Programs in this section

Basic elements of a data type To illustrate the process of implementing a data type in a Java class, we will consider a data type Charge for charged particles. In particular, we are interested in a two-dimensional model that uses *Coulomb's law*, which tells us that the electric potential at a point (x, y) due to a given charged particle is $V = kq/r$, where q is the charge value, r is the distance from the point to the charge, and $k = 8.99 \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2}$ is the electrostatic constant. When there are multiple charged particles, the electric potential at any point is the sum of the potentials due to each charge. For consistency, we use SI (Système International d'Unités): in this formula, N designates newtons (force), m designates meters (distance), and C represent coulombs (electric charge).

API. The application programming interface is the contract with all clients and, therefore, the starting point for any implementation. Here is our API for charged particles:

```
public class Charge
    Charge(double x0, double y0, double q0)
    double potentialAt(double x, double y)   electric potential at (x, y) due to charge
    String toString()                      string representation
```

API for charged particles (see PROGRAM 3.2.1)



Coulomb's law for a charged particle

To implement the Charge data type, we need to define the data-type values and implement the constructor that creates a charged particle, a method potentialAt() that returns the potential at the point (x, y) due to the charge, and a `toString()` method that returns a string representation of the charge.

Class. In Java, you implement a data type in a `class`. As with the libraries of static methods that we have been using, we put the code for a data type in a file with the same name as the class, followed by the `.java` extension. We have been implementing Java classes, but the classes that we have been implementing do not have the key features of data types: *instance variables*, *constructors*, and *instance methods*. Each of these building blocks is also qualified by an *access* (or *visibility*) *modifier*. We next consider these four concepts, with examples, culminating in an implementation of the Charge data type (PROGRAM 3.2.1).

Access modifiers. The keywords `public`, `private`, and `final` that sometimes precede class names, instance variable names, and method names are known as *access modifiers*. The `public` and `private` modifiers control access from client code: we designate every instance variable and method within a class as either `public` (this entity is accessible by clients) or `private` (this entity is not accessible by clients). The `final` modifier indicates that the value of the variable will not change once it is initialized—its access is read-only. Our convention is to use `public` for the constructors and methods in the API (since we are promising to provide them to clients) and `private` for everything else. Typically, our private methods are helper methods used to simplify code in other methods in the class. Java is not so restrictive on its usage of modifiers—we defer to SECTION 3.3 a discussion of our reasons for these conventions.

Instance variables. To write code for the instance methods that manipulate data-type values, first we need to declare *instance variables* that we can use to refer to these values in code. These variables can be any type of data. We declare the types and names of instance variables in the same way as we declare local variables: for `Charge`, we use three `double` variables—two to describe the charge’s position in

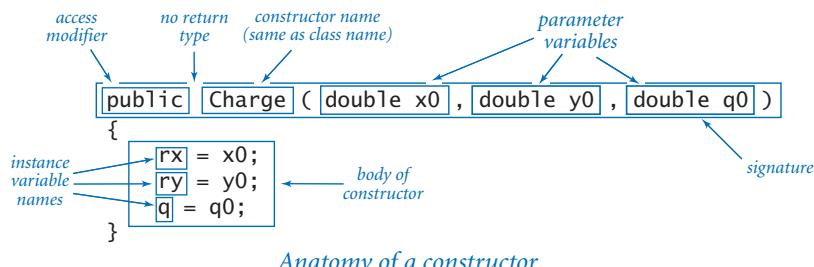
the plane and one to describe the amount of charge. These declarations appear as the first statements in the class, not inside `main()` or any other method. There is a critical distinction between instance variables and the *local variables* defined within a method or a block that you are accustomed to: there is just *one* value corresponding to each local variable at a given time, but there are *numerous* values corresponding to

```
public class Charge
{
    private final double rx, ry;
    private final double q;
    ...
}
```

Instance variables

each instance variable (one for each object that is an instance of the data type). There is no ambiguity with this arrangement, because each time that we invoke an instance method, we do so with an object reference—the referenced object is the one whose value we are manipulating.

Constructors. A constructor is a special method that creates an object and provides a reference to that object. Java automatically invokes a constructor when a client program uses the keyword `new`. Java does most of the work: our code just needs to initialize the instance variables to meaningful values. Constructors always share the same name as the class, but we can overload the name and have multiple

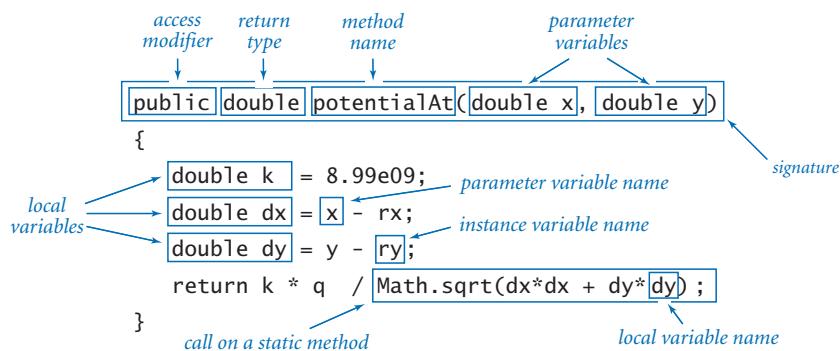


constructors with different signatures, just as with static methods. To the client, the combination of new followed by a constructor name (with arguments enclosed within parentheses) is the same as a function call that returns an object reference of the specified type. A constructor signature has no return type, because constructors always return a reference to an object of its data type (the name of the type, the class, and the constructor are all the same). Each time that a client invokes a constructor, Java automatically

- Allocates memory for the object
- Invokes the constructor code to initialize the instance variables
- Returns a reference to the newly created object

The constructor in `Charge` is typical: it initializes the instance variables with the values provided by the client as arguments.

Instance methods. To implement instance methods, we write code that is precisely like the code that we learned in CHAPTER 2 to implement static methods (functions). Each method has a signature (which specifies its return type and the types and names of its parameter variables) and a body (which consists of a sequence of statements, including a return statement that provides a value of the re-



turn type back to the client). When a client invokes an instance method, the system initializes the parameter variables with client values; executes statements until it reaches a `return` statement; and returns the computed value to the client, with the same effect as if the method invocation in the client were replaced with that return value. All of this is the same as for static methods, but there is one critical distinction for instance methods: *they can perform operations on instance variables.*

Variables within methods. Accordingly, the Java code that we write to implement instance methods uses *three* kinds of variables:

- Parameter variables
- Local variables
- *Instance variables*

The first two are the same as for static methods: parameter variables are specified in the method signature and initialized with client values when the method is called, and local variables are declared and initialized within the method body. The scope of parameter variables is the entire method; the scope of local variables is the following statements in the block where they are defined. Instance variables are completely different: they hold data-type values for objects in a class, and their scope is the entire class. How do we specify which object's value we want to use? If you think for a moment about this question, you will recall the answer. Each object in the class has a value: the code in an instance method refers to the value *for the object that was used to invoke the method*. For example, when we write `c1.potentialAt(x, y)`, the code in `potentialAt()` is referring to the instance variables for `c1`.

The implementation of `potentialAt()` in `Charge` uses all three kinds of variable names, as illustrated in the diagram at the bottom of the previous page and summarized in this table:

Be sure that you understand the distinctions among the three kinds of variables that we use in implementing instance methods. These differences are a key to object-oriented programming.

<i>variable</i>	<i>purpose</i>	<i>example</i>	<i>scope</i>
<i>parameter</i>	to pass value from client to method	x, y	method
<i>local</i>	for temporary use within method	dx, dy	block
<i>instance</i>	to specify data-type value	rx, ry	class

Variables within instance methods

Program 3.2.1 Charged particle

```

public class Charge
{
    private final double rx, ry;
    private final double q;

    public Charge(double x0, double y0, double q0)
    { rx = x0; ry = y0; q = q0; }

    public double potentialAt(double x, double y)
    {
        double k = 8.99e09;
        double dx = x - rx;
        double dy = y - ry;
        return k * q / Math.sqrt(dx*dx + dy*dy);
    }

    public String toString()
    {
        return q + " at (" + rx + ", " + ry + ")";
    }

    public static void main(String[] args)
    {
        double x = Double.parseDouble(args[0]);
        double y = Double.parseDouble(args[1]);
        Charge c1 = new Charge(0.51, 0.63, 21.3);
        Charge c2 = new Charge(0.13, 0.94, 81.9);
        StdOut.println(c1);
        StdOut.println(c2);
        double v1 = c1.potentialAt(x, y);
        double v2 = c2.potentialAt(x, y);
        StdOut.printf("%.2e\n", (v1 + v2));
    }
}

```

rx, ry *query point*
q *charge*

k *electrostatic constant*
dx, dy *delta distances to query point*

x, y *query point*
c1 *first charge*
v1 *potential due to c1*
c2 *second charge*
v2 *potential due to c2*

This implementation of our data type for charged particles contains the basic elements found in every data type: instance variables rx, ry, and q; a constructor Charge(); instance methods potentialAt() and toString(); and a test client main().

```
% java Charge 0.2 0.5
21.3 at (0.51, 0.63)
81.9 at (0.13, 0.94)
2.22e+12
```

```
% java Charge 0.51 0.94
21.3 at (0.51, 0.63)
81.9 at (0.13, 0.94)
2.56e+12
```

Test client. Each class can define its own `main()` method, which we typically reserve for testing the data type. At a minimum, the test client should call every constructor and instance method in the class. For example, the `main()` method in PROGRAM 3.2.1 takes two command-line arguments `x` and `y`, creates two `Charge` objects, and prints the two charged particles along with the total electric potential at (x, y) due to those two particles. When there are multiple charged particles, the electric potential at any point is the sum of the potentials due to each charge.

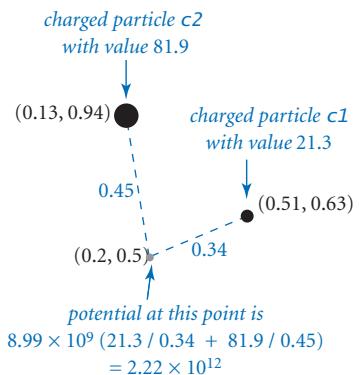
THESE ARE THE BASIC COMPONENTS THAT you need to understand to be able to define your own data types in Java. Every data-type implementation (Java class) that we will develop has the same basic ingredients as this first example: instance variables, constructors, instance methods, and a test client. In each data type that we develop, we go through the same steps. Rather than thinking about which action we need to take next to accomplish a computational goal (as we did when first learning to program), we think about the needs of a client, then accommodate them in a data type.

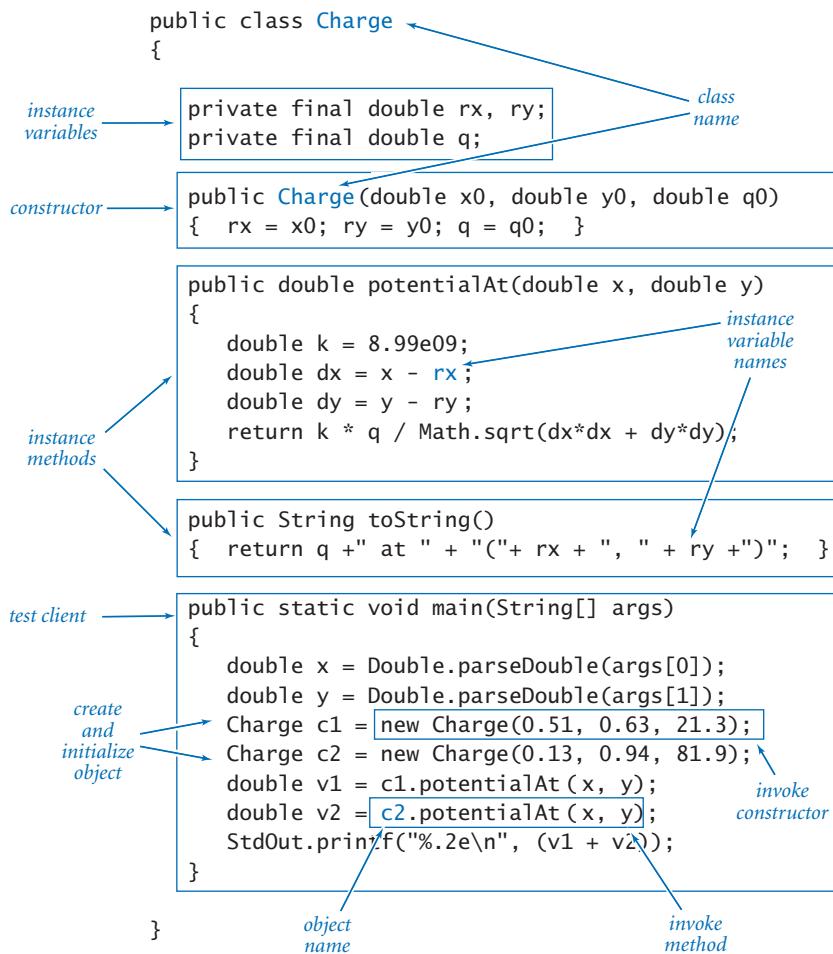
The first step in creating a data type is to specify an API. The purpose of the API is to *separate clients from implementations*, so as to enable modular programming. We have two goals when specifying an API. First, we want to enable clear and correct client code. Indeed, it is a good idea to write some client code before finalizing the API to gain confidence that the specified data-type operations are the ones that clients need. Second, we want to be able to implement the operations. There is no point in specifying operations that we have no idea how to implement.

The second step in creating a data type is to implement a Java class that meets the API specifications. First we choose the instance variables, then we write the code that manipulates the instance variables to implement the specified constructors and instance methods.

The third step in creating a data type is to write test clients, to validate the design decisions made in the first two steps.

What are the values that define the data type, and which operations do clients need to perform on those values? With these basic decisions made, you can create new data types and write clients that use them in the same way as you have been using built-in types. You will find many exercises at the end of this section that are intended to give you experience with data-type creation.





Anatomy of a class

Stopwatch One of the hallmarks of object-oriented programming is the idea of easily modeling real-world objects by creating abstract programming objects. As a simple example, consider `Stopwatch` (PROGRAM 3.3.2), which implements the following API:

```
public class Stopwatch  
  
    Stopwatch()           create a new stopwatch and start it running  
    double elapsedTime() return the elapsed time since creation, in seconds  
  
API for stopwatches (see PROGRAM 3.2.2)
```

In other words, a `Stopwatch` is a stripped-down version of an old-fashioned stopwatch. When you create one, it starts running, and you can ask it how long it has been running by invoking the method `elapsedTime()`. You might imagine adding all sorts of bells and whistles to `Stopwatch`, limited only by your imagination. Do you want to be able to reset the stopwatch? Start and stop it? Include a lap timer? These sorts of things are easy to add (see EXERCISE 3.2.12).

The implementation of `Stopwatch` uses the Java system method `System.currentTimeMillis()`, which returns a `long` value giving the current time in milliseconds (the number of milliseconds since midnight on January 1, 1970 UTC). The data-type implementation could hardly be simpler. A `Stopwatch` saves its creation time in an instance variable, then returns the difference between that time and the current time whenever a client invokes its `elapsedTime()` method. A `Stopwatch` itself does not actually tick (an internal system clock on your computer does all the ticking); it just creates the illusion that it does for clients. Why not just use `System.currentTimeMillis()` in clients? We could do so, but using the `Stopwatch` leads to client code that is easier to understand and maintain.

The test client is typical. It creates two `Stopwatch` objects, uses them to measure the running time of two different computations, then prints the running times. The question of whether one approach to solving a problem is better than another has been lurking since the first few programs that you have run, and plays an essential role in program development. In SECTION 4.1, we will develop a scientific approach to understanding the cost of computation. `Stopwatch` is a useful tool in that approach.

Program 3.2.2 Stopwatch

```
public class Stopwatch
{
    private final long start;                                start | creation time

    public Stopwatch()
    { start = System.currentTimeMillis(); }

    public double elapsedTime()
    {
        long now = System.currentTimeMillis();
        return (now - start) / 1000.0;
    }

    public static void main(String[] args)
    {
        // Compute and time computation using Math.sqrt().
        int n = Integer.parseInt(args[0]);
        Stopwatch timer1 = new Stopwatch();
        double sum1 = 0.0;
        for (int i = 1; i <= n; i++)
            sum1 += Math.sqrt(i);
        double time1 = timer1.elapsedTime();
        StdOut.printf("%e (%.2f seconds)\n", sum1, time1);

        // Compute and time computation using Math.pow().
        Stopwatch timer2 = new Stopwatch();
        double sum2 = 0.0;
        for (int i = 1; i <= n; i++)
            sum2 += Math.pow(i, 0.5);
        double time2 = timer2.elapsedTime();
        StdOut.printf("%e (%.2f seconds)\n", sum2, time2);
    }
}
```

This class implements a simple data type that we can use to compare running times of performance-critical methods (see SECTION 4.1). The test client compares the running times of two functions for computing square roots in Java’s Math library. For the task of computing the sum of the square roots of the numbers from 1 to n, the version that calls Math.sqrt() is more than 10 times faster than the one that calls Math.pow(). Results are likely to vary by system.

```
% java Stopwatch 100000000
6.666667e+11 (0.65 seconds)
6.666667e+11 (8.47 seconds)
```

Histogram Now, we consider a data type to visualize data using a familiar plot known as a *histogram*. For simplicity, we assume that the data consists of a sequence of integer values between 0 and $n - 1$. A histogram counts the number of times each value appears and plots a bar for each value (with height proportional to its frequency). The following API describes the operations:

<code>public class Histogram</code>	
	<i>create a histogram for the integer values 0 to $n-1$</i>
<code> Histogram(int n)</code>	
	<i>add an occurrence of the value i</i>
<code> double addDataPoint(int i)</code>	
	<i>draw the histogram to standard drawing</i>
<code> void draw()</code>	
	<i>API for histograms (see PROGRAM 3.2.3)</i>

To implement a data type, you must first determine which instance variables to use. In this case, we need to use an array as an instance variable. Specifically, `Histogram` (PROGRAM 3.2.3) maintains an instance variable `freq[]` so that `freq[i]` records the number of times the data value i appears in the data, for each i between 0 and $n-1$. `Histogram` also includes an integer instance variable `max` that stores the maximum frequency of any of the values (which corresponds to the height of the tallest bar). The instance method `draw()` method uses the variable `max` to set the y -scale of the standard drawing window and calls the method `StdStats.plotBars()` to draw the histogram of values. The `main()` method is a sample client that performs Bernoulli trials. It is substantially simpler than `Bernoulli` (PROGRAM 2.2.6) because it uses the `Histogram` data type.

By creating a data type such as `Histogram`, we reap the benefits of modular programming (reusable code, independent development of small programs, and so forth) that we discussed in CHAPTER 2, with the additional benefit that we separate the *data*. Without `Histogram`, we would have to mix the code for creating the histogram with the code for managing the data of interest, resulting in a program much more difficult to understand and maintain than the two separate programs. *Whenever you can clearly separate data and associated operations within a program, you should do so.*

Program 3.2.3 Histogram

```

public class Histogram
{
    private final double[] freq;
    private double max;

    public Histogram(int n)
    { // Create a new histogram.
        freq = new double[n];
    }

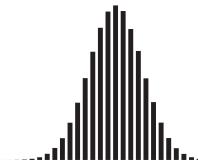
    public void addDataPoint(int i)
    { // Add one occurrence of the value i.
        freq[i]++;
        if (freq[i] > max) max = freq[i];
    }

    public void draw() % java Histogram 50 1000000
    { // Draw (and scale) the histogram.
        StdDraw.setScale(0, max);
        StdStats.plotBars(freq);
    }

    public static void main(String[] args)
    { // See Program 2.2.6.
        int n = Integer.parseInt(args[0]);
        int trials = Integer.parseInt(args[1]);
        Histogram histogram = new Histogram(n+1);
        StdDraw.setCanvasSize(500, 200);
        for (int t = 0; t < trials; t++)
            histogram.addDataPoint(Bernoulli.binomial(n));
        histogram.draw();
    }
}

```

`freq[]` | *frequency counts*
`max` | *maximum frequency*



This data type supports simple client code to create histograms of the frequency of occurrence of integers values between 0 and $n-1$. The frequencies are kept in an instance variable that is an array. An integer instance variable `max` tracks the maximum frequency (for scaling the y-axis when drawing the histogram). The sample client is a version of `Bernoulli` (PROGRAM 2.2.6), but is substantially simpler because it uses the `Histogram` data type.

Turtle graphics Whenever you can clearly separate tasks within a program, you should do so. In object-oriented programming, we extend that mantra to include *data* (or *state*) with the tasks. A small amount of state can be immensely valuable in simplifying a computation. Next, we consider *turtle graphics*, which is based on the data type defined by this API:

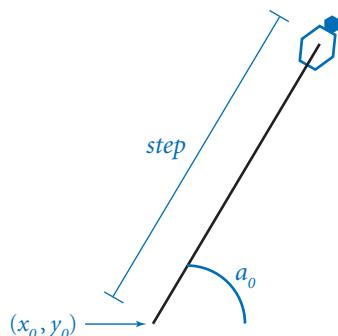
```
public class Turtle
    Turtle(double x0, double y0, double a0)
        create a new turtle at (x0, y0) facing a0
        degrees counterclockwise from the x-axis
    void turnLeft(double delta)
        rotate delta degrees counterclockwise
    void goForward(double step)
        move distance step, drawing a line
```

API for turtle graphics (see PROGRAM 3.2.4)

Imagine a turtle that lives in the unit square and draws lines as it moves. It can move a specified distance in a straight line, or it can rotate left (counterclockwise) a specified number of degrees. According to the API, when we create a turtle, we place it at a specified point, facing a specified direction. Then, we create drawings by giving the turtle a sequence of `goForward()` and `turnLeft()` commands.

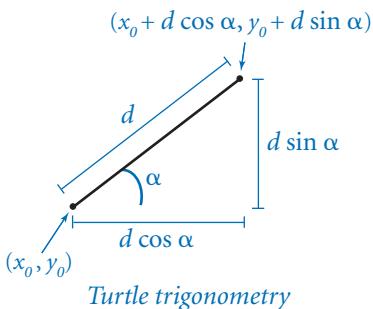
For example, to draw an equilateral triangle, we create a `Turtle` at $(0.5, 0)$ facing at an angle of 60 degrees counterclockwise from the origin, then direct it to take a step forward, then rotate 120 degrees counterclockwise, then take another step forward, then rotate another 120 degrees counterclockwise, and then take a third step forward to complete the triangle. Indeed, all of the turtle clients that we will examine simply create a turtle, then give it an alternating sequence of step and rotate commands, varying the step size and the amount of rotation. As you will see in the next several pages, this simple model allows us to create arbitrarily complex drawings, with many important applications.

```
double x0 = 0.5;
double y0 = 0.0;
double a0 = 60.0;
double step = Math.sqrt(3)/2;
Turtle turtle = new Turtle(x0, y0, a0);
turtle.goForward(step);
```



A turtle's first step

Turtle (PROGRAM 3.2.4) is an implementation of this API that uses StdDraw. It maintains three instance variables: the coordinates of the turtle's position and the current direction it is facing, measured in degrees counterclockwise from the x -axis. Implementing the two methods requires *changing* the values of these variables, so they are not `final`. The necessary updates are straightforward: `turnLeft(delta)` adds `delta` to the current angle, and `goForward(step)` adds the step size times the cosine of its argument to the current x -coordinate and the step size times the sine of its argument to the current y -coordinate.



The test client in `Turtle` takes an integer command-line argument `n` and draws a regular polygon with `n` sides. If you are interested in elementary analytic geometry, you might enjoy verifying that fact. Whether or not you choose to do so, think about what you would need to do to compute the coordinates of all the points in the polygon. The simplicity of the turtle's approach is very appealing. In short, turtle graphics serves as a useful abstraction for describing geometric shapes of all sorts. For example, we obtain a good approximation to a circle by taking `n` to a sufficiently large value.

You can use a `Turtle` as you use any other object. Programs can create arrays of `Turtle` objects, pass them as arguments to functions, and so forth. Our examples will illustrate these capabilities and convince you that creating a data type like `Turtle` is both very easy and very useful. For each of them, as with regular polygons, it is *possible* to compute the coordinates of all the points and draw straight lines to get the drawings, but it is *easier* to do so with a `Turtle`. Turtle graphics exemplifies the value of data abstraction.

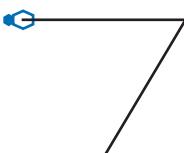
```
turtle.goForward(step);
```



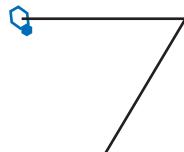
```
turtle.turnLeft(120.0);
```



```
turtle.goForward(step);
```



```
turtle.turnLeft(120.0);
```



```
turtle.goForward(step);
```



Your first turtle graphics drawing

Program 3.2.4 Turtle graphics

```

public class Turtle
{
    private double x, y;
    private double angle;           x, y | position (in unit square)
                                    angle | direction of motion (degrees,
                                         counterclockwise from x-axis)

    public Turtle(double x0, double y0, double a0)
    { x = x0; y = y0; angle = a0; }

    public void turnLeft(double delta)
    { angle += delta; }

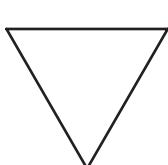
    public void goForward(double step)
    { // Compute new position; move and draw line to it.
        double oldx = x, oldy = y;
        x += step * Math.cos(Math.toRadians(angle));
        y += step * Math.sin(Math.toRadians(angle));
        StdDraw.line(oldx, oldy, x, y);
    }

    public static void main(String[] args)
    { // Draw a regular polygon with n sides.
        int n = Integer.parseInt(args[0]);
        double angle = 360.0 / n;
        double step = Math.sin(Math.toRadians(angle/2));
        Turtle turtle = new Turtle(0.5, 0.0, angle/2);
        for (int i = 0; i < n; i++)
        {
            turtle.goForward(step);
            turtle.turnLeft(angle);
        }
    }
}

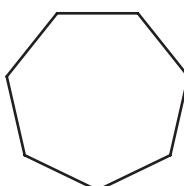
```

This data type supports turtle graphics, which often simplifies the creation of drawings.

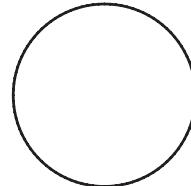
% java Turtle 3



% java Turtle 7



% java Turtle 1000



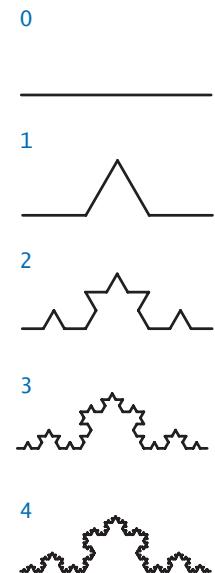
Recursive graphics. A Koch curve of order 0 is a straight line segment. To form a Koch curve of order n , draw a Koch curve of order $n-1$, turn left 60 degrees, draw a second Koch curve of order $n-1$, turn right 120 degrees (left -120 degrees), draw a third Koch curve of order $n-1$, turn left 60 degrees, and draw a fourth Koch curve of order $n-1$. These recursive instructions lead immediately to turtle client code. With appropriate modifications, recursive schemes like this are useful in modeling self-similar patterns found in nature, such as snowflakes.

The client code is straightforward, except for the value of the step size. If you carefully examine the first few examples, you will see (and be able to prove by induction) that the width of the curve of order n is 3^n times the step size, so setting the step size to $1/3^n$ produces a curve of width 1. Similarly, the number of steps in a curve of order n is 4^n , so Koch will not finish if you invoke it for large n .

You can find many examples of recursive patterns of this sort that have been studied and developed by mathematicians, scientists, and artists from many cultures in many contexts. Here, our interest in them is that the turtle graphics abstraction greatly simplifies the client code that draws these patterns.

```
public class Koch
{
    public static void koch(int n, double step, Turtle turtle)
    {
        if (n == 0)
        {
            turtle.goForward(step);
            return;
        }
        koch(n-1, step, turtle);
        turtle.turnLeft(60.0);
        koch(n-1, step, turtle);
        turtle.turnLeft(-120.0);
        koch(n-1, step, turtle);
        turtle.turnLeft(60.0);
        koch(n-1, step, turtle);
    }

    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        double step = 1.0 / Math.pow(3.0, n);
        Turtle turtle = new Turtle(0.0, 0.0, 0.0);
        koch(n, step, turtle);
    }
}
```



Drawing Koch curves with turtle graphics

Spira mirabilis. Perhaps the turtle is a bit tired after taking 4^n steps to draw a Koch curve. Accordingly, imagine that the turtle's step size decays by a tiny constant factor each time that it takes a step. What happens to our drawings? Remarkably, modifying the polygon-drawing test client in PROGRAM 3.2.4 to answer this question leads to a geometric shape known as a *logarithmic spiral*, a curve that is found in many contexts in nature.

`Spiral` (PROGRAM 3.2.5) is an implementation of this curve. It takes n and the decay factor as command-line arguments and instructs the turtle to alternately step and turn until it has wound around itself 10 times. As you can see from the four examples given with the program, if the decay factor is greater than 1, the path spirals into the center of the drawing. The argument n controls the shape of the spiral. You are encouraged to experiment with `Spiral` yourself to develop an understanding of the way in which the parameters control the behavior of the spiral.

The logarithmic spiral was first described by René Descartes in 1638. Jacob Bernoulli was so amazed by its mathematical properties that he named it the *spira mirabilis* (miraculous spiral) and even asked to have it engraved on his tombstone. Many people also consider it to be “miraculous” that this precise curve is clearly present in a broad variety of natural phenomena. Three examples are depicted below: the chambers of a nautilus shell, the arms of a spiral galaxy, and the cloud formation in a tropical storm. Scientists have also observed it as the path followed by a hawk approaching its prey and as the path followed by a charged particle moving perpendicular to a uniform magnetic field.

One of the goals of scientific enquiry is to provide simple but accurate models of complex natural phenomena. Our tired turtle certainly passes that test!

nautilus shell

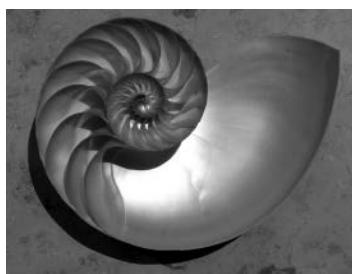


Photo: Chris 73 (CC by-SA license)

spiral galaxy



Photo: NASA and ESA

storm clouds



Photo: NASA

Examples of the spira mirabilis in nature

Program 3.2.5 Spira mirabilis

```

public class Spiral
{
    public static void main(String[] args)
    {
        int n          = Integer.parseInt(args[0]);
        double decay  = Double.parseDouble(args[1]);
        double angle   = 360.0 / n;
        double step    = Math.sin(Math.toRadians(angle/2));
        Turtle turtle = new Turtle(0.5, 0, angle/2);

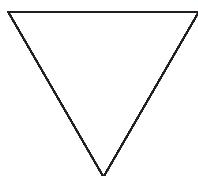
        for (int i = 0; i < 10 * 360 / angle; i++)
        {
            step /= decay;
            turtle.goForward(step);
            turtle.turnLeft(angle);
        }
    }
}

```

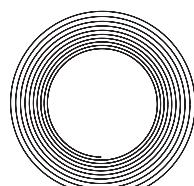
<code>step</code> <code>decay</code> <code>angle</code> <code>turtle</code>	<i>step size</i> <i>decay factor</i> <i>rotation amount</i> <i>tired turtle</i>
--	--

This code is a modification of the test client in PROGRAM 3.2.4 that decreases the step size at each step and cycles around 10 times. The angle controls the shape; the decay controls the nature of the spiral.

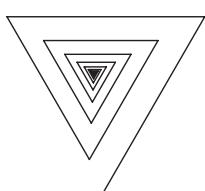
% java Spiral 3 1.0



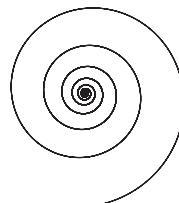
% java Spiral 1440 1.00004



% java Spiral 3 1.2



% java Spiral 1440 1.0004



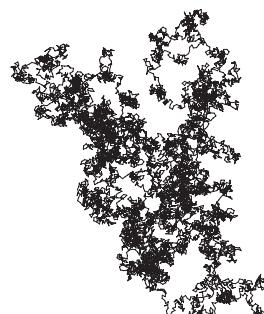
Brownian motion. Or perhaps the turtle has had one too many. Accordingly, imagine that the disoriented turtle (again following its standard alternating turn-and-step regimen) turns in a *random* direction before each step. Again, it is easy to plot the path followed by such a turtle for millions of steps, and again, such paths are found in nature in many contexts. In 1827, the botanist Robert Brown observed through a microscope that tiny particles ejected from pollen grains seemed to move about in just such a random fashion when immersed in water. This process, which later became known as *Brownian motion*, led to Albert Einstein's insights into the atomic nature of matter.

Or perhaps our turtle has friends, all of whom have had one too many. After they have wandered around for a sufficiently long time, their paths merge together and become indistinguishable from a single path. Astrophysicists today are using this model to understand observed properties of distant galaxies.

TURTLE GRAPHICS WAS ORIGINALLY DEVELOPED BY Seymour Papert at MIT in the 1960s as part of an educational programming language, Logo, that is still used today in toys. But turtle graphics is no toy, as we have just seen in numerous scientific examples. Turtle graphics also has numerous commercial applications. For example, it is the basis for POSTSCRIPT, a programming language for creating printed pages that is used for most newspapers, magazines, and books. In the present context, Turtle is a quintessential object-oriented programming example, showing that a small amount of saved state (data abstraction using objects, not just functions) can vastly simplify a computation.

```
public class DrunkenTurtle
{
    public static void main(String[] args)
    {
        int trials = Integer.parseInt(args[0]);
        double step = Double.parseDouble(args[1]);
        Turtle turtle = new Turtle(0.5, 0.5, 0.0);
        for (int t = 0; t < trials; t++)
        {
            turtle.turnLeft(StdRandom.uniform(0.0, 360.0));
            turtle.goForward(step);
        }
    }
}
```

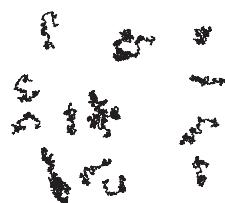
% java DrunkenTurtle 10000 0.01



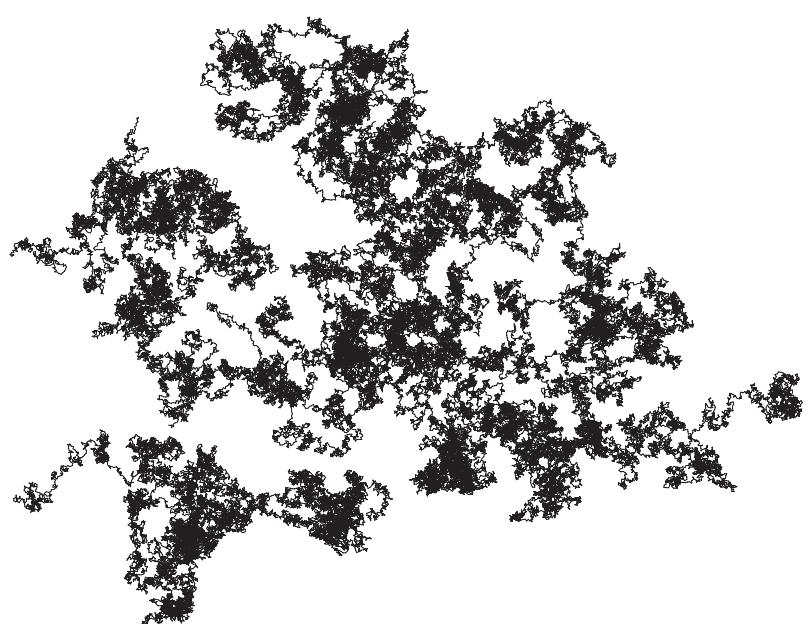
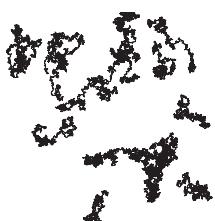
Brownian motion of a drunken turtle (moving a fixed distance in a random direction)

```
public class DrunkenTurtles
{
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);           // number of turtles
        int trials = Integer.parseInt(args[1]);       // number of steps
        double step = Double.parseDouble(args[2]);   // step size
        Turtle[] turtles = new Turtle[n];
        for (int i = 0; i < n; i++)
        {
            double x = StdRandom.uniform(0.0, 1.0);
            double y = StdRandom.uniform(0.0, 1.0);
            turtles[i] = new Turtle(x, y, 0.0);
        }
        for (int t = 0; t < trials; t++)
        { // All turtles take one step.
            for (int i = 0; i < n; i++)
            { // Turtle i takes one step in a random direction.
                turtles[i].turnLeft(StdRandom.uniform(0.0, 360.0));
                turtles[i].goForward(step);
            }
        }
    }
}

% java DrunkenTurtles 20 5000 0.005
20 500 0.005
```



20 1000 0.005



Brownian motion of a bale of drunken turtles

Complex numbers A *complex number* is a number of the form $x + iy$, where x and y are real numbers and i is the square root of -1 . The number x is known as the *real* part of the complex number, and the number y is known as the *imaginary* part. This terminology stems from the idea that the square root of -1 has to be an imaginary number, because no real number can have this value. Complex numbers are a quintessential mathematical abstraction: whether or not one believes that it makes sense physically to take the square root of -1 , complex numbers help us understand the natural world. They are used extensively in applied mathematics and play an essential role in many branches of science and engineering. They are used to model physical systems of all sorts, from circuits to sound waves to electromagnetic fields. These models typically require extensive computations involving manipulating complex numbers according to well-defined arithmetic operations, so we want to write computer programs to do the computations. In short, we need a new data type.

Developing a data type for complex numbers is a prototypical example of object-oriented programming. No programming language can provide implementations of every mathematical abstraction that we might need, but the ability to implement data types gives us not just the ability to write programs to easily manipulate abstractions such as complex numbers, polynomials, vectors, and matrices, but also the freedom to think in terms of new abstractions.

The operations on complex numbers that are needed for basic computations are to add and multiply them by applying the commutative, associative, and distributive laws of algebra (along with the identity $i^2 = -1$); to compute the magnitude; and to extract the real and imaginary parts, according to the following equations:

- *Addition:* $(x + iy) + (v + iw) = (x + v) + i(y + w)$
- *Multiplication:* $(x + iy) \times (v + iw) = (xv - yw) + i(yv + xw)$
- *Magnitude:* $|x + iy| = \sqrt{x^2 + y^2}$
- *Real part:* $\text{Re}(x + iy) = x$
- *Imaginary part:* $\text{Im}(x + iy) = y$

For example, if $a = 3 + 4i$ and $b = -2 + 3i$, then $a + b = 1 + 7i$, $a \times b = -18 + i$, $\text{Re}(a) = 3$, $\text{Im}(a) = 4$, and $|a| = 5$.

With these basic definitions, the path to implementing a data type for complex numbers is clear. As usual, we start with an API that specifies the data-type operations:

```
public class Complex

---



|                                   |                                     |
|-----------------------------------|-------------------------------------|
| Complex(double real, double imag) |                                     |
| Complex plus(Complex b)           | <i>sum of this number and b</i>     |
| Complex times(Complex b)          | <i>product of this number and b</i> |
| double abs()                      | <i>magnitude</i>                    |
| double re()                       | <i>real part</i>                    |
| double im()                       | <i>imaginary part</i>               |
| String toString()                 | <i>string representation</i>        |


```

API for complex numbers (see PROGRAM 3.2.6)

For simplicity, we concentrate in the text on just the basic operations in this API, but EXERCISE 3.2.19 asks you to consider several other useful operations that might be included in such an API.

`Complex` (PROGRAM 3.2.6) is a class that implements this API. It has all of the same components as did `Charge` (and every Java data type implementation): instance variables (`re` and `im`), a constructor, instance methods (`plus()`, `times()`, `abs()`, `re()`, `im()`, and `toString()`), and a test client. The test client first sets z_0 to $1 + i$, then sets z to z_0 , and then evaluates

$$\begin{aligned} z &= z^2 + z_0 = (1 + i)^2 + (1 + i) = (1 + 2i - 1) + (1 + i) = 1 + 3i \\ z &= z^2 + z_0 = (1 + 3i)^2 + (1 + i) = (1 + 6i - 9) + (1 + i) = -7 + 7i \end{aligned}$$

This code is straightforward and similar to code that you have seen earlier in this chapter, with one exception: the code that implements the arithmetic methods makes use of a new mechanism for accessing object values.

Accessing instance variables of other objects of the same type. The instance methods `plus()` and `times()` each need to access values in two objects: the object passed as an argument and the object used to invoke the method. If we call the method with `a.plus(b)`, we can access the instance variables of `a` using the names `re` and `im`, as usual, but to access the instance variables of `b` we use the code `b.re` and `b.im`. Declaring the instance variables as `private` means that you cannot access directly the instance variables from *another* class. However, within a class, you can access directly the instance variables of *any* object from that *same* class, not just the instance variables of the invoking object.

Creating and returning new objects. Observe the manner in which `plus()` and `times()` provide return values to clients: they need to return a `Complex` value, so they each compute the requisite real and imaginary parts, use them to create a new object, and then return a reference to that object. This arrangement allow clients to manipulate complex numbers in a natural manner, by manipulating local variables of type `Complex`.

Chaining method calls. Observe the manner in which `main()` *chains* two method calls into one compact Java expression `z.times(z).plus(z0)`, which corresponds to the mathematical expression $z^2 + z_0$. This usage is convenient because you do not have to invent variable names for intermediate values. That is, you can use any object reference to invoke a method, even one without a name (such as one that is the result of evaluating a subexpression). If you study the expression, you can see that there is no ambiguity: moving from left to right, each method returns a reference to a `Complex` object, which is used to invoke the next instance method in the chain. If desired, we can use parentheses to override the default precedence order (for example, the Java expression `z.times(z.plus(z0))` corresponds to the mathematical expression $z(z + z_0)$).

Final instance variables. The two instance variables in `Complex` are `final`, meaning that their values are set for each `Complex` object when it is created and do not change during the lifetime of that object. We discuss the reasons behind this design decision in SECTION 3.3.

COMPLEX NUMBERS ARE THE BASIS FOR sophisticated calculations from applied mathematics that have many applications. With `Complex` we can concentrate on developing applications programs that use complex numbers without worrying about re-implementing methods such as `times()`, `abs()`, and so forth. Such methods are implemented once, and are *reusable*, as opposed to the alternative of copying this code into any applications program that uses complex numbers. Not only does this approach save debugging, but it also allows for changing or improving the implementation if needed, since it is separate from its clients. *Whenever you can clearly separate data and associated tasks within a computation, you should do so.*

To give you a feeling for the nature of calculations involving complex numbers and the utility of the complex number abstraction, we next consider a famous example of a `Complex` client.

Program 3.2.6 Complex number

```
public class Complex
{
    private final double re;
    private final double im;

    public Complex(double real, double imag)
    {   re = real; im = imag; }

    public Complex plus(Complex b)
    {   // Return the sum of this number and b.
        double real = re + b.re;
        double imag = im + b.im;
        return new Complex(real, imag);
    }

    public Complex times(Complex b)
    {   // Return the product of this number and b.
        double real = re * b.re - im * b.im;
        double imag = re * b.im + im * b.re;
        return new Complex(real, imag);
    }

    public double abs()
    {   return Math.sqrt(re*re + im*im); }

    public double re() { return re; }
    public double im() { return im; }

    public String toString()
    {   return re + " + " + im + "i"; }

    public static void main(String[] args)
    {
        Complex z0 = new Complex(1.0, 1.0);
        Complex z = z0;
        z = z.times(z).plus(z0);
        z = z.times(z).plus(z0);
        StdOut.println(z);
    }
}
```

re | *real part*
im | *imaginary part*

This data type is the basis for writing Java programs that manipulate complex numbers.

```
% java Complex
-7.0 + 7.0i
```

Mandelbrot set The *Mandelbrot set* is a specific set of complex numbers discovered by Benoît Mandelbrot. It has many fascinating properties. It is a fractal pattern that is related to the Barnsley fern, the Sierpinski triangle, the Brownian bridge, the Koch curve, the drunken turtle, and other recursive (self-similar) patterns and programs that we have seen in this book. Patterns of this kind are found in natural phenomena of all sorts, and these models and programs are very important in modern science.

The set of points in the Mandelbrot set cannot be described by a single mathematical equation. Instead, it is defined by an *algorithm*, and therefore is a perfect candidate for a Complex client: we study the set by writing a program to plot it.

The rule for determining whether a complex number z_0 is in the Mandelbrot set is simple. Consider the sequence of complex numbers $z_0, z_1, z_2, \dots, z_t, \dots$, where $z_{t+1} = (z_t)^2 + z_0$. For example, this table shows the first few elements in the sequence corresponding to $z_0 = 1 + i$:

t	z_t	$(z_t)^2$	$(z_t)^2 + z_0$	$=$	z_{t+1}
0	$1 + i$	$1 + 2i + i^2 = 2i$	$2i + (1 + i)$	$=$	$1 + 3i$
1	$1 + 3i$	$1 + 6i + 9i^2 = -8 + 6i$	$-8 + 6i + (1 + i)$	$=$	$-7 + 7i$
2	$-7 + 7i$	$49 - 98i + 49i^2 = -98i$	$-98i + (1 + i)$	$=$	$1 - 97i$

Mandelbrot sequence computation

Now, if the sequence $|z_t|$ diverges to infinity, then z_0 is *not* in the Mandelbrot set; if the sequence is bounded, then z_0 is in the Mandelbrot set. For many points, the test is simple. For many other points, the test requires more computation, as indicated by the examples in this table:

z_0	$0 + 0i$	$2 + 0i$	$1 + i$	$0 + i$	$-0.5 + 0i$	$-0.10 - 0.64i$
z_1	0	6	$1 + 3i$	$-1 + i$	-0.25	$-0.30 - 0.77i$
z_2	0	38	$-7 + 7i$	$-i$	-0.44	$-0.40 - 0.18i$
z_3	0	1446	$1 - 97i$	$-1 + i$	-0.31	$0.23 - 0.50i$
z_4	0	2090918	$-9407 - 193i$	$-i$	-0.40	$-0.09 - 0.87i$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
<i>in set?</i>	yes	no	no	yes	yes	yes

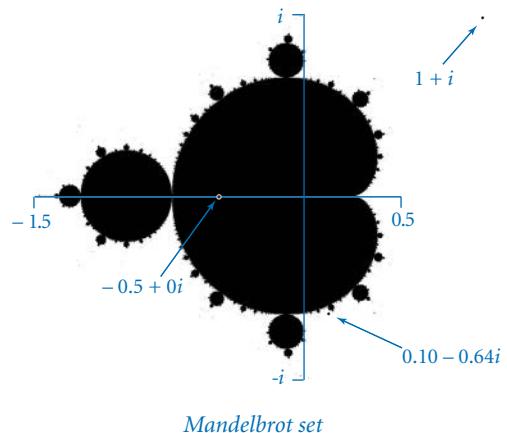
Mandelbrot sequence for several starting points

For brevity, the numbers in the rightmost two columns of this table are given to just two decimal places. In some cases, we can prove whether numbers are in the set. For example, $0 + 0i$ is certainly in the set (since the magnitude of all the numbers in its sequence is 0), and $2 + 0i$ is certainly not in the set (since its sequence dominates the powers of 2, which diverges to infinity). In some other cases, the growth is readily apparent. For example, $1 + i$ does not seem to be in the set. Other sequences exhibit a periodic behavior. For example, i maps to $-1 + i$ to $-i$ to $-1 + i$ to $-i$, and so forth. Still other sequences go on for a very long time before the magnitude of the numbers begins to get large.

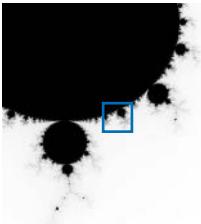
To visualize the Mandelbrot set, we sample *complex* points, just as we sample real-valued points to plot a real-valued function. Each complex number $x + iy$ corresponds to a point (x, y) in the plane, so we can plot the results as follows: for a specified resolution n , we define an evenly spaced n -by- n pixel grid within a specified square and draw a black pixel if the corresponding point is in the Mandelbrot set and a white pixel if it is not. This plot is a strange and wondrous pattern, with all the black dots connected and falling roughly within in the 2-by-2 square centered at the point $-1/2 + 0i$. Large values of n will produce higher-resolution images, at the cost of more computation.

Looking closer reveals self-similarities throughout the plot. For example, the same bulbous pattern with self-similar appendages appears all around the contour of the main black cardioid region, of sizes that resemble the simple ruler function of PROGRAM 1.2.1. When we zoom in near the edge of the cardioid, tiny self-similar cardioids appear!

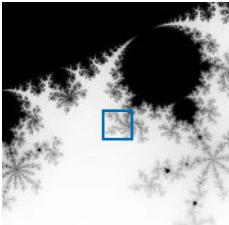
But how, precisely, do we produce such plots? Actually, no one knows for sure, because there is no simple test that would enable us to conclude that a point is surely in the set. Given a complex number, we can compute the terms at the beginning of its sequence, but may not be able to know for sure that the sequence remains bounded. There *is* a test that tells us for sure that a complex number is *not* in the set: if the magnitude of any number in its sequence ever exceeds 2 (such as for $1 + 3i$), then the sequence surely will diverge.



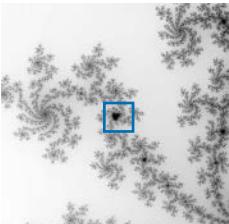
512 .1015 -.633 1.0



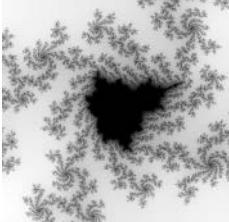
512 .1015 -.633 .10



512 .1015 -.633 .01



512 .1015 -.633 .001



Zooming in on the set

Mandelbrot (PROGRAM 3.2.7) uses this test to plot a visual representation of the Mandelbrot set. Since our knowledge of the set is not quite black-and-white, we use grayscale in our visual representation. It is based on the function `mand()`, which takes a `Complex` argument `z0` and an `int` argument `max` and computes the Mandelbrot iteration sequence starting at `z0`, returning the number of iterations for which the magnitude stays less than (or equal to) 2, up to the limit `max`.

For each pixel, the `main()` method in `Mandelbrot` computes the complex number `z0` corresponding to the pixel and then computes `255 - mand(z0, 255)` to create a grayscale color for the pixel. Any pixel that is not black corresponds to a complex number that we know to be not in the Mandelbrot set because the magnitude of the numbers in its sequence exceeds 2 (and therefore will go to infinity). The black pixels (grayscale value 0) correspond to points that we assume to be in the set because the magnitude did not exceed 2 during the first 255 Mandelbrot iterations.

The complexity of the images that this simple program produces is remarkable, even when we zoom in on a tiny portion of the plane. For even more dramatic pictures, we can use color (see EXERCISE 3.2.35). And the Mandelbrot set is derived from iterating just one function $f(z) = (z^2 + z_0)$: we have a great deal to learn from studying the properties of other functions as well.

The simplicity of the code masks a substantial amount of computation. There are about 0.25 million pixels in a 512-by-512 image, and all of the black ones require 255 Mandelbrot iterations, so producing an image with `Mandelbrot` requires hundreds of millions of operations on `Complex` values.

Fascinating as it is to study, our primary interest in `Mandelbrot` is as an example client of `Complex`, to illustrate that computing with a data type that is not built into Java (complex numbers) is a natural and useful programming activity. `Mandelbrot` is a simple and natural expression of the computation, made so by the design and implementation of `Complex`. You could implement `Mandelbrot` without using `Complex`, but the code would essentially have to merge together the code in PROGRAM 3.2.6 and PROGRAM 3.2.7 and, therefore, would be much more difficult to understand. *Whenever you can clearly separate tasks within a program, you should do so.*

Program 3.2.7 Mandelbrot set

```

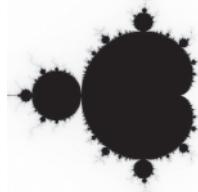
import java.awt.Color;
public class Mandelbrot
{
    private static int mand(Complex z0, int max)
    {
        Complex z = z0;
        for (int t = 0; t < max; t++)
        {
            if (z.abs() > 2.0) return t;
            z = z.times(z).plus(z0);
        }
        return max;
    }

    public static void main(String[] args)
    {
        double xc    = Double.parseDouble(args[0]);
        double yc    = Double.parseDouble(args[1]);
        double size  = Double.parseDouble(args[2]);
        int n = 512;
        Picture picture = new Picture(n, n);
        for (int i = 0; i < n; i++)
            for (int j = 0; j < n; j++)
            {
                double x0 = xc - size/2 + size*i/n;
                double y0 = yc - size/2 + size*j/n;
                Complex z0 = new Complex(x0, y0);
                int gray = 255 - mand(z0, 255);
                Color c = new Color(gray, gray, gray);
                picture.set(i, n-1-j, c);
            }
        picture.show();
    }
}

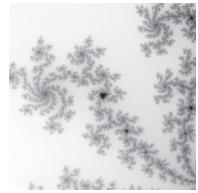
```

x0, y0 point in square
 z0 $x_0 + iy_0$
 max iteration limit
 xc, yc center of square
 size square is size-by-size
 n grid is n-by-n pixels
 pic image for output
 c pixel color for output

-.5 0 2



.1015 -.633 .01



This program takes three command-line arguments that specify the center and size of a square region of interest, and makes a digital image showing the result of sampling the Mandelbrot set in that region at a size-by-size grid of evenly spaced points. It colors each pixel with a grayscale value that is determined by counting the number of iterations before the Mandelbrot sequence for the corresponding complex number exceeds 2.0 in magnitude, up to 255.

Commercial data processing One of the driving forces behind the development of object-oriented programming has been the need for an extensive amount of reliable software for commercial data processing. As an illustration, we consider an example of a data type that might be used by a financial institution to keep track of customer information.

Suppose that a stockbroker needs to maintain customer accounts containing shares of various stocks. That is, the set of values the broker needs to process includes the customer's name, number of different stocks held, number of shares and ticker symbol for each stock, and cash on hand. To process an account, the broker needs at least the operations defined in this API:

public class StockAccount	
StockAccount(String filename)	<i>create a new account from file</i>
double valueOf()	<i>total value of account dollars</i>
void buy(int amount, String symbol)	<i>add shares of stock to account</i>
void sell(int amount, String symbol)	<i>subtract shares of stock from account</i>
void save(String filename)	<i>save account to file</i>
void printReport()	<i>print a detailed report of stocks and values</i>

API for processing stock accounts (see PROGRAM 3.2.8)

The broker certainly needs to buy, sell, and provide reports to the customer, but the first key to understanding this kind of data processing is to consider the `StockAccount()` constructor and the `save()` method in this API. The customer information has a long lifetime and needs to be saved in a *file* or *database*. To process an account, a client program needs to read information from the corresponding file; process the information as appropriate; and, if the information changes, write it back to the file, saving it for later. To enable this kind of processing, we need a *file format* and an *internal representation*, or a *data structure*, for the account information.

As a (whimsical) running example, we imagine that a broker is maintaining a small portfolio of stocks in leading software companies for Alan Turing, the father of computing. *As an aside:* Turing's life story is a fascinating one that is worth investigating further. Among many other things, he worked on computational cryptography that helped to bring about the end of World War II, he developed the basis for the theory of computing, he designed and built one of the first computers, and

he was a pioneer in artificial intelligence research. It is perhaps safe to assume that Turing, whatever his financial situation as an academic researcher in the middle of the last century, would be sufficiently optimistic about the potential impact of computing software in today's world that he would make some small investments.

File format. Modern systems often use text files, even for data, to minimize dependence on formats defined by any one program. For simplicity, we use a direct representation where we list the account holder's name (a string), cash balance (a floating-point number), and number of stocks held (an integer), followed by a line for each stock giving the number of shares and the ticker symbol, as shown in the example at right. It is also wise to use *tags* such as <Name>, <Number of shares>, and so forth to label all the information so as to further minimize dependencies on any one program, but we omit such tags here for brevity.

```
% more Turing.txt
Turing, Alan
10.24
4
100 ADBE
25 GOOG
97 IBM
250 MSFT
```

File format

Data structure. To represent information for processing by Java programs, we use *instance variables*. They specify the type of information and provide the structure that we need to clearly refer to it in code. For our example, we clearly need the following:

- A `String` value for the account name
- A `double` value for the cash balance
- An `int` value for the number of stocks
- An array of `String` values for stock symbols
- An array of `int` values for numbers of shares

We directly reflect these choices in the instance variable declarations in `StockAccount` (PROGRAM 3.2.8). The arrays `stocks[]` and `shares[]` are known as *parallel arrays*. Given an index `i`, `stocks[i]` gives a stock symbol and `shares[i]` gives the number of shares of that stock in the account. An alternative design would be to define a separate data type for stocks to manipulate this information for each stock and maintain an array of objects of that type in `StockAccount`.

```
public class StockAccount
{
    private final String name;
    private double cash;
    private int n;
    private int[] shares;
    private String[] stocks;
    ...
}
```

Data structure blueprint

StockAccount includes a constructor, which reads a file in the specified format and creates an account with this information. Also, our broker needs to provide a periodic detailed report to customers, perhaps using the following code for `printReport()` in StockAccount, which relies on StockQuote (PROGRAM 3.1.8) to retrieve each stock's price from the web.

```
public void printReport()
{
    StdOut.println(name);
    double total = cash;
    for (int i = 0; i < n; i++)
    {
        int amount = shares[i];
        double price = StockQuote.priceOf(stocks[i]);
        total += amount * price;
        StdOut.printf("%4d %5s ", amount, stocks[i]);
        StdOut.printf("%9.2f %11.2f\n", price, amount*price);
    }
    StdOut.printf("%21s %10.2f\n", "Cash: ", cash);
    StdOut.printf("%21s %10.2f\n", "Total:", total);
}
```

Implementations of `valueOf()` and `save()` are straightforward (see EXERCISE 3.2.22). The implementations of `buy()` and `sell()` require the use of basic mechanisms introduced in SECTION 4.4, so we defer them to EXERCISE 4.4.65.

On the one hand, this client illustrates the kind of computing that was one of the primary drivers in the evolution of computing in the 1950s. Banks and other companies bought early computers precisely because of the need to do such financial reporting. For example, formatted writing was developed precisely for such applications. On the other hand, this client exemplifies modern web-centric computing, as it gets information directly from the web, without using a browser.

Beyond these basic methods, an actual application of these ideas would likely use a number of other clients. For example, a broker might want to create an array of all accounts, then process a list of transactions that both modify the information in those accounts and actually carry out the transactions through the web. Of course, such code needs to be developed with great care!

WHEN YOU LEARNED HOW TO DEFINE functions that can be used in multiple places in a program (or in other programs) in CHAPTER 2, you moved from a world where programs are simply sequences of statements in a single file to the world of modular programming, summarized in our mantra: *whenever you can clearly separate subtasks within a program, you should do so*. The analogous capability for data, introduced in this chapter, moves you from a world where data has to be one of a few elementary types of data to a world where you can define your own data types. This profound new capability vastly extends the scope of your programming. As with the concept of a function, once you have learned to implement and use data types, you will marvel at the primitive nature of programs that do not use them.

But object-oriented programming is much more than structuring data. It enables us to associate the data relevant to a subtask with the operations that manipulate that data and to keep both separate in an independent module. With object-oriented programming, our mantra is this: *whenever you can clearly separate data and associated operations for subtasks within a computation, you should do so*.

The examples that we have considered are persuasive evidence that object-oriented programming can play a useful role in a broad range of activities. Whether we are trying to design and build a physical artifact, develop a software system, understand the natural world, or process information, a key first step is to define an appropriate abstraction, such as a geometric description of the physical artifact, a modular design of the software system, a mathematical model of the natural world, or a data structure for the information. When we want to write programs to manipulate instances of a well-defined abstraction, we can just implement it as a data type in a Java class and write Java programs to create and manipulate objects of that type.

Each time that we develop a class that makes use of other classes by creating and manipulating objects of the type defined by the class, we are programming at a higher layer of abstraction. In the next section, we discuss some of the design challenges inherent in this kind of programming.

Q&A

Q. Do instance variables have default initial values that we can depend upon?

A. Yes. They are automatically set to 0 for numeric types, `false` for the boolean type, and the special value `null` for all reference types. These values are consistent with the way Java automatically initializes array elements. This automatic initialization ensures that every instance variable always stores a legal (but not necessarily meaningful) value. Writing code that depends on these values is controversial: some experienced programmers embrace the idea because the resulting code can be very compact; others avoid it because the code is opaque to someone who does not know the rules.

Q. What is `null`?

A. It is a literal value that refers to no object. Using the `null` reference to invoke an instance method is meaningless and results in a `NullPointerException`. Often, this is a sign that you failed to properly initialize an object's instance variables or an array's elements.

Q. Can I initialize an instance variable to a value other than the default value when I declare it?

A. Normally, you initialize instance variables to nondefault values in the constructor. However, you can specify initial values for an instance variables when you declare them, using the same conventions as for inline initialization of local variables. This inline initialization occurs before the constructor is called.

Q. Must every class have a constructor?

A. Yes, but if you do not specify a constructor, Java provides a default (no-argument) constructor automatically. When the client invokes that constructor with `new`, the instance variables are auto-initialized as usual. If you *do* specify a constructor, then the default no-argument constructor disappears.

Q. Suppose I do not include a `toString()` method. What happens if I try to print an object of that type with `StdOut.println()`?

A. The printed output is an integer that is unlikely to be of much use to you.

Q. Can I have a static method in a class that implements a data type?

A. Of course. For example, all of our classes have `main()`. But it is easy to get confused when static methods and instance methods are mixed up in the same class. For example, it is natural to consider using static methods for operations that involve multiple objects where none of them naturally suggests itself as the one that should invoke the method. For example, we write `z.abs()` to get $|z|$, but writing `a.plus(b)` to get the sum is perhaps not so natural. Why not `b.plus(a)`? An alternative is to define a static method like the following within `Complex`:

```
public static Complex plus(Complex a, Complex b)
{
    return new Complex(a.re + b.re, a.im + b.im);
}
```

We generally avoid such usage and live with expressions that do not mix static methods and instance methods to avoid having to write code like this:

```
z = Complex.plus(Complex.times(z, z), z0)
```

Instead, we would write:

```
z = z.times(z).plus(z0)
```

Q. These computations with `plus()` and `times()` seem rather clumsy. Is there some way to use symbols like `+` and `*` in expressions involving objects where they make sense, such as `Complex` and `Vector`, so that we could write more compact expressions like `z = z * z + z0` instead?

A. Some languages (notably C++ and Python) support this feature, which is known as *operator overloading*, but Java does not do so. As usual, this is a decision of the language designers that we just live with, but many Java programmers do not consider this to be much of a loss. Operator overloading makes sense only for types that represent numeric or algebraic abstractions, a small fraction of the total, and many programs are easier to understand when operations have descriptive names such as `plus()` and `times()`. The APL programming language of the 1970s took this issue to the opposite extreme by insisting that *every* operation be represented by a single symbol (including Greek letters).

Q. Are there other kinds of variables besides argument, local, and instance variables in a class?

A. If you include the keyword `static` in a variable declaration (outside of any method), it creates a completely different type of variable, known as a *static variable* or *class variable*. Like instance variables, static variables are accessible to every method in the class; however, they are not associated with any object—there is one variable per class. In older programming languages, such variables are known as *global variables* because of their global scope. In modern programming, we focus on limiting scope, so we rarely use such variables.

Q. `Mandelbrot` creates tens of millions of `Complex` objects. Doesn't all that object-creation overhead slow things down?

A. Yes, but not so much that we cannot generate our plots. Our goal is to make our programs readable and easy to maintain—limiting scope via the complex number abstraction helps us achieve that goal. You certainly could speed up `Mandelbrot` by bypassing the complex number abstraction or by using a different implementation of `Complex`.

Exercises

3.2.1 Consider the following data-type implementation for axis-aligned rectangles, which represents each rectangle with the coordinates of its center point and its width and height:

```
public class Rectangle
{
    private final double x, y;      // center of rectangle
    private final double width;     // width of rectangle
    private final double height;    // height of rectangle

    public Rectangle(double x0, double y0, double w, double h)
    {
        x = x0;
        y = y0;
        width = w;
        height = h;
    }

    public double area()
    {   return width * height;   }

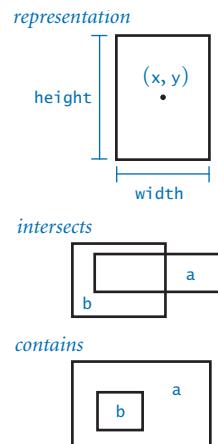
    public double perimeter()
    { /* Compute perimeter. */ }

    public boolean intersects(Rectangle b)
    { /* Does this rectangle intersect b? */ }

    public boolean contains(Rectangle b)
    { /* Is b inside this rectangle? */ }

    public void draw(Rectangle b)
    { /* Draw rectangle on standard drawing. */ }
}
```

Write an API for this class, and fill in the code for `perimeter()`, `intersects()`, and `contains()`. Note: Consider two rectangles to intersect if they share one or more common points (improper intersections). For example, `a.intersects(a)` and `a.contains(a)` are both `true`.



3.2.2 Write a test client for `Rectangle` that takes three command-line arguments `n`, `min`, and `max`; generates `n` random rectangles whose width and height are uniformly distributed between `min` and `max` in the unit square; draws them on standard drawing; and prints their average area and perimeter to standard output.

3.2.3 Add code to your test client from the previous exercise code to compute the average number of rectangles that intersect a given rectangle.

3.2.4 Develop an implementation of your `Rectangle` API from EXERCISE 3.2.1 that represents rectangles with the *x*- and *y*-coordinates of their lower-left and upper-right corners. Do *not* change the API.

3.2.5 What is wrong with the following code?

```
public class Charge
{
    private double rx, ry;    // position
    private double q;         // charge

    public Charge(double x0, double y0, double q0)
    {
        double rx = x0;
        double ry = y0;
        double q = q0;
    }
    ...
}
```

Answer: The assignment statements in the constructor are also *declarations* that create new *local* variables `rx`, `ry`, and `q`, which go out of scope when the constructor completes. The *instance* variables `rx`, `ry`, and `q` remain at their default value of 0. *Note:* A local variable with the same name as an instance variable is said to *shadow* the instance variable—we discuss in the next section a way to refer to shadowed instance variables, which are best avoided by beginners.

3.2.6 Create a data type `Location` that represents a location on Earth using latitudes and longitudes. Include a method `distanceTo()` that computes distances using the great-circle distance (see EXERCISE 1.2.33).

3.2.7 Implement a data type `Rational` for rational numbers that supports addition, subtraction, multiplication, and division.

<code>public class Rational</code>	

<code>Rational(int numerator, int denominator)</code>	
<code>Rational plus(Rational b)</code>	<i>sum of this number and b</i>
<code>Rational minus(Rational b)</code>	<i>difference of this number and b</i>
<code>Rational times(Rational b)</code>	<i>product of this number and b</i>
<code>Rational divides(Rational b)</code>	<i>quotient of this number and b</i>
<code>String toString()</code>	<i>string representation</i>

Use `Euclid.gcd()` (PROGRAM 2.3.1) to ensure that the numerator and the denominator never have any common factors. Include a test client that exercises all of your methods. Do not worry about testing for integer overflow (see EXERCISE 3.3.17).

3.2.8 Write a data type `Interval` that implements the following API:

<code>public class Interval</code>	

<code>Interval(double min, double max)</code>	
<code>boolean contains(double x)</code>	<i>is x in this interval?</i>
<code>boolean intersects(Interval b)</code>	<i>do this interval and b intersect?</i>
<code>String toString()</code>	<i>string representation</i>

An interval is defined to be the set of all points on the line greater than or equal to `min` and less than or equal to `max`. In particular, an interval with `max` less than `min` is empty. Write a client that is a filter that takes a floating-point command-line argument `x` and prints all of the intervals on standard input (each defined by a pair of `double` values) that contain `x`.

3.2.9 Write a client for your `Interval` class from the previous exercise that takes an integer command-line argument `n`, reads `n` intervals (each defined by a pair of `double` values) from standard input, and prints all pairs of intervals that intersect.

3.2.10 Develop an implementation of your `Rectangle` API from EXERCISE 3.2.1 that takes advantage of the `Interval` data type to simplify and clarify the code.

3.2.11 Write a data type `Point` that implements the following API:

```
public class Point
    Point(double x, double y)
    double distanceTo(Point q)      Euclidean distance between this point and q
    String toString()               string representation
```

3.2.12 Add methods to `Stopwatch` that allow clients to stop and restart the stopwatch.

3.2.13 Use `Stopwatch` to compare the cost of computing harmonic numbers with a `for` loop (see PROGRAM 1.3.5) as opposed to using the recursive method given in SECTION 2.3.

3.2.14 Develop a version of `Histogram` that uses `Draw`, so that a client can create multiple histograms. Add to the display a red vertical line showing the sample mean and blue vertical lines at a distance of two standard deviations from the mean. Use a test client that creates histograms for flipping coins (Bernoulli trials) with a biased coin that is heads with probability p , for $p = 0.2, 0.4, 0.6$ and 0.8 , taking the number of flips and the number of trials from the command line, as in PROGRAM 3.2.3.

3.2.15 Modify the test client in `Turtle` to take an odd integer n as a command-line argument and draw a star with n points.

3.2.16 Modify the `toString()` method in `Complex` (PROGRAM 3.2.6) so that it prints complex numbers in the traditional format. For example, it should print the value $3 - i$ as $3 - i$ instead of $3.0 + -1.0i$, the value 3 as 3 instead of $3.0 + 0.0i$, and the value $3i$ as $3i$ instead of $0.0 + 3.0i$.

3.2.17 Write a `Complex` client that takes three floating-point numbers a , b , and c as command-line arguments and prints the two (complex) roots of $ax^2 + bx + c$.

3.2.18 Write a `Complex` client `RootsOfUnity` that takes two `double` values a and b and an integer n from the command line and prints the n th roots of $a + bi$. Note: Skip this exercise if you are not familiar with the operation of taking roots of complex numbers.

3.2.19 Implement the following additions to the `Complex` API:

<code>double theta()</code>	<i>phase (angle) of this number</i>
<code>Complex minus(Complex b)</code>	<i>difference of this number and b</i>
<code>Complex conjugate()</code>	<i>conjugate of this number</i>
<code>Complex divides(Complex b)</code>	<i>result of dividing this number by b</i>
<code>Complex power(int b)</code>	<i>result of raising this number to the bth power</i>

Write a test client that exercises all of your methods.

3.2.20 Suppose you want to add a constructor to `Complex` that takes a `double` value as its argument and creates a `Complex` number with that value as the real part (and no imaginary part). You write the following code:

```
public void Complex(double real)
{
    re = real;
    im = 0.0;
}
```

But then the statement `Complex c = new Complex(1.0);` does not compile. Why?

Solution: Constructors do not have return types, not even `void`. This code defines a method named `Complex`, not a constructor. Remove the keyword `void`.

3.2.21 Find a `Complex` value for which `mand()` returns a number greater than 100, and then zoom in on that value, as in the example in the text.

3.2.22 Implement the `valueOf()` and `save()` methods for `StockAccount` (PROGRAM 3.2.8).

Creative Exercises

3.2.23 *Electric potential visualization.* Write a program `Potential` that creates an array of charged particles from values given on standard input (each charged particle is specified by its x -coordinate, y -coordinate, and charge value) and produces a visualization of the electric potential in the unit square. To do so, sample points in the unit square. For each sampled point, compute the electric potential at that point (by summing the electric potentials due to each charged particle) and plot the corresponding point in a shade of gray proportional to the electric potential.

3.2.24 *Mutable charges.* Modify `Charge` (PROGRAM 3.2.1) so that the charge value `q` is not `final`, and add a method `increaseCharge()` that takes a `double` argument and adds the given value to the charge. Then, write a client that initializes an array with

```
Charge[] a = new Charge[3];
a[0] = new Charge(0.4, 0.6, 50);
a[1] = new Charge(0.5, 0.5, -5);
a[2] = new Charge(0.6, 0.6, 50);
```

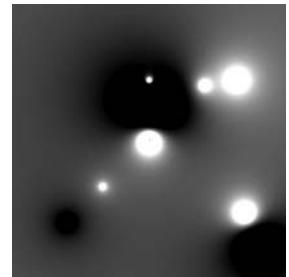
and then displays the result of slowly decreasing the charge value of `a[i]` by wrapping the code that computes the images in a loop like the following:

```
for (int t = 0; t < 100; t++)
{
    // Compute the picture.
    picture.show();
    a[1].increaseCharge(-2.0);
}
```



% `more charges.txt`

```
9
.51 .63 -100
.50 .50 40
.50 .72 10
.33 .33 5
.20 .20 -10
.70 .70 10
.82 .72 20
.85 .23 30
.90 .12 -50
```



% `java Potential < charges.txt`

Potential visualization for a set of charges



-155



-205

Mutating a charge

3.2.25 *Complex timing.* Write a Stopwatch client that compares the cost of using Complex to the cost of writing code that directly manipulates two double values, for the task of doing the calculations in Mandelbrot. Specifically, create a version of Mandelbrot that just does the calculations (remove the code that refers to Picture), then create a version of that program that does not use Complex, and then compute the ratio of the running times.

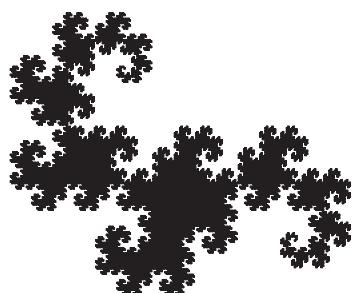
3.2.26 *Quaternions.* In 1843, Sir William Hamilton discovered an extension to complex numbers called quaternions. A *quaternion* is a 4-tuple $a = (a_0, a_1, a_2, a_3)$ with the following operations:

- *Magnitude:* $|a| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$
- *Conjugate:* the conjugate of a is $(a_0, -a_1, -a_2, -a_3)$
- *Inverse:* $a^{-1} = (a_0/|a|^2, -a_1/|a|^2, -a_2/|a|^2, -a_3/|a|^2)$
- *Sum:* $a + b = (a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3)$
- *Product:* $a \times b = (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3, a_0 b_1 - a_1 b_0 + a_2 b_3 - a_3 b_2, a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1, a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0)$
- *Quotient:* $a / b = ab^{-1}$

Create a data type Quaternion for quaternions and a test client that exercises all of your code. Quaternions extend the concept of rotation in three dimensions to four dimensions. They are used in computer graphics, control theory, signal processing, and orbital mechanics.

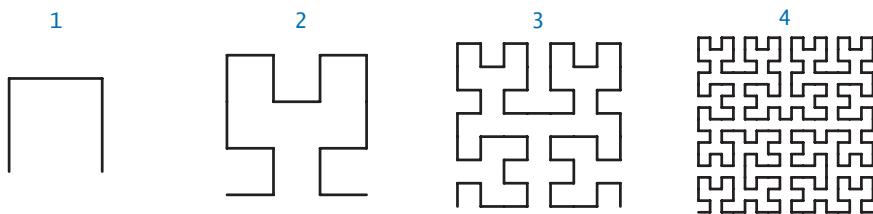
3.2.27 *Dragon curves.* Write a recursive Turtle client Dragon that draws dragon curves (see EXERCISE 1.2.35 and EXERCISE 1.5.9).

```
% java Dragon 15
```



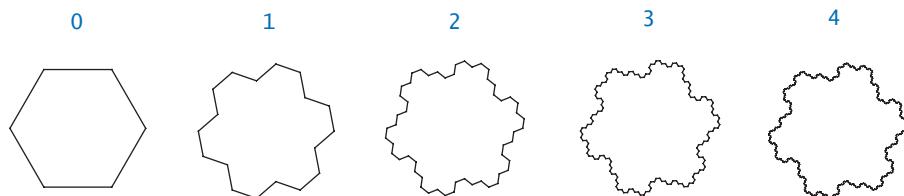
Answer: These curves, which were originally discovered by three NASA physicists, were popularized in the 1960s by Martin Gardner and later used by Michael Crichton in the book and movie *Jurassic Park*. This exercise can be solved with remarkably compact code, based on a pair of mutually recursive methods derived directly from the definition in EXERCISE 1.2.35. One of them, dragon(), should draw the curve as you expect; the other, nogard(), should draw the curve in *reverse* order. See the booksite for details.

3.2.28 *Hilbert curves.* A *space-filling curve* is a continuous curve in the unit square that passes through every point. Write a recursive Turtle client that produces these recursive patterns, which approach a space-filling curve that was defined by the mathematician David Hilbert at the end of the 19th century.



Partial answer: Design a pair of mutually recursive methods: `hilbert()`, which traverses a Hilbert curve, and `treblih()`, which traverses a Hilbert curve *in reverse order*. See the booksite for details.

3.2.29 *Gosper island.* Write a recursive Turtle client that produces these recursive patterns.



3.2.30 *Chemical elements.* Create a data type `ChemicalElement` for entries in the *Periodic Table of Elements*. Include data-type values for element, atomic number, symbol, and atomic weight, and accessor methods for each of these values. Then create a data type `PeriodicTable` that reads values from a file to create an array of `ChemicalElement` objects (you can find the file and a description of its formation on the booksite) and responds to queries on standard input so that a user can type a molecular equation like H2O and the program responds by printing the molecular weight. Develop APIs and implementations for each data type.

3.2.31 *Data analysis.* Write a data type for use in running experiments where the control variable is an integer in the range $[0, n]$ and the dependent variable is a *double* value. (For example, studying the running time of a program that takes an integer argument would involve such experiments.) Implement the following API:

```
public class Data
```

```
    Data(int n, int max)
```

*create a new data analysis object
for the n integer values in $[0, n]$*

```
    double addDataPoint(int i, double x)
```

add a data point (i, x)

```
    void plotPoints()
```

plot all the data points

Use the static methods in `StdStats` to do the statistical calculations and draw the plots. Write a test client that plots the results (percolation probability) of running experiments with `Percolation` as the grid size n increases.

3.2.32 *Stock prices.* The file `DJIA.csv` on the booksite contains all closing stock prices in the history of the Dow Jones Industrial Average, in the comma-separated-value format. Create a data type `DowJonesEntry` that can hold one entry in the table, with values for date, opening price, daily high, daily low, closing price, and so forth. Then, create a data type `DowJones` that reads the file to build an array of `DowJonesEntry` objects and supports methods for computing averages over various periods of time. Finally, create interesting `DowJones` clients to produce plots of the data. Be creative: this path is well trodden.

3.2.33 *Biggest winner and biggest loser.* Write a `StockAccount` client that builds an array of `StockAccount` objects, computes the total value of each account, and prints a report for the accounts with the largest and smallest values. Assume that the information in the accounts is kept in a single file that contains the information for the accounts, one after the other, in the format given in the text.

3.2.34 *Chaos with Newton's method.* The polynomial $f(z) = z^4 - 1$ has four roots: at $1, -1, i$, and $-i$. We can find the roots using Newton's method in the complex plane: $z_{k+1} = z_k - f(z_k) / f'(z_k)$. Here, $f(z) = z^4 - 1$ and $f'(z) = 4z^3$. The method converges to one of the four roots, depending on the starting point z_0 . Write a Complex and Picture client `NewtonChaos` that takes a command-line argument n and creates an n -by- n picture corresponding to the square of size 2 centered at the origin. Color each pixel white, red, green, or blue according to which of the four roots the corresponding complex number converges (black if no convergence after 100 iterations).

3.2.35 *Color Mandelbrot plot.* Create a file of 256 integer triples that represent interesting `Color` values, and then use those colors instead of grayscale values to plot each pixel in `Mandelbrot`. Read the values to create an array of 256 `Color` values, then index into that array with the return value of `mand()`. By experimenting with various color choices at various places in the set, you can produce astonishing images. See `mandel.txt` on the booksite for an example.

3.2.36 *Julia sets.* The *Julia set* for a given complex number c is a set of points related to the Mandelbrot function. Instead of fixing z and varying c , we fix c and vary z . Those points z for which the modified Mandelbrot function stays bounded are in the *Julia set*; those for which the sequence diverges to infinity are not in the set. All points z of interest lie in the 4-by-4 box centered at the origin. The Julia set for c is connected if and only if c is in the Mandelbrot set! Write a program `ColorJulia` that takes two command-line arguments a and b , and plots a color version of the Julia set for $c = a + bi$, using the color-table method described in the previous exercise.

3.3 Designing Data Types

THE ABILITY TO CREATE DATA TYPES turns every programmer into a language designer. You do not have to settle for the types of data and associated operations that are built into the language, because you can create your own data types and write client programs that use them. For example, Java does not have a predefined data type for complex numbers, but you can define `Complex` and write client programs such as `Mandelbrot`. Similarly, Java does not have a built-in facility for turtle graphics, but you can define `Turtle` and write client programs that take immediate advantage of this abstraction. Even when Java does include a particular facility, you might prefer to create separate data types tailored to your specific needs, as we do with `Picture`, `In`, `Out`, and `Draw`.

The first thing that we strive for when creating a program is an understanding of the types of data that we will need. Developing this understanding is a *design* activity. In this section, we focus on developing APIs as a critical step in the development of any program. We need to consider various alternatives, understand their impact on both client programs and implementations, and refine the design to strike an appropriate balance between the needs of clients and the possible implementation strategies.

If you take a course in systems programming, you will learn that this design activity is critical when building large systems, and that Java and similar languages have powerful high-level mechanisms that support code reuse when writing large programs. Many of these mechanisms are intended for use by experts building large systems, but the general approach is worthwhile for every programmer, and some of these mechanisms are useful when writing small programs.

In this section we discuss *encapsulation*, *immutability*, and *inheritance*, with particular attention to the use of these mechanisms in data-type design to enable modular programming, facilitate debugging, and write clear and correct code.

At the end of the section, we discuss Java's mechanisms for use in checking design assumptions against actual conditions at run time. Such tools are invaluable aids in developing reliable software.

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Programs in this section

Designing APIs In SECTION 3.1, we wrote client programs that *use* APIs; in SECTION 3.2, we *implemented* APIs. Now we consider the challenge of *designing* APIs. Treating these topics in this order and with this focus is appropriate because most of the time that you spend programming will be writing client programs.

Often the most important and most challenging step in building software is designing the APIs. This task takes practice, careful deliberation, and many iterations. However, any time spent designing a good API is certain to be repaid in time saved during debugging or with code reuse.

Articulating an API might seem to be overkill when writing a small program, but you should consider writing every program as though you will need to reuse the code someday—not because you know that you will reuse that code, but because you are quite likely to want to reuse *some* of your code and you cannot know *which* code you will need.

Standards. It is easy to understand why writing to an API is so important by considering other domains. From railroad tracks, to threaded nuts and bolts, to MP3s, to radio frequencies, to Internet standards, we know that using a common standard interface enables the broadest usage of a technology. Java itself is another example: your Java programs are clients of the *Java virtual machine*, which is a standard interface that is implemented on a wide variety of hardware and software platforms. By using APIs to separate clients from implementations, we reap the benefits of standard interfaces for every program that we write.

client

```
Charge c1 = new Charge(0.51, 0.63, 21.3);
```

```
c1.potentialAt(x, y)
```

*creates objects
and invokes methods*

API

```
public class Charge
```

 Charge(double x0, double y0, double q0)

 double potentialAt(double x, double y) *p*otential at (x, y)
*d*ue to charge

 String toString()

*s*tring
*r*epresentation

*d*efines signatures
*a*nd describes methods

implementation

```
public class Charge
```

```
{    private final double rx, ry;
    private final double q;
```

```
    public Charge(double x0, double y0, double q0)
    {    ...    }
```

```
    public double potentialAt(double x, double y)
    {    ...    }
```

```
    public String toString()
    {    ...    }
```

*d*efines instance variables
*a*nd implements methods

*O*bject-oriented library abstraction

Specification problem. Our APIs are lists of methods, along with brief English-language descriptions of what the methods are supposed to do. Ideally, an API would clearly articulate behavior for all possible inputs, including side effects, and then we would have software to check that implementations meet the specification. Unfortunately, a fundamental result from theoretical computer science, known as the *specification problem*, says that this goal is actually *impossible* to achieve. Briefly, such a specification would have to be written in a formal language like a programming language, and the problem of determining whether two programs perform the same computation is known, mathematically, to be *unsolvable*. (If you are interested in this idea, you can learn much more about the nature of unsolvable problems and their role in our understanding of the nature of computation in a course in theoretical computer science.) Therefore, we resort to informal descriptions with examples, such as those in the text surrounding our APIs.

Wide interfaces. A *wide interface* is one that has an excessive number of methods. An important principle to follow in designing an API is to *avoid wide interfaces*. The size of an API naturally tends to grow over time because it is easy to add methods to an existing API, whereas it is difficult to remove methods without breaking existing clients. In certain situations, wide interfaces are justified—for example, in widely used systems libraries such as `String`. Various techniques are helpful in reducing the effective width of an interface. One approach is to include methods that are orthogonal in functionality. For example, Java’s `Math` library includes trigonometric functions for sine, cosine, and tangent but not secant and cosecant.

Start with client code. One of the primary purposes of developing a data type is to simplify client code. Therefore, it makes sense to pay attention to client code from the start. Often, it is wise to write the client code *before* working on an implementation. When you find yourself with some client code that is becoming cumbersome, one way to proceed is to write a fanciful simplified version of the code that expresses the computation the way you are thinking about it. Or, if you have done a good job of writing succinct comments to describe your computation, one possible starting point is to think about opportunities to convert the comments into code.

Avoid dependence on representation. Usually when developing an API, we have a representation in mind. After all, a data type is a set of values and a set of operations on those values, and it does not make much sense to talk about the operations without knowing the values. But that is different from knowing the *representation* of the values. One purpose of the data type is to simplify client code by allowing it to avoid details of and dependence on a particular representation. For example, our client programs for Picture and StdAudio work with simple abstract representations of pictures and sound, respectively. The primary value of the APIs for these abstractions is that they allow client code to ignore a substantial amount of detail that is found in the standard representations of those abstractions.

Pitfalls in API design. An API may be *too hard to implement*, implying implementations that are difficult or impossible to develop, or *too hard to use*, creating client code that is more complicated than without the API. An API might be *too narrow*, omitting methods that clients need, or *too wide*, including a large number of methods not needed by any client. An API may be *too general*, providing no useful abstractions, or *too specific*, providing abstractions so detailed or so diffuse as to be useless. These considerations are sometimes summarized in yet another motto: *provide to clients the methods they need and no others.*

WHEN YOU FIRST STARTED PROGRAMMING, YOU typed in `HelloWorld.java` without understanding much about it except the effect that it produced. From that starting point, you learned to program by mimicking the code in the book and eventually developing your own code to solve various problems. You are at a similar point with API design. There are many APIs available in the book, on the booksite, and in online Java documentation that you can study and use, to gain confidence in designing and developing APIs of your own.

Encapsulation The process of separating clients from implementations by hiding information is known as *encapsulation*. Details of the implementation are kept hidden from clients, and implementations have no way of knowing details of client code, which may even be created in the future.

As you may have surmised, we have been practicing encapsulation in our data-type implementations. In SECTION 3.1, we started with the mantra *you do not need to know how a data type is implemented to use it*. This statement describes one of the prime benefits of encapsulation. We consider it to be so important that we have not described to you any other way of designing a data type. Now, we describe our three primary reasons for doing so in more detail. We use encapsulation for the following purposes:

- To enable modular programming
- To facilitate debugging
- To clarify program code

These reasons are tied together (well-designed modular code is easier to debug and understand than code based entirely on primitive types in long programs).

Modular programming. The programming style that we have been developing since CHAPTER 2 has been predicated on the idea of breaking large programs into small modules that can be developed and debugged independently. This approach improves the resiliency of our software by limiting and localizing the effects of making changes, and it promotes code reuse by making it possible to substitute new implementations of a data type to improve performance, accuracy, or memory footprint. The same idea works in many settings. We often reap the benefits of encapsulation when we use system libraries. New versions of the Java system often include new implementations of various data types, but *the APIs do not change*. There is strong and constant motivation to improve data-type implementations because *all* clients can potentially benefit from an improved implementation. The key to success in modular programming is to maintain *independence* among modules. We do so by insisting on the API being the *only* point of dependence between client and implementation. *You do not need to know how a data type is implemented to use it*. The flip side of this mantra is that a data-type implementation can assume that the client knows nothing about the data type except the API.

Example. For example, consider `Complex` (PROGRAM 3.3.1). It has the same name and API as PROGRAM 3.2.6, but uses a different representation for the complex numbers. PROGRAM 3.2.6 uses the *Cartesian* representation, where instance variables `x` and `y` represent a complex number $x + iy$. PROGRAM 3.3.1 uses the *polar* representation, where instance variables `r` and `theta` represent a complex number in the form $r(\cos \theta + i \sin \theta)$. The polar representation is of interest because certain operations on complex number (such as multiplication and division) are more efficient using the polar representation. The idea of encapsulation is that we can substitute one of these programs for the other (for whatever reason) *without changing client code*. The choice between the two implementations depends on the client. Indeed, in principle, the *only* difference to the client should be in different performance properties. This capability is of critical importance for many reasons. One of the most important is that it allows us to improve software constantly: when we develop a better way to implement a data type, all of its clients can benefit. You take advantage of this property every time you install a new version of a software system, including Java itself.

Private. Java's language support for enforcing encapsulation is the `private` access modifier. When you declare an instance variable (or method) to be `private`, you are making it impossible for any client (code in another class) to directly access that instance variable (or method). Clients can access the data type only through the `public` methods and constructors—the API. Accordingly, you can modify the implementation to use different private instance variables (or reorganize the `private` instance method) and know that no client will be directly affected. Java does not *require* that all instance variables be `private`, but we insist on this convention in the programs in this book. For example, if the instance variables `re` and `im` in `Complex` (PROGRAM 3.2.6) were `public`, then a client could write code that directly accesses them. If `z` refers to a `Complex` object, `z.re` and `z.im` refer to those values. But any client code that does so becomes completely dependent on that implementation, violating a basic precept of encapsulation. A switch to a different implementation, such as the one in PROGRAM 3.3.1, would render that code useless. To protect ourselves against such situations, we always make instance variables `private`. Next, we examine some ramifications of this convention.

Program 3.3.1 Complex number (alternate)

```

public class Complex
{
    private final double r;
    private final double theta;

    public Complex(double re, double im)
    {
        r = Math.sqrt(re*re + im*im);
        theta = Math.atan2(im, re);
    }

    public Complex plus(Complex b)
    { // Return the sum of this number and b.
        double real = re() + b.re();
        double imag = im() + b.im();
        return new Complex(real, imag);
    }

    public Complex times(Complex b)
    { // Return the product of this number and b.
        double radius = r * b.r;
        double angle = theta + b.theta;
        // See Q&A.
    }

    public double abs()
    { return r; }

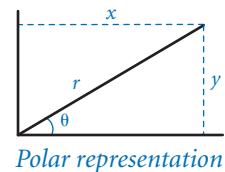
    public double re() { return r * Math.cos(theta); }
    public double im() { return r * Math.sin(theta); }

    public String toString()
    { return re() + " + " + im() + "i"; }

    public static void main(String[] args)
    {
        Complex z0 = new Complex(1.0, 1.0);
        Complex z = z0;
        z = z.times(z).plus(z0);
        z = z.times(z).plus(z0);
        StdOut.println(z);
    }
}

```

r radius
theta angle



Polar representation

This data type implements the same API as PROGRAM 3.2.6. It uses the same instance methods but different instance variables. Since the instance variables are *private*, this program might be used in place of PROGRAM 3.2.6 without changing any client code.

```
% java Complex
-7.000000000000002 + 7.000000000000003i
```

Planning for the future. There have been numerous examples of important applications where significant expense can be directly traced to programmers not encapsulating their data types.

- *Y2K problem.* In the last millennium, many programs represented the year using only two decimal digits to save storage. Such programs could not distinguish between the year 1900 and the year 2000. As January 1, 2000, approached, programmers raced to fix such rollover errors and avert the catastrophic failures that were predicted by many technologists.
- *ZIP codes.* In 1963, The United States Postal Service (USPS) began using a five-digit ZIP code to improve the sorting and delivery of mail. Programmers wrote software that assumed that these codes would remain at five digits forever, and represented them in their programs using a single 32-bit integer. In 1983, the USPS introduced an expanded ZIP code called ZIP+4, which consists of the original five-digit ZIP code plus four extra digits.
- *IPv4 versus IPv6.* The Internet Protocol (IP) is a standard used by electronic devices to exchange data over the Internet. Each device is assigned a unique integer or address. IPv4 uses 32-bit addresses and supports about 4.3 billion addresses. Due to explosive growth of the Internet, a new version, IPv6, uses 128-bit addresses and supports 2^{128} addresses.

In each of these cases, a necessary change to the internal representation meant that a large amount of client code that depended on the current standard (because the data type was not encapsulated) simply would not function as intended. The estimated costs for the changes in each of these cases ran to hundreds of millions of dollars! That is a huge cost for failing to encapsulate a single number. These predicaments might seem distant to you, but you can be sure that every individual programmer (that's you) who does not take advantage of the protection available through encapsulation risks losing significant amounts of time and effort fixing broken code when conventions change.

Our convention to define *all* of our instance variables with the `private` access modifier provides some protection against such problems. If you adopt this convention when implementing a data type for a year, ZIP code, IP address, or whatever, you can change the representation without affecting clients. The *datatype implementation* knows the data representation, and the *object* holds the data; the *client* holds only a reference to the object and does not know the details.

Limiting the potential for error. Encapsulation also helps programmers ensure that their code operates as intended. As an example, we consider yet another horror story: In the 2000 presidential election, Al Gore received *negative* 16,022 votes on an electronic voting machine in Volusia County, Florida. The counter variable was not properly encapsulated in the voting machine software! To understand the problem, consider `Counter` (PROGRAM 3.3.2), which implements a simple counter according to the following API:

<code>public class Counter</code>	
<code>Counter(String id, int max)</code>	<i>create a counter, initialized to 0</i>
<code>void increment()</code>	<i>increment the counter unless its value is max</i>
<code>int value()</code>	<i>return the value of the counter</i>
<code>String toString()</code>	<i>string representation</i>

API for a counter data type (see PROGRAM 3.3.2)

This abstraction is useful in many contexts, including, for example, an electronic voting machine. It encapsulates a single integer and ensures that the only operation that can be performed on the integer is *increment by 1*. Therefore, it can never go negative. The goal of data abstraction is to *restrict* the operations on the data. It also *isolates* operations on the data. For example, we could add a new implementation with a logging capability so that `increment()` saves a timestamp for each vote or some other information that can be used for consistency checks. But without the `private` modifier, there could be client code like the following somewhere in the voting machine:

```
Counter c = new Counter("Volusia", VOTERS_IN_VOLUSIA_COUNTY);
c.count = -16022;
```

With the `private` modifier, code like this will not compile; without it, Gore's vote count was negative. Using encapsulation is far from a complete solution to the voting security problem, but it is a good start.

Program 3.3.2 Counter

```

public class Counter
{
    private final String name;
    private final int maxCount;
    private int count;

    public Counter(String id, int max)
    { name = id; maxCount = max; }

    public void increment()
    { if (count < maxCount) count++; }

    public int value()
    { return count; }

    public String toString()
    { return name + ": " + count; }

    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        int trials = Integer.parseInt(args[1]);
        Counter[] hits = new Counter[n];
        for (int i = 0; i < n; i++)
            hits[i] = new Counter(i + "", trials);

        for (int t = 0; t < trials; t++)
            hits[StdRandom.uniform(n)].increment();
        for (int i = 0; i < n; i++)
            StdOut.println(hits[i]);
    }
}

```

name	<i>counter name</i>
maxCount	<i>maximum value</i>
count	<i>value</i>

This class encapsulates a simple integer counter, assigning it a string name and initializing it to 0 (Java's default initialization), incrementing it each time the client calls `increment()`, reporting the value when the client calls `value()`, and creating a string with its name and value in `toString()`.

```
% java Counter 6 600000
0: 100684
1: 99258
2: 100119
3: 100054
4: 99844
5: 100037
```

Code clarity. Precisely specifying a data type is also good design because it leads to client code that can more clearly express its computation. You have seen many examples of such client code in SECTIONS 3.1 and 3.2, and we already mentioned this issue in our discussion of `Histogram` (PROGRAM 3.2.3). Clients of that program are clearer with it than without it because calls on the instance method `addDataPoint()` clearly identify points of interest in the client. One key to good design is to observe that code written with the proper abstractions can be nearly self-documenting. Some aficionados of object-oriented programming might argue that `Histogram` itself would be easier to understand if it were to use `Counter` (see EXERCISE 3.3.3), but that point is perhaps debatable.

WE HAVE STRESSED THE BENEFITS OF encapsulation throughout this book. We summarize them again here, in the context of designing data types. Encapsulation enables modular programming, allowing us to:

- Independently develop client and implementation code
- Substitute improved implementations without affecting clients
- Support programs not yet written (any client can write to the API)

Encapsulation also isolates data-type operations, which leads to the possibility of:

- Adding consistency checks and other debugging tools in implementations
- Clarifying client code

A properly implemented data type (encapsulated) extends the Java language, allowing any client program to make use of it.

Immutability As defined at the end of Section 3.1, an object from a data type is *immutable* if its data-type value cannot change once created. An *immutable data type* is one in which all objects of that type are immutable. In contrast, a *mutable data type* is one in which objects of that type have values that are designed to change. Of the data types considered in this chapter, `String`, `Charge`, `Color`, and `Complex` are all immutable, and `Turtle`, `Picture`, `Histogram`, `StockAccount`, and `Counter` are all mutable. Whether to make a data type immutable is an important design decision and depends on the application at hand.

Immutable types. The purpose of many data types is to encapsulate values that do not change so that they behave in the same way as primitive types. For example, a programmer implementing a `Complex` client might reasonably expect to write the code `z = z0` for two `Complex` variables, in the same way as for `double` or `int` variables. But if `Complex` objects were mutable and the value of `z` were to change *after* the assignment `z = z0`, then the value of `z0` would *also* change (they are both references to the same object)! This unexpected result, known as an *aliasing* bug, comes as a surprise to many newcomers to object-oriented programming. One very important reason to implement immutable types is that we can use immutable objects in assignment statements (or as arguments and return values from methods) without having to worry about their values changing.

<i>immutable</i>	<i>mutable</i>
<code>String</code>	<code>Turtle</code>
<code>Charge</code>	<code>Picture</code>
<code>Color</code>	<code>Histogram</code>
<code>Complex</code>	<code>StockAccount</code>
<code>Vector</code>	<code>Counter</code>
	Java arrays

Mutable types. For many data types, the very purpose of the abstraction is to encapsulate values as they change. `Turtle` (PROGRAM 3.2.4) is a prime example. Our reason for using `Turtle` is to relieve client programs of the responsibility of tracking the changing values. Similarly, `Picture`, `Histogram`, `StockAccount`, `Counter`, and Java arrays are all data types for which we expect values to change. When we pass a `Turtle` as an argument to a method, as in `Koch`, we expect the value of the `Turtle` object to change.

Arrays and strings. You have already encountered this distinction as a client programmer, when using Java arrays (mutable) and Java's `String` data type (immutable). When you pass a `String` to a method, you do not need to worry about that method changing the sequence of characters in the `String`, but when you pass an array to a method, the method is free to change the values of the elements in the array. The `String` data type is immutable because we generally do *not* want string

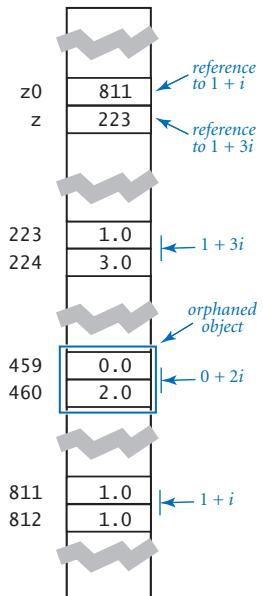
values to change, and Java arrays are mutable because we generally *do* want array values to change. There are also situations where we want to have mutable strings (that is the purpose of Java's `StringBuilder` data type) and where we want to have immutable arrays (that is the purpose of the `Vector` data type that we consider later in this section).

Advantages of immutability. Generally, immutable types are easier to use and harder to misuse because the scope of code that can change their values is far smaller than for mutable types. It is easier to debug code that uses immutable types because it is easier to guarantee that variables in the client code that uses them will remain in a consistent state. When using mutable types, you must always be concerned about where and when their values change.

Cost of immutability. The downside of immutability is that *a new object must be created for every value*. For example, the expression `z = z.times(z).plus(z0)` involves creating a new object (the return value of `z.times(z)`), then using that object to invoke `plus()`, but never saving a reference to it. A program such as `Mandelbrot` (PROGRAM 3.2.7) might create a large number of such intermediate orphans. However, this expense is normally manageable because Java garbage collectors are typically optimized for such situations. Also, as in the case of `Mandelbrot`, when the point of the calculation is to create a large number of values, we expect to pay the cost of representing them. `Mandelbrot` also creates a large number of (immutable) `Color` objects.

Final. You can use the `final` modifier to help enforce immutability in a data type. When you declare an instance variable as `final`, you are promising to assign it a value only once, either in an inline initialization statement or in the constructor. Any other code that could modify the value of a `final` variable leads to a compile-time error. In our code, we use the modifier `final` with instance variables whose values never change. This policy serves as documentation that the value does not change, prevents accidental changes, and makes programs easier to debug. For example, you do not have to include a `final` variable in a trace, since you know that its value never changes.

```
Complex z0;
z0 = new Complex(1.0, 1.0);
Complex z = z0;
z = z.times(z).plus(z0);
```



An intermediate orphan

Reference types. Unfortunately, `final` guarantees immutability only when instance variables are primitive types, not reference types. If an instance variable of a reference type has the `final` modifier, the value of that instance variable (the object reference) will never change—it will always refer to the same object. However, the value of the object itself *can* change. For example, if you have a `final` instance variable that is an array, you cannot change the array (to change its length or type, say), but you *can* change the values of the individual array elements. Thus, aliasing bugs can arise. For example, this code does *not* implement an immutable data type:

```
public class Vector
{
    private final double[] coords;
    public Vector(double[] a)
    {
        coords = a;
    }
    ...
}
```

A client program could create a `Vector` by specifying the elements in an array, and then (bypassing the API) change the elements of the `Vector` after construction:

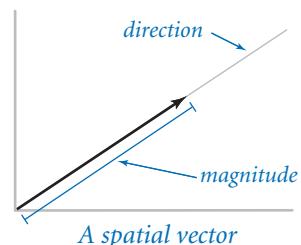
```
double[] a = { 3.0, 4.0 };
Vector vector = new Vector(a);
a[0] = 17.0; // coords[0] is now 17.0
```

The instance variable `coords[]` is `private` and `final`, but `Vector` is mutable because the *client* holds a reference to the *same* array. When the client changes the value of an element in its array, the change also appears in the corresponding `coords[]` array, because `coords[]` and `a[]` are aliases. To ensure immutability of a data type that includes an instance variable of a mutable type, we need to make a local copy, known as a *defensive copy*. Next, we consider such an implementation.

IMMUTABILITY NEEDS TO BE TAKEN INTO account in any data-type design. Ideally, whether a data type is immutable should be specified in the API, so that clients know that object values will not change. Implementing an immutable data type can be a burden in the presence of reference types. For complicated data types, making the defensive copy is one challenge; ensuring that none of the instance methods change values is another.

Example: spatial vectors To illustrate these ideas in the context of a useful mathematical abstraction, we now consider a *vector* data type. Like complex numbers, the basic definition of the vector abstraction is familiar because it has played a central role in applied mathematics for more than 100 years. The field of mathematics known as *linear algebra* is concerned with properties of vectors. Linear algebra is a rich and successful theory with numerous applications, and plays an important role in all fields of social and natural science. Full treatment of linear algebra is certainly beyond the scope of this book, but several important applications are based upon elementary and familiar calculations, so we touch upon vectors and linear algebra throughout the book (for example, the random-surfer example in SECTION 1.6 is based on linear algebra). Accordingly, it is worthwhile to encapsulate such an abstraction in a data type.

A *spatial vector* is an abstract entity that has a *magnitude* and a *direction*. Spatial vectors provide a natural way to describe properties of the physical world, such as force, velocity, momentum, and acceleration. One standard way to specify a vector is as an arrow from the origin to a point in a Cartesian coordinate system: the direction is the ray from the origin to the point and the magnitude is the length of the arrow (distance from the origin to the point). To specify the vector it suffices to specify the point.



This concept extends to any number of dimensions: a sequence of n real numbers (the coordinates of an n -dimensional point) suffices to specify a vector in n -dimensional space. By convention, we use a boldface letter to refer to a vector and numbers or indexed variable names (the same letter in italics) separated by commas within parentheses to denote its value. For example, we might use \mathbf{x} to denote the vector $(x_0, x_1, \dots, x_{n-1})$ and \mathbf{y} to denote the vector $(y_0, y_1, \dots, y_{n-1})$.

API. The basic operations on vectors are to add two vectors, scale a vector, compute the dot product of two vectors, and compute the magnitude and direction, as follows:

- *Addition:* $\mathbf{x} + \mathbf{y} = (x_0 + y_0, x_1 + y_1, \dots, x_{n-1} + y_{n-1})$
- *Vector scaling:* $\alpha \mathbf{x} = (\alpha x_0, \alpha x_1, \dots, \alpha x_{n-1})$
- *Dot product:* $\mathbf{x} \cdot \mathbf{y} = x_0 y_0 + x_1 y_1 + \dots + x_{n-1} y_{n-1}$
- *Magnitude:* $|\mathbf{x}| = (x_0^2 + x_1^2 + \dots + x_{n-1}^2)^{1/2}$
- *Direction:* $\mathbf{x} / |\mathbf{x}| = (x_0 / |\mathbf{x}|, x_1 / |\mathbf{x}|, \dots, x_{n-1} / |\mathbf{x}|)$

The result of addition, vector scaling, and the direction are vectors, but the magnitude and the dot product are scalar quantities (real numbers). For example, if $\mathbf{x} = (0, 3, 4, 0)$, and $\mathbf{y} = (0, -3, 1, -4)$, then $\mathbf{x} + \mathbf{y} = (0, 0, 5, -4)$, $3\mathbf{x} = (0, 9, 12, 0)$, $\mathbf{x} \cdot \mathbf{y} = -5$, $|\mathbf{x}| = 5$, and $\mathbf{x}/|\mathbf{x}| = (0, 3/5, 4/5, 0)$. The direction vector is a *unit vector*: its magnitude is 1. These definitions lead immediately to an API:

```
public class Vector
```

Vector(double[] a)	<i>create a vector with the given Cartesian coordinates</i>
Vector plus(Vector that)	<i>sum of this vector and that</i>
Vector minus(Vector that)	<i>difference of this vector and that</i>
Vector scale(double alpha)	<i>this vector, scaled by alpha</i>
double dot(Vector b)	<i>dot product of this vector and that</i>
double magnitude()	<i>magnitude</i>
Vector direction()	<i>unit vector with same direction as this vector</i>
double cartesian(int i)	<i>ith Cartesian coordinate</i>
String toString()	<i>string representation</i>

API for spatial vectors (see PROGRAM 3.3.3)

As with the Complex API, this API does not explicitly specify that this type is immutable, but we know that client programmers (who are likely to be thinking in terms of the mathematical abstraction) will certainly expect that.

Representation. As usual, our first choice in developing an implementation is to choose a representation for the data. Using an array to hold the Cartesian coordinates provided in the constructor is a clear choice, but not the only reasonable choice. Indeed, one of the basic tenets of linear algebra is that other sets of n vectors can be used as the basis for a coordinate system: any vector can be expressed as a linear combination of a set of n vectors, satisfying a certain condition known as *linear independence*. This ability to change coordinate systems aligns nicely with encapsulation. Most clients do not need to know about the internal representation at all and can work with Vector objects and operations. If warranted, the implementation can change the coordinate system without affecting any client code.

Program 3.3.3 Spatial vectors

```

public class Vector
{
    private final double[] coords;           coords[] | Cartesian coordinates

    public Vector(double[] a)
    { // Make a defensive copy to ensure immutability.
        coords = new double[a.length];
        for (int i = 0; i < a.length; i++)
            coords[i] = a[i];
    }

    public Vector plus(Vector that)
    { // Sum of this vector and that.
        double[] result = new double[coords.length];
        for (int i = 0; i < coords.length; i++)
            result[i] = this.coords[i] + that.coords[i];
        return new Vector(result);
    }

    public Vector scale(double alpha)
    { // Scale this vector by alpha.
        double[] result = new double[coords.length];
        for (int i = 0; i < coords.length; i++)
            result[i] = alpha * coords[i];
        return new Vector(result);
    }

    public double dot(Vector that)
    { // Dot product of this vector and that.
        double sum = 0.0;
        for (int i = 0; i < coords.length; i++)
            sum += this.coords[i] * that.coords[i];
        return sum;
    }

    public double magnitude()
    { return Math.sqrt(this.dot(this)); }

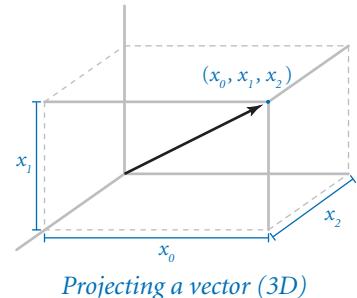
    public Vector direction()
    { return this.scale(1>this.magnitude()); }

    public double cartesian(int i)
    { return coords[i]; }
}

```

This implementation encapsulates the mathematical spatial-vector abstraction in an immutable Java data type. Sketch (PROGRAM 3.3.4) and Body (PROGRAM 3.4.1) are typical clients. The instance methods minus() and toString() are left for exercises (EXERCISE 3.3.4 and EXERCISE 3.3.14), as is the test client (EXERCISE 3.3.5).

Implementation. Given the representation, the code that implements all of these operations (`Vector`, in PROGRAM 3.3.3) is straightforward. The constructor makes a defensive copy of the client array and none of the methods assign values to the copy, so that the `Vector` data type is immutable. The `cartesian()` method is easy to implement in our Cartesian coordinate representation: return the i th coordinate in the array. It actually implements a mathematical function that is defined for any `Vector` representation: the geometric projection onto the i th Cartesian axis.



The `this` reference. Within an instance method (or constructor), the `this` keyword gives us a way to refer to the object whose instance method (or constructor) is being called. You can use `this` in the same way you use any other object reference (for example, to invoke a method, pass as an argument to a method, or access instance variables). For example, the `magnitude()` method in `Vector` uses the `this` keyword in two ways: to invoke the `dot()` method and as an argument to the `dot()` method. Thus, the expression `vector.magnitude()` is equivalent to `Math.sqrt(vector.dot(vector))`. Some Java programmers *always* use `this` to access instance variables. This policy is easy to defend because it clearly indicates when you are referring to an instance variable (as opposed to a local or parameter variable). However, it leads to a surfeit of `this` keywords, so we take the opposite tack and use `this` sparingly in our code.

WHY GO TO THE TROUBLE OF using a `Vector` data type when all of the operations are so easily implemented with arrays? By now the answer to this question should be obvious to you: to enable modular programming, facilitate debugging, and clarify code. A `double` array is a low-level Java mechanism that admits all kinds of operations on its elements. By restricting ourselves to just the operations in the `Vector` API (which are the only ones that we need, for many clients), we simplify the process of designing, implementing, and maintaining our programs. Because the `Vector` data type is immutable, we can use it in the same way we use primitive types. For example, when we pass a `Vector` to a method, we are assured its value will not change (but we do not have that assurance when passing an array). Writing programs that use the `Vector` data type and its associated operations is an easy and natural way to take advantage of the extensive amount of mathematical knowledge that has been developed around this abstract concept.

JAVA PROVIDES LANGUAGE SUPPORT FOR DEFINING relationships among objects, known as *inheritance*. Software developers use these mechanisms widely, so you will study them in detail if you take a course in software engineering. Generally, effective use of such mechanisms is beyond the scope of this book, but we briefly describe the two main forms of inheritance in Java—*interface inheritance* and *implementation inheritance*—here because there are a few situations where you are likely to encounter them.

Interface inheritance (subtyping) Java provides the `interface` construct for declaring a relationship between otherwise unrelated classes, by specifying a common set of methods that each implementing class must include. That is, an interface is a *contract* for a class to implement a certain set of methods. We refer to this arrangement as *interface inheritance* because an implementing class *inherits* a partial API from the interface. Interfaces enable us to write client programs that can manipulate objects of varying types, by invoking common methods from the interface. As with most new programming concepts, it is a bit confusing at first, but will make sense to you after you have seen a few examples.

Defining an interface. As a motivating example, suppose that we want to write code to plot *any* real-valued function. We have previously encountered programs in which we plot *one* specific function by sampling the function of interest at evenly spaced points in a particular interval. To generalize these programs to handle arbitrary functions, we define a Java interface for real-valued functions of a single variable:

```
public interface Function
{
    public abstract double evaluate(double x);
}
```

The first line of the interface declaration is similar to that of a class declaration, but uses the keyword `interface` instead of `class`. The body of the interface contains a list of *abstract methods*. An abstract method is a method that is declared but does not include any implementation code; it contains only the method signature, terminated by a semicolon. The modifier `abstract` designates a method as abstract. As with a Java class, you must save a Java interface in a file whose name matches the name of the interface, with a `.java` extension.

Implementing an interface. An interface is a contract for a class to implement a certain set of methods. To write a class that *implements* an interface, you must do two things. First, you must include an `implements` clause in the class declaration with the name of the interface. You can think of this as signing a contract, promising to implement each of the abstract methods declared in the interface. Second, you must implement each of these abstract methods. For example, you can define a class for computing the square of a real number that implements the `Function` interface as follows:

```
public class Square implements Function
{
    public double evaluate(double x)
    {   return x*x;   }
}
```

Similarly, you can define a class for computing the Gaussian probability density function (see PROGRAM 2.1.2):

```
public class GaussianPDF implements Function
{
    public double evaluate(double x)
    {   return Math.exp(-x*x/2) / Math.sqrt(2 * Math.PI);   }
}
```

If you fail to implement any of the abstract methods specified in the interface, you will get a compile-time error. Conversely, a class implementing an interface may include methods not specified in the interface.

Using an interface. An interface is a reference type. You can use an interface name in the same way that you use any other data-type name. For example, you can declare the type of a variable to be the name of an interface. When you do so, any object you assign to that variable must be an instance of a class that implements the interface. For example, a variable of type `Function` may store an object of type `Square` or `GaussianPDF`, but not of type `Complex`.

```
Function f1 = new Square();
Function f2 = new GaussianPDF();
Function f3 = new Complex(1.0, 2.0); // compile-time error
```

A variable of an interface type may invoke only those methods declared in the interface, even if the implementing class defines additional methods.

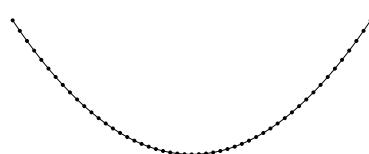
When a variable of an interface type invokes a method declared in the interface, Java knows which method to call because it knows the type of the invoking object. For example, `f1.evaluate()` would call the `evaluate()` method defined in the `Square` class, whereas `f2.evaluate()` would call the `evaluate()` method defined in the `GaussianPDF` class. This powerful programming mechanism is known as *polymorphism* or *dynamic dispatch*.

To see the advantages of using interfaces and polymorphism, we return to the application of plotting the graph of a function f in the interval $[a, b]$. If the function f is sufficiently smooth, we can sample the function at $n + 1$ evenly spaced points in the interval $[a, b]$ and display the results using `StdStats.plotPoints()` or `StdStats.plotLines()`.

```
public static void plot(Function f, double a, double b, int n)
{
    double[] y = new double[n+1];
    double delta = (b - a) / n;
    for (int i = 0; i <= n; i++)
        y[i] = f.evaluate(a + delta*i);
    StdStats.plotPoints(y);
    StdStats.plotLines(y);
}
```

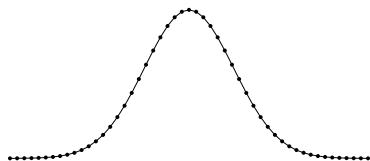
The advantage of declaring the variable `f` using the interface type `Function` is that the same method call `f.evaluate()` works for an object `f` of *any* data type that implements the `Function` interface, including `Square` or `GaussianPDF`. Consequently, we don't need to write overloaded methods for each type—we can reuse the same `plot()` function for many types! This ability to arrange to write a client to plot any function is a persuasive example of interface inheritance.

```
Function f1 = new Square();
plot(f1, -0.6, 0.6, 50);
```



Plotting function graphs

```
Function f2 = new GaussianPDF();
plot(f2, -4.0, 4.0, 50);
```



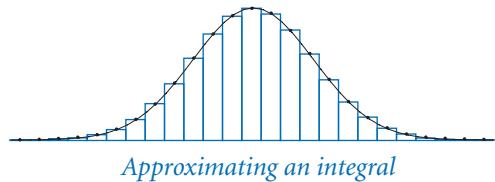
Computing with functions. Often, particularly in scientific computing, we want to compute with *functions*: we want differentiate functions, integrate functions, find roots of functions, and so forth. In some programming languages, known as *functional programming languages*, this desire aligns with the underlying design of the language, which uses computing with functions to substantially simplify client code. Unfortunately, *methods are not first-class objects in Java*. However, as we just saw with `plot()`, we can use Java interfaces to achieve some of the same objectives.

As an example, consider the problem of estimating the *Riemann integral* of a positive real-valued function f (the area under the curve) in an interval (a, b) . This computation is known as *quadrature* or *numerical integration*. A number of methods have been developed for quadrature. Perhaps the simplest is known as the *rectangle rule*, where we approximate the value of the integral by computing the total area of n equal-width rectangles under the curve. The `integrate()` function defined below evaluates the integral of a real-valued function f in the interval (a, b) , using the rectangle rule with n rectangles:

```
public static double integrate(Function f,
                               double a, double b, int n)
{
    double delta = (b - a) / n;
    double sum = 0.0;
    for (int i = 0; i < n; i++)
        sum += delta * f.evaluate(a + delta * (i + 0.5));
    return sum;
}
```

The indefinite integral of x^2 is $x^3/3$, so the definite integral between 0 and 10 is $1,000/3$. The call to `integrate(new Square(), 0, 10, 1000)` returns `333.33324999999996`, which is the correct answer to six significant digits of accuracy. Similarly, the call to `integrate(new GaussianPDF(), -1, 1, 1000)` returns `0.6826895727940137`, which is the correct answer to seven significant digits of accuracy (recall the Gaussian probability density function and PROGRAM 2.1.2).

Quadrature is not always the most efficient or accurate way to evaluate a function. For example, the `Gaussian.cdf()` function in PROGRAM 2.1.2 is a faster and more accurate way to integrate the Gaussian probability density function. However, quadrature has the advantage of being useful for any function whatsoever, subject only to certain technical conditions on smoothness.



Lambda expressions. The syntax that we have just considered for computing with functions is a bit unwieldy. For example, it is awkward to define a new class that implements the `Function` interface for each function that we might want to plot or integrate. To simplify syntax in such situations, Java provides a powerful functional programming feature known as *lambda expressions*. You should think of a lambda expression as a block of code that you can pass around and execute later. In its simplest form, a lambda expression consists of the three elements:

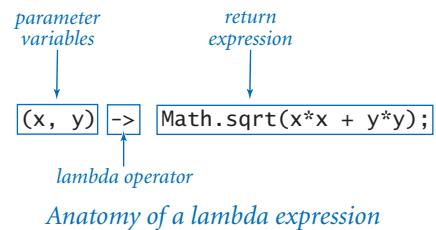
- A list of parameters variables, separated by commas, and enclosed in parentheses
- The *lambda operator* `->`
- A single expression, which is the value returned by the lambda expression

For example, the lambda expression `(x, y) -> Math.sqrt(x*x + y*y)` implements the hypotenuse function. The parentheses are optional when there is only one parameter. So the lambda expression `x -> x*x` implements the square function and `x -> Gaussian.pdf(x)` implements the Gaussian probability density function.

Our primary use of lambda expressions is as a concise way to implement a *functional interface* (an interface with a single abstract method). Specifically, you can use a lambda expression wherever an object from a functional interface is expected.

<i>expression</i>	<pre> new Square() new GaussianPDF() x -> x*x x -> Gaussian.pdf(x) x -> Math.cos(x) </pre>
-------------------	---

Typical expressions that implement the Function interface



Anatomy of a lambda expression

For example, you can integrate the square function with the call `integrate(x -> x*x, 0, 10, 1000)`, thereby bypassing the need to define the `Square` class. You do not need to declare explicitly that the lambda expression implements the `Function` interface; as long as the signature of the single abstract method is compatible with the lambda expression (same number of arguments and types), Java will infer it from context. In this case, the lambda expression `x -> x*x` is compatible with the abstract method `evaluate()`.

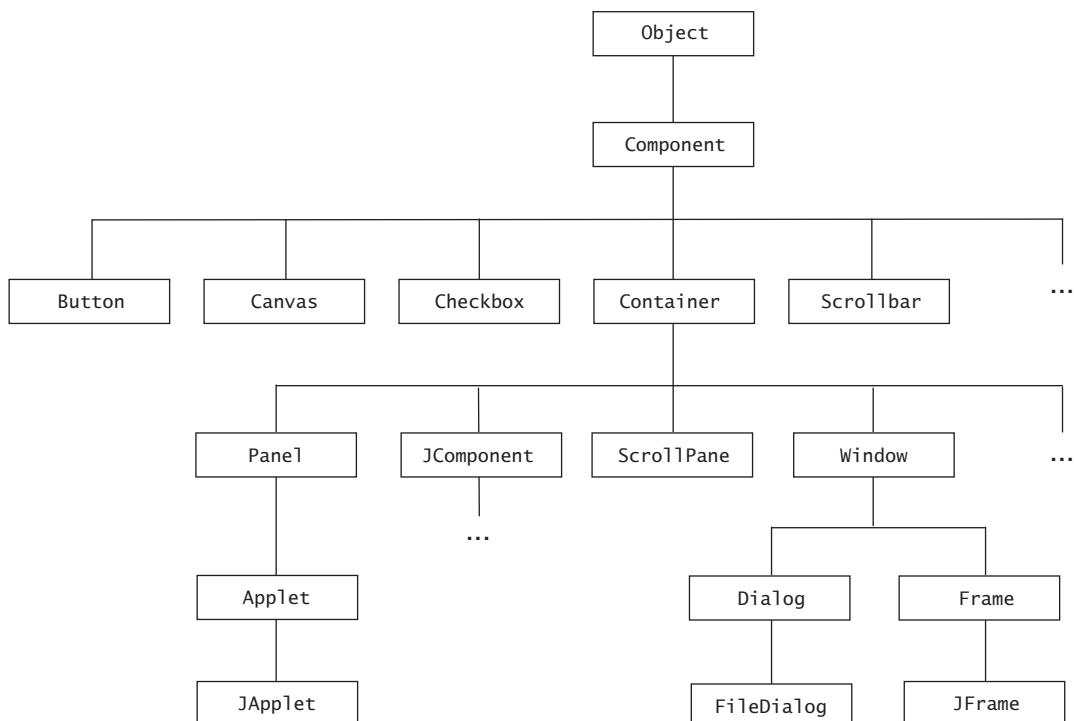
Built-in interfaces. Java includes three interfaces that we will consider later this book. In SECTION 4.2, we will consider Java’s `java.util.Comparable` interface, which contains a single abstract method `compareTo()`. The `compareTo()` method defines a natural order for comparing objects of the same type, such as alphabetical order for strings and ascending order for integers and real numbers. This enables us to write code to sort arrays of objects. In SECTION 4.3, we will use interfaces to enable clients to iterate over the items in a collection, without relying on the underlying representation. Java supplies two interfaces—`java.util.Iterator` and `java.lang.Iterable`—for this purpose.

Event-based programming. Another powerful example of the value of interface inheritance is its use in *event-based programming*. In a familiar setting, consider the problem of extending Draw to respond to user input such as mouse clicks and keystrokes. One way to do so is to define an interface to specify which method or methods Draw should call when user input happens. The descriptive term *callback* is sometimes used to describe a call from a method in one class to a method in another class through an interface. You can find on the booksite an example interface `DrawListener` and information on how to write code to respond to user mouse clicks and keystrokes within Draw. You will find it easy to write code that creates a Draw object and includes a method that the Draw method can invoke (*callback* your code) to tell your method the character typed on a user keystroke event or the mouse position on a mouse click. Writing interactive code is fun but challenging because you have to plan for all possible user input actions.

INTERFACE INHERITANCE IS AN ADVANCED PROGRAMMING concept that is embraced by many experienced programmers because it enables code reuse, without sacrificing encapsulation. The *functional programming* style that it supports is controversial in some quarters, but lambda expressions and similar constructs date back to the earliest days of programming and have found their way into numerous modern programming languages. The style has passionate proponents who believe that we should be using and teaching it exclusively. We have not emphasized it from the start because the preponderance of code that you will encounter was built without it, but we introduce it here because every programmer needs to be aware of the possibility and on the watch for opportunities to exploit it.

Implementation inheritance (subclassing) Java also supports another inheritance mechanism known as *subclassing*. The idea is to define a new class (*subclass*, or *derived class*) that *inherits* instance variables (state) and instance methods (behavior) from another class (*superclass*, or *base class*), enabling code reuse. Typically, the subclass redefines or *overrides* some of the methods in the superclass. We refer to this arrangement as *implementation inheritance* because one class inherits code from another class.

Systems programmers use subclassing to build so-called *extensible* libraries—one programmer (even you) can add methods to a library built by another programmer (or, perhaps, a team of systems programmers), effectively reusing the code in a potentially huge library. This approach is widely used, particularly in the development of user interfaces, so that the large amount of code required to provide all the facilities that users expect (windows, buttons, scrollbars, drop-down menus, cut-and-paste, access to files, and so forth) can be reused.



Subclass inheritance hierarchy for GUI components (partial)

The use of subclassing is controversial among systems programmers because its advantages over subtyping are debatable. In this book, we avoid subclassing because it works against encapsulation in two ways. First, any change in the superclass affects all subclasses. The subclass cannot be developed *independently* of the superclass; indeed, it is *completely dependent* on the superclass. This problem is known as the *fragile base class* problem. Second, the subclass code, having access to instance variables in the superclass, can subvert the intention of the superclass code. For example, the designer of a class such as `Vector` may have taken great care to make the `Vector` immutable, but a subclass, with full access to those instance variables, can recklessly change them.

Java's Object superclass. Certain vestiges of subclassing are built into Java and therefore unavoidable. Specifically, every class is a subclass of Java's `Object` class. This structure enables implementation of the “convention” that every class includes an implementation of `toString()`, `equals()`, `hashCode()`, and several other methods. Every class inherits these methods from `Object` through subclassing. When programming in Java, you will often override one or more of these methods.

<code>public class Object</code>	
<code>String toString()</code>	<i>string representation of this object</i>
<code>boolean equals(Object x)</code>	<i>is this object equal to x?</i>
<code>int hashCode()</code>	<i>hash code of this object</i>
<code>Class getClass()</code>	<i>class of this object</i>

Methods inherited by all classes (used in this book)

String conversion. Every Java class inherits the `toString()` method, so any client can invoke `toString()` for any object. As with Java interfaces, Java knows which `toString()` method to call (polymorphically) because it knows the type of the invoking object. This convention is the basis for Java's automatic conversion of one operand of the string concatenation operator `+` to a string whenever the other operand is a string. For example, if `x` is any object reference, then Java automatically converts the expression `"x = " + x` to `"x = " + x.toString()`. If a class does not override the `toString()` method, then Java invokes the inherited `toString()` implementation, which is normally not helpful (typically a string representation of the memory address of the object). Accordingly, it is good programming practice to override the `toString()` method in every class that you develop.

Equality. What does it mean for two objects to be equal? If we test equality with `(x == y)`, where `x` and `y` are object references, we are testing whether they have the same identity: whether the *object references* are equal. For example, consider the code in the diagram at right, which creates two `Complex` objects (PROGRAM 3.2.6) referenced by three variables `c1`, `c2`, and `c3`. As illustrated in the diagram, `c1` and `c3` both reference the same object, which is different from the object referenced by `c2`. Consequently, `(c1 == c3)` is true but `(c1 == c2)` is false. This is known as *reference equality*, but it is rarely what clients want.

Typical clients want to test whether the *data-type values* (object state) are the same. This is known as *object equality*. Java includes the `equals()` method—which is inherited by all classes—for this purpose. For example, the `String` data type overrides this method in a natural manner: If `x` and `y` refer to `String` objects, then `x.equals(y)` is true if and only if the two strings correspond to the same sequence of characters (and not depending on whether they reference the same `String` object).

Java's convention is that the `equals()` method must implement an *equivalence relation* by satisfying the following three natural properties for all object references `x`, `y`, and `z`:

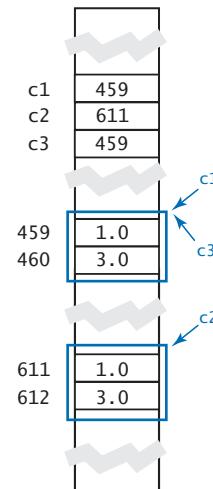
- *Reflexive*: `x.equals(x)` is true.
- *Symmetric*: `x.equals(y)` is true if and only if `y.equals(x)` is true.
- *Transitive*: if `x.equals(y)` is true and `y.equals(z)` is true, then `x.equals(z)` is true.

In addition, the following two properties must hold:

- Multiple calls to `x.equals(y)` return the same truth value, provided neither object is modified between calls.
- `x.equals(null)` returns false.

Typically, when we define our own data types, we override the `equals()` method because the inherited implementation is reference equality. For example, suppose we want to consider two `Complex` objects equal if and only if their real and imaginary components are the same. The implementation at the top of the next page gets the job done:

```
Complex c1, c2, c3;
c1 = new Complex(1.0, 3.0);
c2 = new Complex(1.0, 3.0);
c3 = c1;
```



Three references to two objects

```
public boolean equals(Object x)
{
    if (x == null) return false;
    if (this.getClass() != x.getClass()) return false;
    Complex that = (Complex) x;
    return (this.re == that.re) && (this.im == that.im);
}
```

This code is unexpectedly intricate because the argument to `equals()` can be a reference to an object of *any* type (or `null`), so we summarize the purpose of each statement:

- The first statement returns `false` if the arguments is `null`, as required.
- The second statement uses the inherited method `getClass()` to return `false` if the two objects are of different types.
- The cast in the third statement is guaranteed to succeed because of the second statement.
- The last statement implements the logic of the equality test by comparing the corresponding instance variables of the two objects.

You can use this implementation as a template—once you have implemented one `equals()` method, you will not find it difficult to implement another.

Hashing. We now consider a fundamental operation related to equality testing, known as *hashing*, which maps an object to an integer, known as a *hash code*. This operation is so important that it is handled by a method named `hashCode()`, which is inherited by all classes. Java's convention is that the `hashCode()` method must satisfy the following two properties for all object references `x` and `y`:

- If `x.equals(y)` is `true`, then `x.hashCode()` is equal to `y.hashCode()`.
- Multiple calls of `x.hashCode()` return the same integer, provided the object is not modified between calls.

For example, in the following code fragment, `x` and `y` refer to equal `String` objects—`x.equals(y)` is `true`—so they must have the same hash code; `x` and `z` refer to different `String` objects, so we expect their hash codes to be different.

```
String x = new String("Java"); // x.hashCode() is 2301506
String y = new String("Java"); // y.hashCode() is 2301506
String z = new String("Python"); // z.hashCode() is -1889329924
```

In typical applications, we use the hash code to map an object `x` to an integer in a small range, say between 0 and `m-1`, using this *hash function*:

```
private int hash(Object x)
{   return Math.abs(x.hashCode() % m); }
```

The call to `Math.abs()` ensures that the return value is not a negative integer, which might otherwise be the case if `x.hashCode()` is negative. We can use the hash function value as an integer index into an array of length `m` (the utility of this operation will become apparent in PROGRAM 3.3.4 and PROGRAM 4.4.3). By convention, objects whose values are equal must have the same hash code, so they also have the same hash function value. Objects whose values are not equal can have the same hash function value but we expect the hash function to divide `n` typical objects from the class into `m` groups of roughly equal size. Many of Java's immutable data types (including `String`) include implementations of `hashCode()` that are engineered to distribute objects in a reasonable manner.

Crafting a good implementation of `hashCode()` for a data type requires a deft combination of science and engineering, and is beyond the scope of this book. Instead, we describe a simple recipe for doing so in Java that is effective in a wide variety of situations:

- Ensure that the data type is immutable.
- Import the class `java.util.Objects`.
- Implement `equals()` by comparing all significant instance variables.
- Implement `hashCode()` by using all significant instance variables as arguments to the static method `Objects.hash()`.

The static method `Objects.hash()` generates a hash code for its sequence of arguments. For example, the following `hashCode()` implementation for the `Complex` data type (PROGRAM 3.2.1) accompanies the `equals()` implementation that we just considered:

```
public int hashCode()
{   return Objects.hash(re, im); }
```

```
import java.util.Objects;

public class Complex
{
    private final double re, im;
    ...

    public boolean equals(Object x)
    {
        if (x == null) return false;
        if (this.getClass() != x.getClass())
            return false;
        Complex that = (Complex) x;
        return (this.re == that.re)
            && (this.im == that.im);
    }

    public int hashCode()
    {   return Objects.hash(re, im); }

    public String toString()
    {   return re + " + " + im + "i"; }
}
```

Overriding the `equals()`, `hashCode()`, and `toString()` methods

Wrapper types. One of the main benefits of inheritance is code reuse. However, this code reuse is limited to reference types (and not primitive types). For example, the expression `x.hashCode()` is legal for any object reference `x`, but produces a compile-time error if `x` is a variable of a primitive type. For situations where we wish want to represent a value from a primitive type as an object, Java supplies built-in reference types known as *wrapper types*, one for each of the eight primitive types. For example, the wrapper types `Integer` and `Double` correspond to `int` and `double`, respectively. An object of a wrapper type “wraps” a value from a primitive type into an object, so that you can use instance methods such as `equals()` and `hashCode()`. Each of these wrapper types is immutable and includes both instance methods (such as `compareTo()` for comparing two objects numerically) and static methods (such as `Integer.parseInt()` and `Double.parseDouble()` for converting from strings to primitive types).

Autoboxing and unboxing. Java automatically converts between an object from a wrapper type and the corresponding primitive data-type value—in assignment statements, method arguments, and arithmetic/logic expressions—so that you can write code like the following:

```
Integer x = 17; // Autoboxing (int -> Integer)
int a = x;      // Unboxing   (Integer -> int)
```

In the first statement, Java automatically casts (*autoboxes*) the `int` value 17 to be an object of type `Integer` before assigning it to the variable `x`. Similarly, in the second statement, Java automatically casts (*unboxes*) the `Integer` object to be a value of type `int` before assigning that value to the variable `a`. Autoboxing and unboxing can be convenient features when writing code, but involves a significant amount of processing behind the scenes that can affect performance.

For code clarity and performance, we use primitive types for computing with numbers whenever possible. However, in CHAPTER 4, we will encounter several compelling examples (particularly with data types that store collections of objects), for which wrapper types and autoboxing/unboxing enable us to develop code for use with reference types and reuse that same code (without modification) with primitive types.

primitive type	wrapper type
<code>boolean</code>	<code>Boolean</code>
<code>byte</code>	<code>Byte</code>
<code>char</code>	<code>Character</code>
<code>double</code>	<code>Double</code>
<code>float</code>	<code>Float</code>
<code>int</code>	<code>Integer</code>
<code>long</code>	<code>Long</code>
<code>short</code>	<code>Short</code>

Application: data mining To illustrate some of the concepts discussed in this section in the context of an application, we next consider a software technology that is proving important in addressing the daunting challenges of *data mining*, a term that describes the process of discovering patterns by searching through massive amounts of information. This technology can serve as the basis for dramatic improvements in the quality of web search results, for multimedia information retrieval, for biomedical databases, for research in genomics, for improved scholarship in many fields, for innovation in commercial applications, for learning the plans of evildoers, and for many other purposes. Accordingly, there is intense interest and extensive ongoing research on data mining.

You have direct access to thousands of files on your computer and indirect access to billions of files on the web. As you know, these files are remarkably diverse: there are commercial web pages, music and video, email, program code, and all sorts of other information. For simplicity, we will restrict our attention to *text* documents (though the method we will consider applies to images, music, and all sorts of other files as well). Even with this restriction, there is remarkable diversity in the types of documents. For reference, you can find these documents on the booksite:

<i>file name</i>	<i>description</i>	<i>sample text</i>
Constitution.txt	<i>legal document</i>	... of both Houses shall be determined by ...
TomSawyer.txt	<i>American novel</i>	..."Say, Tom, let ME whitewash a little." ...
HuckFinn.txt	<i>American novel</i>	...was feeling pretty good after breakfast...
Prejudice.txt	<i>English novel</i>	... dared not even mention that gentleman....
Picture.java	<i>Java code</i>	...String suffix = filename.substring(file...
DJIA.csv	<i>financial data</i>	...01-Oct-28,239.43,242.46,350000,240.01 ...
Amazon.html	<i>web page source</i>	...<table width="100%" border="0" cellspac...
ACTG.txt	<i>virus genome</i>	...GTATGGAGCAGCAGACGGCTACTTCGAGCGGAGGCATA...

Some text documents

Our interest is in finding efficient ways to search through the files using their *content* to characterize documents. One fruitful approach to this problem is to associate with each document a vector known as a *sketch*, which is a function of its content. The basic idea is that the sketch should characterize a document, so that documents that are different have sketches that are different and documents that are similar have sketches that are similar. You probably are not surprised to learn that this approach can enable us to distinguish among a novel, a Java program, and

a genome, but you might be surprised to learn that content searches can tell the difference between novels written by different authors and can be effective as the basis for many other subtle search criteria.

To start, we need an abstraction for text documents. What is a text document? Which operations do we want to perform on text documents? The answers to these questions inform our design and, ultimately, the code that we write. For the purposes of data mining, it is clear that the answer to the first question is that a text document is defined by a string. The answer to the second question is that we need to be able to compute a number to measure the similarity between a document and any other document. These considerations lead to the following API:

```
public class Sketch
{
    Sketch(String text, int k, int d)
    double similarTo(Sketch other)      similarity measure between this sketch and other
    String toString()                  string representation
}
```

API for sketches (see PROGRAM 3.3.4)

The arguments of the constructor are a text string and two integers that control the quality and size of the sketch. Clients can use the `similarTo()` method to determine the extent of similarity between this `Sketch` and any other `Sketch` on a scale from 0 (not similar) to 1 (similar). The `toString()` method is primarily for debugging. This data type provides a good separation between implementing a similarity measure and implementing clients that use the similarity measure to search among documents.

Computing sketches. Our first challenge is to compute a sketch of the text string. We will use a sequence of real numbers (or a `Vector`) to represent a document's sketch. But which information should go into computing the sketch and how do we compute the `Vector` sketch? Many different approaches have been studied, and researchers are still actively seeking efficient and effective algorithms for this task. Our implementation `Sketch` (PROGRAM 3.3.4) uses a simple *frequency count* approach. The constructor has two arguments: an integer k and a vector dimension d . It scans the document and examines all of the k -grams in the document—that is, the substrings of length k starting at each position. In its simplest form, the sketch is a vector that gives the relative frequency of occurrence of the k -grams in the string; it is an element for each possible k -gram giving the number of k -grams in the content that have that value. For example, suppose that we use $k = 2$ in genomic

			CTTTCGGTTT
			GGAACCGAAG
			CCGGCGGTCT
	ATAGATGCAT		TGTCTGCTGC
	AGCGCATAGC		AGCATCGTTC
<i>2-gram hash</i>		<i>count</i>	<i>unit</i>
AA	0	0	.139
AC	1	0	.070
AG	2	.397	.139
AT	3	.530	.070
CA	4	.265	.139
CC	5	0	.139
CG	6	.132	.417
CT	7	0	.278
GA	8	.132	.139
GC	9	.530	.417
GG	10	0	.139
GT	11	0	.278
TA	12	.397	0
TC	13	0	.348
TG	14	.132	.278
TT	15	0	.417

Profiling genomic data

data, with $d = 16$ (there are 4 possible character values and therefore $4^2 = 16$ possible 2-grams). The 2-gram AT occurs 4 times in the string ATAGATGCATAGCGATAGC, so, for example, the vector element corresponding to AT would be 4. To build the frequency vector, we need to be able to convert each of the 16 possible k -grams into an integer between 0 and 15 (this function is known as a *hash value*). For genomic data, this is an easy exercise (see EXERCISE 3.3.28). Then, we can compute an array to build the frequency vector in one scan through the text, incrementing the array element corresponding to each k -gram encountered. It would seem that we lose information by disregarding the order of the k -grams, but the remarkable fact is that the information content of that order is lower than that of their frequency. A Markov model paradigm not dissimilar from the one that we studied for the random surfer in SECTION 1.6 can be used to take order into account—such models are effective, but much more work to implement. Encapsulating the computation in Sketch gives us the flexibility to experiment with various designs without needing to rewrite Sketch clients.

Hashing. For ASCII text strings there are 128 different possible values for each character, so there are 128^k possible k -grams, and the dimension d would have to be 128^k for the scheme just described. This number is prohibitively large even for moderately large k . For Unicode, with more than 65,536 characters, even 2-grams lead to huge vector sketches. To ameliorate this problem, we use *hashing*, a fundamental operation related to search algorithms that we just considered in our discussion of inheritance. Recall that all objects inherit a method `hashCode()` that returns an integer between -2^{31} and $2^{31}-1$. Given any string `s`, we use the expression `Math.abs(s.hashCode() % d)` to produce an integer *hash value* between 0 and $d-1$, which we can use as an index into an array of length d to compute frequencies. The sketch that we use is the *direction* of the vector defined by frequencies of these values for all k -grams in the document (the unit vector with the same direction). Since we expect different strings to have different hash values, text documents with similar k -gram distributions will have similar sketches and text documents with different k -gram distributions will very likely have different sketches.

Program 3.3.4 Document sketch

```

public class Sketch
{
    private final Vector profile;                                profile | unit vector

    public Sketch(String text, int k, int d)
    {
        int n = text.length();
        double[] freq = new double[d];
        for (int i = 0; i < n-k-1; i++)
        {
            String kgram = text.substring(i, i+k);
            int hash = kgram.hashCode();
            freq[Math.abs(hash % d)] += 1;
        }
        Vector vector = new Vector(freq);
        profile = vector.direction();
    }

    public double similarTo(Sketch other)
    {   return profile.dot(other.profile);   }

    public static void main(String[] args)
    {
        int k = Integer.parseInt(args[0]);
        int d = Integer.parseInt(args[1]);
        String text = StdIn.readAll();
        Sketch sketch = new Sketch(text, k, d);
        StdOut.println(sketch);
    }
}

```

name k d text n freq[] hash	document name length of gram dimension entire document document length hash frequencies hash for k-gram
---	---

This `Vector` client creates a d -dimensional unit vector from a document's k -grams that clients can use to measure its similarity to other documents (see `text`). The `toString()` method appears as EXERCISE 3.3.15.

```

% more genome20.txt
ATAGATGCATAGCGCATAGC
% java Sketch 2 16 < genome20.txt
(0.0, 0.0, 0.0, 0.620, 0.124, 0.372, ..., 0.496, 0.372, 0.248, 0.0)

```

Comparing sketches. The second challenge is to compute a similarity measure between two sketches. Again, there are many different ways to compare two vectors. Perhaps the simplest is to compute the *Euclidean distance* between them. Given vectors \mathbf{x} and \mathbf{y} , this distance is defined by

$$|\mathbf{x} - \mathbf{y}| = ((x_0 - y_0)^2 + (x_1 - y_1)^2 + \dots + (x_{d-1} - y_{d-1})^2)^{1/2}$$

You are familiar with this formula for $d = 2$ or $d = 3$. With `Vector`, the Euclidean distance is easy to compute. If \mathbf{x} and \mathbf{y} are two `Vector` objects, then `x.minus(y).magnitude()` is the Euclidean distance between them. If documents are similar, we expect their sketches to be similar and the distance between them to be low. Another widely used similarity measure, known as the *cosine similarity measure*, is even simpler: since our sketches are unit vectors with non-negative coordinates, their *dot product*

$$\mathbf{x} \cdot \mathbf{y} = x_0 y_0 + x_1 y_1 + \dots + x_{d-1} y_{d-1}$$

is a real number between 0 and 1. Geometrically, this quantity is the cosine of the angle formed by the two vectors (see EXERCISE 3.3.10). The more similar the documents, the closer we expect this measure to be to 1.

Comparing all pairs. `CompareDocuments` (PROGRAM 3.3.5) is a simple and useful Sketch client that provides the information needed to solve the following problem: given a set of documents, find the two that are most similar. Since this specification is a bit subjective, `CompareDocuments` prints the cosine similarity measure for all pairs of documents on an input list. For moderate-size k and d , the sketches do a remarkably good job of characterizing our sample set of documents. The results say not only that genomic data, financial data, Java code, and web source code are quite different from legal documents and novels, but also that *Tom Sawyer* and *Huckleberry Finn* are much more similar to each other than to *Pride and Prejudice*. A researcher in comparative literature could use this program to discover relationships between texts; a teacher could also use this program to detect plagiarism in a set of student submissions (indeed, many teachers *do* use such programs on a regular basis); and a biologist could use this program to discover relationships among genomes. You can find many documents on the booksite (or gather your own collection) to test the effectiveness of `CompareDocuments` for various parameter settings.

```
% more documents.txt
Constitution.txt
TomSawyer.txt
HuckFinn.txt
Prejudice.txt
Picture.java
DJIA.csv
Amazon.html
ATCG.txt
```

Program 3.3.5 Similarity detection

```

public class CompareDocuments
{
    public static void main(String[] args)
    {
        int k = Integer.parseInt(args[0]);
        int d = Integer.parseInt(args[1]);
        String[] filenames = StdIn.readAllStrings();
        int n = filenames.length;
        Sketch[] a = new Sketch[n];
        for (int i = 0; i < n; i++)
            a[i] = new Sketch(new In(filenames[i]).readAll(), k, d);

        StdOut.print("      ");
        for (int j = 0; j < n; j++)
            StdOut.printf("%8.4s", filenames[j]);
        StdOut.println();
        for (int i = 0; i < n; i++)
        {
            StdOut.printf("%.4s", filenames[i]);
            for (int j = 0; j < n; j++)
                StdOut.printf("%8.2f", a[i].similarTo(a[j]));
            StdOut.println();
        }
    }
}

```

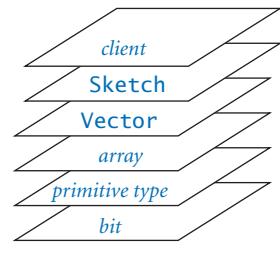
k	<i>length of gram</i>
d	<i>dimension</i>
n	<i>number of documents</i>
a[]	<i>the sketches</i>

This Sketch client reads a document list from standard input, computes sketches based on k-gram frequencies for all the documents, and prints a table of similarity measures between all pairs of documents. It takes two arguments from the command line: the value of k and the dimension d of the sketches.

% java CompareDocuments 5 10000 < documents.txt								
	Cons	TomS	Huck	Prej	Pict	DJIA	Amaz	ATCG
Cons	1.00	0.66	0.60	0.64	0.20	0.18	0.21	0.11
TomS	0.66	1.00	0.93	0.88	0.12	0.24	0.18	0.14
Huck	0.60	0.93	1.00	0.82	0.08	0.23	0.16	0.12
Prej	0.64	0.88	0.82	1.00	0.11	0.25	0.19	0.15
Pict	0.20	0.12	0.08	0.11	1.00	0.04	0.39	0.03
DJIA	0.18	0.24	0.23	0.25	0.04	1.00	0.16	0.11
Amaz	0.21	0.18	0.16	0.19	0.39	0.16	1.00	0.07
ATCG	0.11	0.14	0.12	0.15	0.03	0.11	0.07	1.00

Searching for similar documents. Another natural Sketch client is one that uses sketches to search among a large number of documents to identify those that are similar to a given document. For example, web search engines use clients of this type to present you with pages that are similar to those you have previously visited, online book merchants use clients of this type to recommend books that are similar to ones you have purchased, and social networking websites use clients of this type to identify people whose personal interests are similar to yours. Since In can take web addresses instead of file names, it is feasible to write a program that can surf the web, compute sketches, and return links to web pages that have sketches that are similar to the one sought. We leave this client for a challenging exercise.

THIS SOLUTION IS JUST A SKETCH. Many sophisticated algorithms for efficiently computing sketches and comparing them are still being invented and studied by computer scientists. Our purpose here is to introduce you to this fundamental problem domain while at the same time illustrating the power of abstraction in addressing a computational challenge. Vectors are an essential mathematical abstraction, and we can build a similarity search client by developing *layers of abstraction*: Vector is built with the Java array, Sketch is built with Vector, and client code uses Sketch. As usual, we have spared you from a lengthy account of our many attempts to develop these APIs, but you can see that the data types are designed in response to the needs of the problem, with an eye toward the requirements of implementations. Identifying and implementing appropriate abstractions is the key to effective object-oriented programming. The power of abstraction—in mathematics, physical models, and computer programs—pervades these examples. As you become fluent in developing data types to address your own computational challenges, your appreciation for this power will surely grow.

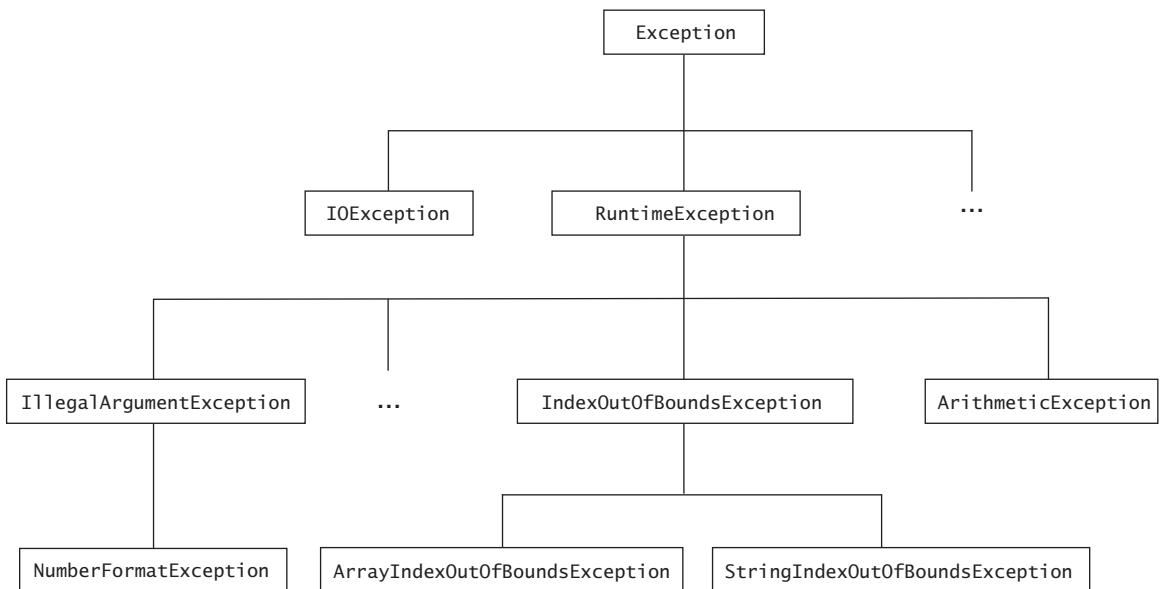


Layers of abstraction

Design by contract To conclude, we briefly discuss Java language mechanisms that enable you to verify assumptions about your program *while it is running*. For example, if you have a data type that represents a particle, you might assert that its mass is positive and its speed is less than the speed of light. Or if you have a method to add two vectors of the same dimension, you might assert that the dimension of the resulting vector is the same.

Exceptions. An *exception* is a disruptive event that occurs *while* a program is running, often to signal an error. The action taken is known as *throwing an exception*. We have already encountered exceptions thrown by Java system methods in the course of learning to program: `ArithmaticException`, `IllegalArgument-Exception`, `NumberFormatException`, and `ArrayIndexOutOfBoundsException` are typical examples.

You can also create and throw your own exceptions. Java includes an elaborate inheritance hierarchy of predefined exceptions; each exception class is a subclasses of `java.lang.Exception`. The diagram at the bottom of this page illustrates a portion of this hierarchy.



Subclass inheritance hierarchy for exceptions (partial)

Perhaps the simplest kind of exception is a `RuntimeException`. The following statement creates a `RuntimeException`; typically it terminates execution of the program and prints a custom error message

```
throw new RuntimeException("Custom error message here.");
```

It is good practice to use exceptions when they can be helpful to the user. For example, in `Vector` (PROGRAM 3.3.3), we should throw an exception in `plus()` if the two `Vectors` to be added have different dimensions. To do so, we insert the following statement at the beginning of `plus()`:

```
if (this.coords.length != that.coords.length)
    throw new IllegalArgumentException("Dimensions disagree.");
```

With this code, the client receives a precise description of the API violation (calling the `plus()` method with vectors of different dimensions), enabling the programmer to identify and fix the mistake. Without this code, the behavior of the `plus()` method is erratic, either throwing an `ArrayIndexOutOfBoundsException` or returning a bogus result, depending on the dimensions of the two vectors (see EXERCISE 3.3.16).

Assertions. An *assertion* is a boolean expression that you are affirming is true at some point *during* the execution of a program. If the expression is false, the program will throw an `AssertionError`, which typically terminates the program and reports an error message. Errors are like exceptions, except that they indicate catastrophic failure; `StackOverflowError` and `OutOfMemoryError` are two examples that we have previously encountered.

Assertions are widely used by programmers to detect bugs and gain confidence in the correctness of programs. They also serve to document the programmer's intent. For example, in `Counter` (PROGRAM 3.3.2), we might check that the counter is never negative by adding the following assertion as the last statement in `increment()`:

```
assert count >= 0;
```

This statement would identify a negative count. You can also add a custom message

```
assert count >= 0 : "Negative count detected in increment();"
```

to help you locate the bug. By default, assertions are disabled, but you can enable them from the command line by using the `-enableassertions` flag (`-ea` for

short). Assertions are for debugging only; your program should not rely on assertions for normal operation since they may be disabled.

When you take a course in systems programming, you will learn to use assertions to ensure that your code *never* terminates in a system error or goes into an infinite loop. One model, known as the *design-by-contract* model of programming, expresses this idea. The designer of a data type expresses a *precondition* (the condition that the client promises to satisfy when calling a method), a *postcondition* (the condition that the implementation promises to achieve when returning from a method), *invariants* (any condition that the implementation promises to satisfy while the method is executing), and *side effects* (any other change in state that the method could cause). During development, these conditions can be tested with assertions. Many programmers use assertions liberally to aid in debugging.

THE LANGUAGE MECHANISMS DISCUSSED THROUGHOUT THIS section illustrate that effective data-type design takes us into deep water in programming-language design. Experts are still debating the best ways to support some of the design ideas that we are discussing. Why does Java not allow functions as arguments to methods? Why does Python not include language support for enforcing encapsulation? Why does Matlab not support mutable data types? As mentioned early in CHAPTER 1, it is a slippery slope from complaining about features in a programming language to becoming a programming-language designer. If you do not plan to do so, your best strategy is to use widely available languages. Most systems have extensive libraries that you certainly should use when appropriate, but you often can simplify your client code and protect yourself by building abstractions that can easily be transferred to other languages. Your main goal is to develop data types so that most of your work is done at a level of abstraction that is appropriate to the problem at hand.

Q&A

Q. What happens if I try to access a `private` instance variable or method from a class in another file?

A. You get a compile-time error that says the given instance variable or method has `private` access in the given class.

Q. The instance variables in `Complex` are `private`, but when I am executing the method `plus()` for a `Complex` object with `a.plus(b)`, I can access not only `a`'s instance variables but also `b`'s. Shouldn't `b`'s instance variables be inaccessible?

A. The granularity of `private` access is at the class level, not the instance level. Declaring an instance variable as `private` means that it is not directly accessible from any other class. Methods within the `Complex` class can access (read or write) the instance variables of any instance in that class. It might be nice to have a more restrictive access modifier—say, `superprivate`—that would impose the granularity at the instance level so that only the invoking object can access its instance variables, but Java does not have such a facility.

Q. The `times()` method in `Complex` (PROGRAM 3.3.1) needs a constructor that takes polar coordinates as arguments. How can we add such a constructor?

A. You cannot, since there is already a constructor that takes two floating-point arguments. An alternative design would be to have two *factory* methods `createRect(x, y)` and `createPolar(r, theta)` in the API that create and return new objects. This design is better because it would provide the *client* with the capability to create objects by specifying either rectangular or polar coordinates. This example demonstrates that it is a good idea to think about more than one implementation when developing a data type.

Q. Is there a relationship between the `Vector` (PROGRAM 3.3.3) data type defined in this section and Java's `java.util.Vector` data type?

A. No. We use the name because the term *vector* properly belongs to linear algebra and vector calculus.

Q. What should the `direction()` method in `Vector` (PROGRAM 3.3.3) do if invoked with the all zero vector?

A. A complete API should specify the behavior of every method for every situation. In this case, throwing an exception or returning `null` would be appropriate.

Q. What is a *deprecated* method?

A. A method that is no longer fully supported, but kept in an API to maintain compatibility. For example, Java once included a method `Character.isSpace()`, and programmers wrote programs that relied on using that method's behavior. When the designers of Java later wanted to support additional Unicode whitespace characters, they could not change the behavior of `isSpace()` without breaking client programs. To deal with this issue, they added a new method `Character.isWhiteSpace()` and *deprecated* the old method. As time wears on, this practice certainly complicates APIs.

Q. What is wrong with the following implementation of `equals()` for `Complex`?

```
public boolean equals(Complex that)
{
    return (this.re == that.re) && (this.im == that.im);
```

A. This code *overloads* the `equals()` method instead of *overriding* it. That is, it defines a new method named `equals()` that takes an argument of type `Complex`. This overloaded method is different from the inherited method `equals()` that takes an argument of type `Object`. There are some situations—such as with the `java.util.HashMap` library that we consider in SECTION 4.4—in which the inherited method gets called instead of the overloaded method, leading to puzzling behavior.

Q. What is wrong with the following of `hashCode()` for `Complex`?

```
public int hashCode()
{   return -17; }
```

A. Technically, it satisfies the contract for `hashCode()`: if two objects are equal, they have the same hash code. However, it will lead to poor performance because we expect `Math.abs(x.hashCode() % m)` to divide `n` typical `Complex` objects into `m` groups of roughly equal size.

Q. Can an interface include constructors?

A. No, because you cannot instantiate an interface; you can instantiate only objects of an implementing class. However, an interface can include constants, method signatures, default methods, static methods, and nested types, but these features are beyond the scope of this book.

Q. Can a class be a direct subclass of more than one class?

A. No. Every class (other than `Object`) is a direct subclass of one and only one superclass. This feature is known as *single inheritance*; some other languages (notably, C++) support *multiple inheritance*, where a class can be a direct subclass of two or more superclasses.

Q. Can a class implement more than one interface?

A. Yes. To do so, list each of the interfaces, separated by commas, after the keyword `implements`.

Q. Can the body of a lambda expression consist of more than a single statement?

A. Yes, the body can be a block of statements and can include variable declarations, loops, and conditionals. In such cases, you must use an explicit `return` statement to specify the value returned by the lambda expression.

Q. In some cases a lambda expression does nothing more than call a named method in another class. Is there any shorthand for doing this?

A. Yes, a *method reference* is a compact, easy-to-read lambda expression for a method that already has a name. For example, you can use the method reference `Gaussian::pdf` as shorthand for the lambda expression `x -> Gaussian.pdf(x)`. See the booksite for more details.

Exercises

3.3.1 Represent a point in time by using an `int` to store the number of seconds since January 1, 1970. When will programs that use this representation face a time bomb? How should you proceed when that happens?

3.3.2 Create a data type `Location` for dealing with locations on Earth using spherical coordinates (latitude/longitude). Include methods to generate a random location on the surface of the Earth, parse a location “25.344 N, 63.5532 W”, and compute the great circle distance between two locations.

3.3.3 Develop an implementation of `Histogram` (PROGRAM 3.2.3) that uses `Counter` (PROGRAM 3.3.2).

3.3.4 Give an implementation of `minus()` for `Vector` solely in terms of the other `Vector` methods, such as `direction()` and `magnitude()`.

Answer:

```
public Vector minus(Vector that)
{   return this.plus(that.scale(-1.0)); }
```

The advantage of such implementations is that they limit the amount of detailed code to check; the disadvantage is that they can be inefficient. In this case, `plus()` and `times()` both create new `Vector` objects, so copying the code for `plus()` and replacing the minus sign with a plus sign is probably a better implementation.

3.3.5 Implement a `main()` method for `Vector` that unit-tests its methods.

3.3.6 Create a data type for a three-dimensional particle with position (r_x, r_y, r_z), mass (m), and velocity (v_x, v_y, v_z). Include a method to return its kinetic energy, which equals $1/2 m (v_x^2 + v_y^2 + v_z^2)$. Use `Vector` (PROGRAM 3.3.3).

3.3.7 If you know your physics, develop an alternate implementation for your data type from the previous exercise based on using the *momentum* (p_x, p_y, p_z) as an instance variable.

3.3.8 Implement a data type `Vector2D` for two-dimensional vectors that has the same API as `Vector`, except that the constructor takes two `double` values as arguments. Use two `double` values (instead of an array) for instance variables.

3.3.9 Implement the `Vector2D` data type from the previous exercise using one `Complex` value as the only instance variable.

3.3.10 Prove that the dot product of two two-dimensional unit-vectors is the cosine of the angle between them.

3.3.11 Implement a data type `Vector3D` for three-dimensional vectors that has the same API as `Vector`, except that the constructor takes three `double` values as arguments. Also, add a *cross-product* method: the cross-product of two vectors is another vector, defined by the equation

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} |\mathbf{a}| |\mathbf{b}| \sin \theta$$

where **c** is the unit normal vector perpendicular to both **a** and **b**, and θ is the angle between **a** and **b**. In Cartesian coordinates, the following equation defines the cross-product:

$$(a_0, a_1, a_2) \times (b_0, b_1, b_2) = (a_1 b_2 - a_2 b_1, a_2 b_0 - a_0 b_2, a_0 b_1 - a_1 b_0)$$

The cross-product arises in the definition of torque, angular momentum, and vector operator curl. Also, $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram with sides **a** and **b**.

3.3.12 Override the `equals()` method for `Charge` (PROGRAM 3.2.6) so that two `Charge` objects are equal if they have identical position and charge value. Override the `hashCode()` method using the `Objects.hash()` technique described in this section.

3.3.13 Override the `equals()` and `hashCode()` methods for `Vector` (PROGRAM 3.3.3) so that two `Vector` objects are equal if they have the same length and the corresponding coordinates are equal.

3.3.14 Add a `toString()` method to `Vector` that returns the vector components, separated by commas, and enclosed in matching parentheses.

3.3.15 Add a `toString()` method to `Sketch` that returns a string representation of the unit vector corresponding to the sketch.

3.3.16 Describe the behavior of the method calls `x.add(y)` and `y.add(x)` in `Vector` (PROGRAM 3.3.3) if `x` corresponds to the vector $(1, 2, 3)$ and `y` corresponds to the vector $(5, 6)$.

3.3.17 Use assertions and exceptions to develop an implementation of `Rational` (see EXERCISE 3.2.7) that is immune to overflow.

3.3.18 Add code to `Counter` (PROGRAM 3.3.2) to throw an `IllegalArgumentException` if the client tries to construct a `Counter` object using a negative value for `max`.

Data-Type Design Exercises

This list of exercises is intended to give you experience in developing data types. For each problem, design one or more APIs with API implementations, testing your design decisions by implementing typical client code. Some of the exercises require either knowledge of a particular domain or a search for information about it on the web.

3.3.19 Statistics. Develop a data type for maintaining statistics for a set of real numbers. Provide a method to add data points and methods that return the number of points, the mean, the standard deviation, and the variance. Develop two implementations: one whose instance values are the number of points, the sum of the values, and the sum of the squares of the values, and another that keeps an array containing all the points. For simplicity, you may take the maximum number of points in the constructor. Your first implementation is likely to be faster and use substantially less space, but is also likely to be susceptible to roundoff error. See the booksite for a well-engineered alternative.

3.3.20 Genome. Develop a data type to store the genome of an organism. Biologists often abstract the genome to a sequence of nucleotides (A, C, G, or T). The data type should support the methods `addNucleotide(char c)` and `nucleotideAt(int i)`, as well as `isPotentialGene()` (see PROGRAM 3.1.1). Develop three implementations. First, use one instance variable of type `String`, implementing `addCodon()` with string concatenation. Each method call takes time proportional to the length of the current genome. Second, use an array of characters, doubling the length of the array each time it fills up. Third, use a boolean array, using two bits to encode each codon, and doubling the length of the array each time it fills up.

3.3.21 Time. Develop a data type for the time of day. Provide client methods that return the current hour, minute, and second, as well as `toString()`, `equals()`, and `hashCode()` methods. Develop two implementations: one that keeps the time as a single `int` value (number of seconds since midnight) and another that keeps three `int` values, one each for seconds, minutes, and hours.

3.3.22 VIN number. Develop a data type for the naming scheme for vehicles known as the Vehicle Identification Number (VIN). A VIN describes the make, model, year, and other attributes of cars, buses, and trucks in the United States.

3.3.23 *Generating pseudo-random numbers.* Develop a data type for generating pseudo-random numbers. That is, convert `StdRandom` to a data type. Instead of using `Math.random()`, base your data type on a *linear congruential generator*. This method traces to the earliest days of computing and is also a quintessential example of the value of maintaining state in a computation (implementing a data type). To generate pseudo-random `int` values, maintain an `int` value x (the value of the last “random” number returned). Each time the client asks for a new value, return $a*x + b$ for suitably chosen values of a and b (ignoring overflow). Use arithmetic to convert these values to “random” values of other types of data. As suggested by D. E. Knuth, use the values 3141592621 for a and 2718281829 for b . Provide a constructor allowing the client to start with an `int` value known as a seed (the initial value of x). This ability makes it clear that the numbers are not at all random (even though they may have many of the properties of random numbers) but that fact can be used to aid in debugging, since clients can arrange to see the same numbers each time.

Creative Exercises

3.3.24 Encapsulation. Is the following class immutable?

```
import java.util.Date;
public class Appointment
{
    private Date date;
    private String contact;

    public Appointment(Date date)
    {
        this.date = date;
        this.contact = contact;
    }
    public Date getDate()
    {   return date;  }
}
```

Answer: No. Java's `java.util.Date` class is mutable. The method `setDate(seconds)` changes the value of the invoking date to the number of milliseconds since January 1, 1970, 00:00:00 GMT. This has the unfortunate consequence that when a client gets a date with `date = getDate()`, the client program can then invoke `date.setDate()` and change the date in an `Appointment` object type, perhaps creating a conflict. In a data type, we cannot let references to mutable objects escape because the caller can then modify its state. One solution is to create a defensive copy of the `Date` before returning it using `new Date(date.getTime())`; and a defensive copy when storing it via `this.date = new Date(date.getTime())`. Many programmers regard the mutability of `Date` as a Java design flaw. (GregorianCalendar is a more modern Java library for storing dates, but it is mutable, too.)

3.3.25 Date. Develop an implementation of Java's `java.util.Date` API that is immutable and therefore corrects the defects of the previous exercise.

3.3.26 Calendar. Develop `Appointment` and `Calendar` APIs that can be used to keep track of appointments (by day) in a calendar year. Your goal is to enable clients to schedule appointments that do not conflict and to report current appointments to clients.

3.3.27 *Vector field.* A vector field associates a vector with every point in a Euclidean space. Write a version of `Potential` (EXERCISE 3.2.23) that takes as input a grid size n , computes the `Vector` value of the potential due to the point charges at each point in an n -by- n grid of evenly spaced points, and draws the unit vector in the direction of the accumulated field at each point. (Modify `Charge` to return a `Vector`.)

3.3.28 *Genome profiling.* Write a function `hash()` that takes as its argument a k -gram (string of length k) whose characters are all A, C, G, or T and returns an `int` value between 0 and $4^k - 1$ that corresponds to treating the strings as base-4 numbers with {A, C, G, T} replaced by {0, 1, 2, 3}, respectively. Next, write a function `unHash()` that reverses the transformation. Use your methods to create a class `Genome` that is like `Sketch` (PROGRAM 3.3.4), but is based on exact counting of k -grams in genomes. Finally, write a version of `CompareDocuments` (PROGRAM 3.3.5) for `Genome` objects and use it to look for similarities among the set of genome files on the booksite.

3.3.29 *Profiling.* Pick an interesting set of documents from the booksite (or use a collection of your own) and run `CompareDocuments` with various values for the command-line arguments k and d , to learn about their effect on the computation.

3.3.30 *Multimedia search.* Develop profiling strategies for sound and pictures, and use them to discover interesting similarities among songs in the music library and photos in the photo album on your computer.

3.3.31 *Data mining.* Write a recursive program that surfs the web, starting at a page given as the first command-line argument, looking for pages that are similar to the page given as the second command-line argument, as follows: to process a name, open an input stream, do a `readAll()`, sketch it, and print the name if its distance to the target page is greater than the threshold value given as the third command-line argument. Then scan the page for all strings that begin with the prefix `http://` and (recursively) process pages with those names. *Note:* This program could read a very large number of pages!

3.4 Case Study: N-Body Simulation

SEVERAL OF THE EXAMPLES THAT WE considered in CHAPTERS 1 AND 2 are better expressed as object-oriented programs. For example, `BouncingBall` (PROGRAM 3.1.9) is naturally implemented as a data type whose values are the position and the velocity of the ball and a client that calls instance methods to move and draw the ball. Such a data type enables, for example, clients that can simulate the motion of several balls at once (see EXERCISE 3.4.1). Similarly, our case study for Percolation in SECTION 2.4 certainly makes an interesting exercise in object-oriented programming, as does our random-surfer case study in SECTION 1.6. We leave the former as EXERCISE 3.4.8 and revisit the latter in SECTION 4.5. In this section, we consider a new example that exemplifies object-oriented programming.

Our task is to write a program that dynamically simulates the motion of n bodies under the influence of mutual gravitational attraction. This problem was first formulated by Isaac Newton more than 350 years ago, and it is still studied intensely today.

What is the set of values, and what are the operations on those values? One reason that this problem is an amusing and compelling example of object-oriented programming is that it presents a direct and natural correspondence between physical objects in the real world and the abstract objects that we use in programming. The shift from solving problems by putting together sequences of statements to be executed to beginning with data-type design is a difficult one for many novices. As you gain more experience, you will appreciate the value in this approach to computational problem-solving.

We recall a few basic concepts and equations that you learned in high school physics. Understanding those equations fully is not required to appreciate the code—because of *encapsulation*, these equations are restricted to a few methods, and because of *data abstraction*, most of the code is intuitive and will make sense to you. In a sense, this is the ultimate object-oriented program.

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Programs in this section

N-body simulation The bouncing ball simulation of SECTION 1.5 is based on Newton's first law of motion: a body in motion remains in motion at the same velocity unless acted on by an outside force. Embellishing that simulation to incorporate Newton's second law of motion (which explains how outside forces affect velocity) leads us to a basic problem that has fascinated scientists for ages. Given a system of n bodies, mutually affected by gravitational forces, the *n-body problem* is to describe their motion. The same basic model applies to problems ranging in scale from astrophysics to molecular dynamics.

In 1687, Newton formulated the principles governing the motion of two bodies under the influence of their mutual gravitational attraction, in his famous *Principia*. However, Newton was unable to develop a mathematical description of the motion of *three* bodies. It has since been shown that not only is there no such description in terms of elementary functions, but also chaotic behavior is possible, depending on the initial values. To study such problems, scientists have no recourse but to develop an accurate simulation. In this section, we develop an object-oriented program that implements such a simulation. Scientists are interested in studying such problems at a high degree of accuracy for huge numbers of bodies, so our solution is merely an introduction to the subject. Nevertheless, you are likely to be surprised at the ease with which we can develop realistic animations depicting the complexity of the motion.

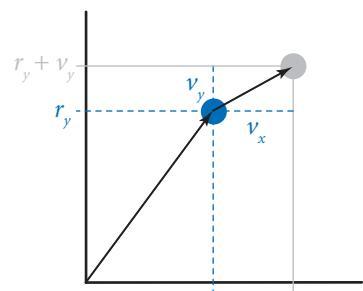
Body data type. In `BouncingBall` (PROGRAM 3.1.9), we keep the displacement from the origin in the double variables `rx` and `ry` and the velocity in the double variables `vx` and `vy`, and displace the ball the amount it moves in one time unit with the statements:

```
rx = rx + vx;
ry = ry + vy;
```

With `Vector` (PROGRAM 3.3.3), we can keep the position in the `Vector` variable `r` and the velocity in the `Vector` variable `v`, and then displace the body by the amount it moves in `dt` time units with a single statement:

```
r = r.plus(v.times(dt));
```

In *n*-body simulation, we have several operations of this kind, so our first design decision is to work with `Vector` objects instead of individual x - and y -components. This decision



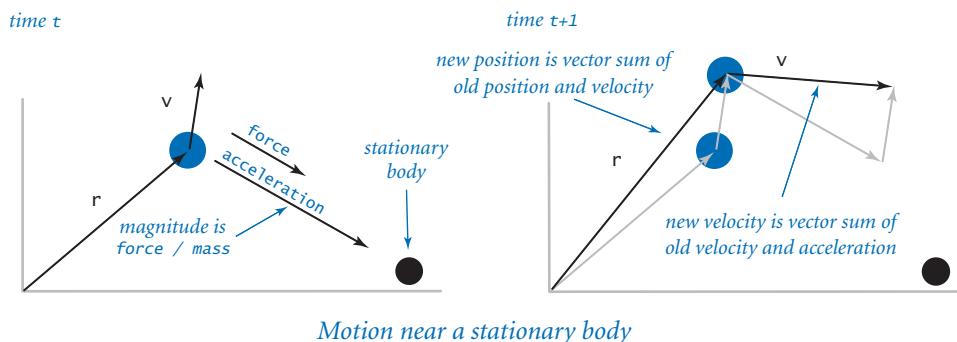
Adding vectors to move a ball

leads to code that is clearer, more compact, and more flexible than the alternative of working with individual components. `Body` (PROGRAM 3.4.1) is a Java class that uses `Vector` to implement a data type for moving bodies. Its instance variables are two `Vector` variables that hold the body's position and velocity, as well as a `double` variable that stores the mass. The data-type operations allow clients to move and to draw the body (and to compute the force vector due to gravitational attraction of another body), as defined by the following API:

```
public class Body
    Body(Vector r, Vector v, double mass)
    void move(Vector f, double dt) apply force f, move body for dt seconds
    void draw() draw the ball
    Vector forceFrom(Body b) force vector between this body and b
    API for bodies moving under Newton's laws (see PROGRAM 3.4.1)
```

Technically, the body's position (displacement from the origin) is not a vector (it is a point in space, rather than a direction and a magnitude), but it is convenient to represent it as a `Vector` because `Vector`'s operations lead to compact code for the transformation that we need to move the body, as just discussed. When we move a `Body`, we need to change not just its position, but also its velocity.

Force and motion. Newton's second law of motion says that the force on a body (a vector) is equal to the product of its mass (a scalar) and its acceleration (also a vector): $\mathbf{F} = m \mathbf{a}$. In other words, to compute the acceleration of a body, we compute the force, then divide by its mass. In `Body`, the force is a `Vector` argument `f` to



`move()`, so that we can first compute the acceleration vector just by dividing by the mass (a scalar value that is stored in a `double` instance variable) and then compute the change in velocity by adding to it the amount this vector changes over the time interval (in the same way as we used the velocity to change the position). This law immediately translates to the following code for updating the position and velocity of a body due to a given force vector \mathbf{f} and amount of time dt :

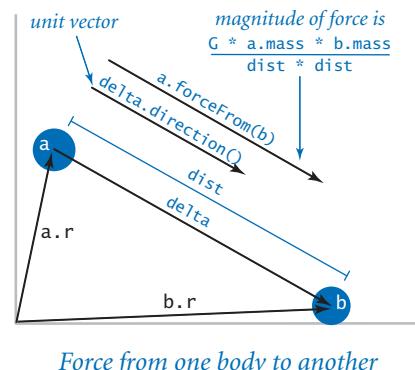
```
Vector a = f.scale(1/mass);
v = v.plus(a.scale(dt));
r = r.plus(v.scale(dt));
```

This code appears in the `move()` instance method in `Body`, to adjust its values to reflect the consequences of that force being applied for that amount of time: the body moves and its velocity changes. This calculation assumes that the acceleration is constant during the time interval.

Forces among bodies. The computation of the force imposed by one body on another is encapsulated in the instance method `forceFrom()` in `Body`, which takes a `Body` object as its argument and returns a `Vector`. *Newton's law of universal gravitation* is the basis for the calculation: it says that the magnitude of the gravitational force between two bodies is given by the product of their masses divided by the square of the distance between them (scaled by the gravitational constant G , which is $6.67 \times 10^{-11} \text{ N m}^2 / \text{kg}^2$) and that the direction of the force is the line between the two particles. This law translates into the following code for computing `a.forceFrom(b)`:

```
double G = 6.67e-11;
Vector delta = b.r.minus(a.r);
double dist = delta.magnitude();
double magnitude = (G * a.mass * b.mass) / (dist * dist);
Vector force = delta.direction().scale(magnitude);
return force;
```

The *magnitude* of the force vector is the `double` variable `magnitude`, and the *direction* of the force vector is the same as the direction of the difference vector between the two body's positions. The force vector `force` is the unit direction vector, scaled by the magnitude.



Program 3.4.1 Gravitational body

```

public class Body
{
    private Vector r;
    private Vector v;
    private final double mass;

    public Body(Vector r0, Vector v0, double m0)
    {   r = r0; v = v0; mass = m0; }

    public void move(Vector force, double dt)
    {   // Update position and velocity.
        Vector a = force.scale(1/mass);
        v = v.plus(a.scale(dt));
        r = r.plus(v.scale(dt));
    }

    public Vector forceFrom(Body b)
    {   // Compute force on this body from b.
        Body a = this;
        double G = 6.67e-11;
        Vector delta = b.r.minus(a.r);
        double dist = delta.magnitude();
        double magnitude = (G * a.mass * b.mass)
                           / (dist * dist);
        Vector force = delta.direction().scale(magnitude);
        return force;
    }

    public void draw()
    {
        StdDraw.setPenRadius(0.0125);
        StdDraw.point(r.cartesian(0), r.cartesian(1));
    }
}

```

<code>r</code> <code>v</code> <code>mass</code>	<i>position</i> <i>velocity</i> <i>mass</i>
<code>force</code> <code>dt</code> <code>a</code>	<i>force on this body</i> <i>time increment</i> <i>acceleration</i>
<code>a</code> <code>b</code> <code>G</code> <code>delta</code> <code>dist</code> <code>magnitude</code>	<i>this body</i> <i>another body</i> <i>gravitational constant</i> <i>vector from b to a</i> <i>distance from b to a</i> <i>magnitude of force</i>

This data type provides the operations that we need to simulate the motion of physical bodies such as planets or atomic particles. It is a mutable type whose instance variables are the position and velocity of the body, which change in the `move()` method in response to external forces (the body's mass is not mutable). The `forceFrom()` method returns a force vector.

Universe data type. Universe (PROGRAM 3.4.2) is a data type that implements the following API:

```
public class Universe
    Universe(String filename)      initialize universe from filename
    void increaseTime(double dt)   simulate the passing of dt seconds
    void draw()                   draw the universe
API for a universe (see PROGRAM 3.4.2)
```

Its data-type values define a universe (its size, number of bodies, and an array of bodies) and two data-type operations: `increaseTime()`, which adjusts the positions (and velocities) of all of the bodies, and `draw()`, which draws all of the bodies. The key to the n -body simulation is the implementation of `increaseTime()` in `Universe`. The main part of the computation is a double nested loop that computes the force vector describing the gravitational force of each body on each other body. It applies the *principle of superposition*, which says that we can add together the force vectors affecting a body to get a single vector representing all the forces. After it has computed all of the forces, it calls `move()` for each body to apply the computed force for a fixed time interval.

File format. As usual, we use a data-driven design, with input taken from a file. The constructor reads the universe parameters and body descriptions from a file that contains the following information:

- The number of bodies
- The radius of the universe
- The position, velocity, and mass of each body

As usual, for consistency, all measurements are in standard SI units (recall also that the gravitational constant G appears in our code). With this defined file format, the code for our `Universe` constructor is straightforward.

```
% more 2body.txt
2
5.0e10
0.0e00 4.5e10 1.0e04 0.0e00 1.5e30
0.0e00 -4.5e10 -1.0e04 0.0e00 1.5e30
```

```
% more 3body.txt
3
1.25e11
0.0e00 0.0e00 0.05e04 0.0e00 5.97e24
0.0e00 4.5e10 3.0e04 0.0e00 1.989e30
0.0e00 -4.5e10 -3.0e04 0.0e00 1.989e30
```

```
% more 4body.txt
4
5.0e10
-3.5e10 0.0e00 0.0e00 1.4e03 3.0e28
-1.0e10 0.0e00 0.0e00 1.4e04 3.0e28
1.0e10 0.0e00 0.0e00 -1.4e04 3.0e28
3.5e10 0.0e00 0.0e00 -1.4e03 3.0e28
```

Universe file format examples

```
public Universe(String filename)
{
    In in = new In(filename);
    n = in.readInt();
    double radius = in.readDouble();
    StdDraw.setScale(-radius, +radius);
    StdDraw.setYscale(-radius, +radius);

    bodies = new Body[n];
    for (int i = 0; i < n; i++)
    {
        double rx = in.readDouble();
        double ry = in.readDouble();
        double[] position = { rx, ry };
        double vx = in.readDouble();
        double vy = in.readDouble();
        double[] velocity = { vx, vy };
        double mass = in.readDouble();
        Vector r = new Vector(position);
        Vector v = new Vector(velocity);
        bodies[i] = new Body(r, v, mass);
    }
}
```

Each Body is described by five double values: the x - and y -coordinates of its position, the x - and y -components of its initial velocity, and its mass.

To summarize, we have in the test client `main()` in `Universe` a data-driven program that simulates the motion of n bodies mutually attracted by gravity. The constructor creates an array of n `Body` objects, reading each body's initial position, initial velocity, and mass from the file whose name is specified as an argument. The `increaseTime()` method calculates the forces for each body and uses that information to update the acceleration, velocity, and position of each body after a time interval dt . The `main()` test client invokes the constructor, then stays in a loop calling `increaseTime()` and `draw()` to simulate motion.

Program 3.4.2 N-body simulation

```

public class Universe
{
    private final int n;
    private final Body[] bodies;

    public void increaseTime(double dt)
    {
        Vector[] f = new Vector[n];
        for (int i = 0; i < n; i++)
            f[i] = new Vector(new double[2]);
        for (int i = 0; i < n; i++)
            for (int j = 0; j < n; j++)
                if (i != j)
                    f[i] = f[i].plus(bodies[i].forceFrom(bodies[j]));
        for (int i = 0; i < n; i++)
            bodies[i].move(f[i], dt);
    }

    public void draw()
    {
        for (int i = 0; i < n; i++)
            bodies[i].draw();                                % java Universe 3body.txt 20000
    }

    public static void main(String[] args)
    {
        Universe newton = new Universe(args[0]);
        double dt = Double.parseDouble(args[1]);
        StdDraw.enableDoubleBuffering();
        while (true)
        {
            StdDraw.clear();
            newton.increaseTime(dt);
            newton.draw();
            StdDraw.show();
            StdDraw.pause(20);
        }
    }
}

```

n number of bodies
bodies[] array of bodies

880 steps



This data-driven program simulates motion in the universe defined by a file specified as the first command-line argument, increasing time at the rate specified as the second command-line argument. See the accompanying text for the implementation of the constructor.

You will find on the booksite a variety of files that define “universes” of all sorts, and you are encouraged to run Universe and observe their motion. When you view the motion for even a small number of bodies, you will understand why Newton had trouble deriving the equations that define their paths. The figures on the following page illustrate the result of running Universe for the 2-body, 3-body, and 4-body examples in the data files given earlier. The 2-body example is a mutually orbiting pair, the 3-body example is a chaotic situation with a moon jumping between two orbiting planets, and the 4-body example is a relatively simple situation where two pairs of mutually orbiting bodies are slowly rotating. The static images on these pages are made by modifying Universe and Body to draw the bodies in white, and then black on a gray background (see EXERCISE 3.4.9): the dynamic images that you get when you run Universe as it stands give a realistic feeling of the bodies orbiting one another, which is difficult to discern in the fixed pictures. When you run Universe on an example with a large number of bodies, you can appreciate why simulation is such an important tool for scientists who are trying to understand a complex problem. The n -body simulation model is remarkably versatile, as you will see if you experiment with some of these files.

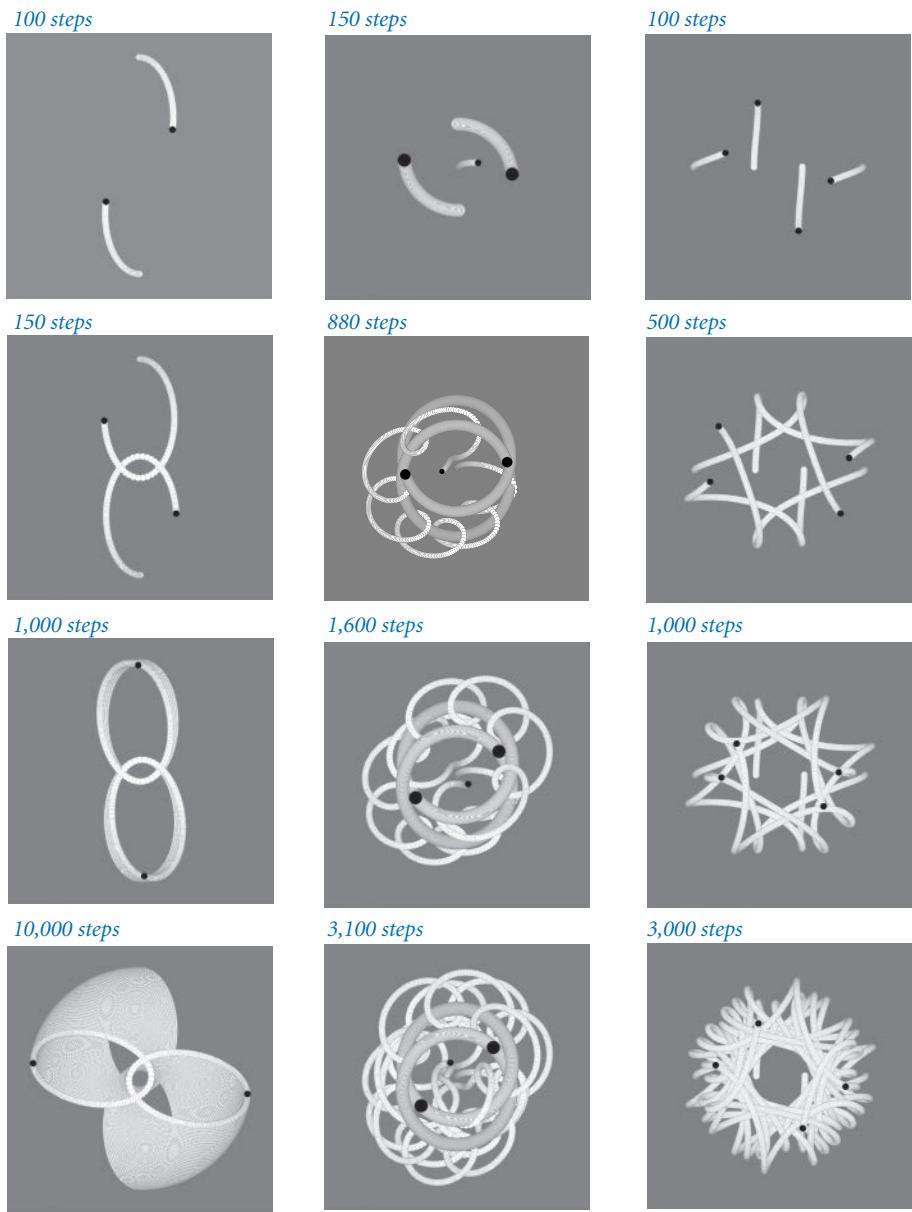
You will certainly be tempted to design your own universe (see EXERCISE 3.4.7). The biggest challenge in creating a data file is appropriately scaling the numbers so that the radius of the universe, time scale, and the mass and velocity of the bodies lead to interesting behavior. You can study the motion of planets rotating around a sun or subatomic particles interacting with one another, but you will have no luck studying the interaction of a planet with a subatomic particle. When you work with your own data, you are likely to have some bodies that will fly off to infinity and some others that will be sucked into others, but enjoy!

planetary scale

```
% more 2body.txt
2
5.0e10
0.0e00 4.5e10 1.0e04 0.0e00 1.5e30
0.0e00 -4.5e10 -1.0e04 0.0e00 1.5e30
```

subatomic scale

```
% more 2bodyTiny.txt
2
5.0e-10
0.0e00 4.5e-10 1.0e-16 0.0e00 1.5e-30
0.0e00 -4.5e-10 -1.0e-16 0.0e00 1.5e-30
```



Simulating 2-body (left column), 3-body (middle column), and 4-body (right column) universes

OUR PURPOSE IN PRESENTING THIS EXAMPLE is to illustrate the utility of data types, not to provide n -body simulation code for production use. There are many issues that scientists have to deal with when using this approach to study natural phenomena. The first is *accuracy*: it is common for inaccuracies in the calculations to accumulate to create dramatic effects in the simulation that would not be observed in nature. For example, our code takes no special action when bodies (nearly) collide. The second is *efficiency*: the `move()` method in `Universe` takes time proportional to n^2 , so it is not usable for huge numbers of bodies. As with genomics, addressing scientific problems related to the n -body problem now involves not just knowledge of the original problem domain, but also understanding core issues that computer scientists have been studying since the early days of computation.

For simplicity, we are working with a *two-dimensional* universe, which is realistic only when we are considering bodies in motion on a plane. But an important implication of basing the implementation of `Body` on `Vector` is that a client could use *three-dimensional* vectors to simulate the motion of bodies in three dimensions (actually, any number of dimensions) without changing the code at all! The `draw()` method projects the position onto the plane defined by the first two dimensions.

The test client in `Universe` is just one possibility; we can use the same basic model in all sorts of other situations (for example, involving different kinds of interactions among the bodies). One such possibility is to observe and measure the current motion of some existing bodies and then run the simulation backward! That is one method that astrophysicists use to try to understand the origins of the universe. In science, we try to understand the past and to predict the future; with a good simulation, we can do both.

Q&A

Q. The Universe API is certainly small. Why not just implement that code in a `main()` test client for `Body`?

A. Our design is an expression of what most people believe about the universe: it was created, and then time moves on. It clarifies the code and allows for maximum flexibility in simulating what goes on in the universe.

Q. Why is `forceFrom()` an instance method? Wouldn't it be better for it to be a static method that takes two `Body` objects as arguments?

A. Yes, implementing `forceFrom()` as an instance method is one of several possible alternatives, and having a static method that takes two `Body` objects as arguments is certainly a reasonable choice. Some programmers prefer to completely avoid static methods in data-type implementations; another option is to maintain the force acting on each `Body` as an instance variable. Our choice is a compromise between these two.

Exercises

3.4.1 Develop an object-oriented version of `BouncingBall` (PROGRAM 3.1.9). Include a constructor that starts each ball moving in a random direction at a random velocity (within reasonable limits) and a test client that takes an integer command-line argument n and simulates the motion of n bouncing balls.

3.4.2 Add a `main()` method to PROGRAM 3.4.1 that unit-tests the `Body` data type.

3.4.3 Modify `Body` (PROGRAM 3.4.1) so that the radius of the circle it draws for a body is proportional to its mass.

3.4.4 What happens in a universe in which there is no gravitational force? This situation would correspond to `forceTo()` in `Body` always returning the zero vector.

3.4.5 Create a data type `Universe3D` to model three-dimensional universes. Develop a data file to simulate the motion of the planets in our solar system around the sun.

3.4.6 Implement a class `RandomBody` that initializes its instance variables with (carefully chosen) random values instead of using a constructor and a client `RandomUniverse` that takes a single command-line argument n and simulates motion in a random universe with n bodies.

Creative Exercises

3.4.7 *New universe.* Design a new universe with interesting properties and simulate its motion with Universe. This exercise is truly an opportunity to be creative!

3.4.8 *Percolation.* Develop an object-oriented version of Percolation (PROGRAM 2.4.5). Think carefully about the design before you begin, and be prepared to defend your design decisions.

3.4.9 *N-body trace.* Write a client UniverseTrace that produces traces of the n -body simulation system like the static images on page 487.

Algorithms and Data Structures

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THIS CHAPTER PRESENTS FUNDAMENTAL DATA TYPES that are essential building blocks for a broad variety of applications. This chapter is also a guide to using them, whether you choose to use Java library implementations or to develop your own variations based on the code given here.

Objects can contain references to other objects, so we can build structures known as *linked structures*, which can be arbitrarily complex. With linked structures and arrays, we can build *data structures* to organize information in such a way that we can efficiently process it with associated *algorithms*. In a data type, we use the set of values to build data structures and the methods that operate on those values to implement algorithms.

The algorithms and data structures that we consider in this chapter introduce a body of knowledge developed over the past 50 years that constitutes the basis for the efficient use of computers for a broad variety of applications. From n -body simulation problems in physics to genetic sequencing problems in bioinformatics, the basic methods we describe have become essential in scientific research; from database systems to search engines, these methods are the foundation of commercial computing. As the scope of computing applications continues to expand, so grows the impact of these basic methods.

Algorithms and data structures themselves are valid subjects of scientific study. Accordingly, we begin by describing a scientific approach for analyzing the performance of algorithms, which we apply throughout the chapter.

4.1 Performance

IN THIS SECTION, YOU WILL LEARN to respect a principle that is succinctly expressed in yet another mantra that should live with you whenever you program: *pay attention to the cost*. If you become an engineer, that will be your job; if you become a biologist or a physicist, the cost will dictate which scientific problems you can address; if you are in business or become an economist, this principle needs no defense; and if you become a software developer, the cost will dictate whether the software that you build will be useful to any of your clients.

To study the cost of running them, we study our programs themselves via the *scientific method*, the commonly accepted body of techniques universally used by scientists to develop knowledge about the natural world. We also apply *mathematical analysis* to derive concise mathematical models of the cost.

Which features of the natural world are we studying? In most situations, we are interested in one fundamental characteristic: *time*. Whenever we run a program, we are performing an experiment involving the natural world, putting a complex system of electronic circuitry through series of state changes involving a huge number of discrete events that we are confident will eventually stabilize to a state with results that we want to interpret. Although developed in the abstract world of Java programming, these events most definitely are happening in the natural world. What will be the elapsed time until we see the result? It makes a great deal of difference to us whether that time is a millisecond, a second, a day, or a week. Therefore, we want to learn, through the scientific method, how to properly control the situation, as when we launch a rocket, build a bridge, or smash an atom.

On the one hand, modern programs and programming environments are complex; on the other hand, they are developed from a simple (but powerful) set of abstractions. It is a small miracle that a program produces the same result each time we run it. To predict the time required, we take advantage of the relative simplicity of the supporting infrastructure that we use to build programs. You may be surprised at the ease with which you can develop cost estimates and predict the performance characteristics of many of the programs that you write.

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Programs in this section

Scientific method. The following five-step approach briefly summarizes the scientific method:

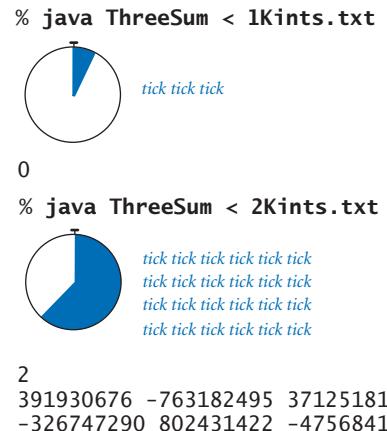
- *Observe* some feature of the natural world.
- *Hypothesize* a model that is consistent with the observations.
- *Predict* events using the hypothesis.
- *Verify* the predictions by making further observations.
- *Validate* by repeating until the hypothesis and observations agree.

One of the key tenets of the scientific method is that the experiments we design must be *reproducible*, so that others can convince themselves of the validity of the hypothesis. In addition, the hypotheses we formulate must be *falsifiable*—we require the possibility of knowing for sure when a hypothesis is wrong (and thus needs revision).

Observations Our first challenge is to make quantitative measurements of the running times of our programs. Although measuring the exact running time of a program is difficult, usually we are happy with approximate estimates. A number of tools can help us obtain such approximations. Perhaps the simplest is a physical stopwatch or the Stopwatch data type (see PROGRAM 3.2.2). We can simply run a program on various inputs, measuring the amount of time to process each input.

Our first qualitative observation about most programs is that there is a *problem size* that characterizes the difficulty of the computational task. Normally, the problem size is either the size of the input or the value of a command-line argument. Intuitively, the running time should increase with the problem size, but the question of *by how much* it increases naturally arises every time we develop and run a program.

Another qualitative observation for many programs is that the running time is relatively insensitive to the input itself; it depends primarily on the problem size. If this relationship does not hold, we need to run more experiments to better understand the running time's sensitivity to the input. Since this relationship does often hold, we focus now on the goal of better quantifying the correspondence between problem size and running time.



Observing the running time of a program

As a concrete example, we start with `ThreeSum` (PROGRAM 4.1.1), which counts the number of (unordered) *triples* in an array of n numbers that sum to 0 (assuming that integer overflow plays no role). This computation may seem contrived to you, but it is deeply related to fundamental tasks in computational geometry, so it is a problem worthy of careful study. What is the relationship between the problem size n and the running time for `ThreeSum`?

Hypotheses In the early days of computer science, Donald Knuth showed that, despite all of the complicating factors in understanding the running time of a program, it is possible *in principle* to create an accurate model that can help us predict precisely how long the program will take. Proper analysis of this sort involves:

- Detailed understanding of the program
- Detailed understanding of the system and the computer
- Advanced tools of mathematical analysis

Thus, it is best left for experts. Every programmer, however, needs to know how to make back-of-the-envelope performance estimates. Fortunately, we can often acquire such knowledge by using a combination of empirical observations and a small set of mathematical tools.

Doubling hypotheses. For a great many programs, we can quickly formulate a hypothesis for the following question: *What is the effect on the running time of doubling the size of the input?* For clarity, we refer to this hypothesis as a *doubling hypothesis*. Perhaps the easiest way to pay attention to the cost is to ask yourself this question about your programs as you develop them. Next, we describe how to answer this question by applying the scientific method.

Empirical analysis. Clearly, we can get a head start on developing a doubling hypothesis by doubling the size of the input and observing the effect on the running time. For example, `DoublingTest` (PROGRAM 4.1.2) generates a sequence of random input arrays for `ThreeSum`, doubling the array length at each step, and prints the ratio of running times of `ThreeSum.countTriples()` for each input to an input of one-half the size. If you run this program, you will find yourself caught in a prediction–verification cycle: It prints several lines very quickly, but then begins to slow down. Each time it prints a line, you find yourself wondering how long it will take to solve a problem of twice the size. If you use a `Stopwatch` to perform the measurements, you will see that the ratio seems to converge to a value around

Program 4.1.1 3-sum problem

```

public class ThreeSum
{
    public static void printTriples(int[] a)           n      number of integers
    { /* See Exercise 4.1.1. */ }
    public static int countTriples(int[] a)            a[]    the n integers
    { // Count triples that sum to 0.                  count   number of triples
                                                       that sum to 0

        int n = a.length;
        int count = 0;
        for (int i = 0; i < n; i++)
            for (int j = i+1; j < n; j++)
                for (int k = j+1; k < n; k++)
                    if (a[i] + a[j] + a[k] == 0)
                        count++;
        return count;
    }

    public static void main(String[] args)
    {
        int[] a = StdIn.readAllInts();
        int count = countTriples(a);
        StdOut.println(count);
        if (count < 10) printTriples(a);
    }
}

```

The `countTriples()` method counts the number of triples in `a[]` whose sum is exactly 0 (ignoring integer overflow). The test client invokes `countTriples()` for the integers on standard input and prints the triples if the count is low. The file `8ints.txt` contains 1,024 random values from the `int` data type. Such a file is not likely to have such a triple (see EXERCISE 4.1.28).

<pre>% more 8ints.txt 30 -30 -20 -10 40 0 10 5</pre>	<pre>% java ThreeSum < 8ints.txt 4 30 -30 0 30 -20 -10 -30 -10 40 -10 0 10</pre>
	<pre>% java ThreeSum < 1Kints.txt 0</pre>

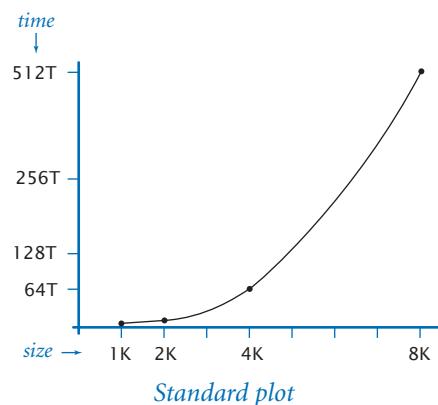
8. This leads immediately to the hypothesis that the running time increases by a factor of 8 when the input size doubles. We might also plot the running times, either on a standard plot (*right*), which clearly shows that the *rate* of increase of the running time increases with input size, or on a log–log plot. In the case of ThreeSum, the log–log plot (*below*) is a straight line with slope 3, which clearly suggests the hypothesis that the running time satisfies a *power law* of the form cn^3 (see EXERCISE 4.1.6).

Mathematical analysis. Knuth's basic insight on building a mathematical model to describe the running time of a program is simple—the total running time is determined by two primary factors:

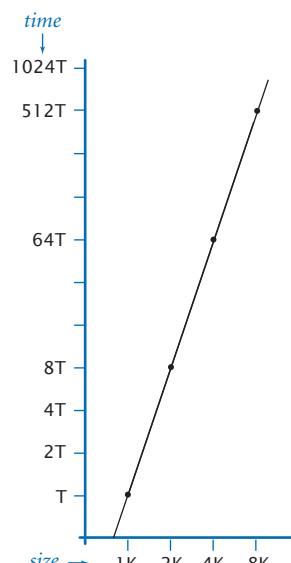
- The cost of executing each statement
- The frequency of executing each statement

The former is a property of the system, and the latter is a property of the algorithm. If we know both for all instructions in the program, we can multiply them together and sum for all instructions in the program to get the running time.

The primary challenge is to determine the frequency of execution of the statements. Some statements are easy to analyze: for example, the statement that sets `count` to 0 in `ThreeSum.countTriples()` is executed only once. Other statements require higher-level reasoning: for example, the `if` statement in `ThreeSum.countTriples()` is executed precisely $n(n-1)(n-2)/6$ times (which is the number of ways to pick three different numbers from the input array—see EXERCISE 4.1.4).



Standard plot



Log-log plot

Program 4.1.2 Validating a doubling hypothesis

```

public class DoublingTest
{
    public static double timeTrial(int n)
    { // Compute time to solve a random input of size n.
        int[] a = new int[n];
        for (int i = 0; i < n; i++)
            a[i] = StdRandom.uniform(2000000) - 1000000;
        Stopwatch timer = new Stopwatch();
        int count = ThreeSum.countTriples(a);
        return timer.elapsedTime();
    }

    public static void main(String[] args)
    { // Print table of doubling ratios.
        for (int n = 512; true; n *= 2)
        { // Print doubling ratio for problem size n.
            double previous = timeTrial(n/2);
            double current = timeTrial(n);           n
            double ratio = current / previous;       previous
            StdOut.printf("%7d %4.2f\n", n, ratio); current
            ratio
        }
    }
}

```

n | problem size
 $a[]$ | random integers
 $timer$ | stopwatch

n | problem size
 $running\ time\ for\ n/2$
 $running\ time\ for\ n$
 $ratio$ | ratio of running times

This program prints to standard output a table of doubling ratios for the three-sum problem. The table shows how doubling the problem size affects the running time of the method call `ThreeSum.countTriples()` for problem sizes starting at 512 and doubling for each row of the table. These experiments lead to the hypothesis that the running time increases by a factor of 8 when the input size doubles. When you run the program, note carefully that the elapsed time between lines printed increases by a factor of about 8, verifying the hypothesis.

```

% java DoublingTest
  512 6.48
  1024 8.30
  2048 7.75
  4096 8.00
  8192 8.05
  ...

```

```

public class ThreeSum
{
    public static int count(int[] a)
    {
        int n = a.length;
        int count = 0;
        for (int i = 0; i < n; i++)
            for (int j = i+1; j < n; j++)
                for (int k = j+1; k < n; k++)
                    if (a[i] + a[j] + a[k] == 0)
                        count++;
        return count;
    }

    public static void main(String[] args)
    {
        int[] a = StdIn.readAllInts();
        int count = count(a);
        StdOut.println(count);
    }
}

```

Anatomy of a program's statement execution frequencies

values. For example, the `if` statement in `ThreeSum` is executed $\sim n^3/6$ times because $n(n-1)(n-2)/6 = n^3/6 - n^2/2 + n/3$, which certainly, when divided by $n^3/6$, approaches 1 as n grows. This notation is useful when the terms after the leading term are relatively insignificant (for example, when $n = 1,000$, this assumption amounts to saying that $-n^2/2 + n/3 \approx -499,667$ is relatively insignificant by comparison with $n^3/6 \approx 166,666,667$, which it is).

Second, we focus on the instructions that are executed most frequently, sometimes referred to as the *inner loop* of the program. In this program it is reasonable to assume that the time devoted to the instructions outside the inner loop is relatively insignificant.

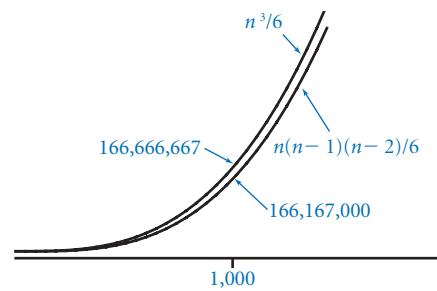
The key point in analyzing the running time of a program is this: for a great many programs, the running time satisfies the relationship

$$T(n) \sim c f(n)$$

where c is a constant and $f(n)$ is a function known as the *order of growth* of the running time. For typical programs, $f(n)$ is a function such as $\log n$, n , $n \log n$, n^2 , or n^3 , as you will soon see

Frequency analyses of this sort can lead to complicated and lengthy mathematical expressions. To substantially simplify matters in the mathematical analysis, we develop simpler *approximate* expressions in two ways.

First, we work with only the *leading term* of a mathematical expression by using a mathematical device known as *tilde notation*. We write $\sim f(n)$ to represent any quantity that, when divided by $f(n)$, approaches 1 as n grows. We also write $g(n) \sim f(n)$ to indicate that $g(n) / f(n)$ approaches 1 as n grows. With this notation, we can ignore complicated parts of an expression that represent small



Leading-term approximation

(customarily, we express order-of-growth functions without any constant coefficient). When $f(n)$ is a power of n , as is often the case, this assumption is equivalent to saying that the running time obeys a power law. In the case of ThreeSum, it is a hypothesis already verified by our empirical observations: *the order of growth of the running time of ThreeSum is n^3* . The value of the constant c depends both on the cost of executing instructions and on the details of the frequency analysis, but we normally do not need to work out the value, as you will now see.

The order of growth is a simple but powerful model of running time. For example, knowing the order of growth typically leads immediately to a doubling hypothesis. In the case of ThreeSum, knowing that the order of growth is n^3 tells us to expect the running time to increase by a factor of 8 when we double the size of the problem because

$$T(2n)/T(n) = c(2n)^3/(cn^3) = 8$$

This matches the value resulting from the empirical analysis, thus validating both the model and the experiments. *Study this example carefully, because you can use the same method to better understand the performance of any program that you write.*

Knuth showed that it is possible to develop an accurate mathematical model of the running time of any program, and many experts have devoted much effort to developing such models. But you do not need such a detailed model to understand the performance of your programs: it is typically safe to ignore the cost of the instructions outside the inner loop (because that cost is negligible by comparison to the cost of the instruction in the inner loop) and not necessary to know the value of the constant in the running-time approximation (because it cancels out when you use a doubling hypothesis to make predictions).

number of instructions	time per instruction in seconds	frequency	total time
6	2×10^{-9}	$n^3/6 - n^2/2 + n/3$	$(2 n^3 - 6 n^2 + 4 n) \times 10^{-9}$
4	3×10^{-9}	$n^2/2 - n/2$	$(6 n^2 + 6 n) \times 10^{-9}$
4	3×10^{-9}	n	$(12 n) \times 10^{-9}$
10	1×10^{-9}	1	10×10^{-9}
<i>grand total:</i>			
$(2 n^3 + 22 n + 10) \times 10^{-9}$			
<i>tilde notation</i>			
$\sim 2 n^3 \times 10^{-9}$			
<i>order of growth</i>			
n^3			

Analyzing the running time of a program (example)

The approximations are such that characteristics of the particular machine that you are using do not play a significant role in the models—the analysis separates the *algorithm* from the *system*. The order of growth of the running time of ThreeSum is n^3 does not depend on whether it is implemented in Java or Python, or whether it is running on your laptop, someone else's cellphone, or a supercomputer; it depends primarily on the fact that it examines all the triples. The properties of the computer and the *system* are all summarized in various assumptions about the relationship between program statements and machine instructions, and in the actual running times that you observe as the basis for the doubling hypothesis. The *algorithm* that you are using determines the order of growth. This separation is a powerful concept because it allows us to develop knowledge about the performance of algorithms and then apply that knowledge to any computer. In fact, much of the knowledge about the performance of classic algorithms was developed decades ago, but that knowledge is still relevant to today's computers.

EMPIRICAL AND MATHEMATICAL ANALYSES LIKE THOSE we have described constitute a model (an explanation of what is going on) that might be formalized by listing all of the assumptions mentioned (each instruction takes the same amount of time each time it is executed, running time has the given form, and so forth). Not many programs are worthy of a detailed model, but you need to have an idea of the running time that you might expect for every program that you write. *Pay attention to the cost.* Formulating a doubling hypothesis—through empirical studies, mathematical analysis, or (preferably) both—is a good way to start. This information about performance is extremely useful, and you will soon find yourself formulating and validating hypotheses every time you run a program. Indeed, doing so is a good use of your time while you wait for your program to finish!

Order-of-growth classifications We use just a few structural primitives (statements, conditionals, loops, and method calls) to build Java programs, so very often the order of growth of our programs is one of just a few functions of the problem size, summarized in the table at right. These functions immediately lead to a doubling hypothesis, which we can verify by running the programs. Indeed, you *have* been running programs that exhibit these orders of growth, as you can see in the following brief discussions.

Constant. A program whose running time's order of growth is *constant* executes a fixed number of statements to finish its job; consequently, its running time does not depend on the problem size. Our first several programs in CHAPTER 1—such as `HelloWorld` (PROGRAM 1.1.1) and `LeapYear` (PROGRAM 1.2.4)—fall into this classification. Each of these programs executes several statements just once. All of Java's operations on primitive types take constant time, as do Java's Math library functions. Note that we do not specify the size of the constant. For example, the constant for `Math.tan()` is much larger than that for `Math.abs()`.

Logarithmic. A program whose running time's order of growth is *logarithmic* is barely slower than a constant-time program. The classic example of a program whose running time is logarithmic in the problem size is looking up a value in sorted array, which we consider in the next section (see `BinarySearch`, in PROGRAM 4.2.3). The base of the logarithm is not relevant with respect to the order of growth (since all logarithms with a constant base are related by a constant factor), so we use $\log n$ when referring to order of growth. When we care about the constant in the leading term (such as when using tilde notation), we are careful to specify the base of the logarithm. We use the notation $\lg n$ for the binary (base-2) logarithm and $\ln n$ for the *natural* (base- e) logarithm.

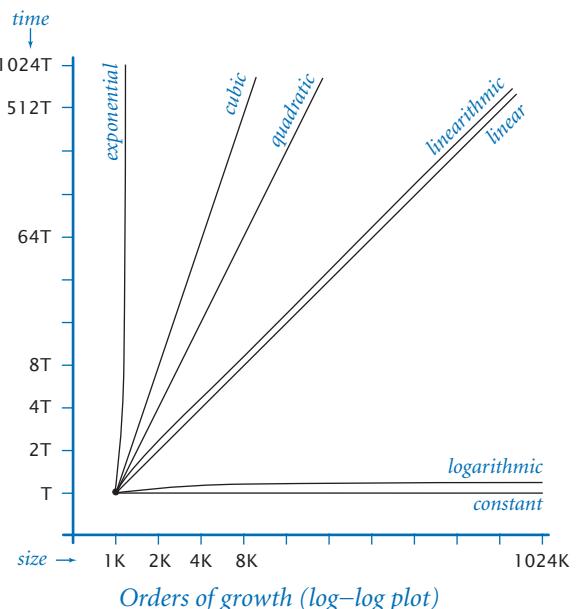
<i>order of growth description</i>	<i>function</i>	<i>factor for doubling hypothesis</i>
<i>constant</i>	1	1
<i>logarithmic</i>	$\log n$	1
<i>linear</i>	n	2
<i>linearithmic</i>	$n \log n$	2
<i>quadratic</i>	n^2	4
<i>cubic</i>	n^3	8
<i>exponential</i>	2^n	2^n

*Commonly encountered
order-of-growth classifications*

Linear. Programs that spend a constant amount of time processing each piece of input data, or that are based on a single `for` loop, are quite common. The order of growth of the running time of such a program is said to be *linear*—its running time is directly proportional to the problem size. Average (PROGRAM 1.5.3), which computes the average of the numbers on standard input, is prototypical, as is our code to shuffle the values in an array in SECTION 1.4. Filters such as `PlotFilter` (PROGRAM 1.5.5) also fall into this classification, as do the various image-processing filters that we considered in SECTION 3.2, which perform a constant number of arithmetic operations per input pixel.

Linearithmic. We use the term *linearithmic* to describe programs whose running time for a problem of size n has order of growth $n \log n$. Again, the base of the logarithm is not relevant. For example, `CouponCollector` (PROGRAM 1.4.2) is linearithmic. The prototypical example is mergesort (see PROGRAM 4.2.6). Several important problems have natural solutions that are quadratic but clever algorithms that are linearithmic. Such algorithms (including mergesort) are critically important in practice because they enable us to address problem sizes far larger than could be addressed with quadratic solutions. In SECTION 4.2, we consider a general design technique known as *divide-and-conquer* for developing linearithmic algorithms.

Quadratic. A typical program whose running time has order of growth n^2 has double nested `for` loops, used for some calculation involving all pairs of n elements. The double nested loop that computes the pairwise forces in `Universe` (PROGRAM 3.4.2) is a prototype of the programs in this classification, as is the insertion sort algorithm (PROGRAM 4.2.4) that we consider in SECTION 4.2.



<i>description</i>	<i>order of growth</i>	<i>example</i>	<i>framework</i>
<i>constant</i>	1	<code>count++;</code>	<i>statement</i> (increment an integer)
<i>logarithmic</i>	$\log n$	<code>for (int i = n; i > 0; i /= 2) count++;</code>	<i>divide in half</i> (bits in binary representation)
<i>linear</i>	n	<code>for (int i = 0; i < n; i++) if (a[i] == 0) count++;</code>	<i>single loop</i> (check each element)
<i>linearithmic</i>	$n \log n$	[see mergesort (PROGRAM 4.2.6)]	<i>divide-and-conquer</i> (mergesort)
<i>quadratic</i>	n^2	<code>for (int i = 0; i < n; i++) for (int j = i+1; j < n; j++) if (a[i] + a[j] == 0) count++;</code>	<i>double nested loop</i> (check all pairs)
<i>cubic</i>	n^3	<code>for (int i = 0; i < n; i++) for (int j = i+1; j < n; j++) for (int k = j+1; k < n; k++) if (a[i] + a[j] + a[k] == 0) count++;</code>	<i>triple nested loop</i> (check all triples)
<i>exponential</i>	2^n	[see Gray code (PROGRAM 2.3.3)]	<i>exhaustive search</i> (check all subsets)

Summary of common order-of-growth hypotheses

Cubic. Our example for this section, ThreeSum, is cubic (its running time has order of growth n^3) because it has *three* nested for loops, to process all triples of n elements. The running time of matrix multiplication, as implemented in SECTION 1.4, has order of growth m^3 to multiply two m -by- m matrices, so the basic matrix multiplication algorithm is often considered to be cubic. However, the size of the input (the number of elements in the matrices) is proportional to $n = m^2$, so the algorithm is best classified as $n^{3/2}$, not cubic.

Exponential. As discussed in SECTION 2.3, both TowersOfHanoi (PROGRAM 2.3.2) and Beckett (PROGRAM 2.3.3) have running times proportional to 2^n because they process all subsets of n elements. Generally, we use the term *exponential* to refer to algorithms whose order of growth is $2^{a \times n^b}$ for any positive constant a and b , even though different values of a and b lead to vastly different running times. Exponential-time algorithms are extremely slow—you will never run one of them for a large problem. They play a critical role in the theory of algorithms because there exists a large class of problems for which it seems that an exponential-time algorithm is the best possible choice.

THESE CLASSIFICATIONS ARE THE MOST COMMON, but certainly not a complete set. Indeed, the detailed analysis of algorithms can require the full gamut of mathematical tools that have been developed over the centuries. Understanding the running time of programs such as Factors (PROGRAM 1.3.9), PrimeSieve (PROGRAM 1.4.3), and Euclid (PROGRAM 2.3.1) requires fundamental results from number theory. Classic algorithms such as HashST (PROGRAM 4.4.3) and BST (PROGRAM 4.4.4) require careful mathematical analysis. The programs Sqrt (PROGRAM 1.3.6) and Markov (PROGRAM 1.6.3) are prototypes for numerical computation: their running time is dependent on the rate of convergence of a computation to a desired numerical result. Simulations such as Gambler (PROGRAM 1.3.8) and its variants are of interest precisely because detailed mathematical models are not always available.

Nevertheless, a great many of the programs that you will write have straightforward performance characteristics that can be described accurately by one of the orders of growth that we have considered. Accordingly, we can usually work with simple higher-level hypotheses, such as *the order of growth of the running time of mergesort is linearithmic*. For economy, we abbreviate such a statement to just say *mergesort is a linearithmic-time algorithm*. Most of our hypotheses about cost are of this form, or of the form *mergesort is faster than insertion sort*. Again, a notable feature of such hypotheses is that they are statements about algorithms, not just about programs.

Predictions You can always try to learn the running time of a program by simply running it, but that might be a poor way to proceed when the problem size is large. In that case, it is analogous to trying to learn where a rocket will land by launching it, how destructive a bomb will be by igniting it, or whether a bridge will stand by building it.

Knowing the order of growth of the running time allows us to make decisions about addressing large problems so that we can invest whatever resources we have to deal with the specific problems that we actually need to solve. We typically use the results of verified hypotheses about the order of growth of the running time of programs in one of the following ways.

Estimating the feasibility of solving large problems. To pay attention to the cost, you need to answer this basic question for every program that you write: *will this program be able to process this input in a reasonable amount of time?* For example, a cubic-time algorithm that runs in a couple of seconds for a problem of size n will require a few weeks for a problem of size $100n$ because it will be a million (100^3)

times slower, and a couple of million seconds is a few weeks. If that is the size of the problem that you need to solve, you have to find a better method. Knowing the order of growth of the running time of an algorithm provides precisely the information that you need to understand limitations on the size of the problems that you can solve. Developing such understanding is the most important reason to study performance. Without it, you are likely to have no idea how much time a program will consume; with it, you can make a back-of-the-envelope calculation to estimate costs and proceed accordingly.

order of growth	predicted running time if problem size is increased by a factor of 100
linear	a few minutes
linearithmic	a few minutes
quadratic	several hours
cubic	a few weeks
exponential	forever

Effect of increasing problem size for a program that runs for a few seconds

Estimating the value of using a faster computer. To pay attention to the cost, you also may be faced with this basic question: *how much faster can I solve the problem if I get a faster computer?* Again, knowing the order of growth of the running time provides precisely the information that you need. A famous rule of thumb known as *Moore's law* implies that you can expect to have a computer with about twice

the speed and double the memory 18 months from now, or a computer with about 10 times the speed and 10 times the memory in about 5 years. It is natural to think that if you buy a new computer that is 10 times faster and has 10 times more memory than your old one, you can solve a problem 10 times the size, but that is *not* the case for quadratic-time or cubic-time algorithms. Whether it is an investment banker running daily financial models or a scientist running a program to analyze experimental data or an engineer running simulations to test a design, it is not unusual for people to regularly run programs that take several hours to complete. Suppose that you are using a program whose running time is cubic, and then buy a new computer that is 10 times faster with 10 times more memory, not just because you need a new computer, but because you face problems that are 10 times larger. The rude awakening is that it will take several weeks to get results, because the larger problems would be a thousand times slower on the old computer and improved by only a factor of 10 on the new computer. This kind of situation is the primary reason that linear and linearithmic algorithms are so valuable: with such an algorithm and a new computer that is 10 times faster with 10 times more memory than an old computer, you can solve a problem that is 10 times larger than could be solved by the old computer in the same amount of time. In other words, you cannot keep pace with Moore's law if you are using a quadratic-time or a cubic-time algorithm.

<i>order of growth</i>	<i>factor of increase in running time</i>
linear	1
linearithmic	1
quadratic	10
cubic	100
exponential	forever

Effect of using a computer that is 10 times as fast to solve a problem that is 10 times as large

Comparing programs. We are always seeking to improve our programs, and we can often extend or modify our hypotheses to evaluate the effectiveness of various improvements. With the ability to predict performance, we can make design decisions during development can guide us toward better, more efficient code. As an example, a novice programmer might have written the nested for loops in ThreeSum (PROGRAM 4.1.1) as follows:

```
for (int i = 0; i < n; i++)
    for (int j = 0; j < n; j++)
        for (int k = 0; k < n; k++)
            if (i < j && j < k)
                if (a[i] + a[j] + a[k] == 0)
                    count++;
```

With this code, the frequency of execution of the instructions in the inner loop would be exactly n^3 (instead of approximately $n^3/6$). It is easy to formulate and verify the hypothesis that this variant is 6 times slower than ThreeSum. Note that improvements like this for code that is *not* in the inner loop will have little or no effect.

More generally, given two algorithms that solve the same problem, we want to know which one will solve our problem using fewer computational resources. In many cases, we can determine the order of growth of the running times and develop accurate hypotheses about comparative performance. The order of growth is extremely useful in this process because it allows us to compare one particular algorithm with whole classes of algorithms. For example, once we have a linearithmic algorithm to solve a problem, we become less interested in quadratic-time or cubic-time algorithms (even if they are highly optimized) to solve the same problem.

Caveats There are many reasons that you might get inconsistent or misleading results when trying to analyze program performance in detail. All of them have to do with the idea that one or more of the basic assumptions underlying our hypotheses might not be quite correct. We can develop new hypotheses based on new assumptions, but the more details that we need to take into account, the more care is required in the analysis.

Instruction time. The assumption that each instruction always takes the same amount of time is not always correct. For example, most modern computer systems use a technique known as *caching* to organize memory, in which case accessing elements in huge arrays can take much longer if they are not close together in the array. You can observe the effect of caching for ThreeSum by letting DoublingTest run for a while. After seeming to converge to 8, the ratio of running times will jump to a larger value for large arrays because of caching.

Nondominant inner loop. The assumption that the inner loop dominates may not always be correct. The problem size n might not be sufficiently large to make the leading term in the analysis so much larger than lower-order terms that we can ignore them. Some programs have a significant amount of code outside the inner loop that needs to be taken into consideration.

System considerations. Typically, there are many, many things going on in your computer. Java is one application of many competing for resources, and Java itself has many options and controls that significantly affect performance. Such considerations can interfere with the bedrock principle of the scientific method that experiments should be reproducible, since what is happening at this moment in your computer will never be reproduced again. Whatever else is going on in your system (that is beyond your control) should *in principle* be negligible.

Too close to call. Often, when we compare two different programs for the same task, one might be faster in some situations, and slower in others. One or more of the considerations just mentioned could make the difference. Again, there is a natural tendency among some programmers (and some students) to devote an extreme amount of energy running such horseraces to find the “best” implementation, but such work is best left for experts.

Strong dependence on input values. One of the first assumptions that we made to determine the order of growth of the program’s running time was that the running time should depend primarily on the problem size (and be relatively insensitive to the input values). When that is not the case, we may get inconsistent results or be unable to validate our hypotheses. Our running example ThreeSum does not have this problem, but many of the programs that we write certainly do. We will see several examples of such programs in this chapter. Often, a prime design goal is to eliminate the dependence on input values. If we cannot do so, we need to more carefully model the kind of input to be processed in the problems that we need to solve, which may be a significant challenge. For example, if we are writing a program to process a genome, how do we know how it will perform on a different genome? But a good model describing the genomes found in nature is precisely what scientists seek, so estimating the running time of our programs on data found in nature actually contributes to that model!

Multiple problem parameters. We have been focusing on measuring performance as a function of a *single* parameter, generally the value of a command-line argument or the size of the input. However, it is not unusual to have several parameters. For example, suppose that $a[]$ is an array of length m and $b[]$ is an array of length n . Consider the following code fragment that counts the number of (unordered) pairs i and j for which $a[i] + b[j]$ equals 0:

```
for (int i = 0; i < m; i++)
    for (int j = 0; j < n; j++)
        if (a[i] + b[j] == 0)
            count++;
```

The order of growth of the running time depends on two parameters— m and n . In such cases, we treat the parameters separately, holding one fixed while analyzing the other. For example, the order of growth of the running time of the preceding code fragment is mn . Similarly, `LongestCommonSubsequence` (PROGRAM 2.3.6) involves two parameters— m (the length of the first string) and n (the length of the second string)—and the order of growth of its running time is mn .

DESPITE ALL THESE CAVEATS, UNDERSTANDING THE order of growth of the running time of each program is valuable knowledge for any programmer, and the methods that we have described are powerful and broadly applicable. Knuth's insight was that we can carry these methods through to the last detail *in principle* to make detailed, accurate predictions. Typical computer systems are extremely complex and close analysis is best left to experts, but the same methods are effective for developing approximate estimates of the running time of any program. A rocket scientist needs to have some idea of whether a test flight will land in the ocean or in a city; a medical researcher needs to know whether a drug trial will kill or cure all the subjects; and any scientist or engineer using a computer program needs to have some idea of whether it will run for a second or for a year.

Performance guarantees For some programs, we demand that the running time of a program is less than a certain bound for *any* input of a given size. To provide such *performance guarantees*, theoreticians take an extremely pessimistic view: what would the running time be in the *worst case*?

For example, such a conservative approach might be appropriate for the software that runs a nuclear reactor or an air traffic control system or the brakes in your car. We must guarantee that such software completes its job within specified bounds because the result could be catastrophic if it does not. Scientists normally do not contemplate the worst case when studying the natural world: in biology, the worst case might be the extinction of the human race; in physics, the worst case might be the end of the universe. But the worst case can be a very real concern in computer systems, where the input is generated by another (potentially malicious) user, rather than by nature. For example, websites that do not use algorithms with performance guarantees are subject to *denial-of-service* attacks, where hackers flood them with pathological requests that degrade performance catastrophically.

Performance guarantees are difficult to verify with the scientific method, because we cannot test a hypothesis such as *mergesort is guaranteed to be linearithmic* without trying all possible inputs, which we cannot do because there are far too many of them. We might falsify such a hypothesis by providing a family of inputs for which mergesort is slow, but how can we prove it to be true? We must do so not with experimentation, but rather with mathematical analysis.

It is the task of the algorithm analyst to discover as much relevant information about an algorithm as possible, and it is the task of the applications programmer to apply that knowledge to develop programs that effectively solve the problems at hand. For example, if you are using a quadratic-time algorithm to solve a problem but can find an algorithm that is guaranteed to be linearithmic time, you will usually prefer the linearithmic one. On rare occasions, you might still prefer the quadratic-time algorithm because it is faster on the kinds of inputs that you need to solve or because the linearithmic algorithm is too complex to implement.

Ideally, we want algorithms that lead to clear and compact code that provides both a good worst-case guarantee and good performance on inputs of interest. Many of the classic algorithms that we consider in this chapter are of importance for a broad variety of applications precisely because they have all of these properties. Using these algorithms as models, you can develop good solutions yourself for the typical problems that you face while programming.

Memory As with running time, a program’s *memory usage* connects directly to the physical world: a substantial amount of your computer’s circuitry enables your program to store values and later retrieve them. The more values you need to have stored at any given instant, the more circuitry you need. To *pay attention to the cost*, you need to be aware of memory usage. You probably are aware of limits on memory usage on your computer (even more so than for time) because you probably have paid extra money to get more memory.

Memory usage is well defined for Java on your computer (every value will require precisely the same amount of memory each time that you run your program), but Java is implemented on a very wide range of computational devices, and memory consumption is implementation dependent. For economy, we use the term *typical* to signal values that are subject to machine dependencies. On a typical 64-bit machine, computer memory is organized into *words*, where each 64-bit word consists of 8 bytes, each *byte* consists of 8 bits, and each *bit* is a single binary digit.

Analyzing memory usage is somewhat different from analyzing time usage, primarily because one of Java’s most significant features is its memory allocation system, which is supposed to *relieve* you of having to worry about memory. Certainly, you are well advised to take advantage of this feature when appropriate. Still, it is your responsibility to know, at least approximately, when a program’s memory requirements will prevent you from solving a given problem.

Primitive types. It is easy to estimate memory usage for simple programs like the ones we considered in CHAPTER 1: count the number of variables and weight them by the number of bytes according to their type. For example, since the Java `int` data type represents the set of integer values between $-2,147,483,648$ and $2,147,483,647$, a grand total of 2^{32} different values, typical Java implementations use 32 bits (4 bytes) to represent each `int` value. Similarly, typical Java implementations represent each `char` value with 2 bytes (16 bits), each `double` value with 8 bytes (64 bits), and each `boolean` value with 1 byte (since computers typically access memory one byte at a time). For example, if you have 1GB of memory on your computer (about 1 billion bytes), you cannot fit more than about 256 million `int` values or 128 million `double` values in memory at any one time.

type	bytes
<code>boolean</code>	1
<code>byte</code>	1
<code>char</code>	2
<code>int</code>	4
<code>float</code>	4
<code>long</code>	8
<code>double</code>	8

Typical memory requirements for primitive types

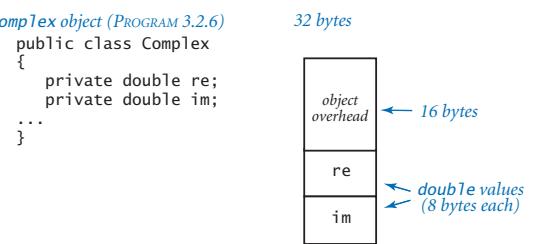
Objects. To determine the memory usage of an object, we add the amount of memory used by each instance variable to the *overhead* associated with each object, typically 16 bytes. The memory is typically *padded* (rounded up) to be a multiple of 8 bytes—an integral number of machine words—if necessary.

For example, on a typical system, a `Complex` (PROGRAM 3.2.6) object uses 32 bytes (16 bytes of overhead and 8 bytes for each of its two double instance variables). Since many programs create millions of `Color` objects, typical Java implementations pack the information needed for them into a single 32-bit `int` value. So, a `Color` object uses 24 bytes (16 bytes of overhead, 4 bytes for the `int` instance variable, and 4 bytes for padding). A `Body` (PROGRAM 3.4.1) object uses 168 bytes: object overhead (16 bytes), one double value (8 bytes), and two references (8 bytes each), plus the memory needed for the `Vector` objects, which we consider next.

An *object reference* typically uses 8 bytes (1 word) of memory. When a class includes an object reference as an instance variable, we must account separately for the memory for the object reference (8 bytes) and the memory needed for the object itself. For example, a `Body` (PROGRAM 3.4.1) object uses 168 bytes: object overhead (16 bytes), one double value (8 bytes), and two references (8 bytes each), plus the memory needed for the `Vector` objects, which we consider next.

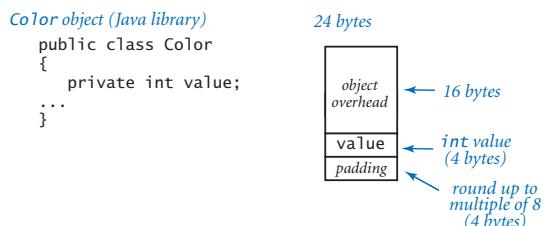
Complex object (PROGRAM 3.2.6)

```
public class Complex
{
    private double re;
    private double im;
}
```



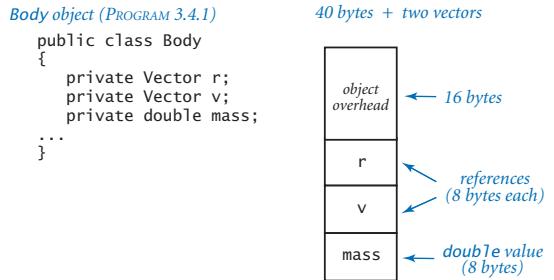
Color object (Java library)

```
public class Color
{
    private int value;
}
```



Body object (PROGRAM 3.4.1)

```
public class Body
{
    private Vector r;
    private Vector v;
    private double mass;
}
```



Typical object memory requirements

Arrays. Arrays in Java are implemented as objects, typically with an `int` instance variable for the length. For *primitive types*, an array of n elements requires 24 bytes of array overhead (16 bytes of object overhead, 4 bytes for the length, and 4 bytes for padding) plus n times the number of bytes needed to store each element. For example, the `int` array in `Sample` (PROGRAM 1.4.1) uses $4n + 24$ bytes; the `boolean` arrays in `Coupon` (PROGRAM 1.4.2) use $n + 24$ bytes. Note that a `boolean` array consumes 1 *byte* of memory per element (wasting 7 of the 8 bits)—with some extra bookkeeping, you could get the job done using only 1 *bit* per element (see EXERCISE 4.1.26).

An array of *objects* is an array of references to the objects, so we need to account for both the memory for the references and the memory for the objects. For example, an array of n `Charge` objects consumes $48n + 24$ bytes: the array overhead (24 bytes), the `Charge` references (8 n bytes), and the memory for the `Charge` objects (40 n bytes). This analysis assumes that all of the objects are different: it is possible that multiple array elements could refer to the same `Charge` object (aliasing).

The class `Vector` (PROGRAM 3.3.3) includes an array as an instance variable. On a typical system, a `Vector` object of length n requires $8n + 48$ bytes: the object overhead (16 bytes), a reference to a `double` array (8 bytes), and the memory for the `double` array ($8n + 24$ bytes). Thus, each of the `Vector` objects in `Body` uses 64 bytes of memory (since $n = 2$).

String objects. We account for memory in a `String` object in the same way as for any other object. A `String` object of length n typically consumes $2n + 56$ bytes: the object overhead (16 bytes), a reference to a `char` array (8 bytes), the memory for the `char` array ($2n + 24$ bytes), one `int` value (4 bytes), and padding (4 bytes). The `int` instance variable in `String` objects is a *hash code* that saves recomputation in certain circumstances that need not concern us now. If the number of characters in the string is not a multiple of 4, memory for the character array would be padded, to make the number of bytes for the `char` array a multiple of 8.

`Vector` object (PROGRAM 3.3.3) 24 bytes + `double` array ($8n + 24$ bytes)

```
public class Vector
{
    private double[] coords;
    ...
}
```

`String` object (Java library)

40 bytes + `char` array ($2n + 24$ bytes)

```
public class String
{
    private int hash;
    private char[] value;
    ...
}
```

Typical memory requirements for `Vector` and `String` objects

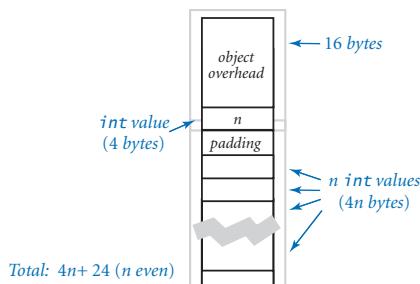
Two-dimensional arrays. As we saw in SECTION 1.4, a two-dimensional array in Java is an array of arrays. As a result, the two-dimensional array in Markov (PROGRAM 1.6.3) consumes $8n^2 + 32n + 24$, or $\sim 8n^2$ bytes: the overhead for the array of arrays (24 bytes), the n references to the row arrays ($8n$ bytes), and the n row arrays ($8n + 24$ bytes each). If the array elements are objects, then a similar accounting gives $\sim 8n^2$ bytes for the array of arrays filled with references to objects, to which we need to add the memory for the objects themselves.

THESE BASIC MECHANISMS ARE EFFECTIVE FOR estimating the memory usage of a great many programs, but there are numerous complicating factors that can make the task significantly more difficult. We have already noted the potential effect of aliasing. Moreover, memory consumption is a complicated dynamic process when function calls are involved because the system memory allocation mechanism plays a more important role, with more system dependencies. For example, when your program calls a method, the system allocates the memory needed for the method (for its local variables) from a special area of memory called the *stack*; when the method returns to the caller, the memory is returned to the stack. For this reason, creating arrays or other large objects in recursive programs is dangerous, since each recursive call implies significant memory usage. When you create an object with `new`, the system allocates the memory needed for the object from another special area of memory known as the *heap*, and you must remember that every object lives until no references to it remain, at which point a system process known as *garbage collection* can reclaim its memory for the heap. Such dynamics can make the task of precisely estimating memory usage of a program challenging.

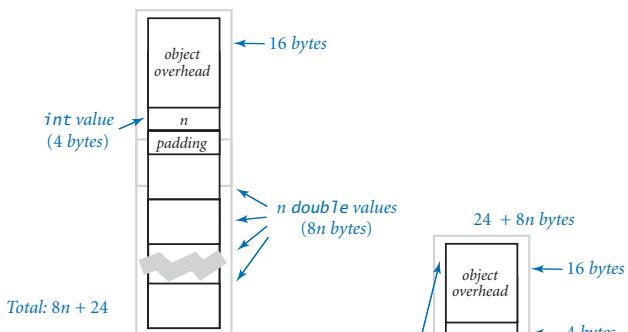
type	bytes
<code>boolean[]</code>	$n + 24 \sim n$
<code>int[]</code>	$4n + 24 \sim 4n$
<code>double[]</code>	$8n + 24 \sim 8n$
<code>Charge[]</code>	$40n + 24 \sim 40n$
<code>Vector</code>	$8n + 48 \sim 8n$
<code>String</code>	$2n + 56 \sim 2n$
<code>boolean[][]</code>	$n^2 + 32n + 24 \sim n^2$
<code>int[][]</code>	$4n^2 + 32n + 24 \sim 4n^2$
<code>double[][]</code>	$8n^2 + 32n + 24 \sim 8n^2$

Typical memory requirements for variable-length data types

Array of int values (PROGRAM 1.4.1)
`int[] perm = new int[n];`

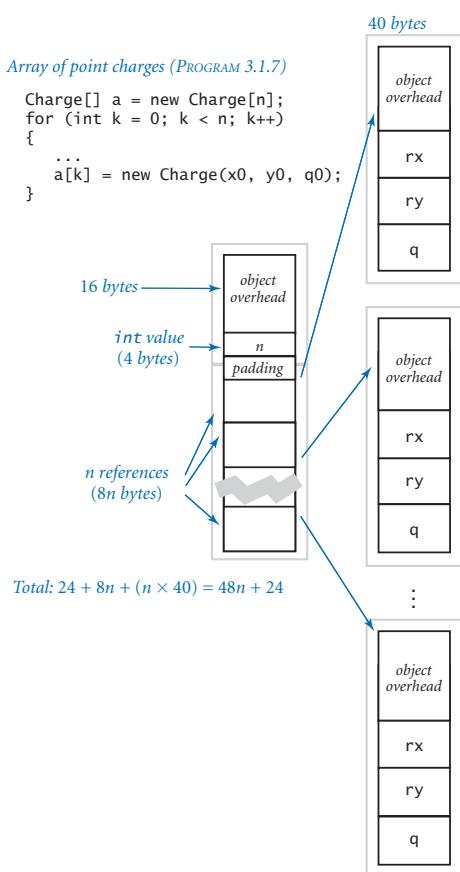


Array of double values (PROGRAM 2.1.4)
`double[] c = new double[n];`



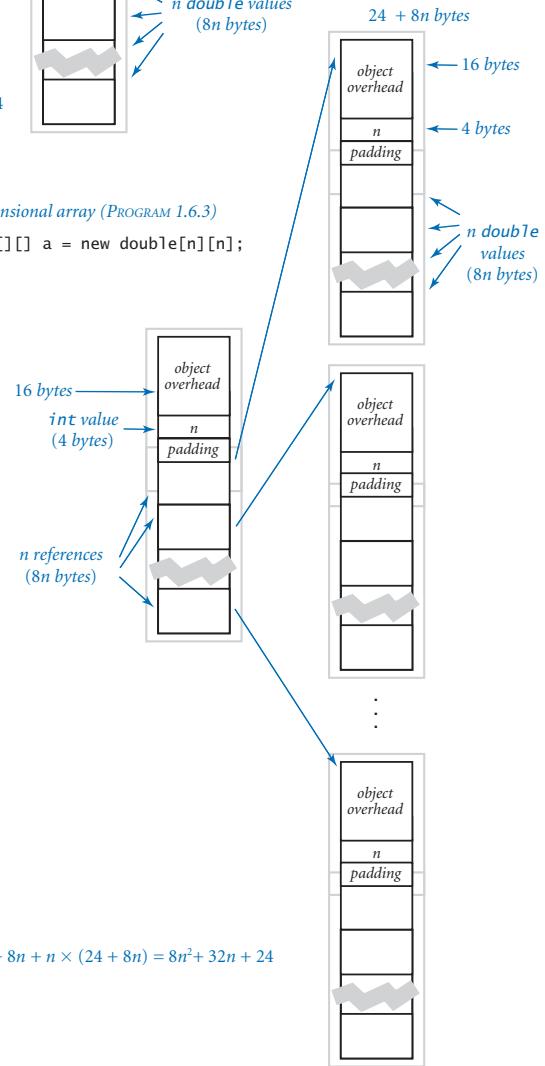
Array of point charges (PROGRAM 3.1.7)

```
Charge[] a = new Charge[n];
for (int k = 0; k < n; k++)
{
    ...
    a[k] = new Charge(x0, y0, q0);
}
```



Two-dimensional array (PROGRAM 1.6.3)

```
double[][] a = new double[n][n];
```



Typical memory requirements for arrays of int values, double values, objects, and arrays

Perspective Good performance is important to the success of a program. An impossibly slow program is almost as useless as an incorrect one, so it is certainly worthwhile to *pay attention to the cost* at the outset, to have some idea of which sorts of problems you might feasibly address. In particular, it is always wise to have some idea of which code constitutes the inner loop of your programs.

Perhaps the most common mistake made in programming is to pay too much attention to performance characteristics. Your first priority is to make your code clear and correct. Modifying a program for the sole purpose of speeding it up is best left for experts. Indeed, doing so is often counterproductive, as it tends to create code that is complicated and difficult to understand. C. A. R. Hoare (the inventor of quicksort and a leading proponent of writing clear and correct code) once summarized this idea by saying that “premature optimization is the root of all evil,” to which Knuth added the qualifier “(or at least most of it) in programming.” Beyond that, improving the running time is not worthwhile if the available cost benefits are insignificant. For example, improving the running time of a program by a factor of 10 is inconsequential if the running time is only an instant. Even when a program takes a few minutes to run, the total time required to implement and debug an improved algorithm might be substantially more than the time required simply to run a slightly slower one—you may as well let the computer do the work. Worse, you might spend a considerable amount of time and effort implementing ideas that should improve a program but actually do not do so.

Perhaps the second most common mistake made in developing an algorithm is to ignore performance characteristics. Faster algorithms are often more complicated than brute-force solutions, so you might be tempted to accept a slower algorithm to avoid having to deal with more complicated code. However, you can sometimes reap huge savings with just a few lines of good code. Users of a surprising number of computer systems lose substantial time waiting for simple quadratic-time algorithms to finish solving a problem, even though linear or logarithmic algorithms are available that are only slightly more complicated and could therefore solve the problem in a fraction of the time. When we are dealing with huge problem sizes, we often have no choice but to seek better algorithms.

Improving a program to make it clearer, more efficient, and elegant should be your goal every time that you work on it. If you *pay attention to the cost* all the way through the development of a program, you will reap the benefits every time you use it.

Q&A

Q. How do I find out how long it takes to add or multiply two floating-point numbers on my system?

A. Run some experiments! The program `TimePrimitives` on the booksite uses `Stopwatch` to test the execution time of various arithmetic operations on primitive types. This technique measures the actual elapsed time as would be observed on a wall clock. If your system is not running many other applications, this can produce accurate results. You can find much more information about refining such experiments on the booksite.

Q. How much time does it take to call functions such as `Math.sin()`, `Math.log()`, and `Math.sqrt()`?

A. Run some experiments! `Stopwatch` makes it easy to write programs such as `TimePrimitives` to answer questions of this sort for yourself, and you will be able to use your computer much more effectively if you get in the habit of doing so.

Q. How much time do string operations take?

A. Run some experiments! (Have you gotten the message yet?) A The `String` data type is implemented to allow the methods `length()` and `charAt()` to run in constant time. Methods such as `toLowerCase()` and `replace()` take time linear in the length of the string. The methods `compareTo()`, `equals()`, `startsWith()`, and `endsWith()` take time proportional to the number of characters needed to resolve the answer (constant time in the best case and linear time in the worst case), but `indexOf()` can be slow. String concatenation and the `substring()` method take time proportional to the total number of characters in the result.

Q. Why does allocating an array of length n take time proportional to n ?

A. In Java, array elements are automatically initialized to default values (0, `false`, or `null`). In principle, this could be a constant-time operation if the system would defer initialization of each element until just before the program accesses that element for the first time, but most Java implementations go through the whole array to initialize each element.

Q. How do I determine how much memory is available for my Java programs?

A. Java will tell you when it runs out of memory, so it is not difficult to run some experiments. For example, if you use `PrimeSieve` (PROGRAM 1.4.3) by typing

```
% java PrimeSieve 100000000
```

and get the result

```
50847534
```

but then type

```
% java PrimeSieve 10000000000
```

and get the result

```
Exception in thread "main"  
java.lang.OutOfMemoryError: Java heap space
```

then you can figure that you have enough room for a boolean array of length 100 million but not for a boolean array of length 1 billion. You can increase the amount of memory allotted to Java with *command-line options*. The following command executes `PrimeSieve` with the command-line argument `1000000000` and the command-line option `-Xmx1110m`, which requests a maximum of 1,100 megabytes of memory (if available).

```
% java -Xmx1110m PrimeSieve 1000000000
```

Q. What does it mean when someone says that the running time is $O(n^2)$?

A. That is an example of a notation known as *big-O* notation. We write $f(n)$ is $O(g(n))$ if there exist constants c and n_0 such that $|f(n)| \leq c |g(n)|$ for all $n > n_0$. In other words, the function $f(n)$ is bounded above by $g(n)$, up to constant factors and for sufficiently large values of n . For example, the function $30n^2 + 10n + 7$ is $O(n^2)$. We say that the *worst-case running time* of an algorithm is $O(g(n))$ if the running time as a function of the input size n is $O(g(n))$ for *all* possible inputs. Big-O notation and worst-case running times are widely used by theoretical computer scientists to prove theorems about algorithms, so you are sure to see this notation if you take a course in algorithms and data structures.

Q. So can I use the fact that the worst-case running time of an algorithm is $O(n^3)$ or $O(n^2)$ to predict performance?

A. Not necessarily, because the actual running time might be much less. For example, the function $30n^2 + 10n + 7$ is $O(n^2)$, but it is also $O(n^3)$ and $O(n^{10})$ because big- O notation provides only an upper bound. Moreover, even if there is some family of inputs for which the running time is proportional to the given function, perhaps these inputs are not encountered in practice. Consequently, you should not use big- O notation to predict performance. The tilde notation and order-of-growth classifications that we use are more precise than big- O notation because they provide matching upper and lower bounds on the growth of the function. Many programmers incorrectly use big- O notation to indicate matching upper and lower bounds.

Exercises

4.1.1 Implement the static method `printTriples()` for `ThreeSum` (PROGRAM 4.1.1), which prints to standard output all of the triples that sum to zero.

4.1.2 Modify `ThreeSum` to take an integer command-line argument `target` and find a triple of numbers on standard input whose sum is closest to `target`.

4.1.3 Write a program `FourSum` that reads long integers from standard input, and counts the number of 4-tuples that sum to zero. Use a quadruple nested loop. What is the order of growth of the running time of your program? Estimate the largest input size that your program can handle in an hour. Then, run your program to validate your hypothesis.

4.1.4 Prove by induction that the number of (unordered) pairs of integers between 0 and $n-1$ is $n(n-1)/2$, and then prove by induction that the number of (unordered) triples of integers between 0 and $n-1$ is $n(n-1)(n-2)/6$.

Answer for pairs: The formula is correct for $n = 1$, since there are 0 pairs. For $n > 1$, count all the pairs that do not include $n-1$, which is $(n-1)(n-2)/2$ by the inductive hypothesis, and all the pairs that do include $n-1$, which is $n-1$, to get the total

$$(n-1)(n-2)/2 + (n-1) = n(n-1)/2$$

Answer for triples: The formula is correct for $n = 2$. For $n > 2$, count all the triples that do not include $n-1$, which is $(n-1)(n-2)(n-3)/6$ by the inductive hypothesis, and all the triples that do include $n-1$, which is $(n-1)(n-2)/2$, to get the total

$$(n-1)(n-2)(n-3)/6 + (n-1)(n-2)/2 = n(n-1)(n-2)/6$$

4.1.5 Show by approximating with integrals that the number of distinct triples of integers between 0 and n is about $n^3/6$.

Answer: $\sum_0^n \sum_0^i \sum_0^j 1 \approx \int_0^n \int_0^i \int_0^j dk dj di = \int_0^n \int_0^i j dj di = \int_0^n (i^2/2) di = n^3/6$

4.1.6 Show that a log–log plot of the function cn^b has slope b and x -intercept $\log c$. What are the slope and x -intercept for $4n^3(\log n)^2$?

4.1.7 What is the value of the variable `count`, as a function of n , after running the following code fragment?

```

long count = 0;
for (int i = 0; i < n; i++)
    for (int j = i + 1; j < n; j++)
        for (int k = j + 1; k < n; k++)
            count++;

```

Answer: $n(n-1)(n-2)/6$

4.1.8 Use tilde notation to simplify each of the following formulas, and give the order of growth of each:

- a. $n(n - 1)(n - 2)(n - 3)/24$
- b. $(n - 2)(\lg n - 2)(\lg n + 2)$
- c. $n(n + 1) - n^2$
- d. $n(n + 1)/2 + n \lg n$
- e. $\ln((n - 1)(n - 2)(n - 3))^2$

4.1.9 Determine the order of growth of the running time of this statement in ThreeSum as a function of the number of integers n on standard input:

```
int[] a = StdIn.readAllInts();
```

Answer: Linear. The bottlenecks are the implicit array initialization and the implicit input loop. Depending on your system, however, the cost of an input loop like this might dominate in a linearithmic-time or even a quadratic-time program unless the input size is sufficiently large.

4.1.10 Determine whether the following code fragment takes linear time, quadratic time, or cubic time (as a function of n).

```

for (int i = 0; i < n; i++)
    for (int j = 0; j < n; j++)
        if (i == j) c[i][j] = 1.0;
        else         c[i][j] = 0.0;

```

4.1.11 Suppose the running times of an algorithm for inputs of size 1,000, 2,000, 3,000, and 4,000 are 5 seconds, 20 seconds, 45 seconds, and 80 seconds, respectively. Estimate how long it will take to solve a problem of size 5,000. Is the algorithm linear, linearithmic, quadratic, cubic, or exponential?

4.1.12 Which would you prefer: an algorithm whose order of growth of running time is quadratic, linearithmic, or linear?

Answer: While it is tempting to make a quick decision based on the order of growth, it is very easy to be misled by doing so. You need to have some idea of the problem size and of the relative value of the leading coefficients of the running times. For example, suppose that the running times are n^2 seconds, $100n \log_2 n$ seconds, and $10,000n$ seconds. The quadratic algorithm will be fastest for n up to about 1,000, and the linear algorithm will never be faster than the linearithmic one (n would have to be greater than 2^{100} , far too large to bother considering).

4.1.13 Apply the scientific method to develop and validate a hypothesis about the order of growth of the running time of the following code fragment, as a function of the argument n .

```
public static int f(int n)
{
    if (n == 0) return 1;
    return f(n-1) + f(n-1);
}
```

4.1.14 Apply the scientific method to develop and validate a hypothesis about the order of growth of the running time of the `collect()` method in `Coupon` (PROGRAM 2.1.3), as a function of the argument n . *Note:* Doubling is not effective for distinguishing between the linear and linearithmic hypotheses—you might try squaring the size of the input.

4.1.15 Apply the scientific method to develop and validate a hypothesis about the order of growth of the running time of `Markov` (PROGRAM 1.6.3), as a function of the command-line arguments `trials` and n .

4.1.16 Apply the scientific method to develop and validate a hypothesis about the order of growth of the running time of each of the following two code fragments as a function of n .

```
String s = "";
for (int i = 0; i < n; i++)
    if (StdRandom.bernoulli(0.5)) s += "0";
    else                            s += "1";

StringBuilder sb = new StringBuilder();
for (int i = 0; i < n; i++)
    if (StdRandom.bernoulli(0.5)) sb.append("0");
    else                            sb.append("1");
String s = sb.toString();
```

4.1.17 Each of the four Java functions given here returns a string of length n whose characters are all x . Determine the order of growth of the running time of each function. Recall that concatenating two strings in Java takes time proportional to the length of the resulting string.

```
public static String method1(int n)
{
    if (n == 0) return "";
    String temp = method1(n / 2);
    if (n % 2 == 0) return temp + temp;
    else            return temp + temp + "x";
}

public static String method2(int n)
{
    String s = "";
    for (int i = 0; i < n; i++)
        s = s + "x";
    return s;
}
```

```

public static String method3(int n)
{
    if (n == 0) return "";
    if (n == 1) return "x";
    return method3(n/2) + method3(n - n/2);
}

public static String method4(int n)
{
    char[] temp = new char[n];
    for (int i = 0; i < n; i++)
        temp[i] = 'x';
    return new String(temp);
}

```

4.1.18 The following code fragment (adapted from a Java programming book) creates a random permutation of the integers from 0 to $n-1$. Determine the order of growth of its running time as a function of n . Compare its order of growth with the shuffling code in SECTION 1.4.

```

int[] a = new int[n];
boolean[] taken = new boolean[n];
int count = 0;
while (count < n)
{
    int r = StdRandom.uniform(n);
    if (!taken[r])
    {
        a[r] = count;
        taken[r] = true;
        count++;
    }
}

```

4.1.19 What is the order of growth of the running time of the following two functions? Each function takes a string as an argument and returns the string reversed.

```
public static String reverse1(String s)
{
    int n = s.length();
    String reverse = "";
    for (int i = 0; i < n; i++)
        reverse = s.charAt(i) + reverse;
    return reverse;
}

public static String reverse2(String s)
{
    int n = s.length();
    if (n <= 1) return s;
    String left = s.substring(0, n/2);
    String right = s.substring(n/2, n);
    return reverse2(right) + reverse2(left);
}
```

4.1.20 Give a linear-time algorithm for reversing a string.

Answer:

```
public static String reverse(String s)
{
    int n = s.length();
    char[] a = new char[n];
    for (int i = 0; i < n; i++)
        a[i] = s.charAt(n-i-1);
    return new String(a);
}
```

4.1.21 Write a program `MooresLaw` that takes a command-line argument n and outputs the increase in processor speed over a decade if microprocessors double every n months. How much will processor speed increase over the next decade if speeds double every $n = 15$ months? 24 months?

4.1.22 Using the 64-bit memory model in the text, give the memory usage for an object of each of the following data types from CHAPTER 3:

- a. Stopwatch
- b. Turtle
- c. Vector
- d. Body
- e. Universe

4.1.23 Estimate, as a function of the grid size n , the amount of space used by `PercolationVisualizer` (PROGRAM 2.4.3) with the vertical percolation detection (PROGRAM 2.4.2). *Extra credit:* Answer the same question for the case where the recursive percolation detection method (PROGRAM 2.4.5) is used.

4.1.24 Estimate the size of the biggest two-dimensional array of `int` values that your computer can hold, and then try to allocate such an array.

4.1.25 Estimate, as a function of the number of documents n and the dimension d , the amount of memory used by `CompareDocuments` (PROGRAM 3.3.5).

4.1.26 Write a version of `PrimeSieve` (PROGRAM 1.4.3) that uses a byte array instead of a boolean array and uses all the bits in each byte, thereby increasing the largest value of n that it can handle by a factor of 8.

4.1.27 The following table gives running times for three programs for various values of n . Fill in the blanks with estimates that you think are reasonable on the basis of the information given.

<i>program</i>	<i>1,000</i>	<i>10,000</i>	<i>100,000</i>	<i>1,000,000</i>
A	0.001 second	0.012 second	0.16 second	? seconds
B	1 minute	10 minutes	1.7 hours	? hours
C	1 second	1.7 minutes	2.8 hours	? days

Give hypotheses for the order of growth of the running time of each program.

Creative Exercises

4.1.28 *Three-sum analysis.* Calculate the probability that no triple among n random 32-bit integers sums to 0. *Extra credit:* Give an approximate formula for the expected number of such triples (as a function of n), and run experiments to validate your estimate.

4.1.29 *Closest pair.* Design a quadratic-time algorithm that, given an array of integers, finds a pair that are closest to each other. (In the next section you will be asked to find a linearithmic algorithm for the problem.)

4.1.30 *The “beck” exploit.* A popular web server supports a function named `no2slash()` whose purpose is to collapse multiple / characters. For example, the string `/d1///d2///d3/test.html` collapses to `/d1/d2/d3/test.html`. The original algorithm was to repeatedly search for a / and copy the remainder of the string:

```
int n = name.length();
int i = 1;
while (i < n)
{
    if ((c[i-1] == '/') && (c[i] == '/'))
    {
        for(int j = i+1; j < n; j++)
            c[j-1] = c[j];
        n--;
    }
    else i++;
}
```

Unfortunately, this code can takes quadratic time (for example, if the string consists of the / character repeated n times). By sending multiple simultaneous requests with large numbers of / characters, a hacker could deluge the server and starve other processes for CPU time, thereby creating a denial-of-service attack. Develop a version of `no2slash()` that runs in linear time and does not allow for this type of attack.

4.1.31 *Subset sum.* Write a program `SubsetSum` that reads `long` integers from standard input, and counts the number of subsets of those integers that sum to exactly zero. Give the order of growth of the running time of your program.

4.1.32 *Young tableaux.* Suppose you have an n -by- n array of integers $a[] []$ such that, for all i and j , $a[i][j] < a[i+1][j]$ and $a[i][j] < a[i][j+1]$, as in the following the 5-by-5 array.

5	23	54	67	89
6	69	73	74	90
10	71	83	84	91
60	73	84	86	92
89	91	92	93	94

A two-dimensional array with this property is known as a *Young tableaux*. Write a function that takes as arguments an n -by- n Young tableaux and an integer, and determines whether the integer is in the Young tableaux. The order of growth of the running time of your function should be linear in n .

4.1.33 *Array rotation.* Given an array of n elements, give a linear-time algorithm to rotate the string k positions. That is, if the array contains a_0, a_1, \dots, a_{n-1} , the rotated array is $a_k, a_{k+1}, \dots, a_{n-1}, a_0, \dots, a_{k-1}$. Use at most a constant amount of extra memory. Hint: Reverse three subarrays.

4.1.34 *Finding a repeated integer.* (a) Given an array of n integers from 1 to n with one value repeated twice and one missing, give an algorithm that finds the missing integer, in linear time and constant extra memory. Integer overflow is not allowed. (b) Given a read-only array of n integers, where each value from 1 to $n-1$ occurs once and one occurs twice, give an algorithm that finds the duplicated value, in linear time and constant extra memory. (c) Given a read-only array of n integers with values between 1 and $n-1$, give an algorithm that finds a duplicated value, in linear time and constant extra memory.

4.1.35 *Factorial.* Design a fast algorithm to compute $n!$ for large values of n , using Java's `BigInteger` class. Use your program to compute the longest run of consecutive 9s in $1000000!$. Develop and validate a hypothesis for the order of growth of the running time of your algorithm.

4.1.36 Maximum sum. Design a linear-time algorithm that finds a contiguous subarray of length at most m in an array of n long integers that has the highest sum among all such subarrays. Implement your algorithm, and confirm that the order of growth of its running time is linear.

4.1.37 Maximum average. Write a program that finds a contiguous subarray of length at most m in an array of n long integers that has the highest average value among all such subarrays, by trying all subarrays. Use the scientific method to confirm that the order of growth of the running time of your program is mn^2 . Next, write a program that solves the problem by first computing the quantity $\text{prefix}[i] = a[0] + \dots + a[i]$ for each i , then computing the average in the interval from $a[i]$ to $a[j]$ with the expression $(\text{prefix}[j] - \text{prefix}[i]) / (j - i + 1)$. Use the scientific method to confirm that this method reduces the order of growth by a factor of n .

4.1.38 Pattern matching. Given an n -by- n subarray of black (1) and white (0) pixels, design a linear-time algorithm that finds the largest square subarray that contains no white pixels. In the following example, the largest such subarray is the 3-by-3 subarray highlighted in blue.

```
1 0 1 1 1 0 0 0  
0 0 0 1 0 1 0 0  
0 0 1 1 1 0 0 0  
0 0 1 1 1 0 1 0  
0 0 1 1 1 1 1 1  
0 1 0 1 1 1 1 0  
0 1 0 1 1 0 1 0  
0 0 0 1 1 1 1 0
```

Implement your algorithm and confirm that the order of growth of its running time is linear in the number of pixels. *Extra credit:* Design an algorithm to find the largest rectangular black subarray.

4.1.39 Sub-exponential function. Find a function whose order of growth is larger than any polynomial function, but smaller than any exponential function. *Extra credit:* Find a program whose running time has that order of growth.

4.2 Sorting and Searching

THE SORTING PROBLEM IS TO REARRANGE an array of items into ascending order. It is a familiar and critical task in many computational applications: the songs in your music library are in alphabetical order, your email messages are displayed in reverse order of the time received, and so forth. Keeping things in some kind of order is a natural desire. One reason that it is so useful is that it is much easier to *search* for something in a sorted array than an unsorted one. This need is particularly acute in computing, where the array to search can be huge and an efficient search can be an important factor in a problem's solution.

Sorting and searching are important for commercial applications (businesses keep customer files in order) and scientific applications (to organize data and computation), and have all manner of applications in fields that may appear to have little to do with keeping things in order, including data compression, computer graphics, computational biology, numerical computing, combinatorial optimization, cryptography, and many others.

We use these fundamental problems to illustrate the idea that *efficient algorithms* are one key to effective solutions for computational problem. Indeed, many different sorting and searching methods have been proposed. Which should we use to address a given task? This question is important because different algorithms can have vastly differing performance characteristics, enough to make the difference between success in a practical situation and not coming close to doing so, even on the fastest available computer.

In this section, we will consider in detail two classical algorithms for sorting and searching—binary search and mergesort—along with several applications in which their efficiency plays a critical role. With these examples, you will be convinced not just of the utility of these methods, but also of the need to *pay attention to the cost* whenever you address a problem that requires a significant amount of computation.

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Programs in this section

Binary search The game of “twenty questions” (see PROGRAM 1.5.2) provides an important and useful lesson in the design of efficient algorithms. The setup is simple: your task is to guess the value of a secret number that is one of the n integers between 0 and $n-1$. Each time that you make a guess, you are told whether your guess is equal to the secret number, too high, or too low. For reasons that will become clear later, we begin by slightly modifying the game to make the questions of the form “is the number greater than or equal to x ?” with *true* or *false* answers, and assume for the moment that n is a power of 2.

interval	length	Q	A
	128	≥ 64 ?	<i>true</i>
	64	≥ 96 ?	<i>false</i>
	32	≥ 80 ?	<i>false</i>
	16	≥ 72 ?	<i>true</i>
	8	≥ 76 ?	<i>true</i>
	4	≥ 78 ?	<i>false</i>
	2	≥ 77 ?	<i>true</i>
	1	$= 77$	

Finding a hidden number with binary search

following recursive strategy:

- *Base case*: If $hi - lo$ equals 1, then the secret number is lo .
- *Reduction step*: Otherwise, ask whether the secret number is greater than or equal to the number $mid = lo + (hi - lo)/2$. If so, look for the number in $[lo, mid]$; if not, look for the number in $[mid, hi]$.

The function `binarySearch()` in Questions (PROGRAM 4.2.1) is an implementation of this strategy. It is an example of the general problem-solving technique known as *binary search*, which has many applications.

As we discussed in SECTION 1.5, an effective strategy for the problem is to maintain an interval that contains the secret number. In each step, we ask a question that enables us to shrink the size of the interval in half. Specifically, we guess the number in the middle of the interval, and, depending on the answer, discard the half of the interval that cannot contain the secret number. More precisely, we use a *half-open interval*, which contains the left endpoint but not the right one. We use the notation $[lo, hi)$ to denote all of the integers greater than or equal to lo and less than (but not equal to) hi . We start with $lo = 0$ and $hi = n$ and use the

Program 4.2.1 Binary search (20 questions)

```

public class Questions
{
    public static int binarySearch(int lo, int hi)
    { // Find number in [lo, hi)
        if (hi - lo == 1) return lo;
        int mid = lo + (hi - lo) / 2;
        StdOut.print("Greater than or equal to " + mid + "? ");
        if (StdIn.readBoolean())
            return binarySearch(mid, hi);
        else
            return binarySearch(lo, mid);
    }
    public static void main(String[] args)
    { // Play twenty questions.
        int k = Integer.parseInt(args[0]);
        int n = (int) Math.pow(2, k);
        StdOut.print("Think of a number ");
        StdOut.println("between 0 and " + (n-1));
        int guess = binarySearch(0, n);
        StdOut.println("Your number is " + guess);
    }
}

```

lo	<i>smallest possible value</i>
hi - 1	<i>largest possible value</i>
mid	<i>midpoint</i>
k	<i>number of questions</i>
n	<i>number of possible values</i>

This code uses binary search to play the same game as PROGRAM 1.5.2, but with the roles reversed: you choose the secret number and the program guesses its value. It takes an integer command-line argument k , asks you to think of a number between 0 and $n-1$, where $n = 2^k$, and always guesses the answer with k questions.

```

% java Questions 7
Think of a number between 0 and 127
Greater than or equal to 64? false
Greater than or equal to 96? true
Greater than or equal to 80? true
Greater than or equal to 72? false
Greater than or equal to 76? false
Greater than or equal to 78? true
Greater than or equal to 77? false
Your number is 77

```

Correctness proof. First, we have to convince ourselves that the algorithm is *correct*: that it always leads us to the secret number. We do so by establishing the following facts:

- The interval always contains the secret number.
- The interval sizes are the powers of 2, decreasing from n .

The first of these facts is enforced by the code; the second follows by noting that if $(hi - lo)$ is a power of 2, then $(hi - lo) / 2$ is the next smaller power of 2 and also the size of both halved intervals $[lo, mid]$ and $[mid, hi]$. These facts are the basis of an induction proof that the algorithm operates as intended. Eventually, the interval size becomes 1, so we are guaranteed to find the number.

Analysis of running time. Let n be the number of possible values. In PROGRAM 4.2.1, we have $n = 2^k$, where $k = \lg n$. Now, let $T(n)$ be the number of questions. The recursive strategy implies that $T(n)$ must satisfy the following recurrence relation:

$$T(n) = T(n/2) + 1$$

with $T(1) = 0$. Substituting 2^k for n , we can telescope the recurrence (apply it to itself) to immediately get a closed-form expression:

$$T(2^k) = T(2^{k-1}) + 1 = T(2^{k-2}) + 2 = \dots = T(1) + k = k$$

Substituting back n for 2^k (and $\lg n$ for k) gives the result

$$T(n) = \lg n$$

This justifies our hypothesis that the running time of binary search is logarithmic.

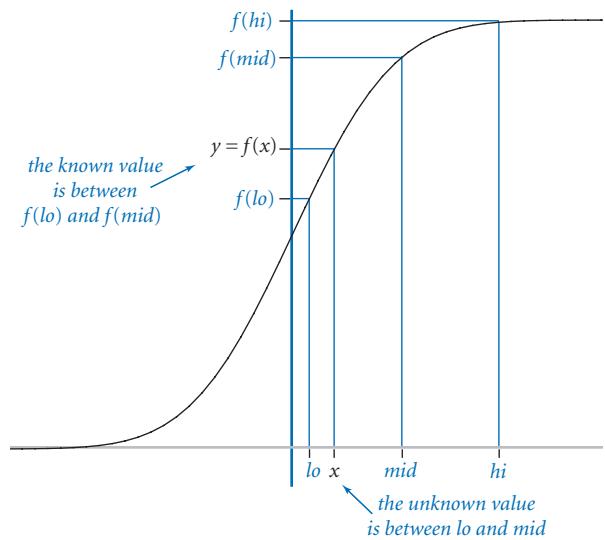
Note: Binary search and TwentyQuestions.binarySearch() work even when n is not a power of 2—we assumed that n is a power of 2 to simplify our proof (see EXERCISE 4.2.1).

Linear-logarithmic chasm. An alternative to using binary search is to guess 0, then 1, then 2, then 3, and so forth, until hitting the secret number. We refer to this algorithm as *sequential search*. It is an example of a *brute-force algorithm*, which seems to get the job done, but without much regard to the cost. The running time of sequential search is sensitive to the secret number: sequential search takes only 1 step if the secret number 0, but it takes n steps if the secret number is $n - 1$. If the secret number is chosen at random, the expected number of steps is $n/2$. Meanwhile, binary search is guaranteed to use no more than $\lg n$ steps. As you will learn to appreciate, the difference between n and $\lg n$ makes a huge difference in practical applications. *Understanding the enormity of this difference is a critical step to under-*

standing the importance of algorithm design and analysis. In the present context, suppose that it takes 1 second to process a guess. With binary search, you can guess the value of any secret number less than 1 million in 20 seconds; with sequential search brute-force algorithm, it might take 1 million seconds, which is more than 1 week. We will see many examples where such a cost difference is the determining factor in whether a practical problem can be feasibly solved.

Binary representation. If you refer back to PROGRAM 1.3.7, you will immediately recognize that binary search is nearly the same computation as converting a number to binary! Each guess determines one bit of the answer. In our example, the information that the number is between 0 and 127 says that the number of bits in its binary representation is 7, the answer to the first question (is the number greater than or equal to 64?) tells us the value of the leading bit, the answer to the second question tells us the value of the next bit, and so forth. For example, if the number is 77, the sequence of answers no yes yes no no yes no immediately yields 1001101, the binary representation of 77. Thinking in terms of the binary representation is another way to understand the linear–logarithmic chasm: when we have a program whose running time is linear in a parameter n , its running time is proportional to the *value* of n , whereas a logarithmic running time is proportional to the *number of digits* in n . In a context that is perhaps slightly more familiar to you, think about the following question, which illustrates the same point: would you rather earn \$6 or a six-figure salary?

Inverting a function. As an example of the utility of binary search in scientific computing, we consider the problem of computing the *inverse* of an increasing function $f(x)$. Given a value y , our task is to find a value x such that $f(x) = y$. In this situation, we use real numbers as the endpoints of our interval,



Binary search (bisection) to invert an increasing function

Program 4.2.2 Bisection search

```

public static double inverseCDF(double y)
{   return bisectionSearch(y, 0.00000001, -8, 8); }

private static double bisectionSearch(double y, double delta,
                                     double lo, double hi)
{   // Compute x with cdf(x) = y.
    double mid = lo + (hi - lo)/2;                      y
    if (hi - lo < delta) return mid;                     delta
    if (cdf(mid) > y)                                     lo
        return bisectionSearch(y, delta, lo, mid);         mid
    else                                                 hi
        return bisectionSearch(y, delta, mid, hi);
}

```

This implementation of `inverseCDF()` for our Gaussian library (PROGRAM 2.1.2) uses bisection search to compute a point x for which $\Phi(x)$ is equal to a given value y , within a given precision δ . It is a recursive function that halves the x -interval containing the desired point, evaluates the function at the midpoint of the interval, and takes advantage of the fact that Φ is increasing to decide whether the desired point is in the left half or the right half, continuing until the interval size is less than the given precision.

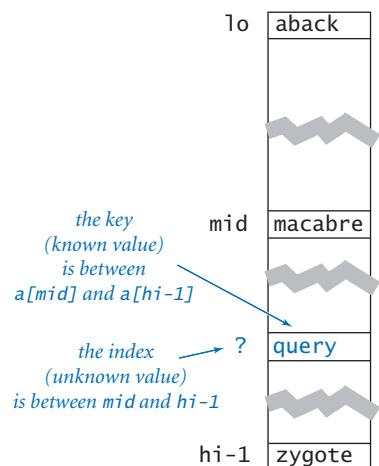
not integers, but we use the same essential algorithm as for guessing a secret number: we halve the size of the interval at each step, keeping x in the interval, until the interval is sufficiently small that we know the value of x to within a desired precision δ . We start with an interval (lo, hi) known to contain x and use the following recursive strategy:

- Compute $mid = lo + (hi - lo)/2$.
- *Base case*: If $hi - lo$ is less than δ , then return mid as an estimate of x .
- *Reduction step*: Otherwise, test whether $f(mid) > y$. If so, look for x in (lo, mid) ; if not, look for x in (mid, hi) .

To fix ideas, PROGRAM 4.2.2 computes the inverse of the Gaussian cumulative distribution function Φ , which we considered in Gaussian (PROGRAM 2.1.2).

The key to this method is the idea that the function is increasing—for any values a and b , knowing that $f(a) < f(b)$ tells us that $a < b$, and vice versa. The recursive step just applies this knowledge: knowing that $y = f(x) < f(mid)$ tells us that $x < mid$, so that x must be in the interval (lo, mid) , and knowing that $y = f(x) > f(mid)$ tells us that $x > mid$, so that x must be in the interval (mid, hi) . You can think of the algorithm as determining which of the $n = (hi - lo) / \delta$ tiny intervals of size δ within (lo, hi) contains x , with running time logarithmic in n . As with number conversion for integers, we determine one bit of x for each iteration. In this context, binary search is often called *bisection search* because we bisect the interval at each stage.

Binary search in a sorted array. One of the most important uses of binary search is to find a piece of information using a key to guide the search. This usage is ubiquitous in modern computing, to the extent that printed artifacts that depend on the same concepts are now obsolete. For example, during the last few centuries, people would use a publication known as a *dictionary* to look up the definition of a word, and during much of the last century people would use a publication known as a *phone book* to look up a person's phone number. In both cases, the basic mechanism is the same: elements appear in order, sorted by a key that identifies it (the word in the case of the dictionary, and the person's name in the case of the phone book, sorted in alphabetical order in both cases). You probably use your computer to reference such information, but think about how you would look up a word in a dictionary. Sequential search would be to start at the beginning, examine each element one at a time, and continue until you find the word. No one uses that algorithm: instead, you open the book to some interior page and look for the word on that page. If it is there, you are done; otherwise, you eliminate either the part of the book before the current page or the part of the book after the current page from consideration, and then repeat. We now recognize this method as binary search (PROGRAM 4.2.3).



Binary search in a sorted array (one step)

Program 4.2.3 Binary search (sorted array)

```

public class BinarySearch
{
    public static int search(String key, String[] a)
    {   return search(key, a, 0, a.length); }

    public static int search(String key, String[] a, int lo, int hi)
    { // Search for key in a[lo, hi].
        if (hi <= lo) return -1;
        int mid = lo + (hi - lo) / 2;
        int cmp = a[mid].compareTo(key);
        if      (cmp > 0) return search(key, a, lo, mid);
        else if (cmp < 0) return search(key, a, mid+1, hi);
        else             return mid;
    }

    public static void main(String[] args)
    { // Print keys from standard input that
      // do not appear in file args[0].
      In in = new In(args[0]);
      String[] a = in.readAllStrings();
      while (!StdIn.isEmpty())
      {
          String key = StdIn.readString();
          if (search(key, a) < 0) StdOut.println(key);
      }
    }
}

```

key a[lo, hi] lo mid hi	<i>search key</i> <i>sorted subarray</i> <i>smallest index</i> <i>middle index</i> <i>largest index</i>
-------------------------------------	---

The `search()` method in this class uses binary search to return the index of a string key in a sorted array (or `-1` if key is not in the array). The test client is an exception filter that reads a (sorted) whitelist from the file given as a command-line argument and prints the words from standard input that are not in the whitelist.

```

% more emails.txt           % more whitelist.txt
bob@office                  alice@home
carl@beach                  bob@office
marvin@spam                 carl@beach
bob@office                  dave@boat
bob@office
mallory@spam
dave@boat
eve@airport
alice@home

% java BinarySearch whitelist.txt < emails.txt
marvin@spam
mallory@spam
eve@airport

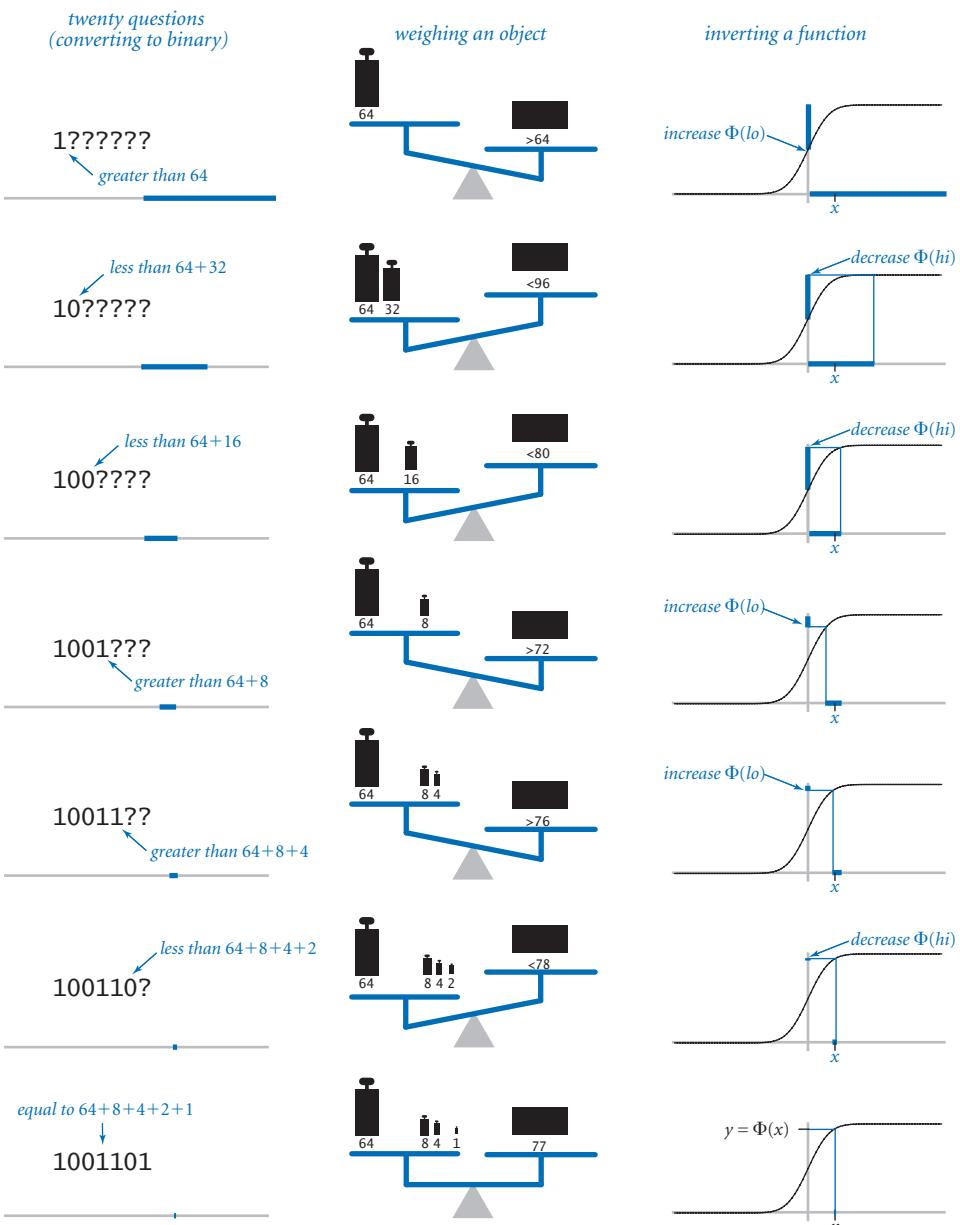
```

Exception filter. We will consider in SECTION 4.3 the details of implementing the kind of computer program that you use in place of a dictionary or a phone book. PROGRAM 4.2.3 uses binary search to solve the simpler *existence problem*: does a given key appear in a sorted array of keys? For example, when checking the spelling of a word, you need only know whether your word is in the dictionary and are not interested in the definition. In a computer search, we keep the information in an array, sorted in order of the key (for some applications, the information comes in sorted order; for others, we have to sort it first, using one of the algorithms discussed later in this section).

The binary search in PROGRAM 4.2.3 differs from our other applications in two details. First, the array length n need not be a power of 2. Second, it has to allow for the possibility that the key sought is not in the array. Coding binary search to account for these details requires some care, as discussed in this section's Q&A and exercises.

The test client in PROGRAM 4.2.3 is known as an *exception filter*: it reads in a sorted list of strings from a file (which we refer to as the *whitelist*) and an arbitrary sequence of strings from standard input, and prints those in the sequence that do not appear in the whitelist. Exception filters have many direct applications. For example, if the whitelist is the words from a dictionary and standard input is a text document, the exception filter prints the misspelled words. Another example arises in web applications: your email application might use an exception filter to reject any email messages that are not on a whitelist that contains the email addresses of your friends. Or, your operating system might have an exception filter that disallows network connections to your computer from any device having an IP address that is not on a preapproved whitelist.

Weighing an object. Binary search has been known since antiquity, perhaps partly because of the following application. Suppose that you need to determine the weight of a given object using only a balancing scale and some weights. With binary search, you can do so with weights that are powers of 2 (you need only one weight of each type). Put the object on the right side of the balance and try the weights in decreasing order on the left side. If a weight causes the balance to tilt to the left, remove it; otherwise, leave it. This process is precisely analogous to determining the binary representation of a number by subtracting decreasing powers of 2, as in PROGRAM 1.3.7.



Three applications of binary search

FAST ALGORITHMS ARE AN ESSENTIAL ELEMENT of the modern world, and binary search is a prototypical example that illustrates the impact of fast algorithms. With a few quick calculations, you can convince yourself that problems like finding all the misspelled words in a document or protecting your computer from intruders using an exception filter *require* a fast algorithm like binary search. *Take the time to do so.* You can find the exceptions in a million-element document to a million-element whitelist in an instant, whereas that task might take days or weeks using a brute-force algorithm. Nowadays, web companies routinely provide services that are based on using binary search *billions* of times in sorted arrays with *billions* of elements—without a fast algorithm like binary search, we could not contemplate such services.

Whether it be extensive experimental data or detailed representations of some aspect of the physical world, modern scientists are awash in data. Binary search and fast algorithms like it are essential components of scientific progress. Using a brute-force algorithm is precisely analogous to searching for a word in a dictionary by starting at the first page and turning pages one by one. With a fast algorithm, you can search among billions of pieces of information in an instant. Taking the time to identify and use a fast algorithm for search certainly can make the difference between being able to solve a problem easily and spending substantial resources trying to do so (and failing).

Insertion sort Binary search requires that the data be sorted, and sorting has many other direct applications, so we now turn to sorting algorithms. We first consider a brute-force method, then a sophisticated method that we can use for huge data sets.

The brute-force algorithm is known as *insertion sort* and is based on a simple method that people often use to arrange hands of playing cards. Consider the cards one at a time and insert each into its proper place among those already considered (keeping them sorted). The following code mimics this process in a Java method that rearranges the strings in an array so that they are in ascending order:

```
public static void sort(String[] a)
{
    int n = a.length;
    for (int i = 1; i < n; i++)
        for (int j = i; j > 0; j--)
            if (a[j-1].compareTo(a[j]) > 0)
                exchange(a, j-1, j);
            else break;
}
```

At the beginning of each iteration of the outer `for` loop, the first i elements in the array are in sorted order; the inner `for` loop moves $a[i]$ into its proper position in the array, as in the following example when i is 6:

		<i>a[]</i>								
<i>i</i>	<i>j</i>	0	1	2	3	4	5	6	7	
6	6	and	had	him	his	was	you	the	but	
6	5	and	had	him	his	was	the	you	but	
6	4	and	had	him	his	the	was	you	but	
		and	had	him	his	the	was	you	but	

Inserting a[6] into position by exchanging it with larger values to its left

Specifically, $a[i]$ is put in its place among the sorted elements to its left by exchanging it (using the `exchange()` method that we first encountered in SECTION 2.1) with each larger value to its left, moving from right to left, until it reaches its proper position. The black elements in the three bottom rows in this trace are the ones that are compared with $a[i]$.

The insertion process just described is executed, first with i equal to 1, then 2, then 3, and so forth, as illustrated in the following trace.

<i>i</i>	<i>j</i>	<i>a[]</i>							
		0	1	2	3	4	5	6	7
		was	had	him	and	you	his	the	but
1	0	had	was	him	and	you	his	the	but
2	1	had	him	was	and	you	his	the	but
3	0	and	had	him	was	you	his	the	but
4	4	and	had	him	was	you	his	the	but
5	3	and	had	him	his	was	you	the	but
6	4	and	had	him	his	the	was	you	but
7	1	and	but	had	him	his	the	was	you
		and	but	had	him	his	the	was	you

Inserting a[1] through a[n-1] into position (insertion sort)

Row *i* of the trace displays the contents of the array when the outer for loop completes, along with the value of *j* at that time. The highlighted string is the one that was in *a[i]* at the beginning of the loop, and the other strings printed in black are the other ones that were involved in exchanges and moved to the right one position within the loop. Since the elements *a[0]* through *a[i-1]* are in sorted order when the loop completes for each value of *i*, they are, in particular, in sorted order the final time the loop completes, when the value of *i* is *a.length*. This discussion again illustrates the first thing that you need to do when studying or developing a new algorithm: convince yourself that it is correct. Doing so provides the basic understanding that you need to study its performance and use it effectively.

Analysis of running time. The inner loop of the insertion sort code is within a double nested for loop, which suggests that the running time is quadratic, but we cannot immediately draw this conclusion because of the break statement. For example, in the best case, when the input array is already in sorted order, the inner for loop amounts to nothing more than a single compare (to learn that *a[j-1]* is less than or equal to *a[j]* for each *j* from 1 to *n-1*) and the break, so the total running time is linear. In contrast, if the input array is in reverse-sorted order, the inner loop fully completes without a break, so the frequency of execution of the instructions in the inner loop is $1 + 2 + \dots + n-1 \sim \frac{1}{2}n^2$ and the running time is quadratic. To understand the performance of insertion sort for randomly ordered input arrays, take a careful look at the trace: it is an *n*-by-*n* array with one black element corresponding to each exchange. That is, the number of black elements is

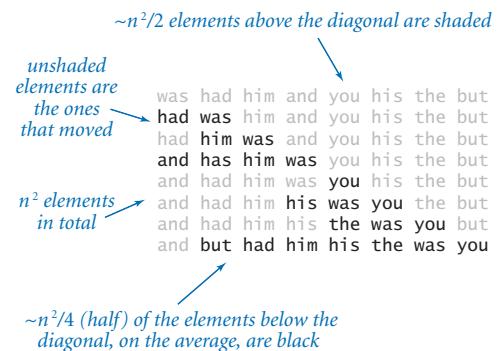
the frequency of execution of instructions in the inner loop. We expect that each new element to be inserted is equally likely to fall into any position, so, on average, that element will move halfway to the left. Thus, on average, we expect only about half of the elements below the diagonal (about $n^2/4$ in total) to be black. This leads immediately to the hypothesis that the expected running time of insertion sort for a randomly ordered input array is quadratic.

Sorting other types of data. We want to be able to sort all types of data, not just strings. In a scientific application, we might wish to sort experimental results by numeric values; in a commercial application, we might wish to use monetary amounts, times, or dates; in systems software, we might wish to use IP addresses or process IDs. The idea of sorting in each of these situations is intuitive, but implementing a sort method that works in all of them is a prime example of the need for a functional abstraction mechanism like the one provided by Java interfaces. For sorting objects in an array, we need only assume that we can *compare* two elements to see whether the first is bigger than, smaller than, or equal to the second. Java provides the `java.util.Comparable` interface for precisely this purpose.

```
public interface Comparable<Key>
    int compareTo(Key b)           compare this object with b for order
    API for Java's java.util.Comparable interface
```

A class that implements the `Comparable` interface promises to implement a method `compareTo()` for objects of its type so that `a.compareTo(b)` returns a negative integer (typically `-1`) if `a` is less than `b`, a positive integer (typically `+1`) if `a` is greater than `b`, and `0` if `a` is equal to `b`. (The `<Key>` notation, which we will introduce in SECTION 4.3, ensures that the two objects being compared have the same type.)

The precise meanings of *less than*, *greater than*, and *equal to* depends on the data type, though implementations that do not respect the natural laws of math-



Analysis of insertion sort

ematics surrounding these concepts will yield unpredictable results. More formally, the `compareTo()` method must define a *total order*. This means that the following three properties must hold (where we use the notation $x \leq y$ as shorthand for `x.compareTo(y) <= 0` and $x = y$ as shorthand for `x.compareTo(y) == 0`):

- *Antisymmetric*: if both $x \leq y$ and $y \leq x$, then $x = y$.
- *Transitive*: if both $x \leq y$ and $y \leq z$, then $x \leq z$.
- *Total*: either $x \leq y$ or $y \leq x$ or both.

These three properties hold for a variety of familiar orderings, including alphabetical order for strings and ascending order for integers and real numbers. We refer to a data type that implements the `Comparable` interface as *comparable* and the associated total order as its *natural order*. Java's `String` type is comparable, as are the primitive wrapper types (such as `Integer` and `Double`) that we introduced in SECTION 3.3.

With this convention, `Insertion` (PROGRAM 4.2.4) implements our `sort` method so that it takes an array of comparable objects as an argument and rearranges the array so that its elements are in ascending order, according to the order specified by the `compareTo()` method. Now, we can use `Insertion.sort()` to sort arrays of type `String[]`, `Integer[]`, or `Double[]`.

It is also easy to make a data type comparable, so that we can sort user-defined types of data. To do so, we must include the phrase `implements Comparable` in the class declaration, and then add a `compareTo()` method that defines a total order. For example, to make the `Counter` data type comparable, we modify PROGRAM 3.3.2 as follows:

```
public class Counter implements Comparable<Counter>
{
    private int count;
    ...
    public int compareTo(Counter b)
    {
        if      (count < b.count) return -1;
        else if (count > b.count) return +1;
        else                      return 0;
    }
    ...
}
```

Now, we can use `Insertion.sort()` to sort an array of `Counter` objects in ascending order of their counts.

Program 4.2.4 Insertion sort

```

public class Insertion
{
    public static void sort(Comparable[] a)
    { // Sort a[] into increasing order.
        int n = a.length;
        for (int i = 1; i < n; i++)
            // Insert a[i] into position.
            for (int j = i; j > 0; j--)
                if (a[j].compareTo(a[j-1]) < 0)
                    exchange(a, j-1, j);
                else break;
    }

    public static void exchange(Comparable[] a, int i, int j)
    { Comparable temp = a[j]; a[j] = a[i]; a[i] = temp; }

    public static void main(String[] args)
    { // Read strings from standard input, sort them, and print.
        String[] a = StdIn.readAllStrings();
        sort(a);
        for (int i = 0; i < a.length; i++)
            StdOut.print(a[i] + " ");
        StdOut.println();
    }
}

```

a[] | *array to sort*
n | *length of array*

The `sort()` function is an implementation of insertion sort. It sorts arrays of any type of data that implements the `Comparable` interface (and, therefore, has a `compareTo()` method). `Insertion.sort()` is appropriate only for small arrays or for arrays that are nearly in order; it is too slow to use for large arrays that are out of order.

```

% more 8words.txt
was had him and you his the but
% java Insertion < 8words.txt
and but had him his the was you
% java Insertion < TomSawyer.txt
tick tick
tick tick tick tick tick tick tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick tick tick tick tick tick tick

```

Empirical analysis. `InsertionDoublingTest` (PROGRAM 4.2.5) tests our hypothesis that insertion sort is quadratic for randomly ordered arrays by running `Insertion.sort()` on n random `Double` objects, computing the ratios of running times as n doubles. This ratio converges to 4, which validates the hypothesis that the running time is quadratic, as discussed in the last section. You are encouraged to run `InsertionDoublingTest` on your own computer. As usual, you might notice the effect of caching or some other system characteristic for some values of n , but the quadratic running time should be quite evident, and you will be quickly convinced that insertion sort is too slow to be useful for large inputs.

Sensitivity to input. Note that `InsertionDoublingTest` takes a command-line argument `trials` and runs `trials` experiments for each array length, not just one. As we have just observed, one reason for doing so is that *the running time of insertion sort is sensitive to its input values*. This behavior is quite different from (for example) `ThreeSum`, and means that we have to carefully interpret the results of our analysis. It is not correct to flatly predict that the running time of insertion sort will be quadratic, because your application might involve input for which the running time is linear. When an algorithm's performance is sensitive to input values, you might not be able to make accurate predictions without taking them into account.

THERE ARE MANY NATURAL APPLICATIONS FOR which insertion sort *is* quadratic, so we need to consider faster sorting algorithms. As we know from SECTION 4.1, a back-of-the-envelope calculation can tell us that having a faster computer is not much help. A dictionary, a scientific database, or a commercial database can contain billions of elements; how can we sort such a large array?

Program 4.2.5 Doubling test for insertion sort

```

public class InsertionDoublingTest
{
    public static double timeTrials(int trials, int n)
    { // Sort random arrays of size n.
        double total = 0.0;                                trials   | number of trials
        Double[] a = new Double[n];                         n       | problem size
        for (int t = 0; t < trials; t++)
        {
            for (int i = 0; i < n; i++)
                a[i] = StdRandom.uniform(0.0, 1.0);          total   | total elapsed time
            Stopwatch timer = new Stopwatch();               timer   | stopwatch
            Insertion.sort(a);                            prev     | running time for n/2
            total += timer.elapsedTime();                  curr     | running time for n
        }                                                 ratio   | ratio of running times
        return total;
    }
    public static void main(String[] args)
    { // Print doubling ratios for insertion sort.
        int trials = Integer.parseInt(args[0]);
        for (int n = 1024; true; n += n)
        {
            double prev = timeTrials(trials, n/2);
            double curr = timeTrials(trials, n);
            double ratio = curr / prev;
            StdOut.printf("%7d %.4f\n", n, ratio);
        }
    }
}

```

The method `timeTrials()` runs `Insertion.sort()` for arrays of random double values. The first argument `n` is the length of the array; the second argument `trials` is the number of trials. Multiple trials produce more accurate results because they dampen system effects and because insertion sort's running time depends on the input.

```
% java InsertionDoublingTest 1
1024 0.71
2048 3.00
4096 5.20
8192 3.32
16384 3.91
32768 3.89
```

```
% java InsertionDoublingTest 10
1024 1.89
2048 5.00
4096 3.58
8192 4.09
16384 4.83
32768 3.96
```

Mergesort To develop a faster sorting method, we use recursion and a *divide-and-conquer* approach to algorithm design that every programmer needs to understand. This nomenclature refers to the idea that one way to solve a problem is to *divide* it into independent parts, *conquer* them independently, and then use the solutions for the parts to develop a solution for the full problem. To sort an array with this strategy, we divide it into two halves, sort the two halves independently, and then *merge* the results to sort the full array. This algorithm is known as *mergesort*.

We process contiguous subarrays of a given array, using the notation $a[lo, hi]$ to refer to $a[lo], a[lo+1], \dots, a[hi-1]$ (adopting the same convention that we used for binary search to denote a half-open interval that excludes $a[hi]$). To sort $a[lo, hi]$, we use the following recursive strategy:

- *Base case*: If the subarray length is 0 or 1, it is already sorted.
- *Reduction step*: Otherwise, compute $mid = lo + (hi - lo)/2$, recursively sort the two subarrays $a[lo, mid]$ and $a[mid, hi]$, and merge them.

Merge (PROGRAM 4.2.6) is an implementation of this algorithm. The values in the array are rearranged by the code that follows the recursive calls, which *merges* the two subarrays that were sorted by the recursive calls. As usual, the easiest way to understand the merge process is to study a trace during the merge. The code maintains one index i into the first subarray, another index j into the second subarray,

i	j	k	$aux[k]$	$a[]$							
				0	1	2	3	4	5	6	7
0	4	0	and	and	had	him	was	but	his	the	you
1	4	1	but	and	had	him	was	but	his	the	you
1	5	2	had	and	had	him	was	but	his	the	you
2	5	3	him	and	had	him	was	but	his	the	you
3	5	4	his	and	had	him	was	but	his	the	you
3	6	5	the	and	had	him	was	but	his	the	you
3	7	6	was	and	had	him	was	but	his	the	you
4	7	7	you	and	had	him	was	but	his	the	you

Trace of the merge of the sorted left subarray with the sorted right subarray

```


was had him and you his the but
sort left
and had him was you his the but
sort right
and had him was but his the you
merge
and but had him his the was you

```

Mergesort overview

and a third index k into an auxiliary array $\text{aux}[]$ that temporarily holds the result. The merge implementation is a single loop that sets $\text{aux}[k]$ to either $a[i]$ or $a[j]$ (and then increments k and the index the subarray that was used). If either i or j has reached the end of its subarray, $\text{aux}[k]$ is set from the other; otherwise, it is set to the smaller of $a[i]$ or $a[j]$. After all of the values from the two subarrays have been copied to $\text{aux}[]$, the sorted result in $\text{aux}[]$ is copied back to the original array. Take a moment to study the trace just given to convince yourself that this code always properly combines the two sorted subarrays to sort the full array.

The recursive method ensures that the two subarrays are each put into sorted order just prior to the merge. Again, the best way to gain an understanding of this process is to study a trace of the contents of the array each time the recursive `sort()` method returns. Such a trace for our example is shown next. First $a[0]$ and $a[1]$ are merged to make a sorted subarray in $a[0, 2]$, then $a[2]$ and $a[3]$ are merged to make a sorted subarray in $a[2, 4]$, then these two subarrays of size 2 are merged to make a sorted subarray in $a[0, 4]$, and so forth. If you are convinced that the merge works properly, you need only convince yourself that the code properly divides the array to be convinced that the sort works properly. Note that when the number of elements in a subarray to be sorted is not even, the left half will have one fewer element than the right half.

<i>a[]</i>								
0	1	2	3	4	5	6	7	
was	had	him	and	you	his	the	but	
sort(<i>a</i> , <i>aux</i> , 0, 8)								
sort(<i>a</i> , <i>aux</i> , 0, 4)								
sort(<i>a</i> , <i>aux</i> , 0, 2)								
return	had	was	him	and	you	his	the	but
sort(<i>a</i> , <i>aux</i> , 2, 4)								
return	had	was	and	him	you	his	the	but
return	and	had	him	was	you	his	the	but
sort(<i>a</i> , <i>aux</i> , 4, 8)								
sort(<i>a</i> , <i>aux</i> , 4, 6)								
return	and	had	him	was	his	you	the	but
sort(<i>a</i> , <i>aux</i> , 6, 8)								
return	and	had	him	was	his	you	but	the
return	and	had	him	was	but	his	the	you
return	and	but	had	him	his	the	was	you

Trace of recursive mergesort calls

Program 4.2.6 Mergesort

```

public class Merge
{
    public static void sort(Comparable[] a)
    {
        Comparable[] aux = new Comparable[a.length];
        sort(a, aux, 0, a.length);
    }

    private static void sort(Comparable[] a, Comparable[] aux,
                           int lo, int hi)
    { // Sort a[lo, hi].
        if (hi - lo <= 1) return;
        int mid = lo + (hi-lo)/2;
        sort(a, aux, lo, mid);
        sort(a, aux, mid, hi);
        int i = lo, j = mid;
        for (int k = lo; k < hi; k++)
            if (i == mid) aux[k] = a[j++];
            else if (j == hi) aux[k] = a[i++];
            else if (a[j].compareTo(a[i]) < 0) aux[k] = a[j++];
            else aux[k] = a[i++];
        for (int k = lo; k < hi; k++)
            a[k] = aux[k];
    }

    public static void main(String[] args)
    { /* See Program 4.2.4. */ }
}

```

$a[lo, hi]$ | subarray to sort
 lo | smallest index
 mid | middle index
 hi | largest index
 $aux[]$ | auxiliary array

The `sort()` function is an implementation of mergesort. It sorts arrays of any type of data that implements the `Comparable` interface. In contrast to `Insertion.sort()`, this implementation is suitable for sorting huge arrays.

```

% java Merge < 8words.txt
was had him and you his the but

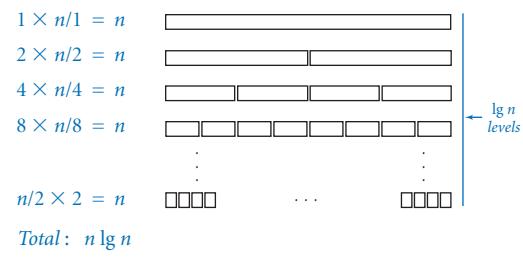
% java Merge < TomSawyer.txt
... achievement aching aching acquire acquired ...

```

Analysis of running time. The inner loop of mergesort is centered on the auxiliary array. The two for loops involve n iterations, so the frequency of execution of the instructions in the inner loop is proportional to the sum of the subarray lengths for all calls to the recursive function. The value of this quantity emerges when we arrange the calls on levels according to their size. For simplicity, suppose that n is a power of 2, with $n = 2^k$. On the first level, we have one call for size n ; on the second level, we have two calls for size $n/2$; on the third level, we have four calls for size $n/4$; and so forth, down to the last level with $n/2$ calls of size 2. There are precisely $k = \lg n$ levels, giving the grand total $n \lg n$ for the frequency of execution of the instructions in the inner loop of mergesort. This equation justifies a hypothesis that the running time of mergesort is linearithmic. Note: When n is not a power of 2, the subarrays on each level are not necessarily all the same size, but the number of levels is still logarithmic, so the linearithmic hypothesis is justified for all n (see EXERCISE 4.2.18 and EXERCISE 4.2.19).

You are encouraged to run a doubling test like PROGRAM 4.2.5 for `Merge.sort()` on your computer. If you do so, you certainly will appreciate that it is much faster for large arrays than is `Insertion.sort()` and that you can sort huge arrays with relative ease. Validating the hypothesis that the running time is linearithmic is a bit more work, but you certainly can see that mergesort makes it possible for us to address sorting problems that we could not contemplate solving with a brute-force algorithm such as insertion sort.

Quadratic–linearithmic chasm. The difference between n^2 and $n \log n$ makes a huge difference in practical applications, just the same as the linear–logarithmic chasm that is overcome by binary search. *Understanding the enormity of this difference is another critical step to understanding the importance of the design and analysis of algorithms.* For a great many important computational problems, a speedup from quadratic to linearithmic—such as we achieve with mergesort—makes the difference between the ability to solve a problem involving a huge amount of data and not being able to effectively address it at all.



Mergesort inner loop count (when n is a power of 2)

Divide-and-conquer algorithms. The same basic divide-and-conquer paradigm is effective for many important problems, as you will learn if you take a course on algorithm design. For the moment, you are particularly encouraged to study the exercises at the end of this section, which describe a host of problems for which divide-and-conquer algorithms provide feasible solutions and which could not be addressed without such algorithms.

Reduction to sorting. A problem A *reduces* to a problem B if we can use a solution to B to solve A. Designing a new divide-and-conquer algorithm from scratch is sometimes akin to solving a puzzle that requires some experience and ingenuity, so you may not feel confident that you can do so at first. But it is often the case that a simpler approach is effective: given a new problem, ask yourself how you would solve it if the data were sorted. It often turns out to be the case that a relatively simple linear pass through the sorted data will do the job. Thus, we get a linearithmic algorithm, with the ingenuity hidden in the mergesort algorithm. For example, consider the problem of determining whether the values of the elements in an array are all distinct. This *element distinctness* problem reduces to sorting because we can sort the array, and then pass through the sorted array to check whether the value of any element is equal to the next—if not, the values are all distinct. For another example, an easy way to implement `StdStats.median()` (see SECTION 2.2) is to reduce selection to sorting. We consider next a more complicated example, and you can find many others in the exercises at the end of this section.

MERGESORT TRACES BACK TO JOHN VON Neumann, an accomplished physicist, who was among the first to recognize the importance of computation in scientific research. Von Neumann made many contributions to computer science, including a basic conception of the computer architecture that has been used since the 1950s. When it came to applications programming, von Neumann recognized that:

- Sorting is an essential ingredient in many applications.
- Quadratic-time algorithms are too slow for practical purposes.
- A divide-and-conquer approach is effective.
- Proving programs correct and knowing their cost is important.

Computers are many orders of magnitude faster and have many orders of magnitude more memory than those available to von Neumann, but these basic concepts remain important today. People who use computers effectively and successfully know, as did von Neumann, that brute-force algorithms are often not good enough to do the job.

Application: frequency counts FrequencyCount (PROGRAM 4.2.7) reads a sequence of strings from standard input and then prints a table of the distinct strings found and the number of times each was found, in decreasing order of frequency. This computation is useful in numerous applications: a linguist might be studying patterns of word usage in long texts, a scientist might be looking for frequently occurring events in experimental data, a merchant might be looking for the customers who appear most frequently in a long list of transactions, or a network analyst might be looking for the most active users. Each of these applications might involve millions of strings or more, so we need a linearithmic algorithm (or better). FrequencyCount is an example of developing such an algorithm by reduction to sorting. It actually does *two* sorts.

Computing the frequencies. Our first step is to sort the strings on standard input. In this case, we are not so much interested in the fact that the strings are put into sorted order, but in the fact that *sorting brings equal strings together*. If the input is

to be or not to be to

then the result of the sort is

be be not or to to to

with equal strings—such as the two occurrences of be and the three occurrences of to—brought together in the array. Now, with equal strings all together in the array, we can make a single pass through the array to compute the frequencies. The Counter data type that we considered in SECTION 3.3 is the perfect tool for the job. Recall that a Counter (PROGRAM 3.3.2) has a string instance variable (initialized to the constructor argument), a count instance variable (initialized to 0), and an increment() instance method, which increments the counter by 1. We maintain an integer m and an array of Counter objects $\text{zipf}[]$ and do the following for each string:

- If the string is not equal to the previous one, create a new Counter object and increment m .
- Increment the most recently created Counter.

At the end, the value of m is the number of different string values, and $\text{zipf}[i]$ contains the i th string value and its frequency.

i	M	a[i]	zipf[i].value()			
			0	1	2	3
	0					
0	1	be	1			
1	1	be	2			
2	2	not	2	1		
3	3	or	2	1	1	
4	4	to	2	1	1	1
5	4	to	2	1	1	2
6	4	to	2	1	1	3
			2	1	1	3

Counting the frequencies

Sorting the frequencies. Next, we sort the Counter objects by frequency. We can do so in client code provided that Counter implements the Comparable interface and its `compareTo()` method compares objects by count (see EXERCISE 4.2.14). Once this is done, we simply sort the array! Note that FrequencyCount allocates `zipf[]` to its maximum possible length and sorts a *subarray*, as opposed to the alternative of making an extra pass through `words[]` to determine the number of distinct strings before allocating `zipf[]`. Modifying Merge (PROGRAM 4.2.6) to support sorting subarrays is left as an exercise (see EXERCISE 4.2.15).

	i	zipf[i]
<i>before</i>		
	0	2 be
	1	1 not
	2	1 or
	3	3 to
<i>after</i>		
	0	1 not
	1	1 or
	2	2 be
	3	3 to

Sorting the frequencies

Zipf's law. The application highlighted in FrequencyCount is elementary linguistic analysis: which words appear most frequently in a text? A phenomenon known as *Zipf's law* says that the frequency of the i th most frequent word in a text of m distinct words is proportional to $1/i$, with its constant of proportionality the inverse of the harmonic number H_m . For example, the second most common word should appear about half as often as the first. This empirical hypothesis holds in a surprising variety of situations, ranging from financial data to web usage statistics. The test client run in PROGRAM 4.2.7 validates Zipf's law for a database containing 1 million sentences drawn randomly from the web (see the booksite).

YOU ARE LIKELY TO FIND YOURSELF writing a program sometime in the future for a simple task that could easily be solved by first using a sort. How many distinct values are there? Which value appears most frequently? What is the median value? With a linearithmic sorting algorithm such as mergesort, you can address these problems and many other problems like them, even for huge data sets. FrequencyCount, which uses two different sorts, is a prime example. If sorting does not apply directly, some other divide-and-conquer algorithm might apply, or some more sophisticated method might be needed. Without a good algorithm (and an understanding of its performance characteristics), you might find yourself frustrated by the idea that your fast and expensive computer cannot solve a problem that seems to be a simple one. With an ever-increasing set of problems that you know how to solve efficiently, you will find that your computer can be a much more effective tool than you now imagine.

Program 4.2.7 Frequency counts

```

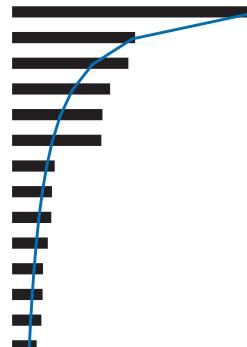
public class FrequencyCount
{
    public static void main(String[] args)
    {   // Print input strings in decreasing order
        // of frequency of occurrence.
        String[] words = StdIn.readAllStrings();
        Merge.sort(words);
        Counter[] zipf = new Counter[words.length];
        int m = 0;
        for (int i = 0; i < words.length; i++)
        {   // Create new counter or increment prev counter.
            if (i == 0 || !words[i].equals(words[i-1]))
                zipf[m++] = new Counter(words[i], words.length);
            zipf[m-1].increment();
        }
        Merge.sort(zipf, 0, m);
        for (int j = m-1; j >= 0; j--)
            StdOut.println(zipf[j]);
    }
}

```

s *input strings in input*
words[] *counter array*
zipf[] *different strings*
m

This program sorts the words on standard input, uses the sorted list to count the frequency of occurrence of each, and then sorts the frequencies. The test file used below has more than 20 million words. The plot compares the i th frequency relative to the first (bars) with $1/i$ (blue).

```
% java FrequencyCount < Leipzig1M.txt
the: 1160105
of: 593492
to: 560945
a: 472819
and: 435866
in: 430484
for: 205531
The: 192296
that: 188971
is: 172225
said: 148915
on: 147024
was: 141178
by: 118429
...
```



Lessons The vast majority of programs that we write involve managing the complexity of addressing a new practical problem by developing a clear and correct solution, breaking the program into modules of manageable size, and testing and debugging our solution. From the very start, our approach in this book has been to develop programs along these lines. But as you become involved in ever more complex applications, you will find that a clear and correct solution is not always sufficient, because the cost of computation can be a limiting factor. The examples in this section are a basic illustration of this fact.

Respect the cost of computation. If you can quickly solve a small problem with a simple algorithm, fine. But if you need to address a problem that involves a large amount of data or a substantial amount of computation, you need to take into account the cost.

Reduce to a known problem. Our use of sorting for frequency counting illustrates the utility of understanding fundamental algorithms and using them for problem solving.

Divide-and-conquer. It is worthwhile for you to reflect a bit on the power of the divide-and-conquer paradigm, as illustrated by developing a linearithmic sorting algorithm (mergesort) that serves as the basis for addressing so many computational problems. Divide-and-conquer is but one approach to developing efficient algorithms.

SINCE THE ADVENT OF COMPUTING, PEOPLE have been developing algorithms such as binary search and mergesort that can efficiently solve practical problems. The field of study known as *design and analysis of algorithms* encompasses the study of design paradigms such as divide-and-conquer and dynamic programming, the invention of algorithms for solving fundamental problems like sorting and searching, and techniques to develop hypotheses about the performance of algorithms. Implementations of many of these algorithms are found in Java libraries or other specialized libraries, but understanding these basic tools of computation is like understanding the basic tools of mathematics or science. You can use a matrix-processing package to find the eigenvalues of a matrix, but you still need a course in linear algebra. Now that you *know* a fast algorithm can make the difference between spinning your wheels and properly addressing a practical problem, you can be on the lookout for situations where algorithm design and analysis can make the difference, and where efficient algorithms such as binary search and mergesort can do the job.

Q&A

Q. Why do we need to go to such lengths to prove a program correct?

A. To spare ourselves considerable pain. Binary search is a notable example. For example, you now understand binary search; a classic programming exercise is to write a version that uses a `while` loop instead of recursion. Try solving EXERCISE 4.2.2 without looking back at the code in the book. In a famous experiment, Jon Bentley once asked several professional programmers to do so, and most of their solutions were *not* correct.

Q. Are there implementations for sorting and searching in the Java library?

A. Yes. The Java package `java.util` contains the static methods `Arrays.sort()` and `Arrays.binarySearch()` that implement mergesort and binary search, respectively. Actually, each represents a family of overloaded methods, one for `Comparable` types, and one for each primitive type.

Q. So why not just use them?

A. Feel free to do so. As with many topics we have studied, you will be able to use such tools more effectively if you understand the background behind them.

Q. Explain why we use `lo + (hi - lo) / 2` to compute the index midway between `lo` and `hi` instead of using `(lo + hi) / 2`.

A. The latter fails when `lo + hi` overflows an `int`.

Q. Why do I get a `unchecked` or `unsafe` operation warning when compiling `Insertion.java` and `Merge.java`?

A. The argument to `sort()` is a `Comparable` array, but nothing, technically, prevents its elements from being of different types. To eliminate the warning, change the signature to:

```
public static <Key extends Comparable<Key>> void sort(Key[] a)
```

We'll learn about the `<Key>` notation in the next section when we discuss *generics*.

Exercises

4.2.1 Develop an implementation of `Questions` (PROGRAM 4.2.1) that takes the maximum number n as a command-line argument. Prove that your implementation is correct.

4.2.2 Develop a nonrecursive version of `BinarySearch` (PROGRAM 4.2.3).

4.2.3 Modify `BinarySearch` (PROGRAM 4.2.3) so that if the search key is in the array, it returns the *smallest* index i for which $a[i]$ is equal to key , and otherwise returns $-i$, where i is the smallest index such that $a[i]$ is greater than key .

4.2.4 Describe what happens if you apply binary search to an unordered array. Why shouldn't you check whether the array is sorted before each call to binary search? Could you check that the elements binary search examines are in ascending order?

4.2.5 Describe why it is desirable to use immutable keys with binary search.

4.2.6 Add code to `Insertion` to produce the trace given in the text.

4.2.7 Add code to `Merge` to produce a trace like the following:

```
% java Merge < tiny.txt
was had him and you his the but
had was
    and him
and had him was
        his you
            but the
            but his the you
and but had him his the was you
```

4.2.8 Give traces of insertion sort and mergesort in the style of the traces in the text, for the input `it was the best of times it was`.

4.2.9 Implement a more general version of PROGRAM 4.2.2 that applies bisection search to any monotonically increasing function. Use functional programming, in the same style as the numerical integration example from SECTION 3.3.

4.2.10 Write a filter `DeDup` that reads strings from standard input and prints them to standard output, with all duplicate strings removed (and in sorted order).

4.2.11 Modify `StockAccount` (PROGRAM 3.2.8) so that it implements the `Comparable` interface (comparing the stock accounts by name). *Hint:* Use the `compareTo()` method from the `String` data type for the heavy lifting.

4.2.12 Modify `Vector` (PROGRAM 3.3.3) so that it implements the `Comparable` interface (comparing the vectors lexicographically by coordinates).

4.2.13 Modify `Time` (EXERCISE 3.3.21) so that it implements the `Comparable` interface (comparing the times chronologically).

4.2.14 Modify `Counter` (PROGRAM 3.3.2) so that it implements the `Comparable` interface (comparing the objects by frequency count).

4.2.15 Add methods to `Insertion` (PROGRAM 4.2.4) and `Merge` (PROGRAM 4.2.6) to support sorting subarrays.

4.2.16 Develop a nonrecursive version of mergesort (PROGRAM 4.2.6). For simplicity, assume that the number of items n is a power of 2. *Extra credit:* Make your program work even if n is not a power of 2.

4.2.17 Find the frequency distribution of words in your favorite novel. Does it obey Zipf's law?

4.2.18 Analyze mathematically the number of compares that mergesort makes to sort an array of length n . For simplicity, assume n is a power of 2.

Answer: Let $M(n)$ be the number of compares to mergesort an array of length n . Merging two subarrays whose total length is n requires between $\frac{1}{2}n$ and $n-1$ compares. Thus, $M(n)$ satisfies the following recurrence relation:

$$M(n) \leq 2M(n/2) + n$$

with $M(1) = 0$. Substituting 2^k for n gives

$$M(2^k) \leq 2M(2^{k-1}) + 2^n$$

which is similar to, but more complicated than, the recurrence that we considered for binary search. But if we divide both sides by 2^n , we get

$$M(2^k)/2^k \leq M(2^{k-1})/2^{k-1} + 1$$

which is *precisely* the recurrence that we had for binary search. That is, $M(2^k)/2^k \leq T(2^k) = n$. Substituting back n for 2^k (and $\lg n$ for k) gives the result $M(n) \leq n \lg n$. A similar argument shows that $M(n) \geq \frac{1}{2} n \lg n$.

4.2.19 Analyze mergesort for the case when n is not a power of 2.

Partial solution. When n is an odd number, one subarray has one more element than the other, so when n is not a power of 2, the subarrays on each level are not necessarily all the same size. Still, every element appears in some subarray, and the number of levels is still logarithmic, so the linearithmic hypothesis is justified for all n .

Creative Exercises

The following exercises are intended to give you experience in developing fast solutions to typical problems. Think about using binary search and mergesort, or devising your own divide-and-conquer algorithm. Implement and test your algorithm.

4.2.20 Median. Add to `StdStats` (PROGRAM 2.2.4) a method `median()` that computes in linearithmic time the median of an array of n integers. *Hint:* Reduce to sorting.

4.2.21 Mode. Add to `StdStats` (PROGRAM 2.2.4) a method `mode()` that computes in linearithmic time the mode (value that occurs most frequently) of an array of n integers. *Hint:* Reduce to sorting.

4.2.22 Integer sort. Write a *linear-time* filter that reads from standard input a sequence of integers that are between 0 and 99 and prints to standard output the same integers in sorted order. For example, presented with the input sequence

```
98 2 3 1 0 0 0 3 98 98 2 2 2 0 0 0 2
```

your program should print the output sequence

```
0 0 0 0 0 0 1 2 2 2 2 3 3 98 98 98
```

4.2.23 Floor and ceiling. Given a sorted array of `Comparable` items, write functions `floor()` and `ceiling()` that return the index of the largest (or smallest) item not larger (or smaller) than an argument item in logarithmic time.

4.2.24 Bitonic maximum. An array is *bitonic* if it consists of an increasing sequence of keys followed immediately by a decreasing sequence of keys. Given a bitonic array, design a logarithmic algorithm to find the index of a maximum key.

4.2.25 Search in a bitonic array. Given a bitonic array of n distinct integers, design a logarithmic-time algorithm to determine whether a given integer is in the array.

4.2.26 Closest pair. Given an array of n real numbers, design a linearithmic-time algorithm to find a pair of numbers that are closest in value.

4.2.27 Furthest pair. Given an array of n real numbers, design a linear-time algorithm to find a pair of numbers that are furthest apart in value.

4.2.28 Two sum. Given an array of n integers, design a linearithmic-time algorithm to determine whether any *two* of them sum to 0.

4.2.29 Three sum. Given an array of n integers, design an algorithm to determine whether any *three* of them sum to 0. The order of growth of the running time of your program should be $n^2 \log n$. *Extra credit:* Develop a program that solves the problem in quadratic time.

4.2.30 Majority. A value in an array of length n is a *majority* if it appears strictly more than $n/2$ times. Given an array of strings, design a linear-time algorithm to identify a majority element (if one exists).

4.2.31 Largest empty interval. Given n timestamps for when a file is requested from a web server, find the largest interval of time in which no file is requested. Write a program to solve this problem in linearithmic time.

4.2.32 Prefix-free codes. In data compression, a set of strings is *prefix-free* if no string is a prefix of another. For example, the set of strings { 01, 10, 0010, 1111 } is prefix-free, but the set of strings { 01, 10, 0010, 1010 } is not prefix-free because 10 is a prefix of 1010. Write a program that reads in a set of strings from standard input and determines whether the set is prefix-free.

4.2.33 Partitioning. Design a linear-time algorithm to sort an array of Comparable objects that is known to have at most two distinct values. *Hint:* Maintain two pointers, one starting at the left end and moving right, and the other starting at the right end and moving left. Maintain the invariant that all elements to the left of the left pointer are equal to the smaller of the two values and all elements to the right of the right pointer are equal to the larger of the two values.

4.2.34 Dutch-national-flag problem. Design a linear-time algorithm to sort an array of Comparable objects that is known to have at most three distinct values. (Edsger Dijkstra named this the *Dutch-national-flag problem* because the result is three “stripes” of values like the three stripes in the flag.)

4.2.35 Quicksort. Write a recursive program that sorts an array of Comparable objects by using, as a subroutine, the partitioning algorithm described in the previous exercise: First, pick a random element v as the partitioning element. Next, partition the array into a left subarray containing all elements less than v , followed by a middle subarray containing all elements equal to v , followed by a right subarray containing all elements greater than v . Finally, recursively sort the left and right subarrays.

4.2.36 Reverse domain name. Write a filter that reads a sequence of domain names from standard input and prints the reverse domain names in sorted order. For example, the reverse domain name of cs.princeton.edu is edu.princeton.cs. This computation is useful for web log analysis. To do so, create a data type Domain that implements the Comparable interface (using reverse-domain-name order).

4.2.37 Local minimum in an array. Given an array of n real numbers, design a logarithmic-time algorithm to identify a *local minimum* (an index i such that both $a[i] < a[i-1]$ and $a[i] < a[i+1]$).

4.2.38 Discrete distribution. Design a fast algorithm to repeatedly generate numbers from the discrete distribution: Given an array $a[]$ of non-negative real numbers that sum to 1, the goal is to return index i with probability $a[i]$. Form an array $\text{sum}[]$ of cumulated sums such that $\text{sum}[i]$ is the sum of the first i elements of $a[]$. Now, generate a random real number r between 0 and 1, and use binary search to return the index i for which $\text{sum}[i] \leq r < \text{sum}[i+1]$. Compare the performance of this approach with the approach taken in RandomSurfer (PROGRAM 1.6.2).

4.2.39 Implied volatility. Typically the volatility σ is the unknown value in the Black–Scholes formula (see EXERCISE 2.1.28). Write a program that reads s , x , r , t , and the current price of the European call option from the command line and uses bisection search to compute σ .

4.2.40 Percolation threshold. Write a Percolation (PROGRAM 2.4.1) client that uses bisection search to estimate the percolation threshold value.

4.3 Stacks and Queues

IN THIS SECTION, WE introduce two closely related data types for manipulating arbitrarily large collections of objects: the *stack* and the *queue*. Stacks and queues are special cases of the idea of a *collection*. We refer to the objects in a collection as *items*.

A collection is characterized by four operations: *create* the collection, *insert* an item, *remove* an item, and test whether the collection is *empty*.

When we insert an item into a collection, our intent is clear. But when we remove an item from the collection, which one do we choose? Each type of collection is characterized by the rule used for *remove*, and each is amenable to various implementations with differing performance characteristics. You have encountered different rules for removing items in various real-world situations, perhaps without thinking about it.

For example, the rule used for a queue is to always remove the item that has been in the collection for the *most* amount of time. This policy is known as *first-in first-out*, or *FIFO*. People waiting in line to buy a ticket follow this discipline: the line is arranged in the order of arrival, so the one who leaves the line has been there longer than any other person in the line.

A policy with quite different behavior is the rule used for a stack: always remove the item that has been in the collection for the *least* amount of time. This policy is known as *last-in first-out*, or *LIFO*. For example, you follow a policy closer to LIFO when you enter and leave the coach cabin in an airplane: people near the front of the cabin board last and exit before those who boarded earlier.

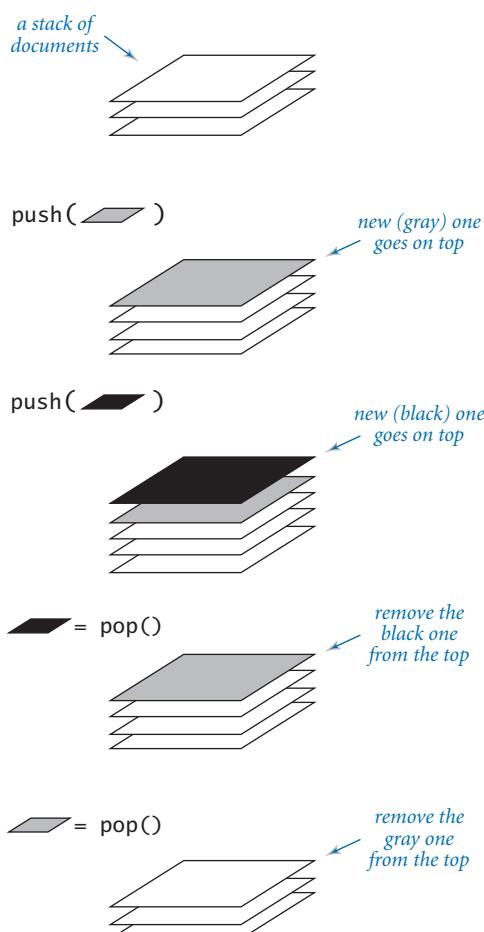
Stacks and queues are broadly useful, so it is important to be familiar with their basic properties and the kind of situation where each might be appropriate. They are excellent examples of fundamental data types that we can use to address higher-level programming tasks. They are widely used in systems and applications programming, as we will see in several examples in this section and in SECTION 4.5.

4.3.1	Stack of strings (array)	570
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Programs in this section

Pushdown stacks A *pushdown stack* (or just a *stack*) is a collection that is based on the last-in first-out (LIFO) policy.

The LIFO policy underlies several of the applications that you use regularly on your computer. For example, many people organize their email as a stack, where messages go on the top when they are received and are taken from the top, with the most recently received message first (last in, first out). The advantage of this strategy is that we see new messages as soon as possible; the disadvantage is that some old messages might never get read if we never empty the stack.



Operations on a pushdown stack

You have likely encountered another common example of a stack when surfing the web. When you click a hyperlink, your browser displays the new page (and inserts it onto a stack). You can keep clicking on hyperlinks to visit new pages, but you can always revisit the previous page by clicking the back button (remove it from a stack). The last-in first-out policy offered by a pushdown stack provides just the behavior that you expect.

Such uses of stacks are intuitive, but perhaps not persuasive. In fact, the importance of stacks in computing is fundamental and profound, but we defer further discussions of applications to later in this section. For the moment, our goal is to make sure that you understand how stacks work and how to implement them.

Stacks have been used widely since the earliest days of computing. By tradition, we name the stack insert operation *push* and the stack remove operation *pop*, as indicated in the following API:

```

public class *StackOfStrings


---


    *StackOfStrings() create an empty stack
    boolean isEmpty() is the stack empty?
    void push(String item) insert a string onto the stack
    String pop() remove and return the most recently inserted string

```

API for a pushdown stack of strings

The asterisk indicates that we will be considering more than one implementation of this API (we consider three in this section: `ArrayStackOfStrings`, `LinkedStackOfStrings`, and `ResizingArrayListOfStrings`). This API also includes a method to test whether the stack is empty, leaving to the client the responsibility of using `isEmpty()` to avoid invoking `pop()` when the stack is empty.

This API has an important restriction that is inconvenient in applications: we would like to have stacks that contain other types of data, not just strings. We describe how to remove this restriction (and the importance of doing so) later in this section.

Array implementation Representing stacks with arrays is a natural idea, but before reading further, it is worthwhile to think for a moment about how you would implement a class `ArrayStackOfStrings`.

The first problem that you might encounter is implementing the constructor `ArrayStackOfStrings()`. You clearly need an instance variable `items[]` with an array of strings to hold the stack items, but how big should the array be? One solution is to start with an array of length 0 and make sure that the array length is always equal to the stack size, but that solution necessitates allocating a new array and copying all of the items into it for each `push()` and `pop()` operation, which is unnecessarily inefficient and cumbersome. We will temporarily finesse this problem by having the client provide an argument for the constructor that gives the maximum stack size.

Your next problem might stem from the natural decision to keep the `n` items in the array in the order they were inserted, with the most recently inserted item in `items[0]` and the least recently inserted item in `items[n-1]`. But then each time you push or pop an item, you would have to move all of the other items to reflect the new state of the stack. A simpler and more efficient way to proceed is to keep

the items in the opposite order, with the most recently inserted item in `items[n-1]` and the least recently inserted item in `items[0]`. This policy allows us to add and remove items at the end of the array, without moving any of the other items in the arrays.

We could hardly hope for a simpler implementation of the stack API than `ArrayStackOfStrings` (PROGRAM 4.3.1)—all of the methods are one-liners! The instance variables are an array `items[]` that holds the items in the stack and an integer `n` that counts the number of items in the stack. To remove an item, we decrement `n` and then return `items[n]`; to insert a new item, we set `items[n]` equal to the new item and then increment `n`. These operations preserve the following properties:

- The number of items in the stack is `n`.
- The stack is empty when `n` is 0.
- The stack items are stored in the array in the order in which they were inserted.
- The most recently inserted item (if the stack is nonempty) is `items[n-1]`.

As usual, thinking in terms of invariants of this sort is the easiest way to verify that an implementation operates as intended. *Be sure that you fully understand this implementation.* Perhaps the best way to do so is to carefully examine a trace of the stack contents for a sequence of `push()` and `pop()` operations. The test client in `ArrayStackOfStrings` allows for testing with an arbitrary sequence of operations: it does a `push()` for each string on standard input except the string consisting of a minus sign, for which it does a `pop()`.

The primary characteristic of this implementation is that *the push and pop operations take constant time*. The drawback is that it requires the client to estimate the maximum size of the stack ahead of time and always uses space proportional to that maximum, which may be unreasonable in some situations. We omit the code in `push()` to test for a full stack, but later we will examine implementations that address this drawback by not allowing the stack to get full (except in an extreme circumstance when there is no memory at all available for use by Java).

StdIn	StdOut	n	items[]				
			0	1	2	3	4
		0					
to		1	to				
be		2	to	be			
or		3	to	be	or		
not		4	to	be	or	not	
to		5	to	be	or	not	to
-	to	4	to	be	or	not	to
be		5	to	be	or	not	be
-	be	4	to	be	or	not	be
-	not	3	to	be	or	not	be
that		4	to	be	or	that	be
-	that	3	to	be	or	that	be
-	or	2	to	be	or	that	be
-	be	1	to	be	or	that	be
is		2	to	is	or	not	to

Trace of `ArrayStackOfStrings` test client

Program 4.3.1 Stack of strings (array)

```

public class ArrayStackOfStrings
{
    private String[] items;
    private int n = 0;

    public ArrayStackOfStrings(int capacity)
    {   items = new String[capacity];   }

    public boolean isEmpty()
    {   return (n == 0);   }           items[] | stack items

    public void push(String item)      n | number of items
    {   items[n++] = item;   }         items[n-1] | item most recently inserted

    public String pop()
    {   return items[--n];   }

    public static void main(String[] args)
    {   // Create a stack of specified capacity; push strings
        // and pop them, as directed on standard input.
        int cap = Integer.parseInt(args[0]);
        ArrayStackOfStrings stack = new ArrayStackOfStrings(cap);
        while (!StdIn.isEmpty())
        {
            String item = StdIn.readString();
            if (!item.equals("-"))
                stack.push(item);
            else
                StdOut.print(stack.pop() + " ");
        }
    }
}

```

Stack methods are simple one-liners, as illustrated in this code. The client pushes or pops strings as directed from standard input (a minus sign indicates pop, and any other string indicates push). Code in push() to test whether the stack is full is omitted (see the text).

```

% more tobe.txt
to be or not to - be - - that - - - is
% java ArrayStackOfStrings 5 < tobe.txt
to be not that or be

```

Linked lists For collections such as stacks and queues, an important objective is to ensure that *the amount of memory used is proportional to the number of items in the collection*. The use of a fixed-length array to implement a stack in `ArrayStackOfStrings` works against this objective: when you create a stack with a specified capacity, you are wasting a potentially huge amount of memory at times when the stack is empty or nearly empty. This property makes our fixed-length array implementation unsuitable for many applications. Now we consider the use of a fundamental data structure known as a *linked list*, which can provide implementations of collections (and, in particular, stacks and queues) that achieve the objective cited at the beginning of this paragraph.

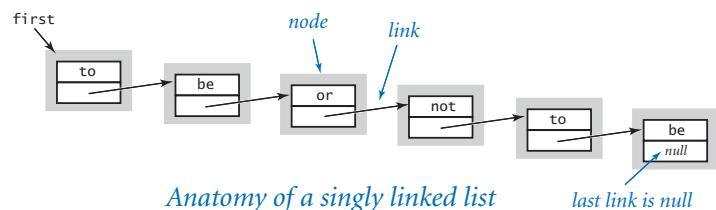
A *singly linked list* comprises a sequence of *nodes*, with each node containing a reference (or *link*) to its successor. By convention, the link in the last node is *null*, to indicate that it terminates the list. A node is an abstract entity that might hold any kind of data, in addition to the link that characterizes its role in building linked lists. When tracing code that uses linked lists and other linked structures, we use a visual representation where:

- We draw a rectangle to represent each linked-list node.
- We put the item and link within the rectangle.
- We use arrows that point to the referenced objects to depict references.

This visual representation captures the essential characteristic of linked lists and focus on the links. For example, the diagram on this page illustrates a singly linked list containing the sequence of items `to`, `be`, `or`, `not`, `to`, and `be`.

With object-oriented programming, implementing linked lists is not difficult. We define a class for the *node* abstraction that is *recursive* in nature. As with recursive functions, the concept of recursive data structures can be a bit mindbending at first.

```
class Node
{
    String item;
    Node next;
}
```



A `Node` object has two instance variables: a `String` and a `Node`. The `String` instance variable is a placeholder for any data that we might want to structure with a linked list (we can use any set of instance variables). The `Node` instance variable `next` characterizes the linked nature of the data structure: it stores a *reference* to

the successor Node in the linked list (or `null` to indicate that there is no such node). Using this recursive definition, we can represent a linked list with a variable of type `Node` by ensuring that its value is either `null` or a reference to a `Node` whose `next` field is a reference to a linked list.

To emphasize that we are just using the `Node` class to structure the data, we do not define any instance methods. As with any class, we can create an object of type `Node` by invoking the (no-argument) constructor with `new Node()`. The result is a reference to a new `Node` object whose instance variables are each initialized to the default value `null`.

For example, to build a linked list that contains the sequence of items `to`, `be`, and `or`, we create a `Node` for each item:

```
Node first = new Node();
Node second = new Node();
Node third = new Node();
```

and assign the `item` instance variable in each of the nodes to the desired value:

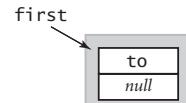
```
first.item = "to";
second.item = "be";
third.item = "or";
```

and set the `next` instance variables to build the linked list:

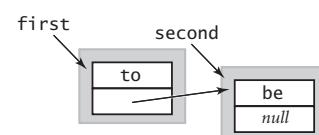
```
first.next = second;
second.next = third;
```

As a result, `first` is a reference to the first node in a three-node linked list, `second` is a reference to the second node, and `third` is a reference to the last node. The code in the accompanying diagram does these same assignment statements, but in a different order.

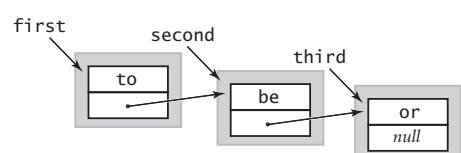
```
Node first = new Node();
first.item = "to";
```



```
Node second = new Node();
second.item = "be";
first.next = second;
```



```
Node third = new Node();
third.item = "or";
second.next = third;
```



Linking together a linked list

A linked list represents a sequence of items. In the example just considered, `first` represents the sequence of items `to`, `be`, and `or`. Alternatively, we can use an array to represent a sequence of items. For example, we could use

```
String[] items = { "to", "be", "or" };
```

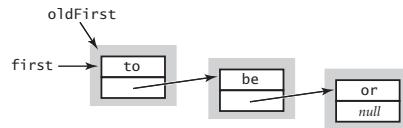
to represent the same sequence of items. The difference is that it is easier to insert items into the sequence and to remove items from the sequence with linked lists. Next, we consider code to accomplish these two tasks.

Suppose that you want to *insert* a new node into a linked list. The easiest place to do so is at the beginning of the list. For example, to insert the string `not` at the beginning of a given linked list whose first node is `first`, we save `first` in a temporary variable `oldFirst`, assign to `first` a new `Node`, and assign its `item` field to `not` and its `next` field to `oldFirst`.

Now, suppose that you want to *remove* the first node from a linked list. This operation is even easier: simply assign to `first` the value `first.next`. Normally, you would retrieve the value of the item (by assigning it to some variable) before doing this assignment, because once you change the value of `first`, you may no longer have any access to the node to which it was referring. Typically, the `Node` object becomes an orphan, and the memory it occupies is eventually reclaimed by the Java memory management system.

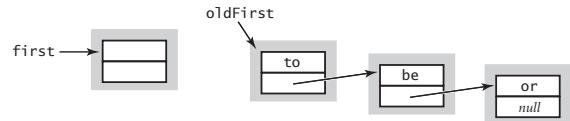
save a link to the first node in the linked list

```
Node oldFirst = first;
```



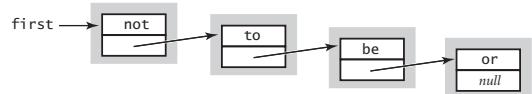
create a new node for the beginning

```
first = new Node();
```



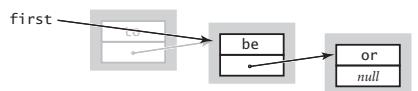
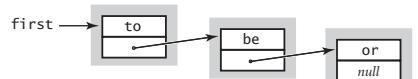
set the instance variables in the new node

```
first.item = "not";
first.next = oldFirst;
```



Inserting a new node at the beginning of a linked list

```
first = first.next;
```



Removing the first node in a linked list

This code for inserting and removing a node from the beginning of a linked list involves just a few assignment statements and thus takes *constant* time (independent of the length of the list). If you hold a reference to a node at an arbitrary position in a list, you can use similar (but more complicated) code to remove the node after it or to insert a node after it, also in constant time. However, we leave those implementations for exercises (see EXERCISE 4.3.24 and EXERCISE 4.3.25) because inserting and removing at the beginning are the only linked-list operations that we need to implement stacks.

Implementing stacks with linked lists. `LinkedStackOfStrings` (PROGRAM 4.3.2) uses a linked list to implement a stack of strings, using little more code than the elementary solution that uses a fixed-length array.

The implementation is based on a *nested* class `Node` like the one we have been using. Java allows us to define and use other classes within class implementations in this natural way. The class is `private` because clients do not need to know any of the details of the linked lists. One characteristic of a `private` nested class is that its instance variables can be directly accessed from within the enclosing class but nowhere else, so there is no need to declare the `Node` instance variables as `public` or `private` (but there is no harm in doing so).

`LinkedStackOfStrings` itself has just one instance variable: a reference to the linked list that represents the stack. That single link suffices to directly access the item at the top of the stack and indirectly access the rest of the items in the stack for `push()` and `pop()`. Again, *be sure that you understand this implementation*—it is the prototype for several implementations using linked structures that we will be examining later in this chapter. Using the abstract visual list representation to trace the code is the best way to proceed.

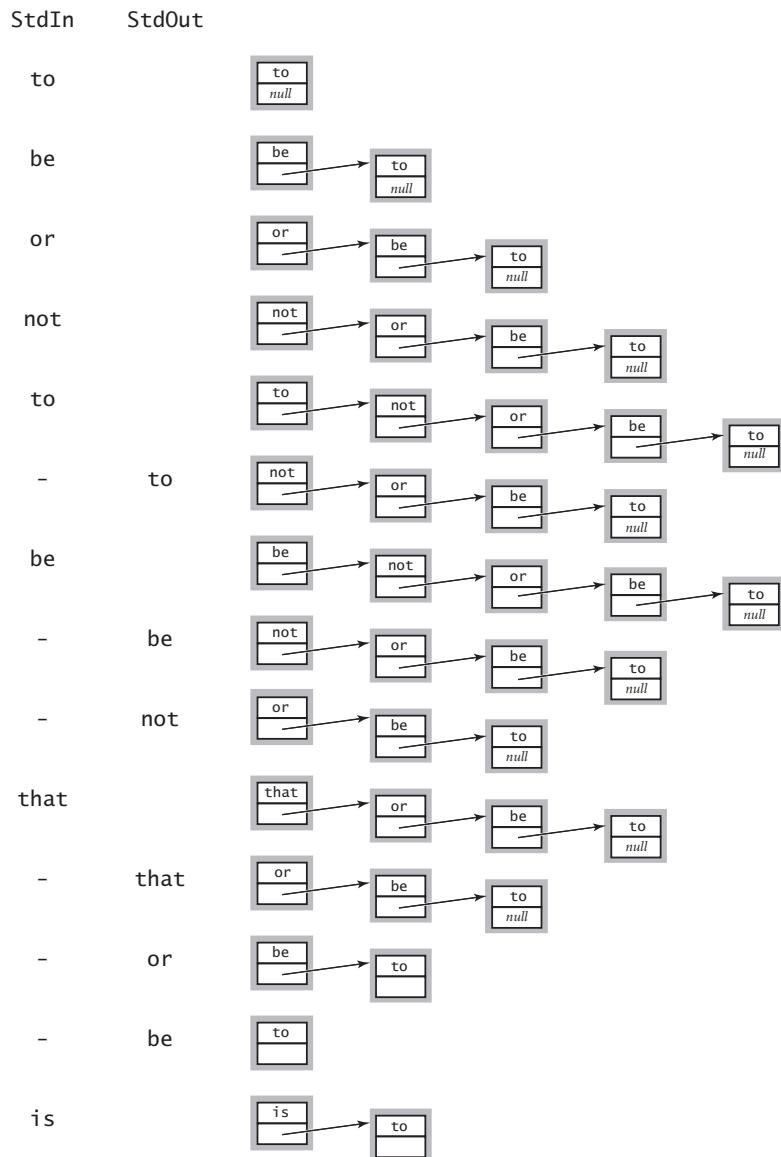
Linked-list traversal. One of the most common operations we perform on collections is to *iterate* over the items in the collection. For example, we might wish to implement the `toString()` method that is inherent in every Java API to facilitate debugging our stack code with traces. For `ArrayStackOfStrings`, this implementation is familiar.

Program 4.3.2 Stack of strings (linked list)

```
public class LinkedStackOfStrings
{
    private Node first;                                first | first node on list
    private class Node
    {
        private String item;                            item | stack item
        private Node next;                            next | next node on list
    }
    public boolean isEmpty()
    {   return (first == null); }
    public void push(String item)
    {   // Insert a new node at the beginning of the list.
        Node oldFirst = first;
        first = new Node();
        first.item = item;
        first.next = oldFirst;
    }
    public String pop()
    {   // Remove the first node from the list and return item.
        String item = first.item;
        first = first.next;
        return item;
    }
    public static void main(String[] args)
    {
        LinkedStackOfStrings stack = new LinkedStackOfStrings();
        // See Program 4.3.1 for the test client.
    }
}
```

This stack implementation uses a private nested class `Node` as the basis for representing the stack as a linked list of `Node` objects. The instance variable `first` refers to the first (most recently inserted) `Node` in the linked list. The `next` instance variable in each `Node` refers to the successor `Node` (the value of `next` in the final node is `null`). No explicit constructors are needed, because Java initializes the instance variables to `null`.

```
% java LinkedStackOfStrings < tobe.txt
to be not that or be
```



Trace of `LinkedStackOfStrings` test client

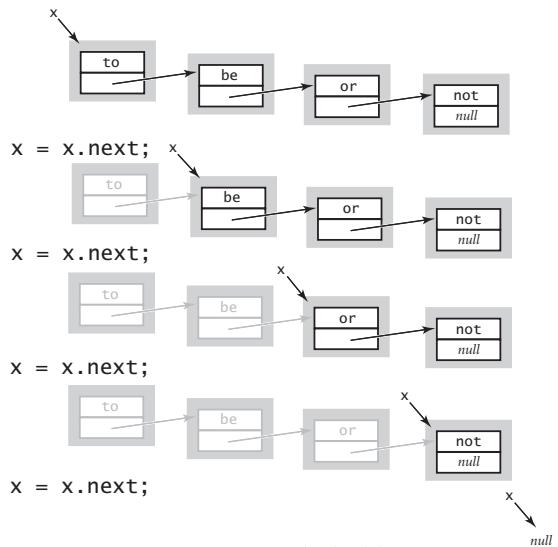
```
public String toString()
{
    String s = "";
    for (int i = 0; i < n; i++)
        s += a[i] + " ";
    return s;
}
```

This solution is intended for use only when n is small—it takes quadratic time because each string concatenation takes linear time.

Our focus now is just on the process of examining every item. There is a corresponding idiom for visiting the items in a linked list: We initialize a loop-index variable x that references the first Node of the linked list. Then, we find the value of the item associated with x by accessing $x.item$, and then update x to refer to the next Node in the linked list, assigning to it the value of $x.next$ and repeating this process until x is `null` (which indicates that we have reached the end of the linked list). This process is known as *traversing* the linked list, and is succinctly expressed in this implementation of `toString()` for `LinkedStackOfStrings`:

```
public String toString()
{
    String s = "";
    for (Node x = first; x != null; x = x.next)
        s += x.item + " ";
    return s;
}
```

When you program with linked lists, this idiom will become as familiar to you as the idiom for iterating over the items in an array. At the end of this section, we consider the concept of an *iterator*, which allows us to write client code to iterate over the items in a collection without having to program at this level of detail.



Traversing a linked list

WITH A LINKED-LIST IMPLEMENTATION we can write client programs that use large numbers of stacks without having to worry much about memory usage. The same principle applies to collections of any sort, so linked lists are widely used in programming. Indeed, typical implementations of the Java memory management system are based on maintaining linked lists corresponding to blocks of memory of various sizes. Before the widespread use of high-level languages like Java, the details of memory management and programming with linked lists were critical parts of any programmer's arsenal. In modern systems, most of these details are encapsulated in the implementations of a few data types like the pushdown stack, including the queue, the symbol table, and the set, which we will consider later in this chapter. If you take a course in algorithms and data structures, you will learn several others and gain expertise in creating and debugging programs that manipulate linked lists. Otherwise, you can focus your attention on understanding the role played by linked lists in implementing these fundamental data types. For stacks, they are significant because they allow us to implement the `push()` and `pop()` methods in constant time while using only a small constant factor of extra memory (for the links).

Resizing arrays Next, we consider an alternative approach to accommodating arbitrary growth and shrinkage in a data structure that is an attractive alternative to linked lists. As with linked lists, we introduce it now because the approach is not difficult to understand in the context of a stack implementation and because it is important to know when addressing the challenges of implementing data types that are more complicated than stacks.

The idea is to modify the array implementation (PROGRAM 4.3.1) to dynamically adjust the length of the array `items[]` so that it is sufficiently large to hold all of the items but not so large as to waste an excessive amount of memory. Achieving these goals turns out to be remarkably easy, and we do so in `ResizingArrayList` (PROGRAM 4.3.3).

First, in `push()`, we check whether the array is too small. In particular, we check whether there is room for the new item in the array by checking whether the stack size `n` is equal to the array length `items.length`. If there is room, we simply insert the new item with the code `items[n++] = item` as before; if not, we *double* the length of the array by creating a new array of twice the length, copying the stack items to the new array, and resetting the `items[]` instance variable to reference the new array.

Program 4.3.3 Stack of strings (resizing array)

```

public class ResizingArrayStackOfStrings
{
    private String[] items = new String[1];           items[] | stack items
    private int n = 0;                                n | number of items on stack

    public boolean isEmpty()
    {   return (n == 0); }

    private void resize(int capacity)
    {   // Move stack to a new array of given capacity.
        String[] temp = new String[capacity];
        for (int i = 0; i < n; i++)
            temp[i] = items[i];
        items = temp;
    }

    public void push(String item)
    {   // Insert item onto stack.
        if (n == items.length) resize(2*items.length);
        items[n++] = item;
    }

    public String pop()
    {   // Remove and return most recently inserted item.
        String item = items[--n];
        items[n] = null; // Avoid loitering (see text).
        if (n > 0 && n == items.length/4) resize(items.length/2);
        return item;
    }

    public static void main(String[] args)
    {
        // See Program 4.3.1 for the test client.
    }
}

```

This implementation achieves the objective of supporting stacks of any size without excessively wasting memory. It doubles the length of the array when full and halves the length of the array to keep it always at least one-quarter full. On average, all operations take constant time (see the text).

```
% java ResizingArrayStackOfStrings < tobe.txt
to be not that or be
```

StdIn	StdOut	n	items. length	items[]							
				0	1	2	3	4	5	6	7
		0	1		null						
to		1	1	to							
be		2	2	to	be						
or		3	4	to	be	or	null				
not		4	4	to	be	or	not				
to		5	8	to	be	or	not	to	null	null	null
-	to	4	8	to	be	or	not	null	null	null	null
be		5	8	to	be	or	not	be	null	null	null
-	be	4	8	to	be	or	not	null	null	null	null
-	not	3	8	to	be	or	null	null	null	null	null
that		4	8	to	be	or	that	null	null	null	null
-	that	3	8	to	be	or	null	null	null	null	null
-	or	2	4	to	be	null	null				
-	be	1	2	to	null						
is		2	2	to	is						

Trace of ResizingArrayListOfStrings test client

Similarly, in `pop()`, we begin by checking whether the array is too large, and we *halve* its length if that is the case. If you think a bit about the situation, you will see that an appropriate test is whether the stack size is less than *one-fourth* the array length. Then, after the array is halved, it will be about half full and can accommodate a substantial number of `push()` and `pop()` operations before having to change the length of the array again. This characteristic is important: for example, if we were to use a policy of halving the array when the stack size is one-half the array length, then the resulting array would be full, which would mean it would be doubled for a `push()`, leading to the possibility of an expensive cycle of doubling and halving.

Amortized analysis. This doubling-and-halving strategy is a judicious tradeoff between wasting space (by setting the length of the array to be too big and leaving empty slots) and wasting time (by reorganizing the array after each insertion). The specific strategy in `ResizingArrayListOfStrings` guarantees that the stack never overflows and never becomes less than one-quarter full (unless the stack is empty, in which case the array length is 1). If you are mathematically inclined, you might enjoy proving this fact with mathematical induction (see EXERCISE 4.3.18). More important, we can prove that the cost of doubling and halving is always ab-

sorbed (to within a constant factor) in the cost of other stack operations. Again, we leave the details to an exercise for the mathematically inclined, but the idea is simple: when `push()` doubles the length of the array to n , it starts with $n/2$ items in the stack, so the length of the array cannot double again until the client has made at least $n/2$ additional calls to `push()` (more if there are some intervening calls to `pop()`). If we *average* the cost of the `push()` operation that causes the doubling with the cost of those $n/2$ `push()` operations, we get a constant. In other words, in `ResizingArrayStackOfStrings`, *the total cost of all of the stack operations divided by the number of operations is bounded by a constant*. This statement is not quite as strong as saying that each operation takes constant time, but it has the same implications in many applications (for example, when our primary interest is in the application's total running time). This kind of analysis is known as *amortized analysis*—the resizing array data structure is a prototypical example of its value.

Orphaned items. Java's garbage collection policy is to reclaim the memory associated with any objects that can no longer be accessed. In the `pop()` implementation in our initial implementation `ArrayListStackOfStrings`, the reference to the popped item remains in the array. The item is an *orphan*—we will never use it again within the class, either because the stack will shrink or because it will be overwritten with another reference if the stack grows—but the Java garbage collector has no way to know this. Even when the client is done with the item, the reference in the array may keep it alive. This condition (holding a reference to an item that is no longer needed) is known as *loitering*, which is not the same as a *memory leak* (where even the memory management system has no reference to the item). In this case, loitering is easy to avoid. The implementation of `pop()` in `ResizingArrayStackOfStrings` sets the array element corresponding to the popped item to `null`, thus overwriting the unused reference and making it possible for the system to reclaim the memory associated with the popped item when the client is finished with it.

With a RESIZING-ARRAY IMPLEMENTATION (as with a linked-list implementation), we can write client programs that use stacks without having to worry much about memory usage. Again, the same principle applies to collections of any sort. For some data types that are more complicated than stacks, resizing arrays are preferred over linked lists because of their ability to access any element in the array in constant time (through indexing), which is critical for implementing certain operations (see, for example, `RandomQueue` in EXERCISE 4.3.37). As with linked lists, it is best to keep resizing-array code local to the implementation of fundamental data types and not worry about using it in client code.

Parameterized data types We have developed stack implementations that allow us to build stacks of one particular type (`String`). But when developing client programs, we need implementations for collections of other types of data, not necessarily strings. A commercial transaction processing system might need to maintain collections of customers, accounts, merchants, and transactions; a university course scheduling system might need to maintain collections of classes, students, and rooms; a portable music player might need to maintain collections of songs, artists, and albums; a scientific program might need to maintain collections of `double` or `int` values. In any program that you write, you should not be surprised to find yourself maintaining collections for any type of data that you might create. How would you do so? After considering two simple approaches (and their shortcomings) that use the Java language constructs we have discussed so far, we introduce a more advanced construct that can help us properly address this problem.

Create a new collection data type for each item data type. We could create classes `StackOfInts`, `StackOfCustomers`, `StackOfStudents`, and so forth to supplement `StackOfStrings`. This approach requires that we duplicate the code for each type of data, which violates a basic precept of software engineering that we should reuse (not copy) code whenever possible. You need a different class for every type of data that you want to put on a stack, so maintaining your code becomes a nightmare: whenever you want or need to make a change, you have to do so in each version of the code. Still, this approach is widely used because many programming languages (including early versions of Java) do not provide any better way to solve the problem. Breaking this barrier is the sign of a sophisticated programmer and programming environment. Can we implement stacks of strings, stacks of integers, and stacks of data of any type whatsoever with just one class?

Use collections of Objects. We could develop a stack whose items are all of type `Object`. Using inheritance, we can legally push an object of any type (if we want to push an object of type `Apple`, we can do so because `Apple` is a subclass of `Object`, as are all other classes). When we pop the stack, we must cast it back to the appropriate type (everything on the stack is an `Object`, but our code is processing objects of type `Apple`). In summary, if we create a class `StackOfObjects` by changing `String` to `Object` everywhere in one of our `*StackOfStrings` implementations, we can write code like

```
StackOfObjects stack = new StackOfObjects();
Apple a = new Apple();
stack.push(a);
...
a = (Apple) (stack.pop());
```

thus achieving our goal of having a single class that creates and manipulates stacks of objects of any type. However, this approach is undesirable because it exposes clients to subtle bugs in client programs that cannot be detected at compile time. For example, there is nothing to stop a programmer from putting different types of objects on the same stack, as in the following example:

```
ObjectStack stack = new ObjectStack();
Apple a = new Apple();
Orange b = new Orange();
stack.push(a);
stack.push(b);
a = (Apple) (stack.pop()); // Throws a ClassCastException.
b = (Orange) (stack.pop());
```

Type casting in this way amounts to *assuming* that clients will cast objects popped from the stack to the proper type, avoiding the protection provided by Java's type system. One reason that programmers use the type system is to protect against errors that arise from such implicit assumptions. The code cannot be type-checked at compile time: there might be an incorrect cast that occurs in a complex piece of code that could escape detection until some particular run-time circumstance arises. We seek to avoid such errors because they can appear long after an implementation is delivered to a client, who would have no way to fix them.

Java generics. A specific mechanism in Java known as *generic types* solves precisely the problem that we are facing. With generics, we can build collections of objects of a type to be specified by client code. The primary benefit of doing so is the ability to discover type-mismatch errors at compile time (when the software is being developed) instead of at run time (when the software is being used by a client). Conceptually, generics are a bit confusing at first (their impact on the programming language is sufficiently deep that they were not included in early versions of Java), but our use of them in the present context involves just a small bit of extra Java syntax and is easy to understand. We name the generic class `Stack` and choose the generic

Program 4.3.4 Generic stack

```

public class Stack<Item>
{
    private Node first;                                first | first node on list

    private class Node
    {
        private Item item;                            item | stack item
        private Node next;                           next | next node on list
    }

    public boolean isEmpty()
    {   return (first == null); }

    public void push(Item item)
    { // Insert item onto stack.
        Node oldFirst = first;
        first = new Node();
        first.item = item;
        first.next = oldFirst;
    }

    public Item pop()
    { // Remove and return most recently inserted item.
        Item item = first.item;
        first = first.next;
        return item;
    }

    public static void main(String[] args)
    {
        Stack<String> stack = new Stack<String>();
        // See Program 4.3.1 for the test client.
    }
}

```

This code is almost identical to PROGRAM 4.3.2, but is worth repeating because it demonstrates how easy it is to use generics to allow clients to make collections of any type of data. The keyword `Item` in this code is a type parameter, a placeholder for an actual type name provided by clients.

```
% java Stack < tobe.txt
to be not that or be
```

name `Item` for the type of the objects in the stack (you can use any name). The code of `Stack` (PROGRAM 4.3.4) is identical to the code of `LinkedStackOfStrings` (we drop the `Linked` modifier because we have a good implementation for clients who do not care about the representation), except that we replace every occurrence of `String` with `Item` and declare the class with the following first line of code:

```
public class Stack<Item>
```

The name `Item` is a *type parameter*, a symbolic placeholder for some actual type to be specified by the client. You can read `Stack<Item>` as *stack of items*, which is precisely what we want. When implementing `Stack`, we do not know the actual type of `Item`, but a client can use our stack for any type of data, including one defined long after we develop our implementation. The client code specifies the *type argument* `Apple` when the stack is created:

```
Stack<Apple> stack = new Stack<Apple>();
Apple a = new Apple();
...
stack.push(a);
```

If you try to push an object of the wrong type on the stack, like this:

```
Stack<Apple> stack = new Stack<Apple>();
Apple a = new Apple();
Orange b = new Orange();
stack.push(a);
stack.push(b);      // Compile-time error.
```

you will get a *compile-time* error:

```
push(Apple) in Stack<Apple> cannot be applied to (Orange)
```

Furthermore, in our `Stack` implementation, Java can use the type parameter `Item` to check for type-mismatch errors—even though no actual type is yet known, variables of type `Item` must be assigned values of type `Item`, and so forth.

Autoboxing. One slight difficulty with generic code like PROGRAM 4.3.4 is that the type parameter stands for a *reference type*. How can we use the code for primitive types such as `int` and `double`? The Java language feature known as *autoboxing* and *unboxing* enables us to reuse generic code with primitive types as well. Java supplies built-in object types known as *wrapper types*, one for each of the primitive types: `Boolean`, `Byte`, `Character`, `Double`, `Float`, `Integer`, `Long`, and `Short`.

respond to `boolean`, `byte`, `char`, `double`, `float`, `int`, `long`, and `short`, respectively. Java automatically converts between these reference types and the corresponding primitive types—in assignment statements, method arguments, and arithmetic/logic expressions—so that we can write code like the following:

```
Stack<Integer> stack = new Stack<Integer>();
stack.push(17);           // Autoboxing (int -> Integer).
int a = stack.pop();     // Unboxing (Integer -> int).
```

In this example, Java automatically casts (autoboxes) the primitive value 17 to be of type `Integer` when we pass it to the `push()` method. The `pop()` method returns an `Integer`, which Java casts (unboxes) to an `int` value before assigning it to the variable `a`. This feature is convenient for writing code, but involves a significant amount of processing behind the scenes that can affect performance. In some performance-critical applications, a class like `StackOfInts` might be necessary, after all.

GENERICs PROVIDE THE SOLUTION THAT WE seek: they enable code reuse and at the same time provide type safety. Carefully studying `Stack` (PROGRAM 4.3.4) and being sure that you understand each line of code will pay dividends in the future, as the ability to parameterize data types is an important high-level programming technique that is well supported in Java. You do not have to be an expert to take advantage of this powerful feature.

Stack applications Pushdown stacks play an essential role in computation. If you study operating systems, programming languages, and other advanced topics in computer science, you will learn that not only are stacks used explicitly in many applications, but they also still serve as the basis for executing programs written in many high-level languages, including Java and Python.

Arithmetic expressions. Some of the first programs that we considered in CHAPTER 1 involved computing the value of arithmetic expressions like this one:

$$(1 + ((2 + 3) * (4 * 5)))$$

If you multiply 4 by 5, add 3 to 2, multiply the result, and then add 1, you get the value 101. But how does Java do this calculation? Without going into the details of how Java is built, we can address the essential ideas just by writing a Java program that can take a string as input (the expression) and produce the number represent-

ed by the expression as output. For simplicity, we begin with the following explicit recursive definition: an *arithmetic expression* is either a number or a left parenthesis followed by an arithmetic expression followed by an operator followed by another arithmetic expression followed by a right parenthesis. For simplicity, this definition is for *fully parenthesized* arithmetic expressions, which specifies precisely which operators apply to which operands—you are a bit more familiar with expressions like $1 + 2 * 3$, in which we use precedence rules instead of parentheses. The same basic mechanisms that we consider can handle precedence rules, but we avoid that complication. For specificity, we support the familiar binary operators $*$, $+$, and $-$, as well as a square-root operator `sqrt` that takes only one argument. We could easily allow more operators to support a larger class of familiar mathematical expressions, including division, trigonometric functions, and exponential functions. Our focus is on understanding how to interpret the string of parentheses, operators, and numbers to enable performing in the proper order the low-level arithmetic operations that are available on any computer.

Arithmetic expression evaluation. Precisely how can we convert an arithmetic expression—a string of characters—to the value that it represents? A remarkably simple algorithm that was developed by Edsger Dijkstra in the 1960s uses two pushdown stacks (one for operands and one for operators) to do this job. An expression consists of parentheses, operators, and operands (numbers). Proceeding from left to right and taking these entities one at a time, we manipulate the stacks according to four possible cases, as follows:

- Push *operands* onto the operand stack.
- Push *operators* onto the operator stack.
- Ignore *left* parentheses.
- On encountering a *right* parenthesis, pop an operator, pop the requisite number of operands, and push onto the operand stack the result of applying that operator to those operands.

After the final right parenthesis has been processed, there is one value on the stack, which is the value of the expression. Dijkstra's two-stack algorithm may seem mysterious at first, but it is easy to convince yourself that it computes the proper value: anytime the algorithm encounters a subexpression consisting of two operands separated by an operator, all surrounded by parentheses, it leaves the result of performing that operation on those operands on the operand stack. The result is the same as if that value had appeared in the input instead of the subexpression, so we can think of replacing the subexpression by the value to get an expression that

Program 4.3.5 Expression evaluation

```

public class Evaluate
{
    public static void main(String[] args)
    {
        Stack<String> ops = new Stack<String>();
        Stack<Double> values = new Stack<Double>();
        while (!StdIn.isEmpty())
        { // Read token, push if operator.
            String token = StdIn.readString();
            if (token.equals("(")) ;
            else if (token.equals("+")) ops.push(token);
            else if (token.equals("-")) ops.push(token);
            else if (token.equals("*")) ops.push(token);
            else if (token.equals("sqrt")) ops.push(token);
            else if (token.equals(")"))
            { // Pop, evaluate, and push result if token is ")".
                String op = ops.pop();
                double v = values.pop();
                if (op.equals("+")) v = values.pop() + v;
                else if (op.equals("-")) v = values.pop() - v;
                else if (op.equals("*")) v = values.pop() * v;
                else if (op.equals("sqrt")) v = Math.sqrt(v);
                values.push(v);
            } // Token not operator or paren: push double value.
            else values.push(Double.parseDouble(token));
        }
        StdOut.println(values.pop());
    }
}

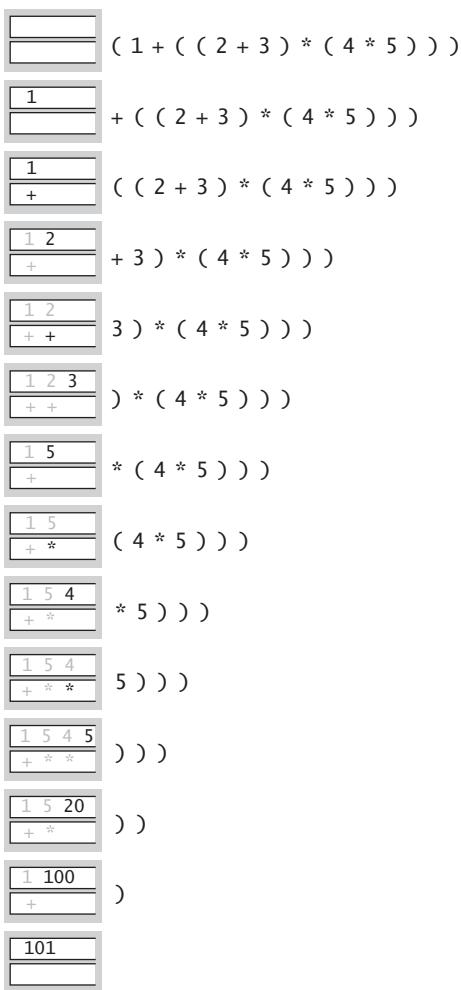
```

ops	operator stack
values	operand stack
token	current token
v	current value

This *Stack* client reads a fully parenthesized numeric expression from standard input, uses Dijkstra's two-stack algorithm to evaluate it, and prints the resulting number to standard output. It illustrates an essential computational process: interpreting a string as a program and executing that program to compute the desired result. Executing a Java program is nothing other than a more complicated version of this same process.

```
% java Evaluate
( 1 + ( ( 2 + 3 ) * ( 4 * 5 ) ) )
101.0
```

```
% java Evaluate
( ( 1 + sqrt ( 5.0 ) ) * 0.5 )
1.618033988749895
```



Trace of expression evaluation (PROGRAM 4.3.5)

would yield the same result. We can apply this argument again and again until we get a single value. For example, the algorithm computes the same value of all of these expressions:

(1 + ((2 + 3) * (4 * 5)))
 (1 + (5 * (4 * 5)))
 (1 + (5 * 20))
 (1 + 100)
 101

Evaluate (PROGRAM 4.3.5) is an implementation of this algorithm. This code is a simple example of an *interpreter*: a program that executes a program (in this case, an arithmetic expression) one step or line at a time. A *compiler* is a program that translates a program from a higher-level language to a lower-level language that can do the job. A compiler's conversion is a more complicated process than the step-by-step conversion used by an interpreter, but it is based on the same underlying mechanism. The Java compiler translates code written in the Java programming language into Java bytecode. Originally, Java was based on using an interpreter. Now, however, Java includes a compiler that converts arithmetic expressions (and, more generally, Java programs) into lower-level code for the *Java virtual machine*, an imaginary machine that is easy to simulate on an actual computer.

Stack-based programming languages. Remarkably, Dijkstra's two-stack algorithm also computes the same value as in our example for this expression:

$$(1((23+)(45**)*)+)$$

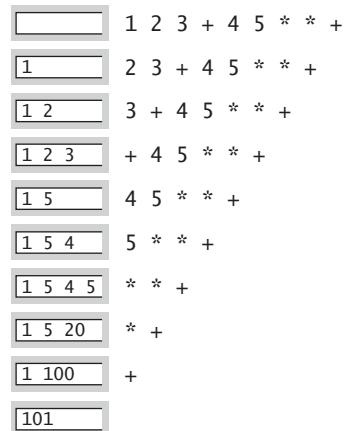
In other words, we can put each operator *after* its two operands instead of *between* them. In such an expression, each right parenthesis immediately follows an operator so we can ignore *both* kinds of parentheses, writing the expressions as follows:

$$1\ 2\ 3\ +\ 4\ 5\ * \ * \ +$$

This notation is known as *reverse Polish notation*, or *postfix*. To evaluate a postfix expression, we use only *one* stack (see EXERCISE 4.3.15). Proceeding from left to right, taking these entities one at a time, we manipulate the stack according to just two possible cases, as follows:

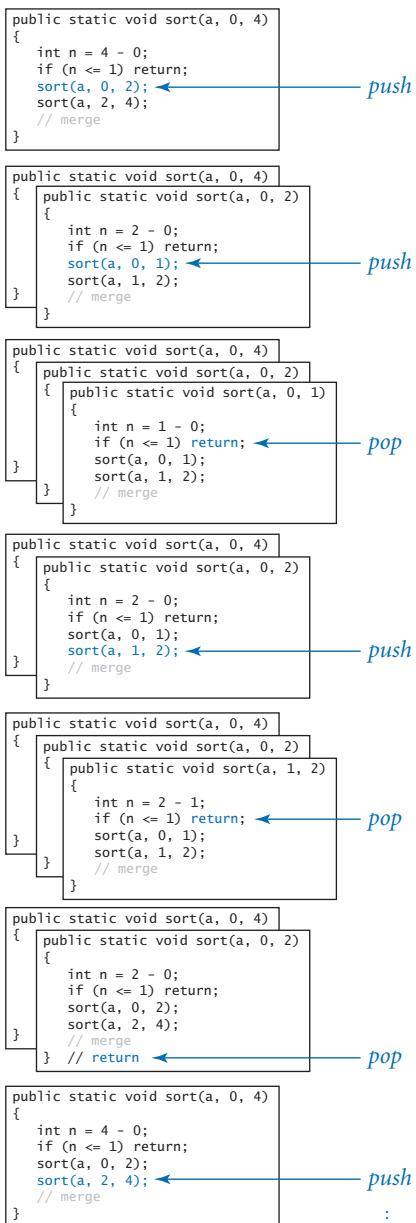
- Push operands onto the stack.
- On encountering an operator, pop the requisite number of operands and push onto the stack the result of applying the operator to those operands.

Again, this process leaves one value on the stack, which is the value of the expression. This representation is so simple that some programming languages, such as Forth (a scientific programming language) and PostScript (a page description language that is used on most printers) use explicit stacks as primary flow-control structures. For example, the string $1\ 2\ 3\ +\ 4\ 5\ * \ * \ +$ is a legal program in both Forth and PostScript that leaves the value 101 on the execution stack. Aficionados of these and similar stack-based programming languages prefer them because they are simpler for many types of computation. Indeed, the Java virtual machine itself is stack based.



Trace of postfix evaluation

Function-call abstraction. Most programs use stacks implicitly because they support a natural way to implement function calls, as follows: at any point during the execution of a function, define its *state* to be the values of all of its variables *and* a pointer to the next instruction to be executed. One of the fundamental characteristics of computing environments is that every computation is fully determined by its state (and the value of its inputs). In particular, the system can suspend a computation by saving away its state, then restart it by restoring the state. If you take a



Using a stack to support function calls

course about operating systems, you will learn the details of this process, because it is critical to much of the behavior of computers that we take for granted (for example, switching from one application to another is simply a matter of saving and restoring state). Now, the natural way to implement the function-call abstraction is to use a stack. To *call* a function, *push* the state on a stack. To *return* from a function call, *pop* the state from the stack to restore all variables to their values before the function call, substitute the function return value (if there is one) in the expression containing the function call (if there is one), and resume execution at the next instruction to be executed (whose location was saved as part of the state of the computation). This mechanism works whenever functions call one another, even recursively. Indeed, if you think about the process carefully, you will see that it is essentially the *same* process that we just examined in detail for expression evaluation. A program is a sophisticated expression.

THE PUSHDOWN STACK IS A FUNDAMENTAL computational abstraction. Stacks have been used for expression evaluation, implementing the function-call abstraction, and other basic tasks since the earliest days of computing. We will examine another (tree traversal) in SECTION 4.4. Stacks are used explicitly and extensively in many areas of computer science, including algorithm design, operating systems, compilers, and numerous other computational applications.

FIFO queues A *FIFO queue* (or just a *queue*) is a collection that is based on the first-in first-out policy.

The policy of doing tasks in the same order that they arrive is one that we encounter frequently in everyday life, from people waiting in line at a theater, to cars waiting in line at a toll booth, to tasks waiting to be serviced by an application on your computer.

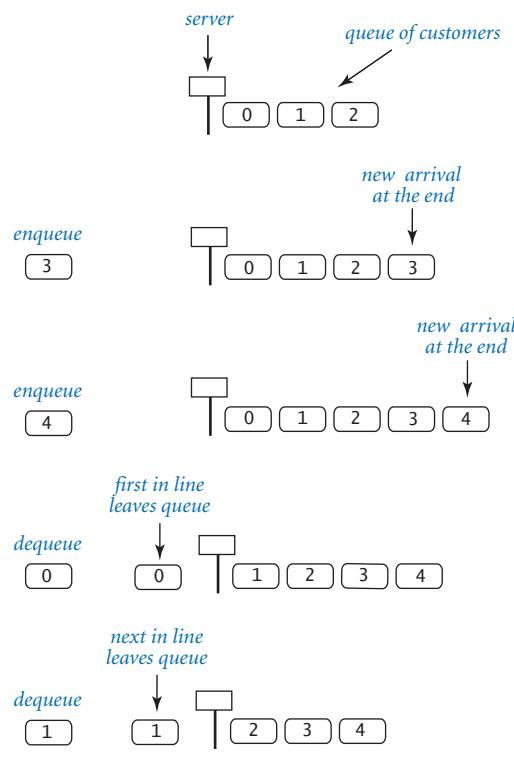
One bedrock principle of any service policy is the perception of fairness. The first idea that comes to mind when most people think about fairness is that whoever has been waiting the longest should be served first. That is precisely the FIFO discipline, so queues play a central role in numerous applications. Queues are a natural model for so many everyday phenomena, and their properties were studied in detail even before the advent of computers.

As usual, we begin by articulating an API. Again by tradition, we name the queue insert operation *enqueue* and the remove operation *dequeue*, as indicated in the following API:

```
public class Queue<Item>
{
    Queue()
    boolean isEmpty()
    void enqueue(Item item)
    Item dequeue()
    int size()
}
```

create an empty queue
is the queue empty?
insert an item into the queue
return and remove the item that was inserted least recently
number of items in the queue

API for a generic FIFO queue



A typical FIFO queue

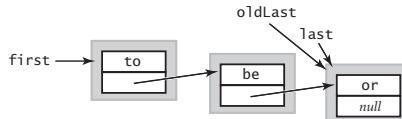
As specified in this API, we will use generics in our implementations, so that we can write client programs that safely build and use queues of any reference type. We include a `size()` method, even though we did not have such a method for stacks because queue clients often do need to be aware of the number of items in the queue, whereas most stack clients do not (see PROGRAM 4.3.8 and EXERCISE 4.3.11).

Applying our knowledge from stacks, we can use linked lists or resizing arrays to develop implementations where the operations take constant time and the memory associated with the queue grows and shrinks with the number of items in the queue. As with stacks, each of these implementations represents a classic programming exercise. You may wish to think about how you might achieve these goals in an implementation before reading further.

Linked-list implementation. To implement a queue with a linked list, we keep the items in order of their arrival (the reverse of the order that we used in Stack). The implementation of `dequeue()` is the same as the `pop()` implementation in Stack (save the item in the first linked-list node, remove that node from the queue, and return the saved item). Implementing `enqueue()`, however, is a bit more challenging: how do we add a node to the *end* of a linked list? To do so, we need a link to the last node in the linked list, because that node's link has to be changed to reference a new node containing the item to be inserted. In Stack, the only instance variable is a reference to the *first* node in the linked list; with only that information, our only recourse is to traverse all the nodes in the linked list to get to the end. That solution is inefficient for long linked lists. A reasonable alternative is to maintain a second instance variable that always references the *last* node in the linked list. Adding an extra instance variable that needs to be maintained is not something that should be taken lightly, particularly in linked-list code, because every method that modifies the list needs code to check whether

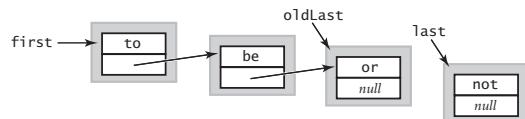
save a link to the last node

```
Node oldLast = last;
```



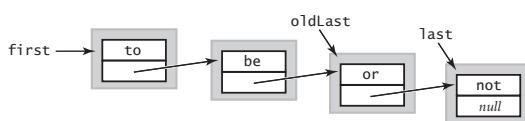
create a new node for the end

```
Node last = new Node();
last.item = "not";
```



link the new node to the end of the list

```
oldLast.next = last;
```



Inserting a new node at the end of a linked list

Program 4.3.6 Generic FIFO queue (linked list)

```

public class Queue<Item>
{
    private Node first;
    private Node last;

    private class Node
    {
        private Item item;
        private Node next;
    }

    public boolean isEmpty()
    {   return (first == null);  }

    public void enqueue(Item item)
    {   // Insert a new node at the end of the list.
        Node oldLast = last;
        last = new Node();
        last.item = item;
        last.next = null;
        if (isEmpty()) first = last;
        else           oldLast.next = last;
    }

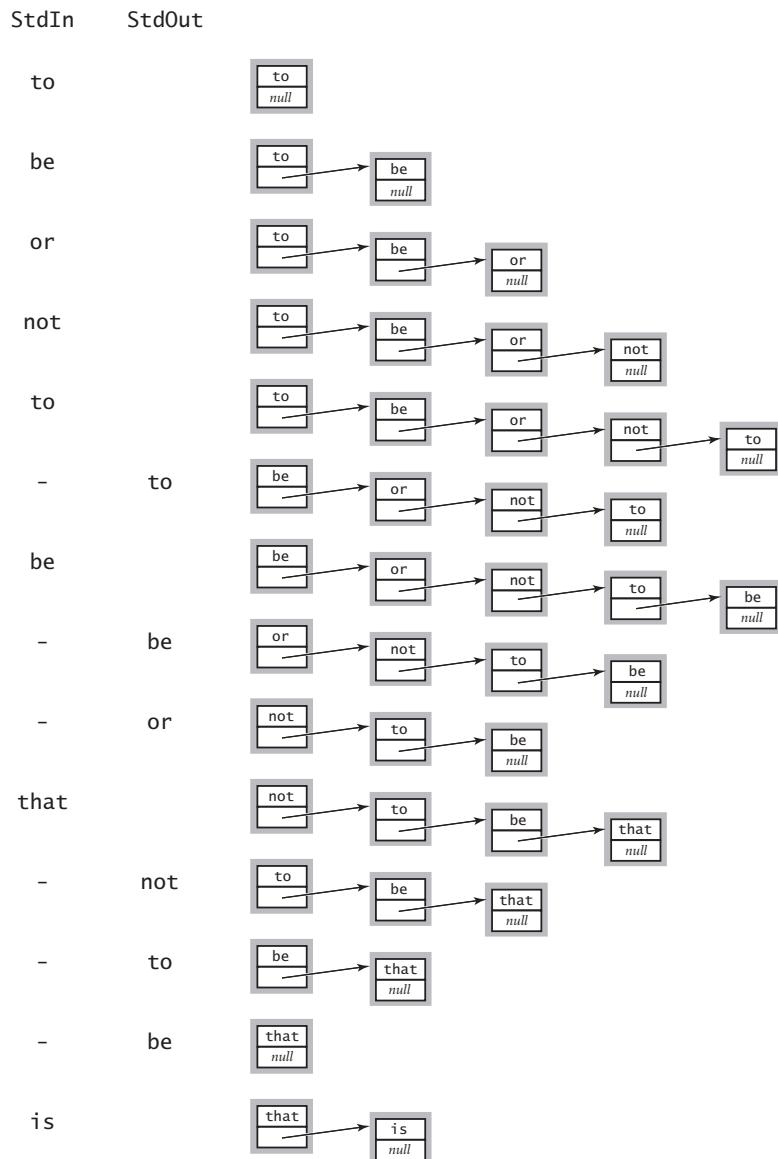
    public Item dequeue()
    {   // Remove the first node from the list and return item.
        Item item = first.item;
        first = first.next;
        if (isEmpty()) last = null;
        return item;
    }

    public static void main(String[] args)
    {   // Test client is similar to Program 4.3.2.
        Queue<String> queue = new Queue<String>();
    }
}

```

This implementation is very similar to our linked-list stack implementation (PROGRAM 4.3.2): `dequeue()` is almost identical to `pop()`, but `enqueue()` links the new node onto the end of the list, not the beginning as in `push()`. To do so, it maintains an instance variable `last` that references the last node in the list. The `size()` method is left for an exercise (see EXERCISE 4.3.11).

```
% java Queue < tobe.txt
to be or not to be
```



Trace of Queue test client (see PROGRAM 4.3.6)

that variable needs to be modified (and to make the necessary modifications). For example, removing the first node in the linked list might involve changing the reference to the last node, since when there is only one node remaining, it is both the first one and the last one! (Details like this make linked-list code notoriously difficult to debug.) `Queue` (PROGRAM 4.3.6) is a linked-list implementation of our FIFO queue API that has the same performance properties as `Stack`: all of the methods are constant time, and memory usage is proportional to the queue size.

[Array implementations](#). It is also possible to develop FIFO queue implementations that use arrays having the same performance characteristics as those that we developed for stacks in `ArrayListOfStrings` (PROGRAM 4.3.1) and `ResizingArrayListOfStrings` (PROGRAM 4.3.3). These implementations are worthy programming exercises that you are encouraged to pursue further (see EXERCISE 4.3.19).

[Random queues](#). Even though they are widely applicable, there is nothing sacred about the FIFO and LIFO policies. It makes perfect sense to consider other rules for removing items. One of the most important to consider is a data type where `dequeue()` removes and returns a *random* item (sampling without replacement), and we have a method `sample()` that returns a random item without removing it from the queue (sampling with replacement). We use the name `RandomQueue` to refer to this data type (see EXERCISE 4.3.37).

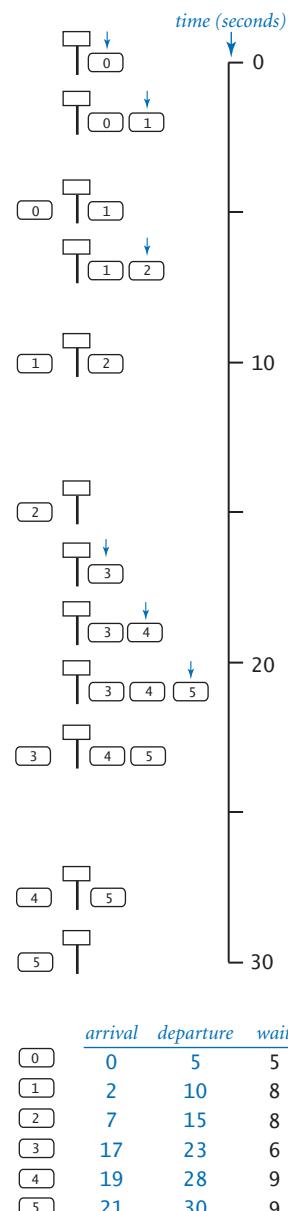
THE STACK, QUEUE, AND RANDOM QUEUE APIs are essentially *identical*—they differ only in the choice of class and method names (which are chosen arbitrarily). The true differences among these data types are in the semantics of the *remove* operation—which item is to be removed? The differences between stacks and queues are in the English-language descriptions of what they do. These differences are akin to the differences between `Math.sin(x)` and `Math.log(x)`, but we might want to articulate them with a formal description of stacks and queues (in the same way as we have mathematical descriptions of the sine and logarithm functions). But precisely describing what we mean by first-in first-out or last-in first-out or random-out is not so simple. For starters, which language would you use for such a description? English? Java? Mathematical logic? The problem of describing how a program behaves is known as the *specification problem*, and it leads immediately to deep issues in computer science. One reason for our emphasis on clear and concise code is that the code itself can serve as the specification for simple data types such as stacks, queues, and random queues.

Queue applications In the past century, FIFO queues proved to be accurate and useful models in a broad variety of applications, ranging from manufacturing processes to telephone networks to traffic simulations. A field of mathematics known as *queueing theory* has been used with great success to help understand and control complex systems of all kinds. FIFO queues also play an important role in computing. You often encounter queues when you use your computer: a queue might hold songs on a playlist, documents to be printed, or events in a game.

Perhaps the ultimate queue application is the Internet itself, which is based on huge numbers of messages moving through huge numbers of queues that have all sorts of different properties and are interconnected in all sorts of complicated ways. Understanding and controlling such a complex system involves solid implementations of the queue abstraction, application of mathematical results of queueing theory, and simulation studies involving both. We consider next a classic example to give a flavor of this process.

M/M/1 queue. One of the most important queueing models is known as an *M/M/1 queue*, which has been shown to accurately model many real-world situations, such as a single line of cars entering a toll booth or patients entering an emergency room. The *M* stands for *Markovian* or *memoryless* and indicates that both arrivals and services are *Poisson processes*: both the interarrival times and the service times obey an *exponential distribution* (see EXERCISE 2.2.8). The *1* indicates that there is one server. An *M/M/1* queue is parameterized by its *arrival rate* λ (for example, the number of cars per minute arriving at the toll booth) and its *service rate* μ (for example, the number of cars per minute that can pass through the toll booth) and is characterized by three properties:

- There is one server—a FIFO queue.
- Interarrival times to the queue obey an exponential distribution with rate λ per minute.
- Service times from a nonempty queue obey an exponential distribution with rate μ per minute.



An *M/M/1* queue

The average time between arrivals is $1/\lambda$ minutes and the average time between services (when the queue is nonempty) is $1/\mu$ minutes. So, the queue will grow without bound unless $\mu > \lambda$; otherwise, customers enter and leave the queue in an interesting dynamic process.

Analysis. In practical applications, people are interested in the effect of the parameters λ and μ on various properties of the queue. If you are a customer, you may want to know the expected amount of time you will spend in the system; if you are designing the system, you might want to know how many customers are likely to be in the system, or something more complicated, such as the likelihood that the queue size will exceed a given maximum size. For simple models, probability theory yields formulas expressing these quantities as functions of λ and μ . For $M/M/1$ queues, it is known that

- The average number of customers in the system L is $\lambda / (\mu - \lambda)$.
- The average time a customer spends in the system W is $1 / (\mu - \lambda)$.

For example, if the cars arrive at a rate of $\lambda = 10$ per minute and the service rate is $\mu = 15$ per minute, then the average number of cars in the system will be 2 and the average time that a customer spends in the system will be $1/5$ minutes or 12 seconds. These formulas confirm that the wait time (and queue length) grows without bound as λ approaches μ . They also obey a general rule known as *Little's law*: the average number of customers in the system is λ times the average time a customer spends in the system ($L = \lambda W$) for many types of queues.

Simulation. `MM1Queue` (PROGRAM 4.3.7) is a Queue client that you can use to validate these sorts of mathematical results. It is a simple example of an *event-based simulation*: we generate *events* that take place at particular times and adjust our data structures accordingly for the events, simulating what happens at the time they occur. In an $M/M/1$ queue, there are two kinds of events: we have either a customer *arrival* or a customer *service*. In turn, we maintain two variables:

- `nextService` is the time of the next service.
- `nextArrival` is the time of the next arrival.

To simulate an arrival event, we enqueue `nextArrival` (the time of arrival); to simulate a service, we dequeue the arrival time of the next customer in the queue, compute that customer's waiting time `wait` (which is the time that the service is completed minus the time that the customer entered the queue), and add the wait time to a histogram (see PROGRAM 3.2.3). The shape that results after a large number

Program 4.3.7 M/M/1 queue simulation

```

public class MM1Queue
{
    public static void main(String[] args)
    {
        double lambda = Double.parseDouble(args[0]);
        double mu     = Double.parseDouble(args[1]);
        Histogram hist = new Histogram(60 + 1);
        Queue<Double> queue = new Queue<Double>();
        double nextArrival = StdRandom.exp(lambda);
        double nextService = nextArrival + StdRandom.exp(mu);
        StdDraw.enableDoubleBuffering();

        while (true)
        { // Simulate arrivals before next service.
            while (nextArrival < nextService)
            {
                queue.enqueue(nextArrival);
                nextArrival += StdRandom.exp(lambda);
            }

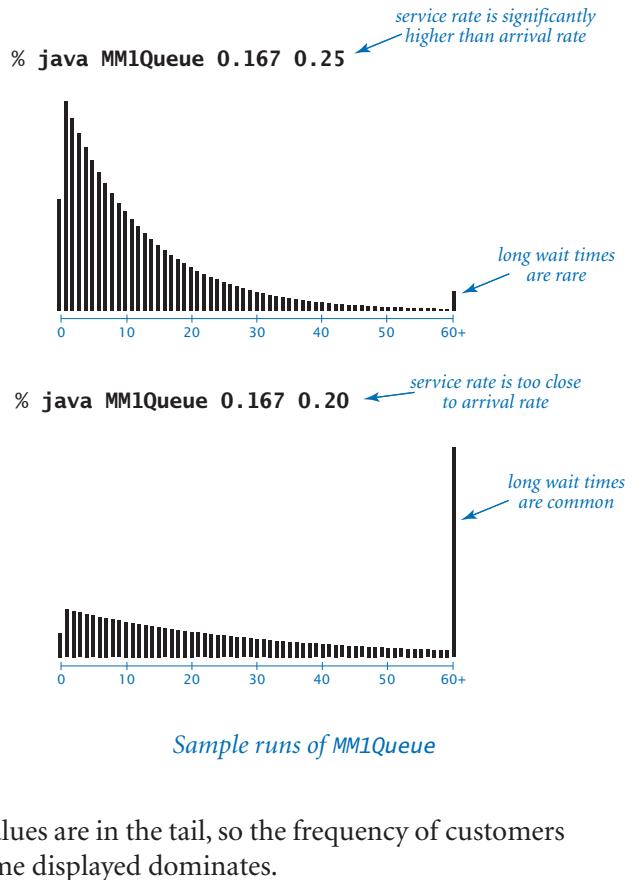
            // Simulate next service.
            double wait = nextService - queue.dequeue();
            hist.addDataPoint(Math.min(60, (int) Math.round(wait)));
            StdDraw.clear();
            hist.draw();
            StdDraw.show();
            StdDraw.wait(20);
            if (queue.isEmpty())
                nextService = nextArrival + StdRandom.exp(mu);
            else
                nextService = nextService + StdRandom.exp(mu);
        }
    }
}

```

<code>lambda</code>	<i>arrival rate</i>
<code>mu</code>	<i>service rate</i>
<code>hist</code>	<i>histogram</i>
<code>queue</code>	<i>M/M/1 queue</i>
<code>wait</code>	<i>time on queue</i>

This simulation of an M/M/1 queue keeps track of time with two variables `nextArrival` and `nextService` and a single Queue of double values to calculate wait times. The value of each item on the queue is the (simulated) time it entered the queue. The waiting times are plotted using Histogram (PROGRAM 3.2.3).

of trials is characteristic of the $M/M/1$ queueing system. From a practical point of view, one of the most important characteristics of the process, which you can discover for yourself by running `MM1Queue` for various values of the parameters λ and μ , is that the average time a customer spends in the system (and the average number of customers in the system) can increase dramatically when the service rate approaches the arrival rate. When the service rate is high, the histogram has a visible tail where the frequency of customers having a given wait time decreases to a negligible duration as the wait time increases. But when the service rate is too close to the arrival rate, the tail of the histogram stretches to the point that most values are in the tail, so the frequency of customers having at least the highest wait time displayed dominates.



AS IN MANY OTHER APPLICATIONS THAT we have studied, the use of simulation to validate a well-understood mathematical model is a starting point for studying more complex situations. In practical applications of queues, we may have multiple queues, multiple servers, multistage servers, limits on queue length, and many other restrictions. Moreover, the distributions of interarrival and service times may not be possible to characterize mathematically. In such situations, we may have no recourse but to use simulations. It is quite common for a system designer to build a computational model of a queuing system (such as `MM1Queue`) and to use it to adjust design parameters (such as the service rate) to properly respond to the outside environment (such as the arrival rate).

Iterable collections As mentioned earlier in this section, one of the fundamental operations on arrays and linked lists is the `for` loop idiom that we use to process each element. This common programming paradigm need not be limited to low-level data structures such as arrays and linked lists. For any collection, the ability to process all of its items (perhaps in some specified order) is a valuable capability. The client's requirement is just to process each of the items in some way, or to *iterate* over the items in the collection. This paradigm is so important that it has achieved first-class status in Java and many other modern programming languages (meaning that the language itself has specific mechanisms to support it, not just the libraries). With it, we can write clear and compact code that is free from dependence on the details of a collection's implementation.

To introduce the concept, we start with a snippet of client code that prints all of the items in a collection of strings, one per line:

```
Stack<String> collection = new Stack<String>();
...
for (String s : collection)
    StdOut.println(s);
...
```

This construct is known as the *foreach* statement: you can read the `for` statement as *for each string s in the collection, print s*. This client code does not need to know anything about the representation or the implementation of the collection; it just wants to process each of the items in the collection. The same foreach loop would work with a Queue of strings or with any other *iterable* collection of strings.

We could hardly imagine code that is clearer and more compact. However, implementing a collection that supports iteration in this way requires some extra work, which we now consider in detail. First, the foreach construct is shorthand for a `while` construct. For example, the foreach statement given earlier is equivalent to the following `while` construct:

```
Iterator<String> iterator = collection.iterator();
while (iterator.hasNext())
{
    String s = iterator.next();
    StdOut.println(s);
}
```

This code exposes the three necessary parts that we need to implement in any iterable collection:

- The collection must implement an `iterator()` method that returns an `Iterator` object.
- The `Iterator` class must include two methods: `hasNext()` (which returns a boolean value) and `next()` (which returns an item from the collection).

In Java, we use the interface inheritance mechanism to express the idea that a class implements a specific set of methods (see SECTION 3.3). For iterable collections, the necessary interfaces are predefined in Java.

To make a class iterable, the first step is to add the phrase `implements Iterable<Item>` to its declaration, matching the interface

```
public interface Iterable<Item>
{
    Iterator<Item> iterator();
}
```

(which is defined in `java.lang.Iterable`), and to add a method to the class that returns an `Iterator<Item>`. Iterators are generic; we can use them to provide clients with the ability to iterate over a specified type of objects (and only objects of that specified type).

What is an iterator? An object from a class that implements the methods `hasNext()` and `next()`, as in the following interface (which is defined in `java.util.Iterator`):

```
public interface Iterator<Item>
{
    boolean hasNext();
    Item next();
    void remove();
}
```

Although the interface requires a `remove()` method, we always use an empty method for `remove()` in this book, because interleaving iteration with operations that modify the data structure is best avoided.

As illustrated in the following two examples, implementing an iterator class is often straightforward for array and linked-list representations of collections.

Making iterable a class that uses an array. As a first example, we will consider all of the steps needed to make `ArrayStackOfStrings` (PROGRAM 4.3.1) iterable. First, change the class declaration to

```
public class ArrayStackOfStrings implements Iterable<String>
```

In other words, we are promising to provide an `iterator()` method so that a client can use a `foreach` statement to iterate over the strings in the stack. The `iterator()` method itself is simple:

```
public Iterator<String> iterator()
{ return new ReverseArrayIterator(); }
```

It just returns an object from a private nested class that implements the `Iterator` interface (which provides `hasNext()`, `next()`, and `remove()` methods):

```
private class ReverseArrayIterator implements Iterator<String>
{
    private int i = n-1;
    public boolean hasNext()
    { return i >= 0; }
    public String next()
    { return items[i--]; }
    public void remove()
    { }
}
```

Note that the nested class `ReverseArrayIterator` can access the instance variables of the enclosing class, in this case `items[]` and `n` (this ability is the main reason we use nested classes for iterators). One crucial detail remains: we have to include

```
import java.util.Iterator;
```

at the beginning of `ArrayStackOfStrings`. Now, since a client can use the `foreach` statement with `ArrayStackOfStrings` objects, it can iterate over the items without being aware of the underlying array representation. This arrangement is of critical importance for implementations of fundamental data types for collections. For example, it frees us to switch to a totally different representation *without having to change any client code*. More important, taking the client's point of view, it allows clients to use iteration *without having to know any details of the implementation*.

Making iterable a class that uses a linked list. The same specific steps (with different code) are effective to make Queue (PROGRAM 4.3.6) iterable, even though it is generic. First, we change the class declaration to

```
public class Queue<Item> implements Iterable<Item>
```

In other words, we are promising to provide an `iterator()` method so that a client can use a `foreach` statement to iterate over the items in the queue, whatever their type. Again, the `iterator()` method itself is simple:

```
public Iterator<Item> iterator()
{   return new ListIterator(); }
```

As before, we have a private nested class that implements the `Iterator` interface:

```
private class ListIterator implements Iterator<Item>
{
    Node current = first;
    public boolean hasNext()
    {   return current != null; }
    public Item next()
    {
        Item item = current.item;
        current = current.next;
        return item;
    }
    public void remove()
    { }
}
```

Again, a client can build a queue of items of any type and then iterate over the items without any awareness of the underlying linked-list representation:

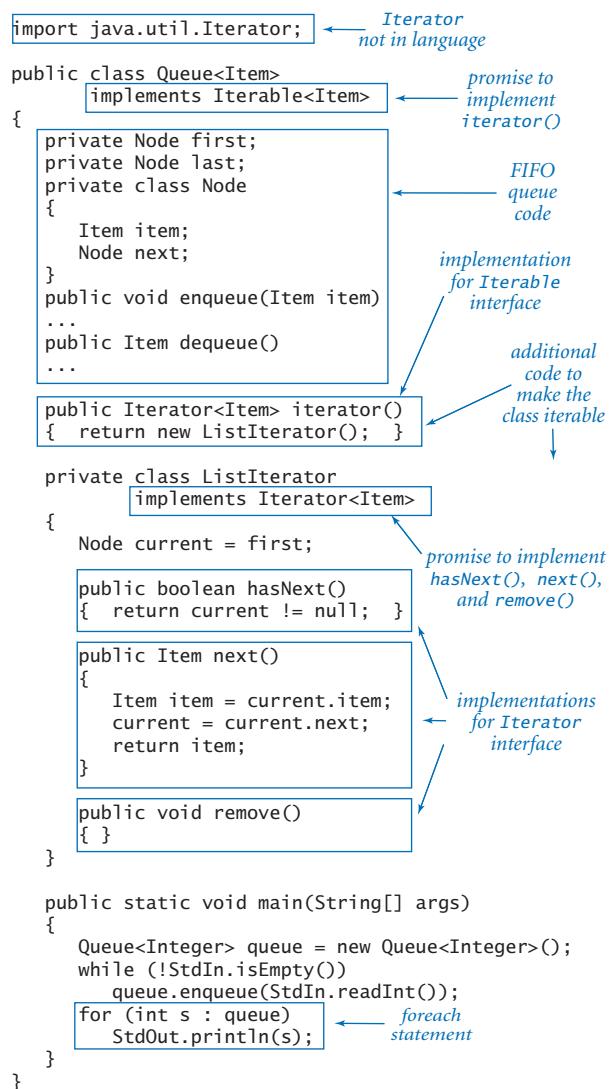
```
Queue<String> queue = new Queue<String>();
...
for (String s : queue)
    StdOut.println(s);
```

This client code is a clearer expression of the computation and therefore easier to write and maintain than code based on the low-level representation.

Our stack iterator iterates over the items in LIFO order and our queue iterator iterates over them in FIFO order, even though there is no requirement to do so: we could return the items in any order whatsoever. However, when developing iterators, it is wise to follow a simple rule: if a data type specification implies a natural iteration order, *use it*.

Iterable implementations may seem a bit complicated to you at first, but they are worth the effort. You will not find yourself implementing them very often, but when you do, you will enjoy the benefits of clear and correct client code and code reuse. Moreover, as with any programming construct, once you begin to enjoy these benefits, you will find yourself taking advantage of them often.

Making a class iterable certainly changes its API, but to avoid overly complicated API tables, we simply use the adjective *iterable* to indicate that we have included the appropriate code to a class, as described in this section, and to indicate that you can use the `foreach` statement in client code. From this point forward we will use in client programs the iterable (and generic) `Stack`, `Queue`, and `RandomQueue` data types described here.



Resource allocation Next, we examine an application that illustrates the data structures and Java language features that we have been considering. A *resource-sharing* system involves a large number of loosely cooperating *servers* that want to share resources. Each server agrees to maintain its own queue of items for sharing, and a central authority distributes the items to the servers (and informs users where they may be found). For example, the items might be songs, photos, or videos to be shared by a large number of users. To fix ideas, we will think in terms of millions of items and thousands of servers.

We will consider the kind of program that the central authority might use to distribute the items, ignoring the dynamics of deleting items from the systems, adding and deleting servers, and so forth.

If we use a *round-robin* policy, cycling through the servers to make the assignments, we get a balanced allocation, but it is rarely possible for a distributor to have such complete control over the situation: for example, there might be a large number of independent distributors, so none of them could have up-to-date information about the servers. Accordingly, such systems often use a *random* policy, where the assignments are based on random choice. An even better policy is to choose a random *sample* of servers and assign a new item to the server that has the fewest items. For small queues, differences among these policies is immaterial, but in a system with millions of items on thousands of servers, the differences can be quite significant, since each server has a fixed amount of resources to devote to this process. Indeed, similar systems are used in Internet hardware, where some queues might be implemented in special-purpose hardware, so queue length translates directly to extra equipment cost. But how big a sample should we take?

`LoadBalance` (PROGRAM 4.3.8) is a simulation of the sampling policy, which we can use to study this question. This program makes good use of the data structures (queues and random queues) and high-level constructs (generics and iterators) that we have been considering to provide an easily understood program that we can use for experimentation. The simulation maintains a random queue of queues and builds the computation around an inner loop where each new request for service goes on the smallest of a sample of queues, using the `sample()` method from `RandomQueue` (EXERCISE 4.3.36) to randomly sample queues. The surprising end result is that samples of size 2 lead to near-perfect balancing, so there is no point in taking larger samples.

Program 4.3.8 Load balancing simulation

```

public class LoadBalance
{
    public static void main(String[] args)
    { // Assign n items to m servers, using
      // shortest-in-a-sample policy.
      int m = Integer.parseInt(args[0]);
      int n = Integer.parseInt(args[1]);
      int size = Integer.parseInt(args[2]);
      // Create server queues.
      RandomQueue<Queue<Integer>> servers;
      servers = new RandomQueue<Queue<Integer>>();
      for (int i = 0; i < m; i++)
          servers.enqueue(new Queue<Integer>());
      for (int j = 0; j < n; j++)
      { // Assign an item to a server.
        Queue<Integer> min = servers.sample();
        for (int k = 1; k < size; k++)
        { // Pick a random server, update if new min.
          Queue<Integer> queue = servers.sample();
          if (queue.size() < min.size()) min = queue;
        } // min is the shortest server queue.
        min.enqueue(j);
      }
      int i = 0;
      double[] lengths = new double[m];
      for (Queue<Integer> queue : servers)
          lengths[i++] = queue.size();
      StdDraw.setScale(0, 2.0 * n / m);
      StdStats.plotBars(lengths);
    }
}

```

m	number of servers
n	number of items
size	sample size
servers	queues
min	shortest in sample
queue	current server

This generic Queue and RandomQueue client simulates the process of assigning n items to a set of m servers. Requests are put on the shortest of a sample of size queues chosen at random.

% java LoadBalance 50 500 1



% java LoadBalance 50 500 2



WE HAVE CONSIDERED IN DETAIL the issues surrounding the space and time usage of basic implementations of the stack and queue APIs not just because these data types are important and useful, but also because you are likely to encounter the very same issues in the context of your own data-type implementations.

Should you use a pushdown stack, a FIFO queue, or a random queue when developing a client that maintains collections of data? The answer to this question depends on a high-level analysis of the client to determine which of the LIFO, FIFO, or random disciplines is appropriate.

Should you use an array, a linked list, or a resizing array to structure your data? The answer to this question depends on low-level analysis of performance characteristics. With an *array*, the advantage is that you can access any element in constant time; the disadvantage is that you need to know the maximum length in advance. A *linked list* has the advantage that there is no limit on the number of items that it can hold; the disadvantage is that you cannot access an arbitrary element in constant time. A *resizing array* combines the advantages of arrays and linked lists (you can access any element in constant time but do not need to know the maximum length in advance) but has the (slight) disadvantage that the running time is constant on an amortized basis. Each data structure is appropriate in certain situations; you are likely to encounter all three in most programming environments. For example, the Java class `java.util.ArrayList` uses a resizing array, and the Java class `java.util.LinkedList` uses a linked list.

The powerful high-level constructs and new language features that we have considered in this section (generics and iterators) are not to be taken for granted. They are sophisticated programming language features that did not come into widespread use in mainstream languages until the turn of the century, and they are still used mostly by professional programmers. Nevertheless, their use is skyrocketing because they are well supported in Java and C++, because newer languages such as Python and Ruby embrace them, and because many people are learning to appreciate the value of using them in client code. By now, you know that learning to use a new language feature is not so different from learning to ride a bicycle or implement `HelloWorld`: it seems completely mysterious until you have done it for the first time, but quickly becomes second nature. Learning to use generics and iterators will be well worth your time.

Q&A

Q. When do I use new with Node?

A. As with any other class, you should use new only when you want to create a new Node object (a new node in the linked list). You should not use new to create a new reference to an existing Node object. For example, the code

```
Node oldFirst = new Node();
oldFirst = first;
```

creates a new Node object, then immediately loses track of the only reference to it. This code does not result in an error, but it is untidy to create orphans for no reason.

Q. Why declare Node as a nested class? Why private?

A. By declaring the nested class Node to be private, methods in the enclosing class can refer to Node objects, but access from other classes is prohibited. *Note for experts:* A nested class that is not static is known as an *inner* class, so technically our Node classes are inner classes, though the ones that are not generic could be static.

Q. When I type javac LinkedStackOfStrings.java to run PROGRAM 4.3.2 and similar programs, I find a file `LinkedStackOfStrings$Node.class` in addition to `LinkedStackOfStrings.class`. What is the purpose of that file?

A. That file is for the nested class Node. Java's naming convention is to use \$ to separate the name of the outer class from the nested class.

Q. Should a client be allowed to insert null items into a stack or queue?

A. This question arises frequently when implementing collections in Java. Our implementation (and Java's stack and queue libraries) do permit the insertion of null values.

Q. Are there Java libraries for stacks and queues?

A. Yes and no. Java has a built-in library called `java.util.Stack`, but you should avoid using it when you want a stack. It has several additional operations that are not normally associated with a stack, such as getting the *i*th item. It also allows

adding an item to the bottom of the stack (instead of the top), so it can implement a queue! Although having such extra operations might appear to be a bonus, it is actually a curse. We use data types not because they provide every available operation, but rather because they allow us to precisely specify the operations we need. The prime benefit of doing so is that the system can prevent us from performing operations that we do not actually want. The `java.util.Stack` API is an example of a *wide interface*, which we generally strive to avoid.

Q. I want to use an array representation for a generic stack, but code like the following will not compile. What is the problem?

```
private Item[] item = new Item[capacity];
```

A. Good try. Unfortunately, Java does not permit the creation of arrays of generics. Experts are still vigorously debating this decision. As usual, complaining too loudly about a programming language feature puts you on the slippery slope toward becoming a language designer. There is a way out, using a cast. You can write:

```
private Item[] item = (Item[]) new Object[capacity];
```

Q. Why do I need to import `java.util.Iterator` but not `java.lang.Iterable`?

A. For historical reasons, the interface `Iterator` is part of the package `java.util`, which is not imported by default. The interface `Iterable` is relatively new and included as part of the package `java.lang`, which is imported by default.

Q. Can I use a `foreach` statement with arrays?

A. Yes (even though, technically, arrays do not implement the `Iterable` interface). The following code prints the command-line arguments to standard output:

```
public static void main(String[] args)
{
    for (String s : args)
        StdOut.println(s);
}
```

Q. When using generics, what happens if I omit the type argument in either the declaration or the constructor call?

```
Stack<String> stack = new Stack();           // unsafe
Stack      stack = new Stack<String>();        // unsafe
Stack<String> stack = new Stack<String>();    // correct
```

A. The first statement produces a compile-time warning. The second statement produces a compile-time warning if you call `stack.push()` with a `String` argument and a compile-time error if you assign the result of `stack.pop()` to a variable of type `String`. As an alternative to the third statement, you can use the *diamond* operator, which enables Java to infer the type argument to the constructor call from context:

```
Stack<String> stack = new Stack<>(); // diamond operator
```

Q. Why not have a single `Collection` data type that implements methods to add items, remove the most recently inserted item, remove the least recently inserted item, remove a random item, iterate over the items, return the number of items in the collection, and whatever other operations we might desire? Then we could get them all implemented in a single class that could be used by many clients.

A. This is an example of a *wide interface*, which, as we pointed out in SECTION 3.3, is to be avoided. One reason to avoid wide interfaces is that it is difficult to construct implementations that are efficient for *all* operations. A more important reason is that narrow interfaces enforce a certain discipline on your programs, which makes client code much easier to understand. If one client uses `Stack<String>` and another uses `Queue<Customer>`, we have a good idea that the LIFO discipline is important to the first and the FIFO discipline is important to the second. Another approach is to use inheritance to try to encapsulate operations that are common to all collections. However, such implementations are for experts, whereas any programmer can learn to build generic implementations such as `Stack` and `Queue`.

Exercises

4.3.1 Add a method `isFull()` to `ArrayStackOfStrings` (PROGRAM 4.3.1) that returns `true` if the stack size equals the array capacity. Modify `push()` to throw an exception if it is called when the stack is full.

4.3.2 Give the output printed by `java ArrayStackOfStrings 5` for this input:

```
it was - the best - of times - - - it was - the - -
```

4.3.3 Suppose that a client performs an intermixed sequence of *push* and *pop* operations on a pushdown stack. The push operations insert the integers 0 through 9 in order onto the stack; the pop operations print the return values. Which of the following sequence(s) could *not* occur?

- a. 4 3 2 1 0 9 8 7 6 5
- b. 4 6 8 7 5 3 2 9 0 1
- c. 2 5 6 7 4 8 9 3 1 0
- d. 4 3 2 1 0 5 6 7 8 9
- e. 1 2 3 4 5 6 9 8 7 0
- f. 0 4 6 5 3 8 1 7 2 9
- g. 1 4 7 9 8 6 5 3 0 2
- h. 2 1 4 3 6 5 8 7 9 0

4.3.4 Write a filter `Reverse` that reads strings one at a time from standard input and prints them to standard output in reverse order. Use either a stack or a queue.

4.3.5 Write a static method that reads floating-point numbers one at a time from standard input and returns an array containing them, in the same order they appear on standard input. *Hint:* Use either a stack or a queue.

4.3.6 Write a stack client `Parentheses` that reads a string of parentheses, square brackets, and curly braces from standard input and uses a stack to determine whether they are properly balanced. For example, your program should print `true` for `[(){}{{[()()]})}` and `false` for `[()])`.

4.3.7 What does the following code fragment print when n is 50? Give a high-level description of what the code fragment does when presented with a positive integer n .

```
Stack<Integer> stack = new Stack<Integer>();
while (n > 0)
{
    stack.push(n % 2);
    n /= 2;
}
while (!stack.isEmpty())
    StdOut.print(stack.pop());
StdOut.println();
```

Answer: Prints the binary representation of n (110010 when n is 50).

4.3.8 What does the following code fragment do to the queue $queue$?

```
Stack<String> stack = new Stack<String>();
while (!queue.isEmpty())
    stack.push(queue.dequeue());
while (!stack.isEmpty())
    queue.enqueue(stack.pop());
```

4.3.9 Add a method `peek()` to `Stack` (PROGRAM 4.3.4) that returns the most recently inserted item on the stack (without removing it).

4.3.10 Give the contents and length of the array for `ResizingArrayListOfStrings` with this input:

```
it was - the best - of times - - - it was - the - -
```

4.3.11 Add a method `size()` to both `Stack` (PROGRAM 4.3.4) and `Queue` (PROGRAM 4.3.6) that returns the number of items in the collection. *Hint:* Make sure that your method takes constant time by maintaining an instance variable n that you initialize to 0, increment in `push()` and `enqueue()`, decrement in `pop()` and `dequeue()`, and return in `size()`.

4.3.12 Draw a memory-usage diagram in the style of the diagrams in SECTION 4.1 for the three-node example used to introduce linked lists in this section.

4.3.13 Write a program that takes from standard input an expression without left parentheses and prints the equivalent infix expression with the parentheses inserted. For example, given the input

1 + 2) * 3 - 4) * 5 - 6)))

your program should print

((1 + 2) * ((3 - 4) * (5 - 6)))

4.3.14 Write a filter `InfixToPostfix` that converts an arithmetic expression from infix to postfix.

4.3.15 Write a program `EvaluatePostfix` that takes a postfix expression from standard input, evaluates it, and prints the value. (Piping the output of your program from the previous exercise to this program gives equivalent behavior to `Evaluate`, in PROGRAM 4.3.5.)

4.3.16 Suppose that a client performs an intermixed sequence of *enqueue* and *dequeue* operations on a FIFO queue. The enqueue operations insert the integers 0 through 9 in order onto the queue; the dequeue operations print the return values. Which of the following sequence(s) could *not* occur?

- a. 0 1 2 3 4 5 6 7 8 9
- b. 4 6 8 7 5 3 2 9 0 1
- c. 2 5 6 7 4 8 9 3 1 0
- d. 4 3 2 1 0 5 6 7 8 9

4.3.17 Write an iterable `Stack` *client* that has a static method `copy()` that takes a stack of strings as its argument and returns a copy of the stack. See EXERCISE 4.3.48 for an alternative approach.

4.3.18 Write a `Queue` client that takes an integer command-line argument *k* and prints the *k*th from the last string found on standard input.

4.3.19 Develop a data type `ResizingArrayQueueOfStrings` that implements a queue with a fixed-length array in such a way that all operations take constant time. Then, extend your implementation to use a resizing array to remove the length restriction. *Hint:* The challenge is that the items will “crawl across” the array as items are added to and removed from the queue. Use modular arithmetic to maintain the array indices of the items at the front and back of the queue.

<code>StdIn</code>	<code>StdOut</code>	<code>n</code>	<code>lo</code>	<code>hi</code>	<code>items[]</code>							
					0	1	2	3	4	5	6	7
		0	0	0	null							
to		1	0	1	to	null						
be		2	0	2	to	be						
or		3	0	3	to	be	or	null				
not		4	0	4	to	be	or	not				
to		5	0	5	to	be	or	not	to	null	null	null
-	to	4	1	4	null	be	or	not	to	null	null	null
be		5	1	6	null	be	or	not	to	be	null	null
-	be	4	2	6	null	null	or	not	to	be	null	null
-	or	3	3	6	null	null	null	not	to	not	null	null
that		4	3	7	null	null	null	not	to	not	that	null

4.3.20 (For the mathematically inclined.) Prove that the array in `ResizingArrayStackOfStrings` is never less than one-quarter full. Then prove that, for any `ResizingArrayStackOfStrings` client, the total cost of all of the stack operations divided by the number of operations is bounded by a constant.

4.3.21 Modify `MM1Queue` (PROGRAM 4.3.7) to make a program `MD1Queue` that simulates a queue for which the service times are fixed (deterministic) at rate of μ . Verify Little’s law for this model.

4.3.22 Develop a class `StackOfInts` that uses a linked-list representation (but no generics) to implement a stack of integers. Write a client that compares the performance of your implementation with `Stack<Integer>` to determine the performance penalty from autoboxing and unboxing on your system.

Linked-List Exercises

These exercises are intended to give you experience in working with linked lists. The easiest way to work them is to make drawings using the visual representation described in the text.

4.3.23 Suppose `x` is a linked-list Node. What is the effect of the following code fragment?

```
x.next = x.next.next;
```

Answer: Deletes from the list the node immediately following `x`.

4.3.24 Write a method `find()` that takes the first Node in a linked list and a string key as arguments and returns `true` if some node in the list has key as its item field, and `false` otherwise.

4.3.25 Write a method `delete()` that takes the first Node in a linked list and an `int` argument `k` and deletes the `k`th node in the linked list, if it exists.

4.3.26 Suppose that `x` is a linked-list Node. What is the effect of the following code fragment?

```
t.next = x.next;
x.next = t;
```

Answer: Inserts node `t` immediately after node `x`.

4.3.27 Why does the following code fragment not have the same effect as the code fragment in the previous question?

```
x.next = t;
t.next = x.next;
```

Answer: When it comes time to update `t.next`, `x.next` is no longer the original node following `x`, but is instead `t` itself!

4.3.28 Write a method `removeAfter()` that takes a linked-list Node as its argument and removes the node following the given one (and does nothing if either the argument is `null` or the `next` field of the argument is `null`).

4.3.29 Write a method `copy()` that takes a linked-list `Node` as its argument and creates a new linked list with the same sequence of items, without destroying the original linked list.

4.3.30 Write a method `remove()` that takes a linked-list `Node` and a string `key` as its arguments and removes every node in the list whose `item` field is equal to `key`.

4.3.31 Write a method `max()` that takes the first `Node` in a linked list as its argument and returns the value of the maximum item in the list. Assume that all items are positive integers, and return 0 if the linked list is empty.

4.3.32 Develop a recursive solution to the previous question.

4.3.33 Write a method that takes the first `Node` in a linked list as its argument and reverses the list, returning the first `Node` in the result.

4.3.34 Write a recursive method to print the items in a linked list in reverse order. Do not modify any of the links. *Easy*: Use quadratic time, constant extra space. *Also easy*: Use linear time, linear extra space. *Not so easy*: Develop a divide-and-conquer algorithm that takes linearithmic time and uses logarithmic extra space.

4.3.35 Write a recursive method to randomly shuffle the nodes of a linked list by modifying the links. *Easy*: Use quadratic time, constant extra space. *Not so easy*: Develop a divide-and-conquer algorithm that takes linearithmic time and uses logarithmic extra memory. See EXERCISE 1.4.40 for the “merging” step.

Creative Exercises

4.3.36 Deque. A double-ended queue or *deque* (pronounced “deck”) is a collection that is a combination of a stack and a queue. Write a class `Deque` that uses a linked list to implement the following API:

public class Deque<Item>	
Deque()	<i>create an empty deque</i>
boolean isEmpty()	<i>is the deque empty?</i>
void enqueue(Item item)	<i>add item to the end</i>
void push(Item item)	<i>add item to the beginning</i>
Item pop()	<i>remove and return the item at the beginning</i>
Item dequeue()	<i>remove and return the item at the end</i>
<i>API for a generic double-ended queue</i>	

4.3.37 Random queue. A *random queue* is a collection that supports the following API:

public class RandomQueue<Item>	
RandomQueue()	<i>create an empty random queue</i>
boolean isEmpty()	<i>is the random queue empty?</i>
void enqueue(Item item)	<i>add item to the random queue</i>
Item dequeue()	<i>remove and return a random item (sample without replacement)</i>
Item sample()	<i>return a random item, but do not remove (sample with replacement)</i>
<i>API for a generic random queue</i>	

Write a class `RandomQueue` that implements this API. *Hint:* Use a resizing array. To remove an item, swap one at a random position (indexed 0 through $n-1$) with the one at the last position (index $n-1$). Then, remove and return the last item, as in `ResizingArrayList`. Write a client that prints a deck of cards in random order using `RandomQueue<Card>`.

4.3.38 *Random iterator.* Write an iterator for RandomQueue<Item> from the previous exercise that returns the items in random order. Different iterators should return the items in different random orders. *Note:* This exercise is more difficult than it looks.

4.3.39 *Josephus problem.* In the Josephus problem from antiquity, n people are in dire straits and agree to the following strategy to reduce the population. They arrange themselves in a circle (at positions numbered from 0 to $n-1$) and proceed around the circle, eliminating every m th person until only one person is left. Legend has it that Josephus figured out where to sit to avoid being eliminated. Write a Queue client Josephus that takes two integer command-line arguments m and n and prints the order in which people are eliminated (and thus would show Josephus where to sit in the circle).

```
% java Josephus 2 7  
1 3 5 0 4 2 6
```

4.3.40 *Generalized queue.* Implement a class that supports the following API, which generalizes both a queue and a stack by supporting removal of the i th most recently inserted item:

<code>public class GeneralizedQueue<Item></code>	
<hr/>	
<code> GeneralizedQueue()</code>	<i>create an empty generalized queue</i>
<code> boolean isEmpty()</code>	<i>is the generalized queue empty?</i>
<code> void add(Item item)</code>	<i>insert item into the generalized queue</i>
<code> Item remove(int i)</code>	<i>remove and return the ith least recently inserted item</i>
<code> int size()</code>	<i>number of items on the queue</i>
	<i>API for a generic generalized queue</i>

First, develop an implementation that uses a resizing array, and then develop one that uses a linked list. (See EXERCISE 4.4.57 for a more efficient implementation that uses a binary search tree.)

4.3.41 Ring buffer. A *ring buffer* (or *circular queue*) is a FIFO collection that stores a sequence of items, up to a prespecified limit. If you insert an item into a ring buffer that is full, the new item replaces the least recently inserted item. Ring buffers are useful for transferring data between asynchronous processes and for storing log files. When the buffer is empty, the consumer waits until data is deposited; when the buffer is full, the producer waits to deposit data. Develop an API for a ring buffer and an implementation that uses a fixed-length array.

4.3.42 Merging two sorted queues. Given two queues with strings in ascending order, move all of the strings to a third queue so that the third queue ends up with the strings in ascending order.

4.3.43 Nonrecursive mergesort. Given n strings, create n queues, each containing one of the strings. Create a queue of the n queues. Then, repeatedly apply the sorted merging operation from the previous exercise to the first two queues and enqueue the merged queue. Repeat until the queue of queues contains only one queue.

4.3.44 Queue with two stacks. Show how to implement a queue using two stacks. *Hint:* If you push items onto a stack and then pop them all, they appear in reverse order. Repeating the process puts them back in FIFO order.

4.3.45 Move-to-front. Read in a sequence of characters from standard input and maintain the characters in a linked list with no duplicates. When you read in a previously unseen character, insert it at the front of the list. When you read in a duplicate character, delete it from the list and reinsert it at the beginning. This implements the well-known *move-to-front* strategy, which is useful for caching, data compression, and many other applications where items that have been recently accessed are more likely to be reaccessed.

4.3.46 Topological sort. You have to sequence the order of n jobs that are numbered from 0 to $n-1$ on a server. Some of the jobs must complete before others can begin. Write a program `TopologicalSorter` that takes a command-line argument n and a sequence on standard input of ordered pairs of jobs $i \ j$, and then prints a sequence of integers such that for each pair $i \ j$ in the input, job i appears before job j . Use the following algorithm: First, from the input, build, for each job, (i) a

queue of the jobs that must follow it and (ii) its *indegree* (the number of jobs that must come before it). Then, build a queue of all nodes whose indegree is 0 and repeatedly delete any job with a 0 indegree, maintaining all the data structures. This process has many applications. For example, you can use it to model course prerequisites for your major so that you can find a sequence of courses to take so that you can graduate.

4.3.47 *Text-editor buffer.* Develop a data type for a buffer in a text editor that implements the following API:

public class Buffer	
Buffer()	<i>create an empty buffer</i>
void insert(char c)	<i>insert c at the cursor position</i>
char delete()	<i>delete and return the character at the cursor</i>
void left(int k)	<i>move the cursor k positions to the left</i>
void right(int k)	<i>move the cursor k positions to the right</i>
int size()	<i>number of characters in the buffer</i>
<i>API for a text buffer</i>	

Hint: Use two stacks.

4.3.48 *Copy constructor for a stack.* Create a new constructor for the linked-list implementation of Stack so that

```
Stack<Item> t = new Stack<Item>(s);
```

makes t a reference to a new and independent copy of the stack s. You should be able to push and pop from either s or t without influencing the other.

4.3.49 *Copy constructor for a queue.* Create a new constructor so that

```
Queue<Item> r = new Queue<Item>(q);
```

makes r a reference to a new and independent copy of the queue q.

4.3.50 *Quote.* Develop a data type `Quote` that implements the following API for quotations:

<code>public class Quote</code>	
<code>Quote()</code>	<i>create an empty quote</i>
<code>void add(String word)</code>	<i>append word to the end of the quote</i>
<code>void add(int i, String word)</code>	<i>insert word to be at index i</i>
<code>String get(int i)</code>	<i>word at index i</i>
<code>int count()</code>	<i>number of words in the quote</i>
<code>String toString()</code>	<i>the words in the quote</i>
	<i>API for a quote</i>

To do so, define a nested class `Card` that holds one word of the quotation and a link to the next word in the quotation:

```
private class Card
{
    private String word;
    private Card next;
    public Card(String word)
    {
        this.word = word;
        this.next = null;
    }
}
```

4.3.51 *Circular quote.* Repeat the previous exercise but uses a *circular linked list*. In a circular linked list, each node points to its successor, and the last node in the list points to the first node (instead of `null`, as in a standard null-terminated linked list).

4.3.52 *Reverse a linked list (iteratively).* Write a nonrecursive function that takes the first Node in a linked list as an argument and reverses the list, returning the first Node in the result.

4.3.53 *Reverse a linked list (recursively).* Write a recursive function that takes the first Node in a linked list as an argument and reverses the list, returning the first Node in the result.

4.3.54 *Queue simulations.* Study what happens when you modify `MM1Queue` to use a stack instead of a queue. Does Little's law hold? Answer the same question for a random queue. Plot histograms and compare the standard deviations of the waiting times.

4.3.55 *Load-balancing simulations.* Modify `LoadBalance` to print the average queue length and the maximum queue length instead of plotting the histogram, and use it to run simulations for 1 million items on 100,000 queues. Print the average value of the maximum queue length for 100 trials each with sample sizes 1, 2, 3, and 4. Do your experiments validate the conclusion drawn in the text about using a sample of size 2?

4.3.56 *Listing files.* A folder is a list of files and folders. Write a program that takes the name of a folder as a command-line argument and prints all of the files contained in that folder, with the contents of each folder recursively listed (indented) under that folder's name. *Hint:* Use a queue, and see `java.io.File`.

4.4 Symbol Tables

A SYMBOL TABLE IS A DATA type that we use to associate *values* with *keys*. Clients can store (*put*) an entry into the symbol table by specifying a key–value pair and then can retrieve (*get*) the value associated with a specified key from the symbol table. For example, a university might associate information such as a student’s name, home address, and grades (the value) with that student’s Social Security number (the key), so that each student’s record can be accessed by specifying a Social Security number. The same approach might be appropriate for a scientist who needs to organize data, a business that needs to keep track of customer transactions, a web search engine that has to associate keywords with web pages, or in countless other ways.

In this section we consider a basic API for the symbol-table data type. In addition to the *put* and *get* operations that characterize a symbol table, our API includes the abilities to test whether any value has been associated with a given key (*contains*), to *remove* a key (and its associated value), to determine the number of key–value pairs in the symbol table (*size*), and to *iterate* over the keys in the symbol table. We also consider other *order-based* operations on symbol tables that arise naturally in various applications.

As motivation, we consider two prototypical clients—*dictionary lookup* and *indexing*—and briefly discuss the use of each in a number of practical situations. Clients like these are fundamental tools, present in some form in every computing environment, easy to take for granted, and easy to misuse. As with any sophisticated tool, it is important for anyone using a dictionary or an index to understand how it is built to know how to use it effectively. That is the reason that we study symbol tables in detail in this section.

Because of their foundational importance, symbol tables have been heavily used and studied since the early days of computing. We consider two classic implementations. The first uses an operation known as *hashing*, which transforms keys into array indices that we can use to access values. The second is based on a data structure known as the *binary search tree* (BST). Both are remarkably simple solutions that serve as the basis for the industrial-strength symbol-table implementa-

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tions that are found in modern programming environments. The code that we consider for hash tables and binary search trees is only slightly more complicated than the linked-list code that we considered for stacks and queues, but it will introduce you to a new dimension in structuring data that has far-reaching impacts.

API A *symbol table* is a collection of key–value pairs. We use a generic type `Key` for keys and a generic type `Value` for values—every symbol-table entry associates a `Value` with a `Key`. These assumptions lead to the following basic API:

<code>public class *ST<Key, Value></code>	
<code>*ST()</code>	<i>create an empty symbol table</i>
<code>void put(Key key, Value val)</code>	<i>associate val with key</i>
<code>Value get(Key key)</code>	<i>value associated with key</i>
<code>void remove(Key key)</code>	<i>remove key (and its associated value)</i>
<code>boolean contains(Key key)</code>	<i>is there a value associated with key?</i>
<code>int size()</code>	<i>number of key–value pairs</i>
<code>Iterable<Key> keys()</code>	<i>all keys in the symbol table</i>
<i>API for a generic symbol table</i>	

As usual, the asterisk is a placeholder to indicate that multiple implementations might be considered. In this section, we provide two classic implementations: `HashST` and `BST`. (We also describe some elementary implementations briefly in the text.) This API reflects several design decisions, which we now enumerate.

Immutable keys. We assume the keys do not change their values while in the symbol table. The simplest and most commonly used types of keys, `String` and built-in wrapper types such as `Integer` and `Double`, are immutable.

Replace-the-old-value policy. If a key–value pair is inserted into the symbol table that already associates another value with the given key, we adopt the convention that the new value replaces the old one (as when assigning a value to an array element with an assignment statement). The `contains()` method gives the client the flexibility to avoid doing so, if desired.

Not found. The method `get()` returns `null` if no value is associated with the specified key. This choice has two implications, discussed next.

Null keys and null values. Clients are not permitted to use `null` as either a key or a value. This convention enables us to implement `contains()` as follows:

```
public boolean contains(Key key)
{   return get(key) != null; }
```

Remove. We also include in the API a method for removing a key (and its associated value) from the symbol table because many applications require such a method. However, for brevity, we defer implementations of the remove functionality to the exercises or a more advanced course in algorithms and data structures.

Iterating over key–value pairs. The `keys()` method provides clients with a way to iterate over the key–value pairs in the data structure. For simplicity, it returns only the keys; clients can use `get` to get the associated value, if desired. This enables client code like the following:

```
ST<String, Double> st = new ST<String, Double>();
...
for (String key : st.keys())
    StdOut.println(key + " " + st.get(key));
```

Hashable keys. Like many languages, Java includes direct language and system support for symbol-table implementations. In particular, every type of object has an `equals()` method (which we can use to test whether two keys are the same, as defined by the key data type) and a `hashCode()` method (which supports a specific type of symbol-table implementation that we will examine later in this section). For the standard data types that we most commonly use for keys, we can depend upon system implementations of these methods. In contrast, for data types that we create, we have to carefully consider implementations, as discussed in SECTION 3.3. Most programmers simply assume that suitable implementations are in place, but caution is advised when working with nonstandard key types.

Comparable keys. In many applications, the keys may be strings, or other data types of data that have a natural order. In Java, as discussed in SECTION 3.3, we expect such keys to implement the `Comparable` interface. Symbol tables with comparable keys are important for two reasons. First, we can take advantage of key

<code>public class *ST<Key extends Comparable<Key>, Value></code>	
<code>*ST()</code>	<i>create an empty symbol table</i>
<code>void put(Key key, Value val)</code>	<i>associate val with key</i>
<code>Value get(Key key)</code>	<i>value associated with key</i>
<code>void remove(Key key)</code>	<i>remove key (and its associated value)</i>
<code>boolean contains(Key key)</code>	<i>is there a value paired with key?</i>
<code>int size()</code>	<i>number of key-value pairs</i>
<code>Iterable<Key> keys()</code>	<i>all keys in sorted order</i>
<code>Key min()</code>	<i>minimum key</i>
<code>Key max()</code>	<i>maximum key</i>
<code>int rank(Key key)</code>	<i>number of keys less than key</i>
<code>Key select(int k)</code>	<i>kth smallest key in symbol table</i>
<code>Key floor(Key key)</code>	<i>largest key less than or equal to key</i>
<code>Key ceiling(Key key)</code>	<i>smallest key greater than or equal to key</i>
<i>API for an ordered symbol table</i>	

ordering to develop implementations of *put* and *get* that can provide performance guarantees. Second, a whole host of new operations come to mind (and can be supported) with comparable keys. A client might want the smallest key, the largest key, the median key, or to iterate over all of the keys in sorted order. Full coverage of this topic is more appropriate for a book on algorithms and data structures, but in this section you will learn about a simple data structure that can easily support the operations detailed in the partial API shown at the top of this page.

SYMBOL TABLES ARE AMONG THE MOST widely studied data structures in computer science, so the impact of these and many alternative design decisions has been carefully studied, as you will learn if you take later courses in computer science. In this section, our approach is to introduce the most important properties of symbol tables by considering two prototypical client programs, developing efficient implementations of two classic approaches, and studying the performance characteristics of those implementations, to convince you that they can effectively meet the needs of typical clients, even when huge numbers of keys and values need to be processed.

Symbol-table clients Once you gain some experience with the idea, you will find that symbol tables are broadly useful. To convince you of this fact, we start with two prototypical examples, each of which arises in a large number of important and familiar practical applications.

Dictionary lookup. The most basic kind of symbol-table client builds a symbol table with successive *put* operations to support *get* requests. That is, we maintain a collection of data in such a way that we can quickly access the data we need. Most applications also take advantage of the idea that a symbol table is a *dynamic* dictionary, where it is easy to look up information *and* to update the information in the table. The following list of familiar examples illustrates the utility of this approach.

- *Phone book.* When keys are people’s names and values are their phone numbers, a symbol table models a phone book. A very significant difference from a printed phone book is that we can add new names or change existing phone numbers. We could also use the phone number as the key and the name as the value. If you have never done so, try typing your phone number (with area code) into the search field in your browser.
- *Dictionary.* Associating a word with its definition is a familiar concept that gives us the name “dictionary.” For centuries people kept printed dictionaries in their homes and offices so that they could check the definitions and spellings (values) of words (keys). Now, because of good symbol-table implementations, people expect built-in spell checkers and immediate access to word definitions on their computers.
- *Account information.* People who own stock now regularly check the current price on the web. Several services on the web associate a ticker symbol (key) with the current price (value), usually along with a great deal of other information (recall PROGRAM 3.1.8). Commercial applications of this sort abound, including financial institutions associating account information

	<i>key</i>	<i>value</i>
<i>phone book</i>	name	phone number
<i>dictionary</i>	word	definition
<i>account</i>	account number	balance
<i>genomics</i>	codon	amino acid
<i>data</i>	data/time	results
<i>Java compiler</i>	variable name	memory location
<i>file share</i>	song name	machine
<i>Internet DNS</i>	website	IP address

Typical dictionary applications

with a name or account number and educational institutions associating grades with a student name or identification number.

- *Genomics.* Symbol tables play a central role in modern genomics. The simplest example is the use of the letters A, C, T, and G to represent the nucleotides found in the DNA of living organisms. The next simplest is the correspondence between codons (nucleotide triplets) and amino acids (TTA corresponds to leucine, TCT to serine, and so forth), then the correspondence between sequences of amino acids and proteins, and so forth. Researchers in genomics routinely use various types of symbol tables to organize this knowledge.
- *Experimental data.* From astrophysics to zoology, modern scientists are awash in experimental data, and organizing and efficiently accessing this data is vital to understanding what it means. Symbol tables are a critical starting point, and advanced data structures and algorithms that are based on symbol tables are now an important part of scientific research.
- *Programming languages.* One of the earliest uses of symbol tables was to organize information for programming. At first, programs were simply sequences of numbers, but programmers very quickly found that using symbolic names for operations and memory locations (variable names) was far more convenient. Associating the names with the numbers requires a symbol table. As the size of programs grew, the cost of the symbol-table operations became a bottleneck in program development time, which led to the development of data structures and algorithms like the one we consider in this section.
- *Files.* We use symbol tables regularly to organize data on computer systems. Perhaps the most prominent example is the *file system*, where we associate a file name (key) with the location of its contents (value). Your music player uses the same system to associate song titles (keys) with the location of the music itself (value).
- *Internet DNS.* The domain name system (DNS) that is the basis for organizing information on the Internet associates URLs (keys) that humans understand (such as www.princeton.edu or www.wikipedia.org) with IP addresses (values) that computer network routers understand (such as 208.216.181.15 or 207.142.131.206). This system is the next-generation “phone book.” Thus, humans can use names that are easy to remember and machines can efficiently process the numbers. The number of symbol-table

```
% more amino.csv
TTT,Phe,F,Phenylalanine
TTC,Phe,F,Phenylalanine
TTA,Leu,L,Leucine
TTG,Leu,L,Leucine
TCT,Ser,S,Serine
TCC,Ser,S,Serine
TCA,Ser,S,Serine
TCG,Ser,S,Serine
TAT,Tyr,Y,Tyrosine
TAC,Tyr,Y,Tyrosine
TA Stop,Stop,Stop
...
GCA,Ala,A,Alanine
GCC,Ala,A,Alanine
GAT,Asp,D,Aspartic Acid
GAC,Asp,D,Aspartic Acid
GAA,Gly,G,Glutamic Acid
GAG,Gly,G,Glutamic Acid
GGT,Gly,G,Glycine
GGC,Gly,G,Glycine
GGA,Gly,G,Glycine
GGG,Gly,G,Glycine

% more DJIA.csv
...
20-Oct-87,1738.74,608099968,1841.01
19-Oct-87,2164.16,604300032,1738.74
16-Oct-87,2355.09,33850000,2246.73
15-Oct-87,2412.70,263200000,2355.09
...
30-Oct-29,230.98,10730000,258.47
29-Oct-29,252.38,16410000,230.07
28-Oct-29,295.18,9210000,260.64
25-Oct-29,299.47,5920000,301.22
...
% more ip.csv
...
www.ebay.com,66.135.192.87
www.princeton.edu,128.112.128.15
www.cs.princeton.edu,128.112.136.35
www.harvard.edu,128.103.60.24
www.yale.edu,130.132.51.8
www.cnn.com,64.236.16.20
www.google.com,216.239.41.99
www.nytimes.com,199.239.136.200
www.apple.com,17.112.152.32
www.slashdot.org,66.35.250.151
www.espn.com,199.181.135.201
www.weather.com,63.111.66.11
www.yahoo.com,216.109.118.65
...
```

Typical comma-separated-value (CSV) files

lookups done each second for this purpose on Internet routers around the world is huge, so performance is of obvious importance. Millions of new computers and other devices are put onto the Internet each year, so these symbol tables on Internet routers need to be dynamic.

Despite its scope, this list is still just a representative sample, intended to give you a flavor of the scope of applicability of the symbol-table abstraction. Whenever you specify something by name, there is a symbol table at work. Your computer's file system or the web might do the work for you, but there is a symbol table behind the scenes.

For example, to build a symbol table that associates amino acid names with codons, we can write code like this:

```
ST<String, String> amino;
amino = new ST<String, String>();
amino.put("TTA", "leucine");
...
...
```

The idea of associating information with a key is so fundamental that many high-level languages have built-in support for *associative arrays*, where you can use standard array syntax but with keys inside the brackets instead of an integer index. In such a language, you could write `amino["TTA"] = "leucine"` instead of `amino.put("TTA", "leucine")`. Although Java does not (yet) support such syntax, thinking in terms of associative arrays is a good way to understand the basic purpose of symbol tables.

`Lookup` (PROGRAM 4.4.1) builds a set of key-value pairs from a file of comma-separated values (see SECTION 3.1) as specified on the command line and then prints values corresponding to keys read

Program 4.4.1 Dictionary lookup

```

public class Lookup
{
    public static void main(String[] args)
    { // Build dictionary, provide values for keys in StdIn.
        In in = new In(args[0]);
        int keyField = Integer.parseInt(args[1]);
        int valField = Integer.parseInt(args[2]);

        String[] database = in.readAllLines();
        StdRandom.shuffle(database);

        ST<String, String> st = new ST<String, String>();
        for (int i = 0; i < database.length; i++)
        { // Extract key, value from one line and add to ST.
            String[] tokens = database[i].split(",");
            String key = tokens[keyField];
            String val = tokens[valField];
            st.put(key, val);
        }

        while (!StdIn.isEmpty())
        { // Read key and provide value.
            String s = StdIn.readString();
            StdOut.println(st.get(s));
        }
    }
}

```

in	<i>input stream (.csv)</i>
keyField	<i>key position</i>
valField	<i>value position</i>
database[]	<i>lines in input</i>
st	<i>symbol table (BST)</i>
tokens	<i>values on a line</i>
key	<i>key</i>
val	<i>value</i>
s	<i>query</i>

This ST client reads key-value pairs from a comma-separated file, then prints values corresponding to keys on standard input. Both keys and values are strings.

```
% java Lookup amino.csv 0 3
TTA
Leucine
ABC
null
TCT
Serine
% java Lookup amino.csv 3 0
Glycine
GGG
```

```
% java Lookup ip.csv 0 1
www.google.com
216.239.41.99
% java Lookup ip.csv 1 0
216.239.41.99
www.google.com
% java Lookup DJIA.csv 0 1
29-Oct-29
252.38
```

from standard input. The command-line arguments are the file name and two integers, one specifying the field to serve as the key and the other specifying the field to serve as the value.

Your first step in understanding symbol tables is to download `Lookup.java` and `ST.java` (the industrial-strength symbol-table implementation that we consider at the end of this section) from the booksite to do some symbol-table searches. You can find numerous comma-separated-value (`.csv`) files that are related to various applications that we have described, including `amino.csv` (codon-to-amino-acid encodings), `DJIA.csv` (opening price, volume, and closing price of the stock market average, for every day in its history), and `ip.csv` (a selection of entries from the DNS database). When choosing which field to use as the key, remember that *each key must uniquely determine a value*. If there are multiple `put` operations to associate values with the same key, the symbol table will remember only the most recent one (think about associative arrays). We will consider next the case where we want to associate multiple values with a key.

Later in this section, we will see that the cost of the `put` operations and the `get` requests in `Lookup` is logarithmic in the size of the table. This fact implies that you may experience a small delay getting the answer to your first request (for all the `put` operations to build the symbol table), but you get immediate response for all the others.

Indexing. `Index` (PROGRAM 4.4.2) is a prototypical example of a symbol-table client that uses an *intermixed* sequence of calls to `get()` and `put()`: it reads a sequence of strings from standard input and prints a sorted list of the distinct strings along with a list of integers specifying the positions where each string appeared in the input. We have a large amount of data and want to know where certain strings of interest occur. In this case, we seem to be associating multiple values with each key, but we are actually associating just one: a queue. `Index` takes two integer command-line arguments to control the output: the first integer is the minimum string length to include in the symbol table, and the second is the minimum number of occurrences (among the words that appear in the text) to include in the printed index. The following list of indexing applications demonstrates their range and scope:

- *Book index.* Every textbook has an index where you can look up a word and find the page numbers containing that word. While no reader wants to see every word in the book in an index, a program like `Index` can provide a starting point for creating a good index.

Program 4.4.2 Indexing

```

public class Index
{
    public static void main(String[] args)
    {
        int minlen = Integer.parseInt(args[0]);
        int minocc = Integer.parseInt(args[1]);

        // Create and initialize the symbol table.
        ST<String, Queue<Integer>> st;
        st = new ST<String, Queue<Integer>>();
        for (int i = 0; !StdIn.isEmpty(); i++)
        {
            String word = StdIn.readString();
            if (word.length() < minlen) continue;
            if (!st.contains(word))
                st.put(word, new Queue<Integer>());
            Queue<Integer> queue = st.get(word);
            queue.enqueue(i);
        }

        // Print words whose occurrence count exceeds threshold.
        for (String s : st)
        {
            Queue<Integer> queue = st.get(s);
            if (queue.size() >= minocc)
                StdOut.println(s + ": " + queue);
        }
    }
}

```

minlen	<i>minimum length</i>
minocc	<i>occurrence threshold</i>
st	<i>symbol table</i>
word	<i>current word</i>
queue	<i>queue of positions for current word</i>

This ST client indexes a text file by word position. Keys are words, and values are queues of positions where the word occurs in the file.

```
% java Index 9 30 < TaleOfTwoCities.txt
confidence: 2794 23064 25031 34249 47907 48268 48577 ...
courtyard: 11885 12062 17303 17451 32404 32522 38663 ...
evremonde: 86211 90791 90798 90802 90814 90822 90856 ...

...
something: 3406 3765 9283 13234 13239 15245 20257 ...
sometimes: 4514 4530 4548 6082 20731 33883 34239 ...
vengeance: 56041 63943 67705 79351 79941 79945 80225 ...
```

- *Programming languages.* In a large program that uses a large number of identifiers, it is useful to know where each name is used. A program like Index can be a valuable tool to help programmers keep track of where identifiers are used in their programs. Historically, an explicit printed symbol table was one of the most important tools used by programmers to manage large programs. In modern systems, symbol tables are the basis of software tools that programmers use to manage names of identifiers in programming systems.

- *Genomics.* In a typical (if oversimplified) scenario in genomics research, a scientist wants to know the positions of a given genetic sequence in an existing genome or set of genomes. Existence or proximity of certain sequences may be of scientific significance. The starting point for such research is an index like the one produced by Index, modified to take into account the fact that genomes are not separated into words.
- *Web search.* When you type a keyword and get a list of websites containing that keyword, you are using an index created by your web search engine. One value (the list of pages) is associated with each key (the query), although the reality is a bit more dynamic and complicated because we often specify multiple keys and the pages are spread through the web, not kept in a table on a single computer.
- *Account information.* One way for a company that maintains customer accounts to keep track of a day's transactions is to keep an index of the list of the transactions. The key is the account number; the value is the list of occurrences of that account number in the transaction list.

	<i>key</i>	<i>value</i>
<i>book</i>	term	page numbers
<i>genomics</i>	DNA substring	locations
<i>web search</i>	keyword	websites
<i>business</i>	customer name	transactions

Typical indexing applications

YOU ARE CERTAINLY ENCOURAGED TO DOWNLOAD Index from the booksite and run it on various input files to gain further appreciation for the utility of symbol tables. If you do so, you will find that it can build large indices for huge files with little delay, because each *put* operation and *get* request is taken care of immediately. Providing this immediate response for huge symbol tables is one of the classic contributions of algorithmic technology.

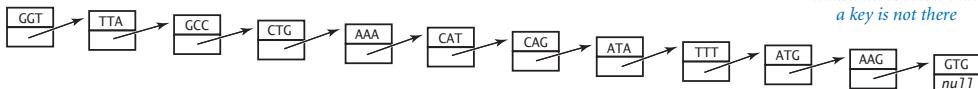
Elementary symbol-table implementations All of these examples are persuasive evidence of the importance of symbol tables. Symbol-table implementations have been heavily studied, many different algorithms and data structures have been invented for this purpose, and modern programming environments (such as Java) include one (or more) symbol-table implementations. As usual, knowing how a basic implementation works will help you appreciate, choose among, and more effectively use the advanced ones, or help implement your own version for some specialized situation that you might encounter.

To begin, we briefly consider two elementary implementations, based on two basic data structures that we have encountered: resizing arrays and linked lists. Our purpose in doing so is to establish that we need a more sophisticated data structure, as each implementation uses linear time for either *put* or *get*, which makes each of them unsuitable for large practical applications.

Perhaps the simplest implementation is to store the key–value pairs in an unordered linked list (or array) and use *sequential search* (see EXERCISE 4.4.6). Sequential search means that, when searching for a key, we examine each node (or element) in sequence until either we find the specified key or we exhaust the list (or array). Such an implementation is not feasible for use by typical clients because, for example, *get* takes linear time when the search key is not in the symbol table.

Alternatively, we might use a sorted (resizing) array for the keys and a parallel array for the values. Since the keys are in sorted order, we can search for a key (and its associated value) using *binary search*, as in SECTION 4.2. It is not difficult to build a symbol-table implementation based on this approach (see EXERCISE 4.4.5). In such an implementation, search is fast (logarithmic time) but insertion is typically slow (linear time) because we must maintain the resizing array in sorted order. Each time a new key is inserted, larger keys must be shifted one position higher in the array, which implies that *put* takes linear time in the worst case.

linked list (unordered)



Sequential search in a linked list takes linear time

AAA	AAA
AAC	AAC
AAG	AAG
AAT	AAT
ACT	ACT
ATA	ATA
ATC	ATC
ATG	ATG
AGG	AGG
AGT	AGT
CAG	CAG
CCT	CAT
CGA	CCT
CGC	CGA
CGG	CGC
CGT	CGG
CTT	CGT
GAA	CTT
GAC	GAA
GAG	GAC
GAT	GAG
GCT	GAT
GGA	GCT
GTC	GGA
GTG	GTC
GTT	GTG
TAA	GTT
TAC	TAA
TAG	TAC
TAT	TAG
TCA	TAT
TGT	TCA
TTA	TGT
TTC	TTA
TTG	TTC
TTT	TTG
	TTT

put CAT into the sorted array

larger keys all have to move

Insertion into a sorted array takes linear time

need to traverse entire linked list to know that a key is not there



TO IMPLEMENT A SYMBOL TABLE THAT is feasible for use with clients such as `Lookup` and `Index`, we need a data structure that is more flexible than either linked lists or resizing arrays. Next, we consider two examples of such data structures: the hash table and the binary search tree.

Hash tables A *hash table* is a data structure in which we divide the keys into small groups that can be quickly searched. We choose a parameter m and divide the keys into m groups, which we expect to be about equal in size. For each group, we keep the keys in an unordered linked list and use sequential search, as in the elementary implementation we just considered.

To divide the keys into the m groups, we use a *hash function* that maps each possible key into a *hash value*—an integer between 0 and $m-1$. This enables us to model the symbol table as an *array of linked lists* and use the hash value as an array index to access the desired list.

Hashing is widely useful, so many programming languages include direct support for it. As we saw in SECTION 3.3, every Java class is supposed to have a `hashCode()` method for this purpose. If you are using a nonstandard type, it is wise to check the `hashCode()` implementation, as the default may not do a good job of dividing the keys into groups of equal size. To convert the hash code into a hash value between 0 and $m-1$, we use the expression `Math.abs(x.hashCode() % m)`.

Recall that whenever two objects are equal—according to the `equals()` method—they must have the same hash code. Objects that are not equal *may* have the same hash code. In the end, hash functions are designed so that it is reasonable to expect the call `Math.abs(x.hashCode() % m)` to return each of the hash values from 0 to $m-1$ with equal likelihood.

The table at right above gives hash codes and hash values for 12 representative `String` keys, with $m = 5$. *Note:* In general, hash codes are integers between -2^{31} and $2^{31}-1$, but for short alphanumeric strings, they happen to be small positive integers.

key	hash code	hash value
GGT	70516	1
TTA	83393	3
GCC	70375	0
CTG	67062	2
AAA	64545	0
CAT	66486	1
CAG	66473	2
ATA	65134	4
TTT	83412	2
ATG	65140	0
AAG	64551	1
GTG	70906	1

*Hash codes and hash values
for $n = 12$ strings ($m = 5$)*

With this preparation, implementing an efficient symbol table with hashing is a straightforward extension of the linked-list code that we considered in SECTION 4.3. We maintain an array of m linked lists, with element i containing a linked list of all keys whose hash value is i (along with their associated values). To search for a key:

- Compute its hash value to identify its linked list.
- Iterate over the nodes in that linked list, checking for the search key.
- If the search key is in the linked list, return the associated value; otherwise, return `null`.

To insert a key–value pair:

- Compute the hash value of the key to identify its linked list.
- Iterate over the nodes in that linked list, checking for the key.
- If the key is in the linked list, replace the value currently associated with the key with the new value; otherwise, create a new node with the specified key and value and insert it at the beginning of the linked list.

`HashST` (PROGRAM 4.4.3) is a full implementation, using a fixed number of $m = 1,024$ linked lists. It relies on the following nested class that represents each node in the linked list:

```
private static class Node
{
    private Object key;
    private Object val;
    private Node next;

    public Node(Object key, Object val, Node next)
    {
        this.key  = key;
        this.val  = val;
        this.next = next;
    }
}
```

The efficiency of `HashST` depends on the value of m and the quality of the hash function. Assuming the hash function reasonably distributes the keys, performance is about m times faster than that for sequential search in a linked list, at the cost of m extra references and linked lists. This is a classic space–time tradeoff: the higher the value of m , the more memory we use, but the less time we spend.

Program 4.4.3 Hash table

```

public class HashST<Key, Value>
{
    private int m = 1024;
    private Node[] lists = new Node[m];
    private class Node
    { /* See accompanying text. */ }

    private int hash(Key key)
    { return Math.abs(key.hashCode() % m); }

    public Value get(Key key)
    {
        int i = hash(key);
        for (Node x = lists[i]; x != null; x = x.next)
            if (key.equals(x.key))
                return (Value) x.val;
        return null;
    }

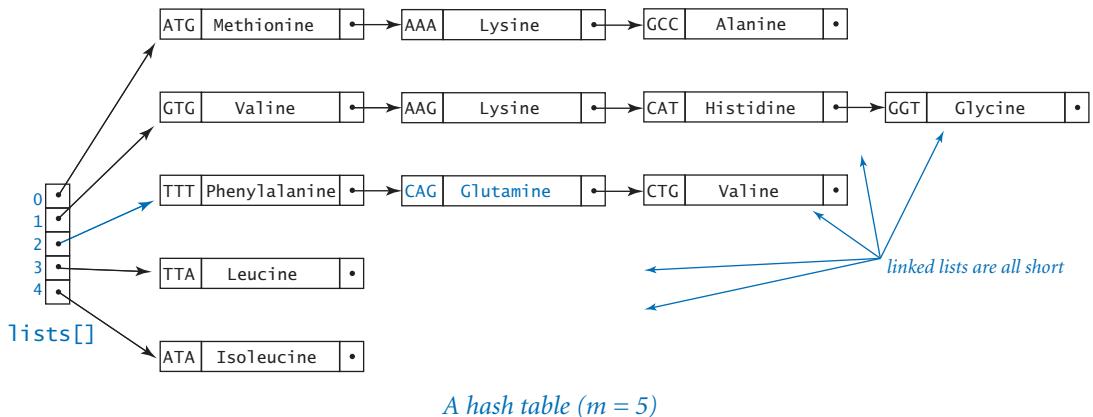
    public void put(Key key, Value val)
    {
        int i = hash(key);
        for (Node x = lists[i]; x != null; x = x.next)
        {
            if (key.equals(x.key))
            {
                x.val = val;
                return;
            }
        }
        lists[i] = new Node(key, val, lists[i]);
    }
}

```

m | number of linked lists
 $\text{lists}[i]$ | linked list for hash value i

This program uses an array of linked lists to implement a hash table. The hash function selects one of the m lists. When there are n keys in the table, the average cost of a $\text{put}()$ or $\text{get}()$ operation is n/m , for suitable $\text{hashCode}()$ implementations. This cost per operation is constant if we use a resizing array to ensure that the average number of keys per list is between 1 and 8 (see EXERCISE 4.4.12). We defer implementations of $\text{contains}()$, $\text{keys}()$, $\text{size}()$, and $\text{remove}()$ to EXERCISE 4.4.8–11.

The figure below shows the hash table built for our sample keys, inserted in the order given on page 636. First, GGT is inserted in linked list 1, then TTA is inserted in linked list 3, then GCC is inserted in linked list 0, and so forth. After the hash table is built, a search for CAG begins by computing its hash value (2) and then sequentially searching linked list 2. After finding the key CAG in the second node of linked list 2, the method `get()` returns the value Glutamine.



Often, programmers choose a large fixed value of m (like the 1,024 default we have chosen) based on a rough estimate of the number of keys to be handled. With more care, we can ensure that the average number of keys per list is a constant, by using a resizing array for `lists[]`. For example, EXERCISE 4.4.12 shows how to ensure that the average number of keys per linked list is between 1 and 8, which leads to constant (amortized) time performance for both `put` and `get`. There is certainly opportunity to adjust these parameters to best fit a given practical situation.

THE PRIMARY ADVANTAGE OF HASH TABLES is that they support the `put` and `get` operations efficiently. A disadvantage of hash tables is that they do not take advantage of order in the keys and therefore cannot provide the keys in sorted order (or support other order-based operations). For example, if we substitute HashST for ST in Index, then the keys will be printed in arbitrary order instead of sorted order. Or, if we want to find the smallest key or the largest key, we have to search through them all. Next, we consider a symbol-table implementation that can support order-based operations when the keys are comparable, without sacrificing much performance for `put()` and `get()`.

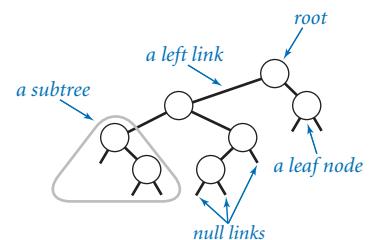
Binary search trees The *binary tree* is a mathematical abstraction that plays a central role in the efficient organization of information. We define a binary tree recursively: it is either empty (null) or a node containing links to two disjoint binary trees. Binary trees play an important role in computer programming because they strike an efficient balance between flexibility and ease of implementation. Binary trees have many applications in science, mathematics, and computational applications, so you are certain to encounter this model on many occasions.

We often use tree-based terminology when discussing binary trees. We refer to the node at the top as the *root* of the tree, the node referenced by its left link as the *left subtree*, and the node referenced by its right link as the *right subtree*. Traditionally, computer scientists draw trees upside down, with the root at the top. Nodes whose links are both null are called *leaf nodes*. The *height* of a tree is the maximum number of links on any path from the root node to a leaf node.

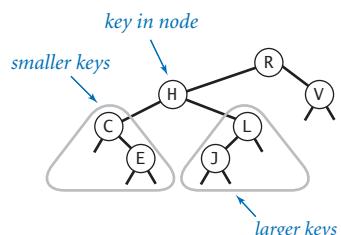
As with arrays, linked lists, and hash tables, we use binary trees to store collections of data. For symbol-table implementations, we use a special type of binary tree known as a *binary search tree (BST)*. A binary search tree is a binary tree that contains a key–value pair in each node and for which the keys are in *symmetric order*: The key in a node is larger than the key of every node in its left subtree and smaller than the key of every node in its right subtree. As you will soon see, symmetric ordering enables efficient implementations of the *put* and *get* operations.

To implement BSTs, we start with a nested class for the *node* abstraction, which has references to a key, a value, and left and right BSTs. The key type must implement Comparable (to specify an ordering of the keys) but the value type is arbitrary.

```
private class Node
{
    private Key key;
    private Value val;
    private Node left, right;
}
```



Anatomy of a binary tree



Symmetric order

This definition is like our definition of nodes for linked lists, except that it has *two* links, instead of one. As with linked lists, the idea of a recursive data structure can be a bit mind-bending, but all we are doing is adding a second link (and imposing an ordering restriction) to our linked-list definition.

To (slightly) simplify the code, we add a constructor to `Node` that initializes the `key` and `val` instance variables:

```
Node(Key key, Value val)
{
    this.key = key;
    this.val = val;
}
```

The result of `new Node(key, val)` is a reference to a `Node` object (which we can assign to any variable of type `Node`) whose `key` and `val` instance variables are set to the specified values and whose `left` and `right` instance variables are both initialized to `null`.

As with linked lists, when tracing code that uses BSTs, we can use a visual representation of the changes:

- We draw a rectangle to represent each object.
- We put the values of instance variables within the rectangle.
- We depict references as arrows that point to the referenced object.

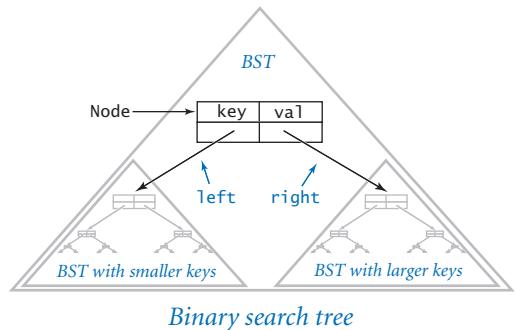
Most often, we use an even simpler abstract representation where we draw rectangles (or circles) containing keys to represent nodes (suppressing the values) and connect the nodes with arrows that represent links. This abstract representation allows us to focus on the linked structure.

As an example, we consider a BST with string keys and integer values. To build a one-node BST that associates the value 0 with the key `it`, we create a `Node`:

```
Node first = new Node("it", 0);
```

Since the left and right links are both `null`, this node represents a BST containing one node. To add a node that associates the value 1 with the key `was`, we create another `Node`:

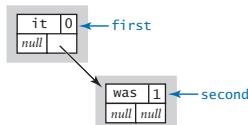
```
Node second = new Node("was", 1);
```



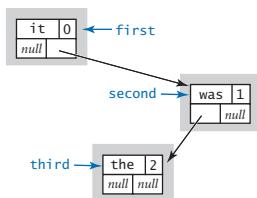
```
Node first = new Node("it", 0);
```



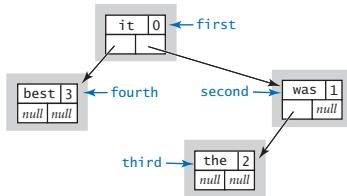
```
Node second = new Node("was", 1);
first.right = second;
```



```
Node third= new Node("the", 2);
second.left = third;
```



```
Node fourth = new Node("best", 2);
first.left = fourth;
```



Linking together a BST

abuse our nomenclature, using ST to signify both “symbol table” and “search tree” because search trees play such a central role in symbol-table implementations.

A BST represents an *ordered* sequence of items. In the example just considered, `first` represents the sequence `best it the was`. We can also use an array to represent a sequence of items. For example, we could use

```
String[] a = { "best", "it", "the", "was" };
```

(which itself is a BST) and link to it from the right field of the first Node:

```
first.right = second;
```

The second node goes to the right of the first because `was` comes after `it` in alphabetical order. (Alternatively, we could have chosen to set `second.left` to `first`.) Now we can add a third node that associates the value 2 with the key `the` with the code:

```
Node third = new Node("the", 2);
second.left = third;
```

and a fourth node that associates the value 3 with the key `best` with the code:

```
Node fourth = new Node("best", 3);
first.left = fourth;
```

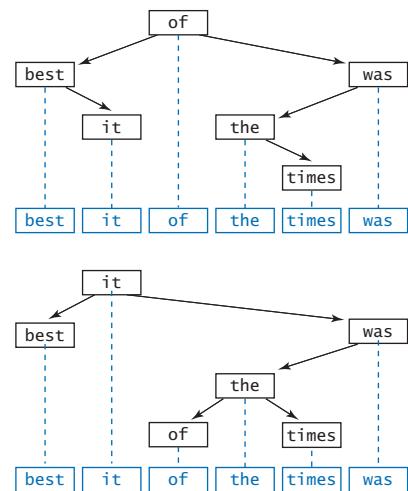
Note that each of our links—`first`, `second`, `third`, and `fourth`—are, by definition, BSTs (each is either null or refers to a BST, and the ordering condition is satisfied at each node).

In the present context, we take care to ensure that we always link together nodes such that *every* Node that we create is the root of a BST (has a key, a value, a link to a left BST with smaller values, and a link to a right BST with a larger value). From the standpoint of the BST data structure, the value is immaterial, so we often ignore it in our figures, but we include it in the definition because it plays such a central role in the symbol-table concept. We slightly

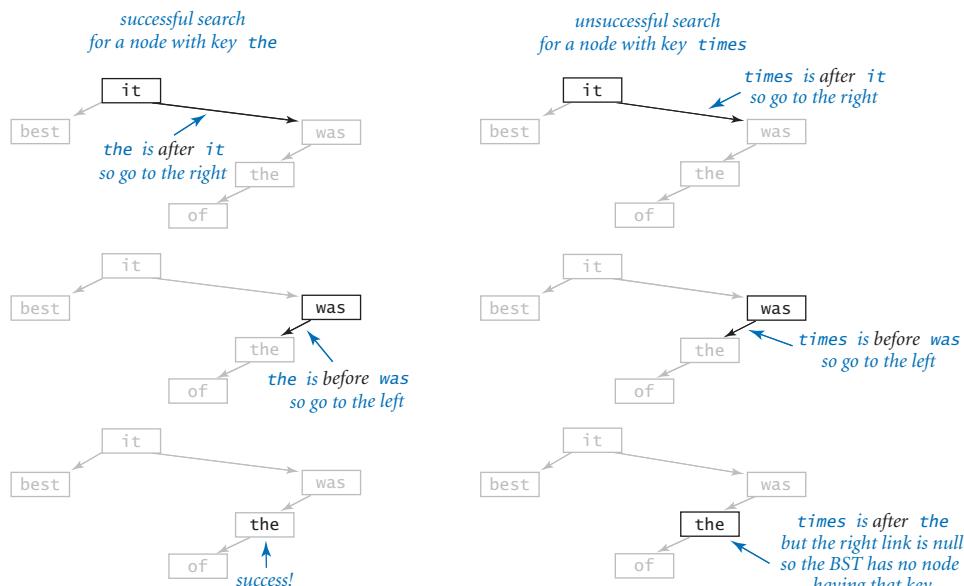
abuse our nomenclature, using ST to signify both “symbol table” and “search tree” because search trees play such a central role in symbol-table implementations.

to represent the same ordered sequence of strings. Given a set of distinct keys, there is only one way to represent them in an ordered array, but there are many ways to represent them in a BST (see EXERCISE 4.4.7). This flexibility allows us to develop efficient symbol-table implementations. For instance, in our example we were able to insert each new key–value pair by creating a new node and changing just one link. As it turns out, it is always possible to do so. Equally important, we can easily find the node in a BST containing a specified key or find the node whose link must change when we insert a new key–value pair. Next, we consider symbol-table code that accomplishes these two tasks.

Search. Suppose that you want to *search* for a node with a given key in a BST (or *get* a value with a given key in a symbol table). There are two possible outcomes: the search might be *successful* (we find the key in the BST; in a symbol-table implementation, we return the associated value) or it might be *unsuccessful* (there is no key in the BST with the given key; in a symbol-table implementation, we return null).



Two BSTs representing the same sequence



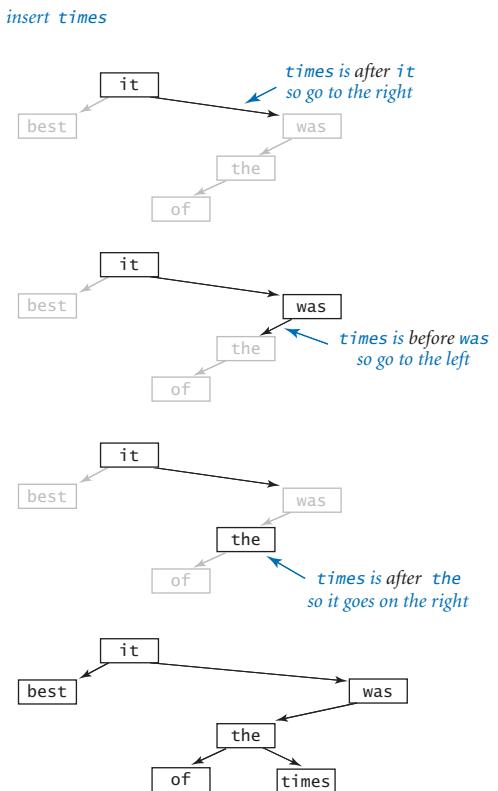
Searching in a BST

A recursive searching algorithm is immediately evident: Given a BST (a reference to a Node), first check whether the tree is empty (the reference is `null`). If so, then terminate the search as unsuccessful (in a symbol-table implementation, return `null`). If the tree is nonempty, check whether the key in the node is equal to the search key. If so, then terminate the search as successful (in a symbol-table implementation, return the value associated with the key). If not, compare the search key with the key in the node. If it is smaller, search (recursively) in the left subtree; if it is greater, search (recursively) in the right subtree.

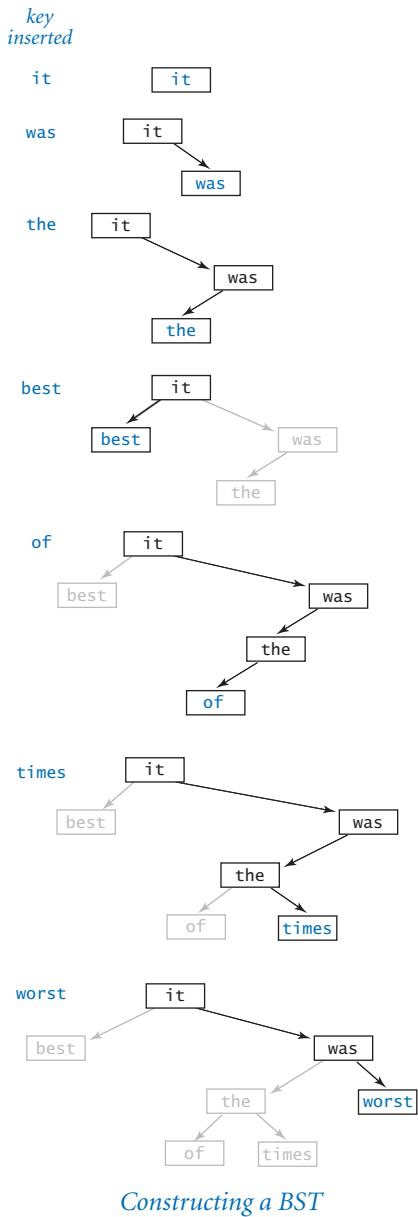
Thinking recursively, it is not difficult to become convinced that this algorithm behaves as intended, based upon the invariant that the key is in the BST if and only if it is in the current subtree. The crucial property of the recursive method is that we always have only one node to examine to decide what to do next. Moreover, we typically examine only a small number of the nodes in the tree: whenever we go to one of the subtrees at a node, we never examine any of the nodes in the other subtree.

Insert. Suppose that you want to *insert* a new node into a BST (in a symbol-table implementation, *put* a new key–value pair into the data structure). The logic is similar to searching for a key, but the implementation is trickier. The key to understanding it is to realize that only one link must be changed to point to the new node, and that link is precisely the link that would be found to be `null` in an unsuccessful search for that key.

If the tree is empty, we create and return a new `Node` containing the key–value pair; if the search key is less than the key at the root, we set the left link to the result of inserting the key–value pair into the left subtree; if the search key is greater, we set



Inserting a new node into a BST



the right link to the result of inserting the key-value pair into the right subtree; otherwise, if the search key is equal, we replace the existing value with the new value. Resetting the left or right link after the recursive call in this way is usually unnecessary, because the link changes only if the subtree is empty, but it is as easy to set the link as it is to test to avoid setting it.

Implementation. BST (PROGRAM 4.4.4) is a symbol-table implementation based on these two recursive algorithms. If you compare this code with our binary search implementation `BinarySearch` (PROGRAM 4.2.3) and our stack and queue implementations `Stack` (PROGRAM 4.3.4) and `Queue` (PROGRAM 4.3.6), you will appreciate the elegance and simplicity of this code. *Take the time to think recursively and convince yourself that this code behaves as intended.* Perhaps the simplest way to do so is to trace the construction of an initially empty BST from a sample set of keys. Your ability to do so is a sure test of your understanding of this fundamental data structure.

Moreover, the `put()` and `get()` methods in `BST` are remarkably efficient: typically, each accesses a small number of the nodes in the BST (those on the path from the root to the node sought or to the null link that is replaced by a link to the new node). Next, we show that `put` operations and `get` requests take logarithmic time (under certain assumptions). Also, `put()` only creates one new `Node` and adds one new link. If you make a drawing of a BST built by inserting some keys into an initially empty tree, you certainly will be convinced of this fact—you can just draw each new node somewhere at the bottom of the tree.

Program 4.4.4 Binary search tree

```

public class BST<Key extends Comparable<Key>, Value>
{
    private Node root;                                root | root of BST

    private class Node
    {
        private Key key;                             key
        private Value val;                          val
        private Node left, right;                   left
        public Node(Key key, Value val)           right
        {   this.key = key; this.val = val;  }
    }

    public Value get(Key key)
    {   return get(root, key);  }

    private Value get(Node x, Key key)
    {
        if (x == null) return null;
        int cmp = key.compareTo(x.key);
        if      (cmp < 0) return get(x.left,  key);
        else if (cmp > 0) return get(x.right, key);
        else                return x.val;
    }

    public void put(Key key, Value val)
    {   root = put(root, key, val);  }

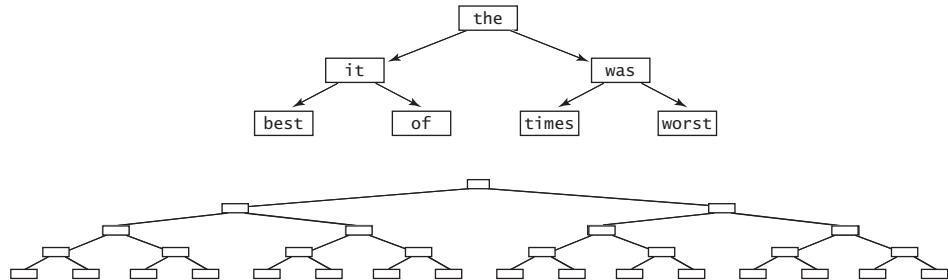
    private Node put(Node x, Key key, Value val)
    {
        if (x == null) return new Node(key, val);
        int cmp = key.compareTo(x.key);
        if      (cmp < 0) x.left = put(x.left,  key, val);
        else if (cmp > 0) x.right = put(x.right, key, val);
        else                x.val = val;
        return x;
    }
}

```

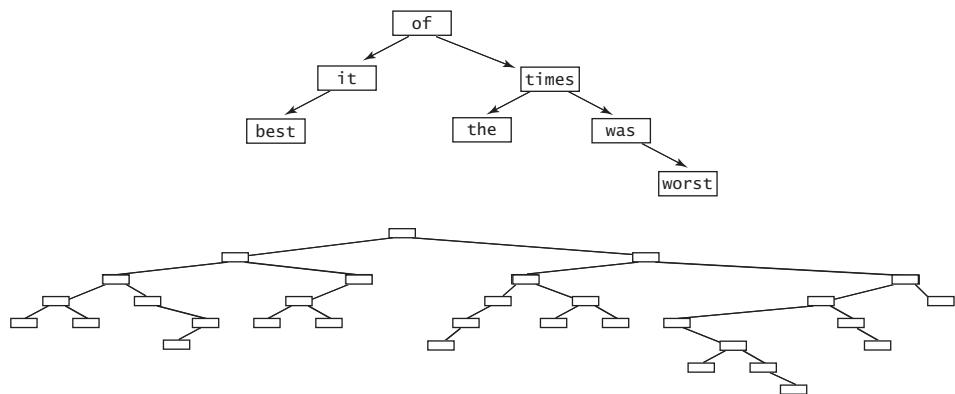
This implementation of the symbol-table data type is centered on the recursive BST data structure and recursive methods for traversing it. We defer implementations of `contains()`, `size()`, and `remove()` to EXERCISE 4.4.18–20. We implement `keys()` at the end of this section.

Performance characteristics of BSTs The running times of BST algorithms are ultimately dependent on the shape of the trees, and the shape of the trees is dependent on the order in which the keys are inserted. Understanding this dependence is a critical factor in being able to use BSTs effectively in practical situations.

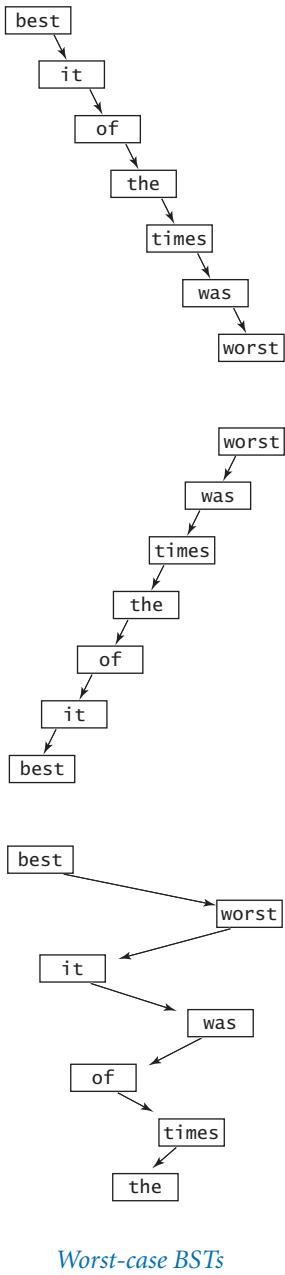
Best case. In the best case, the tree is perfectly balanced (each Node has exactly two non-null children), with about $\lg n$ links between the root and each leaf node. In such a tree, it is easy to see that the cost of an unsuccessful search is logarithmic, because that cost satisfies the same recurrence relation as the cost of binary search (see SECTION 4.2) so that the cost of every *put* operation and *get* request is proportional to $\lg n$ or less. You would have to be quite lucky to get a perfectly balanced tree like this by inserting keys one by one in practice, but it is worthwhile to know the best-case performance characteristics.



Best case (perfectly balanced) BSTs



Typical BSTs constructed from randomly ordered keys



Average case. If we insert random keys, we might expect the search times to be logarithmic as well, because the first key becomes the root of the tree and should divide the keys roughly in half. Applying the same argument to the subtrees, we expect to get about the same result as for the best case. This intuition is, indeed, validated by careful analysis: a classic mathematical derivation shows that the time required for *put* and *get* in a tree constructed from randomly ordered keys is logarithmic (see the booksite for references). More precisely, the *expected number of key compares* is $\sim 2 \ln n$ for a random *put* or *get* in a tree built from n randomly ordered keys. In a practical application such as *Lookup*, when we can explicitly randomize the order of the keys, this result suffices to (probabilistically) guarantee logarithmic performance. Indeed, since $2 \ln n$ is about $1.39 \lg n$, the *average* case is only about 39% greater than the *best* case. In an application like *Index*, where we have no control over the order of insertion, there is no guarantee, but typical data gives logarithmic performance (see EXERCISE 4.4.26). As with binary search, this fact is very significant because of the enormity of the logarithmic–linear chasm: with a BST-based symbol table implementation, we can perform millions of operations per second (or more), even in a huge symbol table.

Worst case. In the worst case, each node (except one) has exactly one null link, so the BST is essentially a linked list with an extra wasted link, where *put* operations and *get* requests take linear time. Unfortunately, this worst case is not rare in practice—it arises, for example, when we insert the keys in order.

Thus, good performance of the basic BST implementation is dependent on the keys being sufficiently similar to random keys that the tree is not likely to contain many long paths. If you are not sure that assumption is justified, *do not use a simple BST*. Your only clue that something is amiss will be slow response time as the problem size increases. (*Note:* It is not unusual to encounter software of this sort!) Remarkably, some BST variants eliminate this worst case and guarantee logarithmic performance per operation, by making all trees nearly perfectly balanced. One popular variant is known as the *red–black tree*.

Traversing a BST Perhaps the most basic tree-processing function is known as *tree traversal*: given a (reference to) a tree, we want to systematically process every node in the tree. For linked lists, we accomplish this task by following the single link to move from one node to the next. For trees, however, we have decisions to make, because there are *two* links to follow. Recursion comes immediately to the rescue. To process every node in a BST:

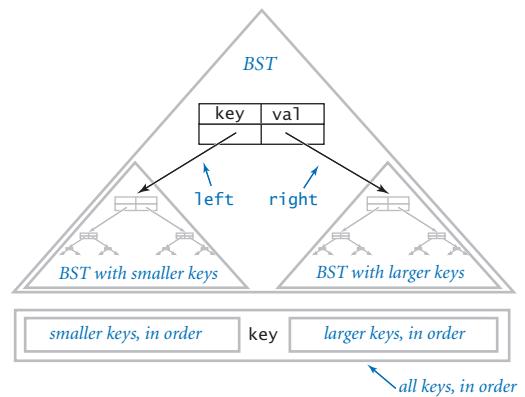
- Process every node in the left subtree.
- Process the node at the root.
- Process every node in the right subtree.

This approach is known as *inorder* tree traversal, to distinguish it from *preorder* (do the root first) and *postorder* (do the root last), which arise in other applications. Given a BST, it is easy to convince yourself with mathematical induction that not only does this approach process every node in the BST, but it also processes them in *key-sorted order*. For example, the following method prints the keys in the BST rooted at its argument in ascending order of the keys in the nodes:

```
private void traverse(Node x)
{
    if (x == null) return;
    traverse(x.left);
    StdOut.println(x.key);
    traverse(x.right);
}
```

First, we print all the keys in the left subtree, in key-sorted order. Then we print the root, which is next in the key-sorted order, and then we print all the keys in the right subtree, in key-sorted order.

This remarkably simple method is worthy of careful study. It can be used as a basis for a `toString()` implementation for BSTs (see EXERCISE 4.4.21). It also serves as the basis for implementing the `keys()` method, which enables clients to use a Java `foreach` loop to iterate over the keys in a BST, in sorted order (recall that this functionality is not available in a hash table, where there is no order). We consider this fundamental application of inorder traversal next.



Recursive inorder traversal of a binary search tree

Iterating over the keys. A close look at the recursive `traverse()` method just considered leads to a way to process all of the key–value pairs in our BST data type. For simplicity, we need only process the keys because we can *get* the values when we need them. Our goal is implement a method `keys()` to enable client code like the following:

```
BST<String, Double> st = new BST<String, Double>();
...
for (String key : st.keys())
    StdOut.println(key + " " + st.get(key));
...
```

Index (PROGRAM 4.4.2) is another example of client code that uses a `foreach` loop to iterate over key–value pairs.

The easiest way to implement `keys()` is to collect all of the keys in an iterable collection—such as a `Stack` or `Queue`—and return that iterable to the client.

```
public Iterable<Key> keys()
{
    Queue<Key> queue = new Queue<Key>();
    inorder(root, queue);
    return queue;
}

private void inorder(Node x, Queue<Key> queue)
{
    if (x == null) return;
    inorder(x.left, queue);
    queue.enqueue(x.key);
    inorder(x.right, queue);
}
```

THE FIRST TIME THAT ONE SEES it, tree traversal seems a bit magical. Ordered iteration essentially comes for free in a data structure designed for fast search and fast insert. Note that we can use a similar technique (i.e., collecting the keys in an iterable collection) to implement the `keys()` method for `HashST` (see EXERCISE 4.4.10). Once again, however, the keys in such an implementation will appear in arbitrary order, since there is no order in hash tables.

Ordered symbol table operations The flexibility of BSTs and the ability to compare keys enable the implementation of many useful operations beyond those that can be supported efficiently in hash tables. This list is representative; numerous other important operations have been invented for BSTs that are broadly useful in applications. We leave implementations of these operations for exercises and leave further study of their performance characteristics and applications for a course in algorithms and data structures.

Minimum and maximum. To find the smallest key in a BST, follow the left links from the root until `null` is reached. The last key encountered is the smallest in the BST. The same procedure, albeit following the right links, leads to the largest key in the BST (see EXERCISE 4.4.27).

Size and subtree sizes. To keep track of the number of nodes in a BST, keep an extra instance variable `n` in `BST` that counts the number of nodes in the tree. Initialize it to 0 and increment it whenever a new `Node` is created. Alternatively, keep an extra instance variable `n` in each `Node` that counts the number of nodes in the subtree rooted at that node (see EXERCISE 4.4.29).

Range search and range count. With a recursive method like `inorder()`, we can return an iterable for the keys falling between two given values in time proportional to the height of the BST plus the number of keys in the range (see EXERCISE 4.4.31). If we maintain an instance variable in each node having the size of the subtree rooted at each node, we can *count* the number of keys falling between two given values in time proportional to the height of the BST (see EXERCISE 4.4.31).

Order statistics and ranks. If we maintain an instance variable in each node having the size of the subtree rooted at each node, we can implement a recursive method that returns the *k*th smallest key in time proportional to the height of the BST (see EXERCISE 4.4.55). Similarly, we can compute the *rank* of a key, which is the number of keys in the BST that are strictly smaller than the key (see EXERCISE 4.4.56).

HENCEFORTH, WE WILL USE THE REFERENCE implementation `ST` that implements our ordered symbol-table API using Java's `java.util.TreeMap`, a symbol-table implementation based on red–black trees. You will learn more about red–black trees if you take an advanced course in data structures and algorithms. They support a logarithmic-time guarantee for `get()`, `put()`, and many of the other operations just described.

Set data type As a final example, we consider a data type that is simpler than a symbol table, still broadly useful, and easy to implement with either hash tables or BSTs. A *set* is a collection of distinct keys, like a symbol table with no values. We could use ST and ignore the values, but client code that uses the following API is simpler and clearer:

public class SET<Key extends Comparable<Key>>	
SET()	<i>create an empty set</i>
boolean isEmpty()	<i>is the set empty?</i>
void add(Key key)	<i>add key to the set</i>
void remove(Key key)	<i>remove key from set</i>
boolean contains(Key key)	<i>is key in the set?</i>
int size()	<i>number of elements in set</i>

Note: Implementations should also implement the Iterable<Key> interface to enable clients to access keys with foreach loops

API for a generic set

As with symbol tables, there is no intrinsic reason that the key type should be comparable. However, processing comparable keys is typical and enables us to support various order-based operations, so we include Comparable in the API. Implementing SET by deleting references to val in our BST code is a straightforward exercise (see EXERCISE 4.4.23). Alternatively, it is easy to develop a SET implementation based on hash tables.

DeDup (PROGRAM 4.4.5) is a SET client that reads a sequence of strings from standard input and prints the first occurrence of each string (thereby removing duplicates). You can find many other examples of SET clients in the exercises at the end of this section.

In the next section, you will see the importance of identifying such a fundamental abstraction, illustrated in the context of a case study.

Program 4.4.5 Dedup filter

```
public class DeDup
{
    public static void main(String[] args)
    { // Filter out duplicate strings.
        SET<String> distinct = new SET<String>();
        while (!StdIn.isEmpty())
        { // Read a string, ignore if duplicate.
            String key = StdIn.readString();
            if (!distinct.contains(key))
            { // Save and print new string.
                distinct.add(key);
                StdOut.print(key);
            }
            StdOut.println();
        }
    }
}
```

distinct *set of distinct strings
on standard input*
key *current string*

This SET client is a filter that reads strings from standard input and writes the strings to standard output, ignoring duplicate strings. For efficiency, it uses a SET containing the distinct strings encountered so far.

```
% java DeDup < TaleOfTwoCities.txt
it was the best of times worst age wisdom...
```

Perspective Symbol-table implementations are a prime topic of further study in algorithms and data structures. Examples include balanced BSTs, hashing, and tries. Implementations of many of these algorithms and data structures are found in Java and most other computational environments. Different APIs and different assumptions about keys call for different implementations. Researchers in algorithms and data structures still study symbol-table implementations of all sorts.

Which symbol-table implementation is better—hashing or BSTs? The first point to consider is whether the client has comparable keys and needs symbol-table operations that involve ordered operations such as selection and rank. If so, then you need to use BSTs. If not, most programmers are likely to use hashing, because symbol tables based on hash tables are typically faster than those based on BSTs, assuming you have access to a good hash function for the key type.

The use of binary search trees to implement symbol tables and sets is a sterling example of exploiting the tree abstraction, which is ubiquitous and familiar. We are accustomed to many tree structures in everyday life, including family trees, sports tournaments, the organization chart of a company, and parse trees in grammar. Trees also arise in numerous computational applications, including function-call trees, parse trees for programming languages, and file systems. Many important applications of trees are rooted in science and engineering, including phylogenetic trees in computational biology, multidimensional trees in computer graphics, minimax game trees in economics, and quad trees in molecular-dynamics simulations. Other, more complicated, linked structures can be exploited as well, as you will see in SECTION 4.5.

People use dictionaries, indexes, and other kinds of symbol tables every day. Within a short amount of time, applications based on symbol tables replaced phone books, encyclopedias, and all sorts of physical artifacts that served us well in the last millennium. Without symbol-table implementations based on data structures such as hash tables and BSTs, such applications would not be feasible; with them, we have the feeling that anything that we need is instantly accessible online.

Q&A

Q. Why use immutable symbol-table keys?

A. If we changed a key while it was in the hash table or BST, it could invalidate the data structure's invariants.

Q. Why is the `val` instance variable in the nested `Node` class in `HashST` declared to be of type `Object` instead of `Value`?

A. Good question. Unfortunately, as we saw in the Q&A at the end of SECTION 3.1, Java does not permit the creation of arrays of generics. One consequence of this restriction is that we need a cast in the `get()` method, which generates a compile-time warning (even though the cast is guaranteed to succeed at run time). Note that we can declare the `val` instance variable in the nested `Node` class in `BST` to be of type `Value` because it does not use arrays.

Q. Why not use the Java libraries for symbol tables?

A. Now that you understand how a symbol table works, you are certainly welcome to use the industrial-strength versions `java.util.TreeMap` and `java.util.HashMap`. They follow the same basic API as `ST`, but allow `null` keys and use the names `containsKey()` and `keySet()` instead of `contains()` and `iterator()`, respectively. They also contain a variety of additional utility methods, but they do not support some of the other methods that we mentioned, such as order statistics. You can also use `java.util.TreeSet` and `java.util.HashSet`, which implement an API like our `SET`.

Exercises

4.4.1 Modify `Lookup` to make a program `LookupAndPut` that allows *put* operations to be specified on standard input. Use the convention that a plus sign indicates that the next two strings typed are the key–value pair to be inserted.

4.4.2 Modify `Lookup` to make a program `LookupMultiple` that handles multiple values having the same key by storing all such values in a queue, as in `Index`, and then printing them all on a *get* request, as follows:

```
% java LookupMultiple amino.csv 3 0
Leucine
TTA TTG CTT CTC CTA CTG
```

4.4.3 Modify `Index` to make a program `IndexByKeyword` that takes a file name from the command line and makes an index from standard input using only the keywords in that file. *Note:* Using the same file for indexing and keywords should give the same result as `Index`.

4.4.4 Modify `Index` to make a program `IndexLines` that considers only consecutive sequences of letters as keys (no punctuation or numbers) and uses line number instead of word position as the value. This functionality is useful for programs, as follows:

```
% java IndexLines 6 0 < Index.java
continue 12
enqueue 15
Integer 4 5 7 8 14
parseInt 4 5
println 22
```

4.4.5 Develop an implementation `BinarySearchST` of the symbol-table API that maintains parallel arrays of keys and values, keeping them in key-sorted order. Use binary search for *get*, and move larger key–value pairs to the right one position for *put* (use a resizing array to keep the array length proportional to the number of key–value pairs in the table). Test your implementation with `Index`, and validate the hypothesis that using such an implementation for `Index` takes time proportional to the product of the number of strings and the number of distinct strings in the input.

4.4.6 Develop an implementation `SequentialSearchST` of the symbol-table API that maintains a linked list of nodes containing keys and values, keeping them in arbitrary order. Test your implementation with `Index`, and validate the hypothesis that using such an implementation for `Index` takes time proportional to the product of the number of strings and the number of distinct strings in the input.

4.4.7 Compute `x.hashCode() % 5` for the single-character strings

E A S Y Q U E S T I O N

In the style of the drawing in the text, draw the hash table created when the i th key in this sequence is associated with the value i , for i from 0 to 11.

4.4.8 Implement the method `contains()` for `HashST`.

4.4.9 Implement the method `size()` for `HashST`.

4.4.10 Implement the method `keys()` for `HashST`.

4.4.11 Modify `HashST` to add a method `remove()` that takes a `Key` argument and removes that key (and the corresponding value) from the symbol table, if it exists.

4.4.12 Modify `HashST` to use a resizing array so that the average length of the list associated with each hash value is between 1 and 8.

4.4.13 Draw the BST that results when you insert the keys

E A S Y Q U E S T I O N

in that order into an initially empty tree. What is the height of the resulting BST?

4.4.14 Suppose we have integer keys between 1 and 1000 in a BST and search for 363. Which of the following *cannot* be the sequence of keys examined?

- a. 2 252 401 398 330 363
- b. 399 387 219 266 382 381 278 363
- c. 3 923 220 911 244 898 258 362 363
- d. 4 924 278 347 621 299 392 358 363
- e. 5 925 202 910 245 363

4.4.15 Suppose that the following 31 keys appear (in some order) in a BST of height 4:

```
10 15 18 21 23 24 30 31 38 41 42 45 50 55 59  
60 61 63 71 77 78 83 84 85 86 88 91 92 93 94 98
```

Draw the top three nodes of the tree (the root and its two children).

4.4.16 Draw all the different BSTs that can represent the sequence of keys

```
best of it the time was
```

4.4.17 True or false: Given a BST, let x be a leaf node, and let p be its parent. Then either (1) the key of p is the smallest key in the BST larger than the key of x or (2) the key of p is the largest key in the BST smaller than the key of x .

4.4.18 Implement the method `contains()` for BST.

4.4.19 Implement the method `size()` for BST.

4.4.20 Modify BST to add a method `remove()` that takes a Key argument and removes that key (and the corresponding value) from the symbol table, if it exists. *Hint:* Replace the key (and its associated value) with the next largest key in the BST (and its associated value); then remove from the BST the node that contained the next largest key.

4.4.21 Implement the method `toString()` for BST, using a recursive helper method like `traverse()`. As usual, you can accept quadratic performance because of the cost of string concatenation. *Extra credit:* Write a linear-time `toString()` method for BST that uses `StringBuilder`.

4.4.22 Modify the symbol-table API to handle values with duplicate keys by having `get()` return an *iterable* for the values having a given key. Implement BST and Index as dictated by this API. Discuss the pros and cons of this approach versus the one given in the text.

4.4.23 Modify BST to implement the SET API given at the end of this section.

4.4.24 Modify HashST to implement the SET API given at the end of this section (remove the Comparable restriction from the API).

4.4.25 A *concordance* is an alphabetical list of the words in a text that gives all word positions where each word appears. Thus, `java Index 0 0` produces a concordance. In a famous incident, one group of researchers tried to establish credibility while keeping details of the Dead Sea Scrolls secret from others by making public a concordance. Write a program `InvertConcordance` that takes a command-line argument `n`, reads a concordance from standard input, and prints the first `n` words of the corresponding text on standard output.

4.4.26 Run experiments to validate the claims in the text that the *put* operations and *get* requests for `Lookup` and `Index` are logarithmic in the size of the table when using ST. Develop test clients that generate random keys and also run tests for various data sets, either from the booksite or of your own choosing.

4.4.27 Modify BST to add methods `min()` and `max()` that return the smallest (or largest) key in the table (or `null` if no such key exists).

4.4.28 Modify BST to add methods `floor()` and `ceiling()` that take as an argument a key and return the largest (smallest) key in the symbol table that is no larger (no smaller) than the specified key (or `null` if no such key exists).

4.4.29 Modify BST to add a method `size()` that returns the number of key–value pairs in the symbol table. Use the approach of storing within each `Node` the number of nodes in the subtree rooted there.

4.4.30 Modify BST to add a method `rangeSearch()` that takes two keys as arguments and returns an iterable over all keys that are between the two given keys. The running time should be proportional to the height of the tree plus the number of keys in the range.

4.4.31 Modify BST to add a method `rangeCount()` that takes two keys as arguments and returns the number of keys in a BST between the two specified keys. Your method should take time proportional to the height of the tree. *Hint:* First work the previous exercise.

4.4.32 Write an ST client that creates a symbol table mapping letter grades to numerical scores, as in the table below, and then reads from standard input a list of letter grades and computes their average (GPA).

A+	A	A-	B+	B	B-	C+	C	C-	D	F
4.33	4.00	3.67	3.33	3.00	2.67	2.33	2.00	1.67	1.00	0.00

Binary Tree Exercises

These exercises are intended to give you experience in working with binary trees that are not necessarily BSTs. They all assume a `Node` class with three instance variables: a positive `double` value and two `Node` references. As with linked lists, you will find it helpful to make drawings using the visual representation shown in the text.

4.4.33 Implement the following methods, each of which takes as its argument a `Node` that is the root of a binary tree.

<code>int size()</code>	<i>number of nodes in the tree</i>
<code>int leaves()</code>	<i>number of nodes whose links are both null</i>
<code>double total()</code>	<i>sum of the key values in all nodes</i>

Your methods should all run in linear time.

4.4.34 Implement a linear-time method `height()` that returns the maximum number of links on any path from the root to a leaf node (the height of a one-node tree is 0).

4.4.35 A binary tree is *heap ordered* if the key at the root is larger than the keys in all of its descendants. Implement a linear-time method `heapOrdered()` that returns `true` if the tree is heap ordered, and `false` otherwise.

4.4.36 A binary tree is *balanced* if both its subtrees are balanced and the height of its two subtrees differ by at most 1. Implement a linear-time method `balanced()` that returns `true` if the tree is balanced, and `false` otherwise.

4.4.37 Two binary trees are *isomorphic* if only their key values differ (they have the same shape). Implement a linear-time static method `isomorphic()` that takes two tree references as arguments and returns `true` if they refer to isomorphic trees, and `false` otherwise. Then, implement a linear-time static method `eq()` that takes two tree references as arguments and returns `true` if they refer to identical trees (isomorphic with the same key values), and `false` otherwise.

4.4.38 Implement a linear-time method `isBST()` that returns `true` if the tree is a BST, and `false` otherwise.

Solution: This task is a bit more difficult than it might seem. Use an overloaded recursive method `isBST()` that takes two additional arguments `lo` and `hi` and returns `true` if the tree is a BST and all its values are between `lo` and `hi`, and use `null` to represent both the smallest possible and largest possible keys.

```
public static boolean isBST()
{   return isBST(root, null, null);  }

private boolean isBST(Node x, Key lo, Key hi)
{
    if (x == null) return true;
    if (lo != null && x.key.compareTo(lo) <= 0) return false;
    if (hi != null && x.key.compareTo(hi) >= 0) return false;
    if (!isBST(x.left, lo, x.key))  return false;
    if (!isBST(x.right, x.key, hi)) return false;
}
```

4.4.39 Write a method `levelOrder()` that prints BST keys in *level order*: first print the root; then the nodes one level below the root, left to right; then the nodes two levels below the root (left to right); and so forth. *Hint:* Use a `Queue<Node>`.

4.4.40 Compute the value returned by `mystery()` on some sample binary trees and then formulate a hypothesis about its behavior and prove it.

```
public int mystery(Node x)
{
    if (x == null) return 0;
    return mystery(x.left) + mystery(x.right);
}
```

Answer: Returns 0 for any binary tree.

Creative Exercises

4.4.41 Spell checking. Write a SET client `SpellChecker` that takes as a command-line argument the name of a file containing a dictionary of words, and then reads strings from standard input and prints any string that is not in the dictionary. You can find a dictionary file on the booksite. *Extra credit:* Augment your program to handle common suffixes such as `-ing` or `-ed`.

4.4.42 Spell correction. Write an ST client `SpellCorrector` that serves as a filter that replaces commonly misspelled words on standard input with a suggested replacement, printing the result to standard output. Take as a command-line argument the name of a file that contains common misspellings and corrections. You can find an example on the booksite.

4.4.43 Web filter. Write a SET client `WebBlocker` that takes as a command-line argument the name of a file containing a list of objectionable websites, and then reads strings from standard input and prints only those websites not on the list.

4.4.44 Set operations. Add methods `union()` and `intersection()` to SET that take two sets as arguments and return the union and intersection, respectively, of those two sets.

4.4.45 Frequency symbol table. Develop a data type `FrequencyTable` that supports the following operations: `click()` and `count()`, both of which take string arguments. The data type keeps track of the number of times the `click()` operation has been called with a given string as an argument. The `click()` operation increments the count by 1, and the `count()` operation returns the count, possibly 0. Clients of this data type might include a web-traffic analyzer, a music player that counts the number of times each song has been played, phone software for counting calls, and so forth.

4.4.46 One-dimensional range searching. Develop a data type that supports the following operations: insert a date, search for a date, and count the number of dates in the data structure that lie in a particular interval. Use Java's `java.util.Date` data type.

4.4.47 Non-overlapping interval search. Given a list of non-overlapping inter-

vals of integers, write a function that takes an integer argument and determines in which, if any, interval that value lies. For example, if the intervals are 1643–2033, 5532–7643, 8999–10332, and 5666653–5669321, then the query point 9122 lies in the third interval and 8122 lies in no interval.

4.4.48 *IP lookup by country.* Write a BST client that uses the data file `ip-to-country.csv` found on the booksite to determine the source country of a given IP address. The data file has five fields: beginning of IP address range, end of IP address range, two-character country code, three-character country code, and country name. The IP addresses are non-overlapping. Such a database tool can be used for credit card fraud detection, spam filtering, auto-selection of language on a website, and web-server log analysis.

4.4.49 *Inverted index of web.* Given a list of web pages, create a symbol table of words contained in those web pages. Associate with each word a list of web pages in which that word appears. Write a program that reads in a list of web pages, creates the symbol table, and supports single-word queries by returning the list of web pages in which that query word appears.

4.4.50 *Inverted index of web.* Extend the previous exercise so that it supports multi-word queries. In this case, output the list of web pages that contain at least one occurrence of each of the query words.

4.4.51 *Multiple word search.* Write a program that takes k words from the command line, reads in a sequence of words from standard input, and identifies the smallest interval of text that contains all of the k words (not necessarily in the same order). You do not need to consider partial words.

Hint: For each index i , find the smallest interval $[i, j]$ that contains the k query words. Keep a count of the number of times each of the k query words appears. Given $[i, j]$, compute $[i+1, j']$ by decrementing the counter for word i . Then, gradually increase j until the interval contains at least one copy of each of the k words (or, equivalently, word i).

4.4.52 *Repetition draw in chess.* In the game of chess, if a board position is repeated three times with the same side to move, the side to move can declare a draw.

Describe how you could test this condition using a computer program.

4.4.53 Registrar scheduling. The registrar at a prominent northeastern university recently scheduled an instructor to teach two different classes at the same exact time. Help the registrar prevent future mistakes by describing a method to check for such conflicts. For simplicity, assume all classes run for 50 minutes and start at 9, 10, 11, 1, 2, or 3.

4.4.54 Random element. Add to BST a method `random()` that returns a random key. Maintain subtree sizes in each node (see EXERCISE 4.4.29). The running time should be proportional to the height of the tree.

4.4.55 Order statistics. Add to BST a method `select()` that takes an integer argument k and returns the k th smallest key in the BST. Maintain subtree sizes in each node (see EXERCISE 4.4.29). The running time should be proportional to the height of the tree.

4.4.56 Rank query. Add to BST a method `rank()` that takes a key as an argument and returns the number of keys in the BST that are strictly smaller than `key`. Maintain subtree sizes in each node (see EXERCISE 4.4.29). The running time should be proportional to the height of the tree.

4.4.57 Generalized queue. Implement a class that supports the following API, which generalizes both a queue and a stack by supporting removal of the i th least recently inserted item (see EXERCISE 4.3.40):

<code>public class GeneralizedQueue<Item></code>	
<hr/>	
<code>GeneralizedQueue()</code>	<i>create an empty generalized queue</i>
<code>boolean isEmpty()</code>	<i>is the generalized queue empty?</i>
<code>void add(Item item)</code>	<i>insert item into the generalized queue</i>
<code>Item remove(int i)</code>	<i>remove and return the ith least recently inserted item</i>
<code>int size()</code>	<i>number of items in the queue</i>

API for a generic generalized queue

Use a BST that associates the k th item inserted into the data structure with the key k and maintains in each node the total number of nodes in the subtree rooted at that node. To find the i th least recently inserted item, search for the i th smallest key in the BST.

4.4.58 Sparse vectors. A d -dimensional vector is *sparse* if its number of nonzero values is small. Your goal is to represent a vector with space proportional to its number of nonzeros, and to be able to add two sparse vectors in time proportional to the total number of nonzeros. Implement a class that supports the following API:

<code>public class SparseVector</code>	
<code>SparseVector()</code>	<i>create a vector</i>
<code>void put(int i, double v)</code>	<i>set a_i to v</i>
<code>double get(int i)</code>	<i>return a_i</i>
<code>double dot(SparseVector b)</code>	<i>vector dot product</i>
<code>SparseVector plus(SparseVector b)</code>	<i>vector addition</i>

API for a sparse vector of double values

4.4.59 Sparse matrices. An n -by- n matrix is *sparse* if its number of nonzeros is proportional to n (or less). Your goal is to represent a matrix with space proportional to n , and to be able to add and multiply two sparse matrices in time proportional to the total number of nonzeros (perhaps with an extra $\log n$ factor). Implement a class that supports the following API:

<code>public class SparseMatrix</code>	
<code>SparseMatrix()</code>	<i>create a matrix</i>
<code>void put(int i, int j, double v)</code>	<i>set a_{ij} to v</i>
<code>double get(int i, int j)</code>	<i>return a_{ij}</i>
<code>SparseMatrix plus(SparseMatrix b)</code>	<i>matrix addition</i>
<code>SparseMatrix times(SparseMatrix b)</code>	<i>matrix product</i>

API for a sparse matrix of double values

4.4.60 *Queue with no duplicates items.* Create a data type that is a queue, except that an item may appear on the queue at most once at any given time. Ignore any request to insert an item if it is already on the queue.

4.4.61 *Mutable string.* Create a data type that supports the following API on a string. Use an ST to implement all operations in logarithmic time.

```
public class MutableString
```

MutableString()	<i>create an empty string</i>
char get(int i)	<i>return the <i>i</i>th character in the string</i>
void insert(int i, char c)	<i>insert <i>c</i> and make it the <i>i</i>th character</i>
void delete(int i)	<i>delete the <i>i</i>th character</i>
int length()	<i>return the length of the string</i>

API for a mutable string

4.4.62 *Assignment statements.* Write a program to parse and evaluate programs consisting of assignment and print statements with fully parenthesized arithmetic expressions (see PROGRAM 4.3.5). For example, given the input

```
A = 5
B = 10
C = A + B
D = C * C
print(D)
```

your program should print the value 225. Assume that all variables and values are of type `double`. Use a symbol table to keep track of variable names.

4.4.63 *Entropy.* We define the relative entropy of a text corpus with n words, k of which are distinct as

$$E = 1 / (n \lg n) (p_0 \lg(k/p_0) + p_1 \lg(k/p_1) + \dots + p_{k-1} \lg(k/p_{k-1}))$$

where p_i is the fraction of times that word i appears. Write a program that reads in a text corpus and prints the relative entropy. Convert all letters to lowercase and treat punctuation marks as whitespace.

4.4.64 *Dynamic discrete distribution.* Create a data type that supports the following two operations: `add()` and `random()`. The `add()` method should insert a new item into the data structure if it has not been seen before; otherwise, it should increase its frequency count by 1. The `random()` method should return an item at random, where the probabilities are weighted by the frequency of each item. Maintain subtree sizes in each node (see EXERCISE 4.4.29). The running time should be proportional to the height of the tree.

4.4.65 *Stock account.* Implement the two methods `buy()` and `sell()` in `StockAccount` (PROGRAM 3.2.8). Use a symbol table to store the number of shares of each stock.

4.4.66 *Codon usage table.* Write a program that uses a symbol table to print summary statistics for each codon in a genome taken from standard input (frequency per thousand), like the following:

UUU	13.2	UCU	19.6	UAU	16.5	UGU	12.4
UUC	23.5	UCC	10.6	UAC	14.7	UGC	8.0
UUA	5.8	UCA	16.1	UAA	0.7	UGA	0.3
UUG	17.6	UCG	11.8	UAG	0.2	UGG	9.5
CUU	21.2	CCU	10.4	CAU	13.3	CGU	10.5
CUC	13.5	CCC	4.9	CAC	8.2	CGC	4.2
CUA	6.5	CCA	41.0	CAA	24.9	CGA	10.7
CUG	10.7	CCG	10.1	CAG	11.4	CGG	3.7
AUU	27.1	ACU	25.6	AAU	27.2	AGU	11.9
AUC	23.3	ACC	13.3	AAC	21.0	AGC	6.8
AUA	5.9	ACA	17.1	AAA	32.7	AGA	14.2
AUG	22.3	ACG	9.2	AAG	23.9	AGG	2.8
GUU	25.7	GCU	24.2	GAU	49.4	GGU	11.8
GUC	15.3	GCC	12.6	GAC	22.1	GGC	7.0
GUU	8.7	GCA	16.8	GAA	39.8	GGA	47.2

4.4.67 *Unique substrings of length k.* Write a program that takes an integer command-line argument k , reads in text from standard input, and calculates the number of unique substrings of length k that it contains. For example, if the input is CGCCGGCCCG, then there are five unique substrings of length 3: CGC, CGG, GCG, GGC, and GGG. This calculation is useful in data compression. *Hint:* Use the string method `substring(i, i+k)` to extract the i th substring and insert into a symbol table. Test your program on a large genome from the booksite and on the first 10 million digits of π .

4.4.68 *Random phone numbers.* Write a program that takes an integer command-line argument n and prints n random phone numbers of the form (xxx) xxx-xxxx. Use a SET to avoid choosing the same number more than once. Use only legal area codes (you can find a file of such codes on the booksite).

4.4.69 *Password checker.* Write a program that takes a string as a command-line argument, reads a dictionary of words from standard input, and checks whether the command-line argument is a “good” password. Here, assume “good” means that it (1) is at least eight characters long, (2) is not a word in the dictionary, (3) is not a word in the dictionary followed by a digit 0-9 (e.g., hello5), (4) is not two words separated by a digit (e.g., hello2world), and (5) none of (2) through (4) hold for reverses of words in the dictionary.

4.5 Case Study: Small-World Phenomenon

THE MATHEMATICAL MODEL THAT WE USE for studying the nature of pairwise connections among entities is known as the *graph*. Graphs are important for studying the natural world and for helping us to better understand and refine the networks that we create. From models of the nervous system in neurobiology, to the study of the spread of infectious diseases in medical science, to the development of the telephone system, graphs have played a critical role in science and engineering over the past century, including the development of the Internet itself.

Some graphs exhibit a specific property known as the *small-world phenomenon*. You may be familiar with this property, which is sometimes known as *six degrees of separation*. It is the basic idea that, even though each of us has relatively few acquaintances, there is a relatively short chain of acquaintances (the six degrees of separation) separating us from one another. This hypothesis was validated experimentally by Stanley Milgram in the 1960s and modeled mathematically by Duncan Watts and Stephen Strogatz in the 1990s. In recent years, the principle has proved important in a remarkable variety of applications. Scientists are interested in small-world graphs because they model natural phenomena, and engineers are interested in building networks that take advantage of the natural properties of small-world graphs.

In this section, we address basic computational questions surrounding the study of small-world graphs. Indeed, the simple question

Does a given graph exhibit the small-world phenomenon?

can present a significant computational burden. To address this question, we will consider a graph-processing data type and several useful graph-processing clients. In particular, we will examine a client for computing *shortest paths*, a computation that has a vast number of important applications in its own right.

A persistent theme of this section is that the algorithms and data structures that we have been studying play a central role in graph processing. Indeed, you will see that several of the fundamental data types introduced earlier in this chapter help us to develop elegant and efficient code for studying the properties of graphs.

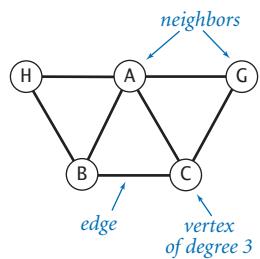
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Programs in this section

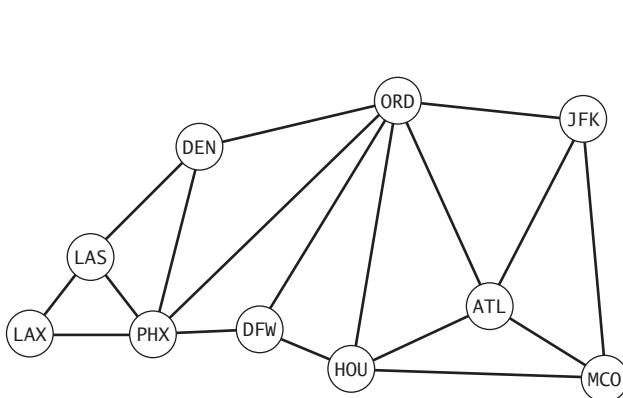
Graphs To nip in the bud any terminological confusion, we start right away with some definitions. A *graph* comprises of a set of *vertices* and a set of *edges*. Each edge represents a connection between two vertices. Two vertices are *adjacent* if they are connected by an edge, and the *degree* of a vertex is its number of adjacent vertices (or *neighbors*). Note that there is no relationship between a graph and the idea of a function graph (a plot of a function values) or the idea of graphics (drawings). We often visualize graphs by drawing labeled circles (vertices) connected by lines (edges), but it is always important to remember that it is the connections that are essential, not the way we depict them.

The following list suggests the diverse range of systems where graphs are appropriate starting points for understanding structure.

Transportation systems. Train tracks connect stations, roads connect intersections, and airline routes connect airports, so all of these systems naturally admit a simple graph model. No doubt you have used applications that are based on such models when getting directions from an interactive mapping program or a GPS device, or when using an online service to make travel reservations. What is the best way to get from here to there?



Graph terminology



Graph model of a transportation system

vertices	edges
JFK	JFK MCO
MCO	ORD DEN
ATL	ORD HOU
ORD	DFW PHX
HOU	JFK ATL
DFW	ORD DFW
PHX	ORD PHX
DEN	ATL HOU
LAX	DEN PHX
LAS	PHX LAX
	JFK ORD
	DEN LAS
	DFW HOU
	ORD ATL
	LAS LAX
	ATL MCO
	HOU MCO
	LAS PHX

<i>system</i>	<i>vertex</i>	<i>edge</i>
<i>natural phenomena</i>		
<i>circulatory</i>	organ	blood vessel
<i>skeletal</i>	joint	bone
<i>nervous</i>	neuron	synapse
<i>social</i>	person	relationship
<i>epidemiological</i>	person	infection
<i>chemical</i>	molecule	bond
<i>n-body</i>	particle	force
<i>genetic</i>	gene	mutation
<i>biochemical</i>	protein	interaction
<i>engineered systems</i>		
<i>transportation</i>	airport	route
	intersection	road
<i>communication</i>	telephone	wire
	computer	cable
	web page	link
<i>distribution</i>	power station	power line
	home	
	reservoir	
	home	
	warehouse	
	retail outlet	
<i>mechanical</i>	joint	beam
<i>software</i>	module	call
<i>financial</i>	account	transaction
<i>Typical graph models</i>		

Human biology. Arteries and veins connect organs, synapses connect neurons, and joints connect bones, so an understanding of the human biology depends on understanding appropriate graph models. Perhaps the largest and most important such modeling challenge in this arena is the human brain. How do local connections among neurons translate to consciousness, memory, and intelligence?

Social networks. People have relationships with other people. From the study of infectious diseases to the study of political trends, graph models of these relationships are critical to our understanding of their implications. Another fascinating problem is understanding how information propagates in online social networks.

Physical systems. Atoms connect to form molecules, molecules connect to form a material or a crystal, and particles are connected by mutual forces such as gravity or magnetism. For example, graph models are appropriate for studying the percolation problem that we considered in SECTION 2.4. How do local interactions propagate through such systems as they evolve?

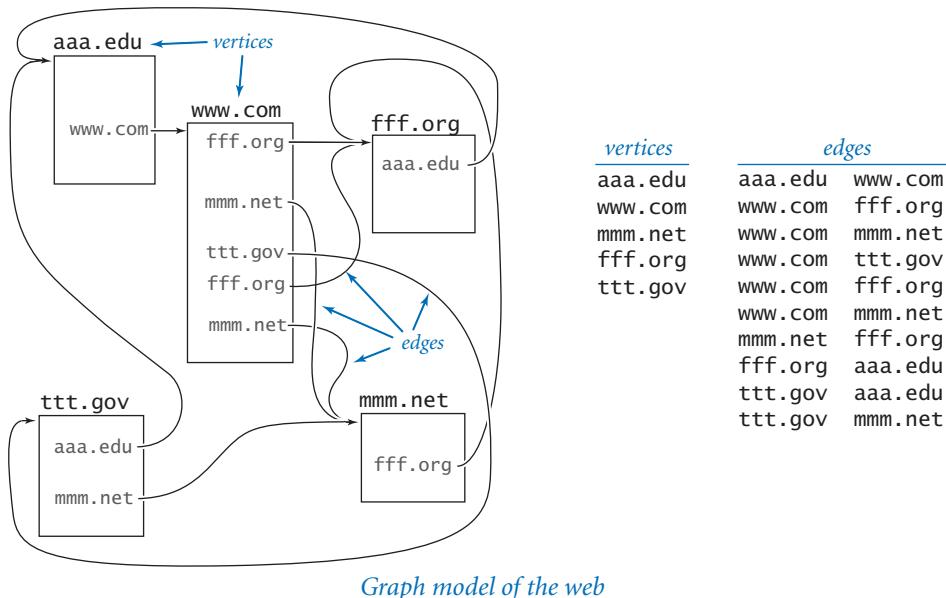
Communications systems. From electric circuits, to the telephone system, to the Internet, to wireless services, communications systems are all based on the idea of connecting devices. For at least the past century, graph models have played a critical role in the development of such systems. What is the best way to connect the devices?

Resource distribution. Power lines connect power stations and home electrical systems, pipes connect reservoirs and home plumbing, and truck routes connect warehouses and retail outlets. The study of effective and reliable means of distributing resources depends on accurate graph models. Where are the bottlenecks in a distribution system?

Mechanical systems. Trusses or steel beams connect joints in a bridge or a building. Graph models help us to design these systems and to understand their properties. Which forces must a joint or a beam withstand?

Software systems. Methods in one program module invoke methods in other modules. As we have seen throughout this book, understanding relationships of this sort is a key to success in software design. Which modules will be affected by a change in an API?

Financial systems. Transactions connect accounts, and accounts connect customers to financial institutions. These are but a few of the graph models that people use to study complex financial transactions, and to profit from better understanding them. Which transactions are routine and which are indicative of a significant event that might translate into profits?

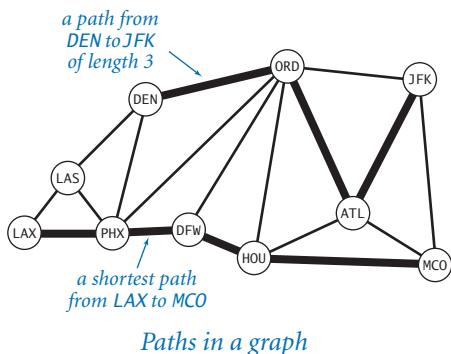


SOME OF THESE ARE MODELS OF natural phenomena, where our goal is to gain a better understanding of the natural world by developing simple models and then using them to formulate hypotheses that we can test. Other graph models are of networks that we engineer, where our goal is to design a better network or to better maintain a network by understanding its basic characteristics.

Graphs are useful models whether they are small or massive. A graph having just dozens of vertices and edges (for example, one modeling a chemical compound, where vertices are molecules and edges are bonds) is already a complicated combinatorial object because there are a huge number of possible graphs, so understanding the structures of the particular ones at hand is important. A graph having billions or trillions of vertices and edges (for example, a government database containing all phone-call metadata or a graph model of the human nervous system) is vastly more complex, and presents significant computational challenges.

Processing graphs typically involves building a graph from information in files and then answering questions about the graph. Beyond the application-specific questions in the examples just cited, we often need to ask basic questions about graphs. How many vertices and edges does the graph have? What are the neighbors of a given vertex? Some questions depend on an understanding of the structure of a graph. For example, a *path* in a graph is a sequence of adjacent vertices connected by edges. Is there a path connecting two given vertices? What is the *length* (number of edges) of the *shortest path* connecting two vertices? We have already seen in this book several examples of questions from scientific applications that are much more complicated than these. What is the probability that a random surfer will land on each vertex? What is the probability that a system represented by a certain graph percolates?

As you encounter complex systems in later courses, you are certain to encounter graphs in many different contexts. You may also study their properties in detail in later courses in mathematics, operations research, or computer science. Some graph-processing problems present insurmountable computational challenges; others can be solved with relative ease with data-type implementations of the sort we have been considering.



Graph data type Graph-processing algorithms generally first build an internal representation of a graph by adding edges, then process it by iterating over the vertices and over the vertices adjacent to a given vertex. The following API supports such processing:

```
public class Graph
```

Graph()	<i>create an empty graph</i>
Graph(String file, String delimiter)	<i>create graph from a file</i>
void addEdge(String v, String w)	<i>add edge v-w</i>
int V()	<i>number of vertices</i>
int E()	<i>number of edges</i>
Iterable<String> vertices()	<i>vertices in the graph</i>
Iterable<String> adjacentTo(String v)	<i>neighbors of v</i>
int degree(String v)	<i>number of neighbors of v</i>
boolean hasVertex(String v)	<i>is v a vertex in the graph?</i>
boolean hasEdge(String v, String w)	<i>is v-w an edge in the graph?</i>

API for a graph with String vertices

As usual, this API reflects several design choices, each made from among various alternatives, some of which we now briefly discuss.

Undirected graph. Edges are *undirected*: an edge that connects v to w is the same as one that connects w to v . Our interest is in the connection, not the direction. Directed edges (for example, one-way streets in road maps) require a slightly different data type (see EXERCISE 4.5.41).

String vertex type. We might use a generic vertex type, to allow clients to build graphs with objects of any type. We leave this sort of implementation for an exercise, however, because the resulting code becomes a bit unwieldy (see EXERCISE 4.5.9). The `String` vertex type suffices for the applications that we consider here.

Invalid vertex names. The methods `adjacentTo()`, `degree()`, and `hasEdge()` all throw an exception if called with a string argument that does not correspond to a vertex name. The client can call `hasVertex()` to detect such situations.

Implicit vertex creation. When a string is used as an argument to `addEdge()`, we assume that it is a vertex name. If no vertex using that name has yet been added, our implementation adds such a vertex. The alternative design of having an `addVertex()` method requires more client code (to create the vertices) and more cumbersome implementation code (to check that edges connect vertices that have previously been created).

Self-loops and parallel edges. Although the API does not explicitly address the issue, we assume that implementations *do* allow self-loops (edges connecting a vertex to itself) but *do not* allow parallel edges (two copies of the same edge). Checking for self-loops and parallel edges is easy; our choice is to omit both checks.

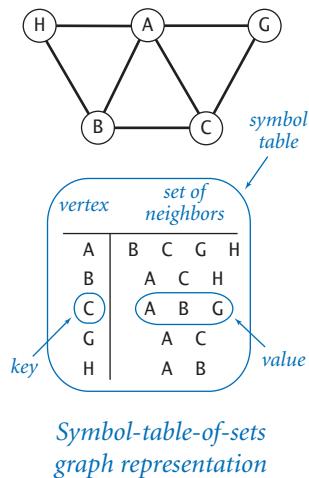
Client query methods. We also include the methods `V()` and `E()` in our API to provide to the client the number of vertices and edges in the graph. Similarly, the methods `degree()`, `hasVertex()`, and `hasEdge()` are useful in client code. We leave the implementation of these methods as exercises, but assume them to be in our Graph API.

NONE OF THESE DESIGN DECISIONS ARE sacrosanct; they are simply the choices that we have made for the code in this book. Some other choices might be appropriate in various situations, and some decisions are still left to implementations. It is wise to carefully consider the choices that you make for design decisions like this *and to be prepared to defend them*.

Graph (PROGRAM 4.5.1) implements this API. Its internal representation is a *symbol table of sets*: the keys are vertices and the values are the sets of neighbors—the vertices adjacent to the key. This representation uses the two data types ST and SET that we introduced in SECTION 4.4. It has three important properties:

- Clients can efficiently iterate over the graph vertices.
- Clients can efficiently iterate over a vertex’s neighbors.
- Memory usage is proportional to the number of edges.

These properties follow immediately from basic properties of ST and SET. As you will see, these two iterators are at the heart of graph processing.



Program 4.5.1 Graph data type

```

public class Graph
{
    private ST<String, SET<String>> st;           st | symbol table of vertex
                                                       neighbor sets
    public Graph()
    {   st = new ST<String, SET<String>>();   }

    public void addEdge(String v, String w)
    {   // Put v in w's SET and w in v's SET.
        if (!st.contains(v)) st.put(v, new SET<String>());
        if (!st.contains(w)) st.put(w, new SET<String>());
        st.get(v).add(w);
        st.get(w).add(v);
    }

    public Iterable<String> adjacentTo(String v)
    {   return st.get(v);   }

    public Iterable<String> vertices()
    {   return st.keys();   }

    // See Exercises 4.5.1-4 for V(), E(), degree(),
    // hasVertex(), and hasEdge().

    public static void main(String[] args)
    {   // Read edges from standard input; print resulting graph.
        Graph G = new Graph();
        while (!StdIn.isEmpty())
            G.addEdge(StdIn.readString(), StdIn.readString());
        StdOut.print(G);
    }
}

```

This implementation uses *ST* and *SET* (see SECTION 4.4) to implement the graph data type. Clients build graphs by adding edges and process them by iterating over the vertices and then over the set of vertices adjacent to each vertex. See the text for *toString()* and a matching constructor that reads a graph from a file.

```
% more tinyGraph.txt
A B
A C
C G
A G
H A
B C
B H
```

```
% java Graph < tinyGraph.txt
A B C G H
B A C H
C A B G
G A C
H A B
```

As a simple example of client code, consider the problem of printing a Graph. A natural way to proceed is to print a list of the vertices, along with a list of the neighbors of each vertex. We use this approach to implement `toString()` in `Graph`, as follows:

```
public String toString()
{
    String s = "";
    for (String v : vertices())
    {
        s += v + " ";
        for (String w : adjacentTo(v))
            s += w + " ";
        s += "\n";
    }
    return s;
}
```

This code prints two representations of each edge—once when discovering that w is a neighbor of v , and once when discovering that v is a neighbor of w . Many graph algorithms are based on this basic paradigm of processing each edge in the graph in this way, and it is important to remember that they process each edge twice. As usual, this implementation is intended for use only for small graphs, as the running time is quadratic in the string length because string concatenation is linear time.

The output format just considered defines a reasonable file format: each line is a vertex name followed by the names of neighbors of that vertex. Accordingly, our basic graph API includes a constructor for building a graph from a file in this format (list of vertices with neighbors). For flexibility, we allow for the use of other delimiters besides spaces for vertex names (so that, for example, vertex names may contain spaces), as in the following implementation:

```
public Graph(String filename, String delimiter)
{
    st = new ST<String, SET<String>>();
    In in = new In(filename);
    while (in.hasNextLine())
    {
        String line = in.readLine();
        String[] names = line.split(delimiter);
        for (int i = 1; i < names.length; i++)
            addEdge(names[0], names[i]);
    }
}
```

Adding this constructor and `toString()` to `Graph` provides a complete data type suitable for a broad variety of applications, as we will now see. Note that this same constructor (with a space delimiter) works properly when the input is a list of edges, one per line, as in the test client for PROGRAM 4.5.1.

Graph client example As a first graph-processing client, we consider an example of social relationships, one that is certainly familiar to you and for which extensive data is readily available.

On the booksite you can find the file `movies.txt` (and many similar files), which contains a list of movies and the performers who appeared in them. Each line gives the name of a movie followed by the cast (a list of the names of the performers who appeared in that movie). Since names have spaces and commas in them, the `/` character is used as a delimiter. (Now you can see why our second `Graph` constructor takes the delimiter as an argument.)

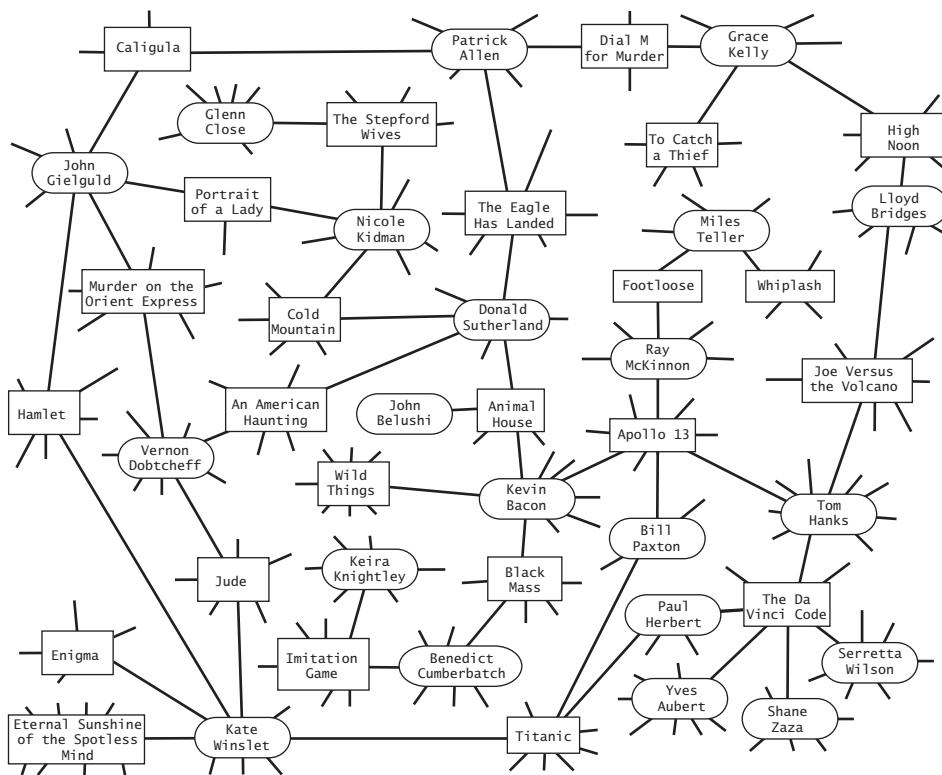
If you study `movies.txt`, you will notice a number of characteristics that, though minor, need attention when working with the database:

- Movies always have the year in parentheses after the title.
- Special characters are present.
- Multiple performers with the same name are differentiated by Roman numerals within parentheses.
- Cast lists are not in alphabetical order.

Depending on your terminal window and operating system settings, special characters may be replaced by blanks or question marks. These types of anomalies are common when working with large amounts of real-world data. You can either choose to live with them or configure your environment properly (see the booksite for details).

```
% more movies.txt
...
Tin Men (1987)/DeBoy, David/Blumenfeld, Alan/... /Geppi, Cindy/Hershey, Barbara
Tirez sur le pianiste (1960)/Heymann, Claude/.../Berger, Nicole (I)
Titanic (1997)/Mazin, Stan/...DiCaprio, Leonardo/.../Winslet, Kate/...
Titus (1999)/Weisskopf, Hermann/Rhys, Matthew/.../McEwan, Geraldine
To Be or Not to Be (1942)/Verebes, Ernö (I)/.../Lombard, Carole (I)
To Be or Not to Be (1983)/.../Brooks, Mel (I)/.../Bancroft, Anne/...
To Catch a Thief (1955)/París, Manuel/.../Grant, Cary/.../Kelly, Grace/...
To Die For (1995)/Smith, Kurtwood/.../Kidman, Nicole/.../ Tucci, Maria
...
```

Movie database example



Using Graph, we can write a simple and convenient client for extracting information from the file `movies.txt`. We begin by building a Graph to better structure the information. What should the vertices and edges model? Should the vertices be movies with edges connecting two movies if a performer has appeared in both? Should the vertices be performers with edges connecting two performers if both have appeared in the same movie? Both choices are plausible, but which should we use? This decision affects both client and implementation code. Another way to proceed (which we choose because it leads to simple implementation code) is to have vertices for *both* the movies and the performers, with an edge connecting each movie to each performer in that movie. As you will see, programs that process this graph can answer a great variety of interesting questions. `IndexGraph` (PROGRAM 4.5.2) is a first example that takes a query, such as the name of a movie, and prints the list of performers who appear in that movie.

Program 4.5.2 Using a graph to invert an index

```

public class IndexGraph
{
    public static void main(String[] args)
    { // Build a graph and process queries.
        String filename = args[0];
        String delimiter = args[1];
        Graph G = new Graph(filename, delimiter);
        while (StdIn.hasNextLine())
        { // Read a vertex and print its neighbors.
            String v = StdIn.readLine();
            for (String w : G.adjacentTo(v))
                StdOut.println(" " + w);
        }
    }
}

```

filename delimiter G v w	<i>filename</i> <i>input delimiter</i> <i>graph</i> <i>query</i> <i>neighbor of v</i>
---	---

This *Graph* client creates a graph from the file specified on the command line, then reads vertex names from standard input and prints its neighbors. When the file corresponds to a movie–cast list, the graph is bipartite and this program amounts to an interactive inverted index.

```

% java IndexGraph tinyGraph.txt " "
C
A
B
G
A
B
C
G
H
% java IndexGraph movies.txt "/"
Da Vinci Code, The (2006)
Aubert, Yves
...
Herbert, Paul
...
Wilson, Serretta
Zaza, Shane
Bacon, Kevin
Animal House (1978)
Apollo 13 (1995)
...
Wild Things (1998)
River Wild, The (1994)
Woodsman, The (2004)

```

Typing a movie name and getting its cast is not much more than regurgitating the corresponding line in `movies.txt` (though `IndexGraph` prints the cast list sorted by last name, as that is the default iteration order provided by `SET`). A more interesting feature of `IndexGraph` is that you can type the name of a *performer* and get the list of *movies* in which that performer has appeared. Why does this work? Even though `movies.txt` seems to connect movies to performers and not the other way around, the edges in the graph are *connections* that also connect performers to movies.

A graph in which connections all connect one kind of vertex to another kind of vertex is known as a *bipartite* graph. As this example illustrates, bipartite graphs have many natural properties that we can often exploit in interesting ways.

As we saw at the beginning of SECTION 4.4, the indexing paradigm is general and very familiar. It is worth reflecting on the fact that building a bipartite graph provides a simple way to automatically invert *any* index! The file `movies.txt` is indexed by movie, but we can query it by performer. You could use `IndexGraph` in precisely the same way to print the index words appearing on a given page or the codons corresponding to a given amino acid, or to invert any of the other indices discussed at the beginning of SECTION 4.2. Since `IndexGraph` takes the delimiter as a command-line argument, you can use it to create an interactive inverted index for a `.csv`.

This inverted-index functionality is a direct benefit of the graph *data structure*. Next, we examine some of the added benefits to be derived from *algorithms* that process the data structure.

```
% more amino.csv
TTT,Phe,F,Phenylalanine
TTC,Phe,F,Phenylalanine
TTA,Leu,L,Leucine
TTG,Leu,L,Leucine
TCT,Ser,S,Serine
TCC,Ser,S,Serine
TCA,Ser,S,Serine
TCG,Ser,S,Serine
TAT,Tyr,Y,Tyrosine
...
GGA,Gly,G,Glycine
GGG,Gly,G,Glycine

% java IndexGraph amino.csv ","
TTA
  Lue
    L
      Leucine
Serine
  TCT
  TCC
  TCA
  TCG
```

Inverting an index

Shortest paths in graphs Given two vertices in a graph, a *path* is a sequence of edges connecting them. A *shortest path* is one with the minimal *length* or *distance* (number of edges) over all such paths (there typically are multiple shortest paths). Finding a shortest path connecting two vertices in a graph is a fundamental problem in computer science. Shortest paths have been famously and successfully applied to solve large-scale problems in a broad variety of applications, from Internet routing to financial transactions to the dynamics of neurons in the brain.

As an example, imagine that you are a customer of an imaginary no-frills airline that serves a limited number of cities with a limited number of routes. Assume that the best way to get from one place to another is to minimize your number of flight segments, because delays in transferring from one flight to another are likely to be lengthy. A shortest-path algorithm is just what you need to plan a trip. Such an application appeals to our intuition in understanding the basic problem and our approach to solving it. After covering these topics in the context of this example, we will consider an application where the graph model is more abstract.

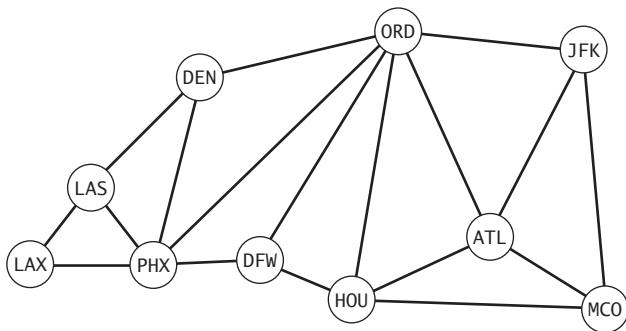
Depending upon the application, clients have various needs with regard to shortest paths. Do we want the shortest path connecting two given vertices? Or just the length of such a path? Will we have a large number of such queries? Is one particular vertex of special interest? In huge graphs or for huge numbers of queries, we have to pay particular attention to such questions because the cost of computing shortest paths might prove to be prohibitive. We start with the following API:

```
public class PathFinder
    PathFinder(Graph G, String s)
        constructor
    int distanceTo(String v)
        length of shortest path
        from s to v in G
    Iterable<String> pathTo(String v)
        shortest path
        from s to v in G
    
```

API for single-source shortest paths in a Graph

Clients can construct a `PathFinder` object for a given graph G and *source* vertex s , and then use that object either to find the length of a shortest path or to iterate over the vertices on a shortest path from s to any other vertex in G . An implementation of these methods is known as a *single-source shortest-path algorithm*. We will consider a classic algorithm for the problem, known as *breadth-first search*, which provides a direct and elegant solution.

Single-source client. Suppose that you have available to you the graph of vertices and connections for your no-frills airline's route map. Then, using your home city as the source, you can write a client that prints your route anytime you want to go on a trip. PROGRAM 4.5.3 is a client for `PathFinder` that provides this functionality for any graph. This sort of client is particularly useful in applications where we anticipate numerous queries from the same source. In this situation, the cost of building a `PathFinder` object is amortized over the cost of all the queries. You are encouraged to explore the properties of shortest paths by running `PathFinder` on our sample input file `routes.txt`.



source	destination	distance	a shortest path
JFK	LAX	3	JFK-ORD-PHX-LAX
LAS	MCO	4	LAS-PHX-DFW-HOU-MCO
HOU	JFK	2	HOU-ATL-JFK

Examples of shortest paths in a graph

former (except Bacon) who has been in the same cast as a performer whose number is 1 has a Kevin Bacon number of 2, and so forth. For example, Meryl Streep has a Kevin Bacon number of 1 because she appeared in *The River Wild* with Kevin Bacon. Nicole Kidman's number is 2: although she did not appear in any movie with Kevin Bacon, she was in *Cold Mountain* with Donald Sutherland, and Sutherland appeared in *Animal House* with Kevin Bacon. Given the name of a performer, the simplest version of the game is to find some alternating sequence of movies and performers that leads back to Kevin Bacon. For example, a movie buff might know that Tom Hanks was in *Joe Versus the Volcano* with Lloyd Bridges, who was in

Degrees of separation. One of the classic applications of shortest-paths algorithms is to find the *degrees of separation* of individuals in social networks. To fix ideas, we discuss this application in terms of a popular pastime known as the *Kevin Bacon game*, which uses the movie–performer graph that we just considered. Kevin Bacon is a prolific actor who has appeared in many movies. We assign every performer who has appeared in a movie a *Kevin Bacon number*: Bacon himself is 0, any performer who has been in the same cast as Bacon has a Kevin Bacon number of 1, any other per-

Program 4.5.3 Shortest-paths client

```

public class PathFinder
{
    // See Program 4.5.4 for implementation.

    public static void main(String[] args)
    {
        // Read graph and compute shortest paths from s.
        String filename = args[0];
        String delimiter = args[1];
        Graph G = new Graph(filename, delimiter);
        String s = args[2];
        PathFinder pf = new PathFinder(G, s);

        // Process queries.
        while (StdIn.hasNextLine())
        {
            String t = StdIn.readLine();
            int d = pf.distanceTo(t);
            for (String v : pf.pathTo(t))
                StdOut.println(" " + v);
            StdOut.println("distance " + d);
        }
    }
}

```

filename	<i>filename</i>
delimiter	<i>input delimiter</i>
G	<i>graph</i>
s	<i>source</i>
pf	<i>PathFinder from s</i>
t	<i>destination query</i>
v	<i>vertex on path</i>

This *PathFinder* client takes the name of a file, a delimiter, and a source vertex as command-line arguments. It builds a graph from the file, assuming that each line of the file specifies a vertex and a list of vertices connected to that vertex, separated by the delimiter. When you type a destination on standard input, you get the shortest path from the source to that destination.

<pre>% more routes.txt JFK MCO ORD DEN PHX LAX ORD HOU DFW PHX ORD DFW ... JFK ORD HOU MCO LAS PHX</pre>	<pre>% java PathFinder routes.txt " " JFK LAX JFK ORD PHX LAX distance 3 DFW JFK ORD DFW distance 2</pre>
--	---

High Noon with Grace Kelly, who was in *Dial M for Murder* with Patrick Allen, who was in *The Eagle Has Landed* with Donald Sutherland, who we know was in *Animal House* with Kevin Bacon. But this knowledge does not suffice to establish Tom Hanks's Bacon number (it is actually 1 because he was in *Apollo 13* with Kevin Bacon). You can see that the Kevin Bacon number has to be defined by counting the movies in the *shortest* such sequence, so it is hard to be sure whether someone wins the game without using a computer. Remarkably, the PathFinder test client in PROGRAM 4.5.3 is just the program you need to find a shortest path that establishes the Kevin Bacon number of any performer in `movies.txt`—the number is precisely half the distance. You might enjoy using this program, or extending it to answer some entertaining questions about the movie business or in one of many other domains. For example, mathematicians play this same game with the graph defined by paper co-authorship and their connection to Paul Erdős, a prolific 20th-century mathematician. Similarly, everyone in New Jersey seems to have a Bruce Springsteen number of 2, because everyone in the state seems to know someone who claims to know Bruce.

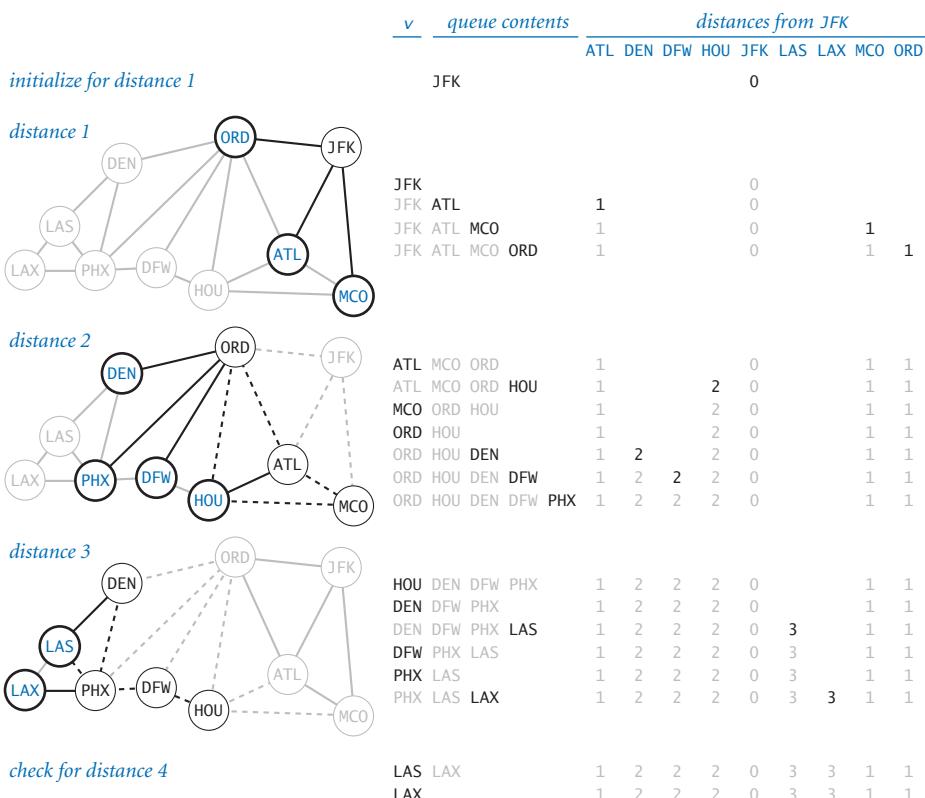
```
% java PathFinder movies.txt "/" "Bacon, Kevin"
Kidman, Nicole
Bacon, Kevin
Animal House (1978)
Sutherland, Donald (I)
Cold Mountain (2003)
Kidman, Nicole
distance 4
Hanks, Tom
Bacon, Kevin
Apollo 13 (1995)
Hanks, Tom
distance 2
```

Degrees of separation from Kevin Bacon

Other clients. PathFinder is a versatile data type that can be put to many practical uses. For example, it is easy to develop a client that handles arbitrary source-destination requests on standard input, by building a PathFinder for each vertex (see EXERCISE 4.5.17). Travel services use precisely this approach to handle requests at a very high service rate. Since this client builds a PathFinder for each vertex (each of which might consume memory proportional to the number of vertices), memory usage might be a limiting factor in using it for huge graphs. For an even more performance-critical application that is conceptually the same, consider an Internet router that has a graph of connections among machines available and must decide the best next stop for packets heading to a given destination. To do so, it can build a PathFinder with itself as the source; then, to send a packet to destination w , it computes $\text{pf}.\text{pathTo}(w)$ and sends the packet to the first vertex on that

path—the next stop on the shortest path to w . Or a central authority might build a `PathFinder` object for each of several dependent routers and use them to issue routing instructions. The ability to handle such requests at a high service rate is one of the prime responsibilities of Internet routers, and shortest-paths algorithms are a critical part of the process.

Shortest-path distances. The first step in understanding breadth-first search is to consider the problem of computing the lengths of the shortest paths from the source to each other vertex. Our approach is to compute and save away all the distances in the `PathFinder` constructor, and then just return the requested value



Using breadth-first search to compute shortest-path distances in a graph

when a client invokes `distanceTo()`. To associate an integer distance with each vertex name, we use a symbol table:

```
ST<String, Integer> dist = new ST<String, Integer>();
```

The purpose of this symbol table is to associate with each vertex an integer: the length of the shortest path (the distance) from `s` to that vertex. We begin by associating the distance 0 with `s` via the call `dist.put(s, 0)`, and we associate the distance 1 with `s`'s neighbors using the following code:

```
for (String v : G.adjacentTo(s))
    dist.put(v, 1)
```

But then what do we do? If we blindly set the distances to all the neighbors of each of those neighbors to 2, then not only would we face the prospect of unnecessarily setting many values twice (neighbors may have many common neighbors), but also we would set `s`'s distance to 2 (it is a neighbor of each of its neighbors), and we clearly do not want that outcome. The solution to these difficulties is simple:

- Consider the vertices in order of their distance from `s`.
- Ignore vertices whose distance to `s` is already known.

To organize the computation, we use a FIFO queue. Starting with `s` on the queue, we perform the following operations until the queue is empty:

- Dequeue a vertex `v`.
- Assign all of `v`'s unknown neighbors a distance 1 greater than `v`'s distance.
- Enqueue all of the unknown neighbors.

Breadth-first search dequeues the vertices in nondecreasing order of their distance from the source `s`. Tracing this algorithm on a sample graph will help to persuade you that it is correct. Showing that breadth-first search labels each vertex `v` with its distance to `s` is an exercise in mathematical induction (see EXERCISE 4.5.12).

Shortest-paths tree. We want not only the lengths of the shortest paths, but also the shortest paths themselves. To implement `pathTo()`, we use a subgraph known as the *shortest-paths tree*, defined as follows:

- Put the source at the root of the tree.
- Put vertex `v`'s neighbors in the tree if they are added to the queue when processing vertex `v`, with an edge connecting each to `v`.

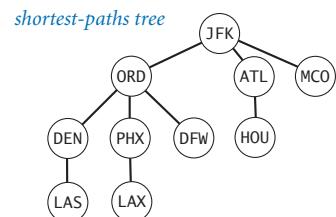
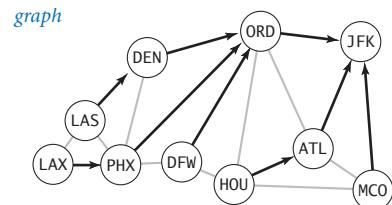
Since we enqueue each vertex only once, this structure is a proper tree: it consists of a root (the source) connected to one subtree for each neighbor of the source. Studying such a tree, you can see immediately that the distance from each vertex to

the root in the tree is the same as the length of the shortest path from the source in the graph. More importantly, each path in the tree is a shortest path in the graph. This observation is important because it gives us an easy way to provide clients with the shortest paths themselves. First, we maintain a symbol table associating each vertex with the vertex one step nearer to the source on the shortest path:

```
ST<String, String> prev;
prev = new ST<String, String>();
```

To each vertex w , we want to associate the previous stop on the shortest path from the source to w . Augmenting breadth-first search to compute this information is easy: when we enqueue w because we first discover it as a neighbor of v , we do so precisely because v is the previous stop on the shortest path from the source to w , so we can call `prev.put(w, v)` to record this information. The `prev` data structure is nothing more than a representation of the shortest-paths tree: it provides a link from each node to its parent in the tree. Then, to respond to a client request for a shortest path from the source to v , we

follow these links *up* the tree from v , which traverses the path in reverse order, so we push each vertex encountered onto a stack and then return that stack (an `Iterable`) to the client. At the top of the stack is the source s ; at the bottom of the stack is v ; and the vertices on the path from s to v are in between, so the client gets the path from s to v when using the return value from `pathTo()` in a `foreach` statement.



parent-link representation

ATL	DEN	DFW	HOU	JFK	LAS	LAX	MCO	ORD	PHX
JFK	ORF	ORD	ATL		DEN	PHX	JFK	JFK	ORD

Shortest-paths tree

shortest-paths tree (parent-link representation)

ATL	DEN	DFW	HOU	JFK	LAS	LAX	MCO	ORD	PHX
JFK	ORF	ORD	ATL		DEN	PHX	JFK	JFK	ORD

ATL	DEN	DFW	HOU	JFK	LAS	LAX	MCO	ORD	PHX
JFK	ORF	ORD	ATL		DEN	PHX	JFK	JFK	ORD

ATL	DEN	DFW	HOU	JFK	LAS	LAX	MCO	ORD	PHX
JFK	ORF	ORD	ATL		DEN	PHX	JFK	JFK	ORD

ATL	DEN	DFW	HOU	JFK	LAS	LAX	MCO	ORD	PHX
JFK	ORF	ORD	ATL		DEN	PHX	JFK	JFK	ORD

stack contents

LAX ← *destination*

PHX LAX

ORD PHX LAX

JFK ORD PHX LAX

source

path

Recovering a path from the shortest-paths tree with a stack

Breadth-first search. PathFinder (PROGRAM 4.5.4) is an implementation of the single-source shortest paths API that is based on the ideas just discussed. It maintains two symbol tables: one for the distance from the source to each vertex and the other for the previous stop on the shortest path from the source to each vertex. The constructor uses a FIFO queue to keep track of vertices that have been encountered (neighbors of vertices to which the shortest path has been found but whose neighbors have not yet been examined). This process is referred to as *breadth-first search* (BFS) because it searches broadly in the graph. By contrast, another important graph-search method known as *depth-first search* is based on a recursive method like the one we used for percolation in PROGRAM 2.4.5 and searches deeply into the graph. Depth-first search tends to find long paths; breadth-first search is guaranteed to find shortest paths.

Performance. The cost of graph-processing algorithms typically depends on two graph parameters: the number of vertices V and the number of edges E . As implemented in PathFinder, the time required by breadth-first search is linearithmic in the size of the input, proportional to $E \log V$ in the worst case. To convince yourself of this fact, first observe that the outer (while) loop iterates at most V times, once for each vertex, because we are careful to ensure that each vertex is enqueued at most once. Then observe that the inner (for) loop iterates a total of at most $2E$ times over all iterations, because we are careful to ensure that each edge is examined at most twice, once for each of the two vertices it connects. Each iteration of the loop requires at least one `contains()` operation and perhaps two `put()` operations, on symbol tables of size at most V . This linearithmic-time performance depends upon using a symbol table based on binary search trees (such as ST or `java.util.TreeMap`), which have logarithmic-time search and insert. Substituting a symbol table based on hash tables (such as `java.util.HashMap`) reduces the running time to be linear in the input size, proportional to E for typical graphs.

Program 4.5.4 Shortest-paths implementation

```

public class PathFinder
{
    private ST<String, Integer> dist;
    private ST<String, String> prev;

    public PathFinder(Graph G, String s)
    { // Use BFS to compute shortest path from source
        // vertex s to each other vertex in graph G.
        prev = new ST<String, String>();
        dist = new ST<String, Integer>();
        Queue<String> queue = new Queue<String>();
        queue.enqueue(s);
        dist.put(s, 0);
        while (!queue.isEmpty())
        { // Process next vertex on queue.
            String v = queue.dequeue();
            for (String w : G.adjacentTo(v))
            { // Check whether distance is already known.
                if (!dist.contains(w))
                { // Add to queue; save shortest-path information.
                    queue.enqueue(w);
                    dist.put(w, 1 + dist.get(v));
                    prev.put(w, v);
                }
            }
        }
    }

    public int distanceTo(String v)
    { return dist.get(v); }

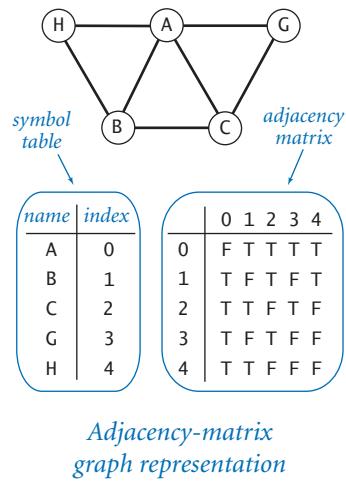
    public Iterable<String> pathTo(String v)
    { // Vertices on a shortest path from s to v.
        Stack<String> path = new Stack<String>();
        while (v != null && dist.contains(v))
        { // Push current vertex; move to previous vertex on path.
            path.push(v);
            v = prev.get(v);
        }
        return path;
    }
}

```

dist *distance from s*
prev *previous vertex on shortest path from s*
G *graph*
s *source*
q *queue of vertices*
v *current vertex*
w *neighbors of v*
PathFinder() *constructor for s in G*
distanceTo() *distance from s to v*
pathTo() *path from s to v*

This class uses breadth-first search to compute the shortest paths from a specified source vertex s to every vertex in graph G. See PROGRAM 4.5.3 for a sample client.

Adjacency-matrix representation. Without proper data structures, fast performance for graph-processing algorithms is sometimes not easy to achieve, and so should not be taken for granted. For example, an alternative graph representation, known as the *adjacency-matrix representation*, uses a symbol table to map vertex names to integers between 0 and $V - 1$, then maintains a V -by- V boolean array with `true` in the element in row i and column j (and the element in row j and column i) if there is an edge connecting the vertex corresponding to i with the vertex corresponding to j , and `false` if there is no such edge. We have already used similar representations in this book, when studying the random-surfer model for ranking web pages in SECTION 1.6. The adjacency-matrix representation is simple, but infeasible for use with huge graphs—a graph with a million vertices would require an adjacency matrix with a *trillion* elements. Understanding this distinction for graph-processing problems makes the difference between solving a problem that arises in a practical situation and not being able to address it at all.



Adjacency-matrix
graph representation

BREADTH-FIRST SEARCH IS A FUNDAMENTAL ALGORITHM that you could use to find your way around an airline route map or a city subway system (see EXERCISE 4.5.38) or in numerous similar situations. As indicated by our degrees-of-separation example, it also is used for countless other applications, from crawling the web and routing packets on the Internet to studying infectious disease, models of the brain, and relationships among genomic sequences. Many of these applications involve huge graphs, so an efficient algorithm is essential.

An important generalization of the shortest-paths problem is to associate a weight (which may represent distance or time) with each edge and seek to find a path that minimizes the sum of the edge weights. If you take later courses in algorithms or in operations research, you will learn a generalization of breadth-first search known as *Dijkstra's algorithm* that solves this problem in linearithmic time. When you get directions from a GPS device or a map application on the web, Dijkstra's algorithm is the basis for solving the associated shortest-path problems. These important and omnipresent applications are just the tip of an iceberg, because graph models are much more general than maps.

Small-world graphs Scientists have identified a particularly interesting class of graphs, known as *small-world graphs*, that arise in numerous applications in the natural and social sciences. Small-world graphs are characterized by the following three properties:

- They are *sparse*: the number of edges is much smaller than the total potential number of edges for a graph with the specified number of vertices.
- They have *short average path lengths*: if you pick two random vertices, the length of the shortest path between them is short.
- They exhibit *local clustering*: if two vertices are neighbors of a third vertex, then the two vertices are likely to be neighbors of each other.

We refer to graphs having these three properties collectively as exhibiting the *small-world phenomenon*. The term *small world* refers to the idea that the preponderance of vertices have both local clustering and short paths to other vertices. The modifier *phenomenon* refers to the unexpected fact that so many graphs that arise in practice are sparse, exhibit local clustering, and have short paths. Beyond the social-relationships applications just considered, small-world graphs have been used to study the marketing of products or ideas, the formation and spread of fame and fads, the analysis of the Internet, the construction of secure peer-to-peer networks, the development of routing algorithms and wireless networks, the design of electrical power grids, modeling information processing in the human brain, the study of phase transitions in oscillators, the spread of infectious viruses (in both living organisms and computers), and many other applications. Starting with the seminal work of Watts and Strogatz in the 1990s, an intensive amount of research has gone into quantifying the small-world phenomenon.

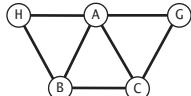
A key question in such research is the following: *given a graph, how can we tell whether it is a small-world graph?* To answer this question, we begin by imposing the conditions that the graph is not small (say, 1,000 vertices or more) and that it is *connected* (there exists *some* path connecting each pair of vertices). Then, we need to settle on specific thresholds for each of the small-world properties:

- By *sparse*, we mean the average vertex degree is less than $20 \lg V$.
- By *short average path length*, we mean the average length of the shortest path between two vertices is less than $10 \lg V$.
- By *locally clustered*, we mean that a certain quantity known as the *clustering coefficient* should be greater than 10%.

The definition of locally clustered is a bit more complicated than the definitions of sparsity and average path length. Intuitively, the clustering coefficient of a vertex

represents the probability that if you pick two of its neighbors at random, they will also be connected by an edge. More precisely, if a vertex has t neighbors, then there are $t(t-1)/2$ possible edges that connect those neighbors; its *local clustering coefficient* is the fraction of those edges that are in the graph. 0 if the vertex has degree 0 or 1. The *clustering coefficient of a graph* is the average of the local clustering coefficients of its vertices. If that average is greater than 10%, we say that the graph is locally clustered. The diagram below calculates these three quantities for a tiny graph.

average vertex degree		average path length			clustering coefficient			
vertex	degree	vertex pair	shortest path	length	vertex	degree	edges in neighborhood	
						actual	possible	
A	4	A B	A-B	1	A	4	3	
B	3	A C	A-C	1	B	3	2	
C	3	A G	A-G	1	C	3	2	
G	2	A H	A-H	1	G	2	1	
H	2	B C	B-C	1	H	2	1	
<u>total</u>		B G	B-A-G	2			6	
average degree = $14/5 = 2.8$		B H	B-H	1				
		C G	C-G	1				
		C H	C-A-H	2				
		G H	G-A-H	2				
		<u>total</u>		<u>13</u>	$\frac{\text{total of lengths}}{\text{number of pairs}} = \frac{13}{10} = 1.3$			
		$\frac{3/6 + 2/3 + 2/3 + 1/1 + 1/1}{5} \approx 0.767$						



Calculating small-world graph characteristics

To better familiarize you with these definitions, we next define some simple graph models, and consider whether they describe small-world graphs by checking the three requisite properties.

Complete graphs. A *complete graph* with V vertices has $V(V-1)/2$ edges, one connecting each pair of vertices. Complete graphs are *not* small-world graphs. They have short average path length (every shortest path has length 1) and they exhibit local clustering (the cluster coefficient is 1), but they are *not* sparse (the average vertex degree is $V-1$, which is much greater than $20 \lg V$ for large V).

Ring graphs. A *ring graph* is a set of V vertices equally spaced on the circumference of a circle, with each vertex adjacent to its neighbor on either side. In a k -*ring graph*, each vertex is adjacent to its k nearest neighbors on either side. The diagram

at right illustrates a 2-ring graph with 16 vertices. Ring graphs are also *not* small-world graphs. For example, 2-ring graphs are sparse (every vertex has degree 4) and are locally clustered (the cluster coefficient is 1/2), but their average path length is not short (see EXERCISE 4.5.20).

Random graphs. The *Erdős–Renyi model* is a well-studied model for generating random graphs. In this model, we build a *random graph* on V vertices by including each possible edge with probability p . Random graphs with a sufficient number of edges are very likely to be connected and have short average path lengths, but they are *not* small-world graphs because they are not locally clustered (see EXERCISE 4.5.46).

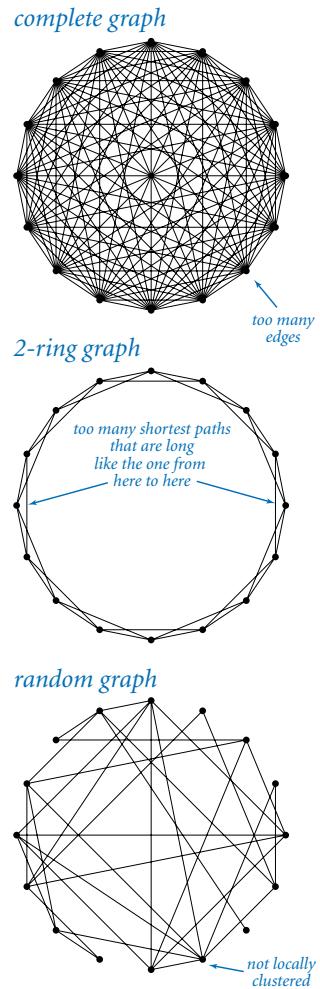
THESE EXAMPLES ILLUSTRATE THAT DEVELOPING A graph model that satisfies all three properties simultaneously is a puzzling challenge. Take a moment to try to design a graph model that you think might do so. After you have thought about this problem, you will realize that you are likely to need a program to help with calculations. Also, you may agree that it is quite surprising that they are found so often in practice. Indeed, you might be wondering if *any* graph is a small-world graph!

Choosing 10% for the clustering threshold instead of some other fixed percentage is somewhat arbitrary, as is the choice of $20 \lg V$ for the sparsity threshold and $10 \lg V$ for the short paths threshold, but we often do not come close to these borderline values. For example, consider the *web graph*, which has a vertex for each web page and an edge connecting two web pages if they are connected by a link. Scientists estimate that

the number of clicks to get from one web page to another is rarely more than about 30. Since there are billions of web pages, this estimate implies that the average path length is very short, much lower than our $10 \lg V$ threshold (which would be about 300 for 1 billion vertices).

model	sparse?	short paths?	locally clustered?
complete	○	●	●
2-ring	●	○	●
random	●	●	○

Small-world properties of graph models



Three graph models

Program 4.5.5 Small-world test

```

public class SmallWorld
{
    public static double averageDegree(Graph G)
    {   return 2.0 * G.E() / G.V();   }

    public static double averagePathLength(Graph G)
    { // Compute average vertex distance.
        int sum = 0;
        for (String v : G.vertices())
        { // Add to total distances from v.
            PathFinder pf = new PathFinder(G, v);
            for (String w : G.vertices())
                sum += pf.distanceTo(w);
        }
        return (double) sum / (G.V() * (G.V() - 1));
    }

    public static double clusteringCoefficient(Graph G)
    { // Compute clustering coefficient.
        double total = 0.0;
        for (String v : G.vertices())
        { // Cumulate local clustering coefficient of vertex v.
            int possible = G.degree(v) * (G.degree(v) - 1);
            int actual = 0;
            for (String u : G.adjacentTo(v))
                for (String w : G.adjacentTo(v))
                    if (G.hasEdge(u, w)) actual++;
            if (possible > 0)
                total += 1.0 * actual / possible;
        }
        return total / G.V();
    }

    public static void main(String[] args)
    { /* See Exercise 4.5.24. */   }
}

```

G graph
sum cumulative sum of
v distances between vertices
w vertex iterator variable
u, w neighbors of v

G graph
possible cumulative sum of
actual possible local edges
v cumulative sum of
u, w actual local edges
u, w vertex iterator variable
u, w neighbors of v

This client reads a graph from a file and computes the values of various graph parameters to test whether the graph exhibits the small-world phenomenon.

```
% java SmallWorld tinyGraph.txt " "
5 vertices, 7 edges
average degree      = 2.800
average path length = 1.300
clustering coefficient = 0.767
```

Having settled on the definitions, testing whether a graph is a small-world graph can still be a significant computational burden. As you probably have suspected, the graph-processing data types that we have been considering provide precisely the tools that we need. `SmallWorld` (PROGRAM 4.5.5) is a `Graph` and `PathFinder` client that implements these tests. Without the efficient data structures and algorithms that we have been considering, the cost of this computation would be prohibitive. Even so, for large graphs (such as `movies.txt`), we must resort to statistical sampling to estimate the average path length and the cluster coefficient in a reasonable amount of time (see EXERCISE 4.5.44) because the functions `averagePathLength()` and `clusteringCoefficient()` take quadratic time.

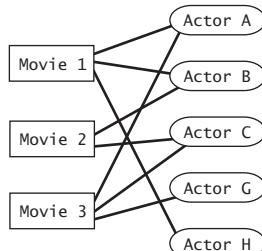
A classic small-world graph. Our movie–performer graph is not a small-world graph, because it is bipartite and therefore has a clustering coefficient of 0. Also, some pairs of performers are not connected to each other by any paths. However, the simpler *performer–performer* graph defined by connecting two performers by an edge if they appeared in the same movie is a classic example of a small-world graph (after discarding performers not connected to Kevin Bacon). The diagram below illustrates the movie–performer and performer–performer graphs associated with a tiny movie-cast file.

`Performer` (PROGRAM 4.5.6) is a program that creates a performer–performer graph from a file in our movie-cast input format. Recall that each line in a movie-cast file consists of a movie followed by all of the performers who appeared in that movie, delimited by slashes. `Performer` adds an edge connecting each pair of performers who appear in that movie. Doing so for each movie in the input produces a graph that connects the performers, as desired.

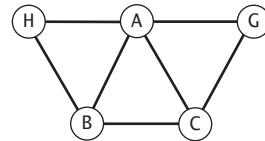
movie-cast file

```
% more tinyMovies.txt
Movie 1/Actor A/Actor B/Actor H
Movie 2/Actor B/Actor C
Movie 3/Actor A/Actor C/Actor G
```

movie–performer graph



performer–performer graph



Two different graph representations of a movie-cast file

Program 4.5.6 Performer–performer graph

```

public class Performer
{
    public static void main(String[] args)
    {
        String filename = args[0];
        String delimiter = args[1];
        Graph G = new Graph();
        In in = new In(filename);
        while (in.hasNextLine())
        {
            String line = in.readLine();
            String[] names = line.split(delimiter);
            for (int i = 1; i < names.length; i++)
                for (int j = i+1; j < names.length; j++)
                    G.addEdge(names[i], names[j]);
        }

        double degree = SmallWorld.averageDegree(G);
        double length = SmallWorld.averagePathLength(G);
        double cluster = SmallWorld.clusteringCoefficient(G);
        StdOut.printf("number of vertices      = %7d\n", G.V());
        StdOut.printf("average degree          = %7.3f\n", degree);
        StdOut.printf("average path length     = %7.3f\n", length);
        StdOut.printf("clustering coefficient = %7.3f\n", cluster);
    }
}

```

`G` | *graph*
`in` | *input stream for file*
`line` | *one line of movie-cast file*
`names[]` | *movie and actors*
`i, j` | *indices of two actors*

This program is a SmallWorld client takes the name of a movie-cast file and a delimiter as command-line arguments and creates the associated performer–performer graph. It prints to standard output the number of vertices, the average degree, the average path length, and the clustering coefficient of this graph. It assumes that the performer–performer graph is connected (see EXERCISE 4.5.29) so that the average page length is defined.

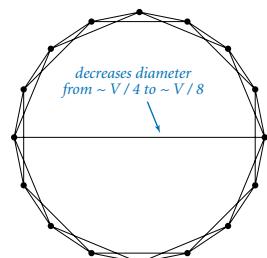
% java Performer tinyMovies.txt "/"	% java Performer moviesG.txt "/"
number of vertices = 5	number of vertices = 19044
average degree = 2.800	average degree = 148.688
average path length = 1.300	average path length = 3.494
clustering coefficient = 0.767	clustering coefficient = 0.911

Since a performer–performer graph typically has many more edges than the corresponding movie–performer graph, we will work for the moment with the smaller performer–performer graph derived from the file `moviesG.txt`, which contains 1,261 G-rated movies and 19,044 performers (all of which are connected to Kevin Bacon). Now, `Performer` tells us that the performer–performer graph associated with `moviesG.txt` has 19,044 vertices and 1,415,808 edges, so the average vertex degree is 148.7 (about half of $20 \lg V = 284.3$), which means it is sparse; its average path length is 3.494 (much less than $10 \lg V = 142.2$), so it has short paths; and its clustering coefficient is 0.911, so it has local clustering. We have found a small-world graph! These calculations validate the hypothesis that social-relationship graphs of this sort exhibit the small-world phenomenon. You are encouraged to find other real-world graphs and to test them with `SmallWorld`.

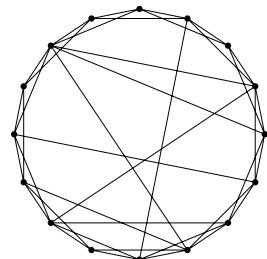
One approach to understanding something like the small-world phenomenon is to develop a mathematical model that we can use to test hypotheses and to make predictions. We conclude by returning to the problem of developing a graph model that can help us to better understand the small-world phenomenon. The trick to developing such a model is to combine two sparse graphs: a 2-ring graph (which has a high cluster coefficient) and a random graph (which has a small average path length).

Ring graphs with random shortcuts. One of the most surprising facts to emerge from the work of Watts and Strogatz is that adding a relatively small number of random edges to a sparse graph with local clustering produces a small-world graph. To gain some insight into why this is the case, consider a 2-ring graph, where the diameter (the length of the path between the farthest pair of vertices) is $\sim V/4$ (see the figure at right). Adding a single edge connecting antipodal vertices decreases the diameter to $\sim V/8$ (see EXERCISE 4.5.21). Adding $V/2$ random “shortcut” edges to a 2-ring graph is extremely likely to significantly lower the average path length, making it logarithmic (see EXERCISE 4.5.25). Moreover, it does so while increasing the average degree by only 1 and without lowering the cluster coefficient much below 1/2. That is, a 2-ring graph with $V/2$ random shortcut edges is extremely likely to be a small-world graph!

2-ring with antipodal edge



2-ring with random shortcuts



A new graph model

GENERATORS THAT CREATE GRAPHS DRAWN FROM such models are simple to develop, and we can use `SmallWorld` to determine whether the graphs exhibit the small-world phenomenon (see EXERCISE 4.5.24). We also can verify the analytic results that we derived for simple graphs such as `tinyGraph.txt`, complete graphs, and ring graphs. As with most scientific research, however, new questions arise as quickly as we answer the old ones. How many random shortcuts do we need to add to get a short average path length? What is the average path length and the clustering coefficient in a random connected graph? Which other graph models might be appropriate for study? How many samples do we need to accurately estimate the clustering coefficient or the average path length in a huge graph? You can find in the exercises many suggestions for addressing such questions and for further investigations of the small-world phenomenon. With the basic tools and the approach to programming developed in this book, you are well equipped to address this and many other scientific questions.

<i>model</i>	<i>average degree</i>	<i>average path length</i>	<i>clustering coefficient</i>
<i>complete</i>	999 ○	1 ●	1.0 ●
<i>2-ring</i>	4 ●	125.38 ○	0.5 ●
<i>random connected graph with $p = 10/V$</i>	10 ●	3.26 ●	0.010 ○
<i>2-ring with $V/2$</i>	5 ●	5.71 ●	0.343 ●
<i>random shortcuts</i>			

*Small-world parameters
for various 1,000-vertex graphs*

Lessons This case study illustrates the importance of algorithms and data structures in scientific research. It also reinforces several of the lessons that we have learned throughout this book, which are worth repeating.

Carefully design your data type. One of our most persistent messages throughout this book is that effective programming is based on a precise understanding of the possible set of data-type values and the set of operations defined on those values. Using a modern object-oriented programming language such as Java provides a path to this understanding because we design, build, and use our own data types. Our `Graph` data type is a fundamental one, the product of many iterations and experience with the design choices that we have discussed. The clarity and simplicity of our client code are testimony to the value of taking seriously the design and implementation of basic data types in any program.

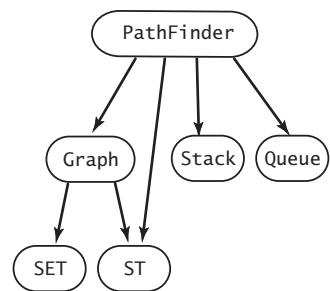
Develop code incrementally. As with all of our other case studies, we build software one module at a time, testing and learning about each module before moving to the next.

Solve problems that you understand before addressing the unknown. Our shortest-paths example involving air routes between a few cities is a simple one that is easy to understand. It is just complicated enough to hold our interest while debugging and following through a trace, but not so complicated as to make these tasks unnecessarily laborious.

Keep testing and check results. When working with complex programs that process huge amounts of data, you cannot be too careful in checking your results. Use common sense to evaluate every bit of output that your program produces. Novice programmers have an optimistic mindset (“If the program produces an answer, it must be correct”); experienced programmers know that a pessimistic mindset (“There must be something wrong with this result”) is far better.

Use real-world data. The `movies.txt` file from the *Internet Movie Database* is just one example of the data files that are now omnipresent on the web. In past years, such data was often cloaked behind private or parochial formats, but most people are now realizing that simple text formats are much preferred. The various methods in Java’s `String` data type make it easy to work with real data, which is the best way to formulate hypotheses about real-world phenomena. Start working with small files in the real-world format, so that you can test and learn about performance before attacking huge files.

Reuse software. Another of our most persistent messages in this book is that effective programming is based on an understanding of the fundamental data types available for our use, so that we do not have to rewrite code for basic functionality. Our use of `ST` and `SET` in `Graph` is a prime example—most programmers still use lower-level representations and implementations that use linked lists or arrays for graphs, which means, inevitably, that they are rewriting code for simple operations such as maintaining and traversing linked lists. Our shortest-paths class `PathFinder` uses `Graph`, `ST`, `SET`, `Stack`, and `Queue`—an all-star lineup of fundamental data structures.



Code reuse for PathFinder

Maintain flexibility. Reusing software often means using classes in various Java libraries. These classes are generally very wide interfaces (i.e., they contain many methods), so it is always wise to define and implement your own APIs with narrow interfaces between clients and implementations, even if your implementations are all calls on Java library methods. This approach provides the flexibility that you need to switch to more effective implementations when warranted and avoids dependence on changes to parts of the library that you do not use. For example, using ST in our Graph implementation (PROGRAM 4.5.1) gives us the flexibility to use any of our symbol-table implementations (such as HashST or BST) or to use Java’s symbol-table implementations (`java.util.TreeMap` and `java.util.HashMap`) without having to change Graph at all.

Performance matters. Without good algorithms and data structures, many of the problems that we have addressed in this chapter would go unsolved, because naïve methods require an impossible amount of time or space. Maintaining an awareness of the approximate resource needs of our programs is essential.

THIS CASE STUDY IS AN APPROPRIATE place to end this chapter because it well illustrates that the programs we have considered are a starting point, not a complete study. The programming skills that we have covered so far are a starting point, too, for your further study in science, mathematics, engineering, or any field of study where computation plays a significant role (almost any field, nowadays). The approach to programming and the tools that you have learned here should prepare you well for addressing any computational problem whatsoever.

Having developed familiarity and confidence with programming in a modern language, you are now well prepared to be able to appreciate important intellectual ideas around computation. These can take you to new levels of engagement with computation that are certain to serve you well however you encounter it in the future. Next, we embark on that journey.

Q&A

Q. How many different graphs are there with V given vertices?

A. With no self-loops or parallel edges, there are $V(V-1)/2$ possible edges, each of which can be present or not present, so the grand total is $2^{V(V-1)/2}$. The number grows to be huge quite quickly, as shown in the following table:

V	1	2	3	4	5	6	7	8	9
$2^{V(V-1)/2}$	1	2	8	64	1,024	32,768	2,097,152	268,435,456	68,719,476,736

These huge numbers provide some insight into the complexities of social relationships. For example, if you just consider the next nine people whom you see on the street, there are more than 68 *trillion* mutual-acquaintance possibilities!

Q. Can a graph have a vertex that is not adjacent to any other vertex?

A. Good question. Such vertices are known as *isolated vertices*. Our implementation disallows them. Another implementation might choose to allow isolated vertices by including an explicit `addVertex()` method for the *add-a-vertex* operation.

Q. Why not just use a linked-list representation for the neighbors of each vertex?

A. You can do so, but you are likely to wind up reimplementing basic linked-list code as you discover that you need the size, an iterator, and so forth.

Q. Why do the `V()` and `E()` query methods need to have constant-time implementations?

A. It might seem that most clients would call such methods only once, but an extremely common idiom is to use code like

```
for (int i = 0; i < G.E(); i++)
{ ... }
```

which would take quadratic time if you were to use a lazy algorithm that counts the edges instead of maintaining an instance variable with the number of edges. See EXERCISE 4.5.1.

Q. Why are `Graph` and `PathFinder` in separate classes? Wouldn't it make more sense to include the `PathFinder` methods in the `Graph` API?

A. Finding shortest paths is just one of many graph-processing problems. It would be poor software design to include all of them in a single API. Please reread the discussion of wide interfaces in SECTION 3.3.

Exercises

4.5.1 Add to `Graph` the implementations of `V()` and `E()` that return the number of vertices and edges in the graph, respectively. Make sure that your implementations take constant time. *Hint:* For `V()`, you may assume that the `size()` method in `ST` takes constant time; for `E()`, maintain an instance variable that holds the current number of edges in the graph.

4.5.2 Add to `Graph` a method `degree()` that takes a string argument and returns the degree of the specified vertex. Use this method to find the performer in the file `movies.txt` who has appeared in the most movies.

Answer:

```
public int degree(String v)
{
    if (st.contains(v)) return st.get(v).size();
    else                  return 0;
}
```

4.5.3 Add to `Graph` a method `hasVertex()` that takes a string argument and returns `true` if it names a vertex in the graph, and `false` otherwise.

4.5.4 Add to `Graph` a method `hasEdge()` that takes two string arguments and returns `true` if they specify an edge in the graph, and `false` otherwise.

4.5.5 Create a copy constructor for `Graph` that takes as its argument a graph `G`, then creates and initializes a new, independent copy of the graph. Any future changes to `G` should not affect the newly created graph.

4.5.6 Write a version of `Graph` that supports explicit vertex creation and allows self-loops, parallel edges, and isolated vertices. *Hint:* Use a Queue for the adjacency lists instead of a SET.

4.5.7 Add to `Graph` a method `remove()` that takes two string arguments and deletes the specified edge from the graph, if present.

4.5.8 Add to `Graph` a method `subgraph()` that takes a `SET<String>` as its argument and returns the *induced subgraph* (the graph comprising the specified vertices together with all edges from the original graph that connect any two of them).

4.5.9 Write a version of `Graph` that supports generic comparable vertex types (easy). Then, write a version of `PathFinder` that uses your implementation to support finding shortest paths using generic comparable vertex types (more difficult).

4.5.10 Create a version of `Graph` from the previous exercise to support bipartite graphs (graphs whose edges all connect a vertex of one generic comparable type to a vertex of another generic comparable type).

4.5.11 *True or false:* At some point during breadth-first search the queue can contain two vertices, one whose distance from the source is 7 and one whose distance is 9.

Answer: False. The queue can contain vertices of at most two distinct distances d and $d+1$. Breadth-first search examines the vertices in increasing order of distance from the source. When examining a vertex at distance d , only vertices of distance $d+1$ can be enqueued.

4.5.12 Prove by induction that `PathFinder` computes shortest paths (and shortest-path distances) from the source to each vertex.

4.5.13 Suppose you use a stack instead of a queue for breadth-first search in `PathFinder`. Does it still compute a path from the source to each vertex? Does it still compute shortest paths? In each case, prove that it does or give a counterexample.

4.5.14 What would be the effect of using a queue instead of a stack when forming the shortest path in `pathTo()`?

4.5.15 Add a method `isReachable(v)` to `PathFinder` that returns `true` if there exists *some* path from the source to v , and `false` otherwise.

4.5.16 Write a `Graph` client that reads a `Graph` from a file (in the file format specified in the text), then prints the edges in the graph, one per line.

4.5.17 Implement a `PathFinder` client `AllShortestPaths` that creates a `PathFinder` object for each vertex, with a test client that takes from standard input two-vertex queries and prints the shortest path connecting them. Support a delimiter, so that you can type the two-string queries on one line (separated by the delimiter) and get as output a shortest path between them. *Note:* For `movies.txt`, the query strings may both be performers, both be movies, or be a performer and a movie.

4.5.18 Write a program that plots average path length versus the number of random edges as random shortcuts are added to a 2-ring graph on 1,000 vertices.

4.5.19 Add an overloaded function `clusterCoefficient()` that takes an integer argument k to `SmallWorld` (PROGRAM 4.5.5) so that it computes a local cluster coefficient for the graph based on the total edges present and the total edges possible among the set of vertices within distance k of each vertex. When k is equal to 1, the function produces results identical to the no-argument version of the function.

4.5.20 Show that the cluster coefficient in a k -ring graph is $(2k-2) / (2k-1)$. Derive a formula for the average path length in a k -ring graph on V vertices as a function of both V and k .

4.5.21 Show that the diameter in a 2-ring graph on V vertices is $\sim V/4$. Show that if you add one edge connecting two antipodal vertices, the diameter decreases to $\sim V/8$.

4.5.22 Perform computational experiments to verify that the average path length in a ring graph on V vertices is $\sim 1/4 V$. Then, repeat these experiments, but add one random edge to the ring graph and verify that the average path length decreases to $\sim 3/16 V$.

4.5.23 Add to `SmallWorld` (PROGRAM 4.5.5) the function `isSmallWorld()` that takes a graph as an argument and returns `true` if the graph exhibits the small-world phenomenon (as defined by the specific thresholds given in the text) and `false` otherwise.

4.5.24 Implement a test client `main()` for `SmallWorld` (PROGRAM 4.5.5) that produces the output given in the text. Your program should take the name of a graph file and a delimiter as command-line arguments; print the number of vertices, the average degree, the average path length, and the clustering coefficient for the graph; and indicate whether the values are too large or too small for the graph to exhibit the small-world phenomenon.

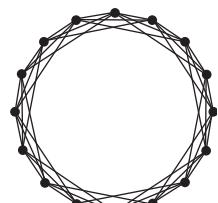
4.5.25 Write a program to generate random connected graphs and 2-ring graphs with random shortcuts. Using `SmallWorld`, generate 500 random graphs from both models (with 1,000 vertices each) and compute their average degree, average path length, and clustering coefficient. Compare your results to the corresponding values in the table on page 700.

4.5.26 Write a `SmallWorld` and `Graph` client that generates k -ring graphs and tests whether they exhibit the small-world phenomenon (first do EXERCISE 4.5.23).

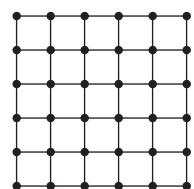
4.5.27 In a *grid graph*, vertices are arranged in an n -by- n grid, with edges connecting each vertex to its neighbors above, below, to the left, and to the right in the grid. Compose a `SmallWorld` and `Graph` client that generates grid graphs and tests whether they exhibit the small-world phenomenon (first do EXERCISE 4.5.23).

4.5.28 Extend your solutions to the previous two exercises to also take a command-line argument m and to add m random edges to the graph. Experiment with your programs for graphs with approximately 1,000 vertices to find small-world graphs with relatively few edges.

4.5.29 Write a `Graph` and `PathFinder` client that takes the name of a movie-cast file and a delimiter as arguments and writes a new movie-cast file, but with all movies not connected to Kevin Bacon removed.



3-ring graph



6-by-6 grid graph

Creative Exercises

4.5.30 *Large Bacon numbers.* Find the performers in `movies.txt` with the largest, but finite, Kevin Bacon number.

4.5.31 *Histogram.* Write a program `BaconHistogram` that prints a histogram of Kevin Bacon numbers, indicating how many performers from `movies.txt` have a Bacon number of 0, 1, 2, 3, Include a category for those who have an infinite number (not connected at all to Kevin Bacon).

4.5.32 *Performer–performer graph.* As mentioned in the text, an alternative way to compute Kevin Bacon numbers is to build a graph where there is a vertex for each performer (but not for each movie), and where two performers are adjacent if they appear in a movie together (see PROGRAM 4.5.6). Calculate Kevin Bacon numbers by running breadth-first search on the performer–performer graph. Compare the running time with the running time on `movies.txt`. Explain why this approach is so much slower. Also explain what you would need to do to include the movies along the path, as happens automatically with our implementation.

4.5.33 *Connected components.* A *connected component* in a graph is a maximal set of vertices that are mutually connected. Write a `Graph` client `CCFinder` that computes the connected components of a graph. Include a constructor that takes a `Graph` as an argument and computes all of the connected components using breadth-first search. Include a method `areConnected(v, w)` that returns `true` if `v` and `w` are in the same connected component and `false` otherwise. Also add a method `components()` that returns the number of connected components.

4.5.34 *Flood fill / image processing.* A `Picture` is a two-dimensional array of `Color` values (see SECTION 3.1) that represent pixels. A *blob* is a collection of neighboring pixels of the same color. Write a `Graph` client whose constructor creates a grid graph (see EXERCISE 4.5.27) from a given image and supports the *flood fill* operation. Given pixel coordinates `col` and `row` and a color `color`, change the color of that pixel and all the pixels in the same blob to `color`.

4.5.35 Word ladders. Write a program `WordLadder` that takes two 5-letter strings as command-line arguments, reads in a list of 5-letter words from standard input, and prints a shortest *word ladder* using the words on standard input connecting the two strings (if it exists). Two words are adjacent in a word ladder chain if they differ in exactly one letter. As an example, the following word ladder connects `green` and `brown`:

```
green greet great groat groan grown brown
```

Write a filter to get the 5-letter words from a system dictionary for standard input or download a list from the booksite. (This game, originally known as *doublet*, was invented by Lewis Carroll.)

4.5.36 All paths. Write a `Graph` client `AllPaths` whose constructor takes a `Graph` as argument and supports operations to count or print *all* simple paths between two given vertices `s` and `t` in the graph. A *simple path* is a path that does not repeat any vertices. In two-dimensional grids, such paths are referred to as *self-avoiding walks* (see SECTION 1.4). Enumerating paths is a fundamental problem in statistical physics and theoretical chemistry—for example, to model the spatial arrangement of linear polymer molecules in a solution. *Warning:* There might be exponentially many paths.

4.5.37 Percolation threshold. Develop a graph model for percolation, and write a `Graph` client that performs the same computation as `Percolation` (PROGRAM 2.4.5). Estimate the percolation threshold for triangular, square, and hexagonal grids.

4.5.38 Subway graphs. In the Tokyo subway system, routes are labeled by letters and stops by numbers, such as G-8 or A-3. Stations allowing transfers are sets of stops. Find a Tokyo subway map on the web, develop a simple file format, and write a `Graph` client that reads a file and can answer shortest-path queries for the Tokyo subway system. If you prefer, do the Paris subway system, where routes are sequences of names and transfers are possible when two stations have the same name.

4.5.39 *Center of the Hollywood universe.* We can measure how good a center Kevin Bacon is by computing each performer's *Hollywood number* or average path length. The Hollywood number of Kevin Bacon is the average Bacon number of all the performers (in its connected component). The Hollywood number of another performer is computed the same way, making that performer the source instead of Kevin Bacon. Compute Kevin Bacon's Hollywood number and find a performer with a better Hollywood number than Kevin Bacon. Find the performers (in the same connected component as Kevin Bacon) with the best and worst Hollywood numbers.

4.5.40 *Diameter.* The *eccentricity* of a vertex is the greatest distance between it and any other vertex. The *diameter* of a graph is the greatest distance between any two vertices (the maximum eccentricity of any vertex). Write a Graph client `Diameter` that can compute the eccentricity of a vertex and the diameter of a graph. Use it to find the diameter of the performer–performer graph associated with `movies.txt`.

4.5.41 *Directed graphs.* Implement a Digraph data type that represents *directed* graphs, where the direction of edges is significant: `addEdge(v, w)` means to add an edge from `v` to `w` but *not* from `w` to `v`. Replace `adjacentTo()` with two methods: one to give the set of vertices having edges directed to them *from* the argument vertex, and the other to give the set of vertices having edges directed from them *to* the argument vertex. Explain how `PathFinder` would need to be modified to find shortest paths in directed graphs.

4.5.42 *Random surfer.* Modify your `Digraph` class from the previous exercise to make a `MultiDigraph` class that allows parallel edges. For a test client, run a random- surfer simulation that matches `RandomSurfer` (PROGRAM 1.6.2).

4.5.43 *Transitive closure.* Write a `Digraph` client `TransitiveClosure` whose constructor takes a `Digraph` as an argument and whose method `isReachable(v, w)` returns `true` if there exists some directed path from `v` to `w`, and `false` otherwise. *Hint:* Run breadth-first search from each vertex.

4.5.44 Statistical sampling. Use statistical sampling to estimate the average path length and clustering coefficient of a graph. For example, to estimate the clustering coefficient, pick `trials` random vertices and compute the average of the clustering coefficients of those vertices. The running time of your functions should be orders of magnitude faster than the corresponding functions from `SmallWorld`.

4.5.45 Cover time. A *random walk* in an undirected connected graph moves from a vertex to one of its neighbors, where each possibility has equal probability of being chosen. (This process is the random surfer analog for undirected graphs.) Write programs to run experiments that support the development of hypotheses about the number of steps used to visit every vertex in the graph. What is the cover time for a complete graph with V vertices? A ring graph? Can you find a family of graphs where the cover time grows proportionally to V^3 or 2^V ?

4.5.46 Erdős–Renyi random graph model. In the classic Erdős–Renyi random graph model, we build a random graph on V vertices by including each possible edge with probability p , independently of the other edges. Compose a `Graph` client to verify the following properties:

- *Connectivity thresholds:* If $p < 1/V$ and V is large, then most of the connected components are small, with the largest being logarithmic in size. If $p > 1/V$, then there is almost surely a giant component containing almost all vertices. If $p < \ln V / V$, the graph is disconnected with high probability; if $p > \ln V / V$, the graph is connected with high probability.
- *Distribution of degrees:* The distribution of degrees follows a binomial distribution, centered on the average, so most vertices have similar degrees. The probability that a vertex is adjacent to k other vertices decreases exponentially in k .
- *No hubs:* The maximum vertex degree when p is a constant is at most logarithmic in V .
- *No local clustering:* The cluster coefficient is close to 0 if the graph is sparse and connected. Random graphs are not small-world graphs.
- *Short path lengths:* If $p > \ln V / V$, then the diameter of the graph (see EXERCISE 4.5.40) is logarithmic.

4.5.47 *Power law of web links.* The indegrees and outdegrees of pages in the web obey a power law that can be modeled by a *preferred attachment* process. Suppose that each web page has exactly one outgoing link. Each page is created one at a time, starting with a single page that points to itself. With probability $p < 1$, it links to one of the existing pages, chosen uniformly at random. With probability $1 - p$, it links to an existing page with probability proportional to the number of incoming links of that page. This rule reflects the common tendency for new web pages to point to popular pages. Compose a program to simulate this process and plot a histogram of the number of incoming links.

Partial solution. The fraction of pages with indegree k is proportional to $k^{-1/(1-p)}$.

4.5.48 *Global clustering coefficient.* Add a function to `SmallWorld` that computes the global clustering coefficient of a graph. The *global clustering coefficient* is the conditional probability that two random vertices that are neighbors of a common vertex are neighbors of each other. Find graphs for which the local and global clustering coefficients are different.

4.5.49 *Watts–Strogatz graph model.* (See EXERCISE 4.5.27 and EXERCISE 4.5.28.) Watts and Strogatz proposed a hybrid model that contains typical links of vertices near each other (people know their geographic neighbors), plus some random long-range connection links. Plot the effect of adding random edges to an n -by- n grid graph on the average path length and on the cluster coefficient, for $n = 100$. Do the same for k -ring graphs on V vertices, for $V = 10,000$ and various values of k up to $10 \log V$.

4.5.50 *Bollobás–Chung graph model.* Bollobás and Chung proposed a hybrid model that combines a 2-ring on V vertices (V is even), plus a *random matching*. A *matching* is a graph in which every vertex has degree 1. To generate a random matching, shuffle the V vertices and add an edge between vertex i and vertex $i+1$ in the shuffled order. Determine the degree of each vertex for graphs in this model. Using `SmallWorld`, estimate the average path length and local clustering coefficient for graphs generated according to this model for $V = 1,000$.

Context

TO CLOSE, WE BRIEFLY SUMMARIZE IN these few pages your newly acquired exposure to programming and then describe a few aspects of the world of computing that you might encounter next. It is our hope that this information will whet your appetite to use the knowledge gained from this book for learning more about the role of computation in the world around you.

You now know how to program. Just as learning to drive an SUV is not difficult when you know how to drive a car, learning to program in a different language will not be difficult for you. Many people regularly use several different languages, for different purposes. The primitive data types, conditionals, loops, arrays, and functional abstraction described in CHAPTERS 1 AND 2 (which served programmers well for the first couple of decades of computing) and the object-oriented programming approach explored in CHAPTER 3 (which is used by modern programmers) are basic models found in many programming languages. Your skill in using them and the fundamental data types introduced in CHAPTER 4 will prepare you to cope with libraries, program development environments, and specialized applications of all sorts. You are also well positioned to appreciate the power of abstraction in designing complex systems and understanding how they work.

The study of *computer science* entails much more than learning to program. Now that you are familiar with programming and conversant with computing, you are well prepared to learn about not just the way in which computers operate, but also some of the outstanding intellectual achievements of the past century, some of the most important unsolved problems of our time, and their role in the evolution of the computational infrastructure that surrounds us. These topics are treated in our book *Computer Science: An Interdisciplinary Approach*, which consists of the first four chapters of this book and three additional chapters, one each on theory of computing, machine architecture, and logical design. These three topics are briefly described in the next three paragraphs.

Theory of computing. In contrast to the opportunities we have emphasized, fundamental limits on computation have been apparent from the beginning of the computer age and continue to play an important role in determining the kinds of problems that we can address. You may be surprised to learn that there are some problems that no computer program can solve and many other problems, which arise commonly in practice, that are thought to be too difficult to solve on any conceivable computer. Everyone who depends on computation for problem solving, creative work, or research needs to understand and respect these facts.

Machine architecture. One of our most important early promises was that we would *demystify* computation for you. Our hope is that Java programming is now much less mysterious to you than before you began reading this book, but a full understanding of how a computer works requires a closer look. Remarkably, virtually all computers use the same basic approach, known as *von Neumann architecture*, and can be programmed in a *machine language* that is not difficult to learn. Insights gained from writing a few programs in machine language can be valuable indeed.

Logical design. Fundamentally, programming in machine language is not much different than programming in Java, but an important reason to learn machine language is that it opens the door to see how computers are actually built. Starting with a few simple abstractions (wires that carry 0–1 values and switches controlled by wires) it is surprisingly easy to design a complete computational engine that is not so different from the one that powers your laptop or your mobile device. Learning the details is not difficult, and certainly does demystify computation.

OF COURSE, ALL OF THE ABOVE is merely an *introduction* to computer science. The field has exploded in all directions, and we conclude with a list (in no particular order) of other aspects of the field that you might encounter as your exposure to computer science widens.

Programming libraries. The Java system provides extensive resources for your use. We have made extensive use of some Java libraries, such as `Math` and `String`, but have ignored most of them. One of Java’s unique features is that a great deal of information about the libraries is readily available online. If you have not yet browsed through the Java libraries, now is the time to do so. You will find that much of this code is intended for use by professional developers, but you are likely to find a number of these libraries useful for your own work. When studying a library, your

attitude should be not that you *need* to use it, but that you *can* use it. When you find an API that seems useful, take advantage of it!

Programming environments. You will certainly find yourself using other programming environments besides Java in the future. Many programmers—even experienced professionals—are caught between the past, because of huge amounts of legacy code in old languages such as C, C++, and Fortran, and the future, because of the availability of modern tools like Ruby, Python, and Scala. If you want to learn Python, you might enjoy our book *An Introduction to Programming in Python*, a twin of this book. Again, perhaps the most important thing for you to keep in mind when using a programming language is that you do not *need* to use it. If some other language might better meet your needs, take advantage of it, by all means. People who insist on staying within a single programming environment, for whatever reason, are missing out on valuable opportunities.

Scientific computing. In particular, computing with numbers can be very tricky (because of accuracy and precision) so the use of libraries of mathematical functions is certainly justified. Many scientists use Fortran, an old scientific language; many others use Matlab, a language that was developed specifically for computing with matrices. The combination of good libraries and built-in matrix operations makes Matlab an attractive choice for many problems. However, since Matlab lacks support for mutable types and other modern facilities, Java is a better choice for many other problems. *You can use both!* The same mathematical libraries used by Matlab and Fortran programmers are accessible from Java (and through use of modern scripting languages).

Apps and cloud computing. A great deal of engagement with computing nowadays involves building and using programs intended to be run from a browser or on a mobile device, perhaps on a virtual computer in the cloud. This state of affairs is remarkable because it has vastly extended the number of people whose lives are positively affected by computing. If you find yourself engaged in this kind of computing, you are likely to be struck by the effectiveness of the basic approaches that we have discussed in this book. You can write programs that process data that is maintained elsewhere, write programs that interact with programs executing elsewhere, and take advantage of many other properties of the extensive and evolving computational infrastructure. In particular, our focus on using a scientific approach to understand performance prepares you to be able to compute on a giant scale.

Computer systems. Properties of specific computer systems once completely determined the nature and extent of problems that could be solved, but now they hardly intrude on this scope. You can still count on having a faster machine with much more memory next year at this time. Strive to keep your code machine independent, but also be prepared to learn and exploit new technologies, from GPUs to massively parallel computers and networks.

Machine learning. The field of *artificial intelligence* has long captured the imagination of computer scientists. The vast scale of modern computing has meant that the dreams of early researchers are being realized, to the extent that we are beginning to depend on computers to learn from their environments, whether the goal is to guide a self-driving car, lead us to the products we want to buy, or teach us what we want to learn. Harnessing computation at this level is certainly more profound than learning another set of APIs, and something that you are certain to exploit in the future.

YOU HAVE CERTAINLY COME A LONG way since you tentatively created, compiled, and ran `HelloWorld`, but you still have a great deal to learn. Keep programming, and keep learning about programming environments, scientific computing, apps and cloud computing, computer systems, theory of computing, and machine learning. By doing so, you will open opportunities for yourself that people who do not program cannot even conceive. Perhaps even more significant, as we have hinted throughout the book, is the reality that computation is playing an ever-increasing role in our understanding of nature, from genomics to molecular dynamics to astrophysics. Further study of the fascinating world of computer science is certain to pay dividends, whatever the future holds for you.

Glossary

algorithm A step-by-step procedure for solving a problem, such as Euclid’s algorithm mergesort, or binary search.

alias Two (or more) variables that refer to the same object.

API (application programming interface) Specification of the set of operations that characterize how a client can use a data type.

array A data structure that holds a sequence of values of the same type, with support for creation, indexed access, indexed assignment, and iteration.

argument An expression that Java evaluates and passes by value to a method.

ASCII (American Standard Code for Information Interchange) A widely used standard for encoding English text, which is incorporated into Unicode.

assignment statement A Java statement consisting of a variable name followed by the equals sign (=) followed by an expression, which directs Java to evaluate the expression and to assign the value produced to the variable.

bit A binary digit (0 or 1).

booksite library A library created by the authors for use in the book, such as StdIn, StdOut, StdDraw, and StdAudio.

boolean expression An expression that evaluates to a value of type boolean.

boolean value 0 or 1; true or false.

built-in type A data type built into the Java language, such as int, double, boolean, char, and String.

class The Java construct to implement a user-defined data type, providing a template to create and manipulate objects holding values of the type, as specified by an API.

.class file A file with a .class extension that contains Java bytecode, suitable for execution on the Java virtual machine.

class variable See static variable.

client A program that uses an implementation via an API.

command line The active line in the terminal application; used to invoke system commands and to run programs.

command-line argument A string passed to a program at the command line.

comment Explanatory text (ignored by the compiler) to help a reader understand the purpose of code.

comparable data type A Java data type that implements the Comparable interface and defines a total order.

compile-time error An error in syntax found by the compiler.

compiler A program that translates a program from a high-level language into a low-level language.

The Java compiler translates a .java file (containing Java source code) to a .class file (containing Java bytecode).

conditional statement A statement that performs a different computation depending on the value of one or more boolean expressions, such as an if, if-else, or switch statement.

constant variable A variable whose value is known at compile time and does not change during execution of the program (or from one execution of the program to the next).

constructor A special data-type method that creates and initializes a new object.

data structure A way to organize data in a computer (usually to save time or space), such as an array, a resizing array, a linked list, or a binary search tree.

data type A set of values and a set of operations defined on those values.

declaring a variable Specifying the name and type of a variable.

element One of the components in an array.

evaluate an expression Simplify an expression to a value by applying operators to the operands in the expression. Operator precedence, operator associativity, and order of evaluation determine the order in which to apply the operators to the operands.

exception An exceptional condition or error at run time.

exponential-time algorithm An algorithm that runs in time bounded below by an exponential function of the input size.

expression A combination of literals, variables, operators, and method calls that Java evaluates to produce a value.

floating point Generic description of the use of “scientific notation” to represent real numbers on a computer (see IEEE 754).

function See static method.

functional interface An interface with exactly one method.

garbage collection The process of automatically identifying and freeing memory when it is no longer in use.

generic class A class that is parameterized by one or more type parameter, such as Queue, Stack, ST, or SET.

global variable A variable whose scope is the entire program or file. See also *static variable*.

hash table A symbol-table implementation based on hashing.

hashing Transforming a data-type value into an integer in a given range, so that different keys are unlikely to map to the same integer.

identifier A name used to identify a variable, method, class, or other entity.

IEEE 754 International standard for floating-point computations, which is used in modern computer hardware (see *floating point*).

immutable data type A data type for which the data-type value of any instance cannot change, such as Integer, String, or Complex.

immutable object An object whose data-type value cannot change.

implementation A program that implements a set of methods defined in an API, for use by a client.

import statement A Java statement that enables you to refer to code in another package without using the fully qualified name.

initializing a variable Assigning a value to a variable for the first time in a program.

instance An object of a particular class.

instance method The implementation of a data-type operation (a method that is invoked with respect to a particular object).

instance variable A variable defined inside a class (but outside any method) that represents a data-type value (data associated with each instance of the class).

interface A contract for a class to implement a certain set of methods.

interpreter A program that executes a program written in a high-level language, one line at a time. The Java virtual machine interprets Java bytecode and executes it on your computer.

item One of the objects in a collection.

iterable data type A data type that implements the Iterable interface and can be used with a foreach loop, such as Stack, Queue, or SET.

iterator A data type that implements the Iterator interface. Used to implement iterable data types.

Java bytecode The low-level, machine-independent language used by the Java virtual machine.

.java file A file that contains a program written in the Java programming language.

Java programming language A general-purpose, object-oriented programming language.

Java virtual machine (JVM) The program that executes Java bytecode on a microprocessor, using both an interpreter and a just-in-time compiler.

just-in-time-compiler A compiler that continuously translates a program in a high-level language to a lower-level language, while the program executes. Java's just-in-time compiler translates from Java bytecode to native machine language.

lambda expression An anonymous function that you can pass around and execute later.

library A .java file structured so that its features can be reused in other Java programs.

linked list A data structure that consists of a sequence of nodes, where each node contains a reference to the next node in the sequence.

literal Source-code representation of a data-type value for built-in types, such as 123, "Hello", or true.

local variable A variable defined within a method, whose scope is limited to that method.

loop A statement that repeatedly performs a computation depending on the value of some boolean expression, such as a for or while statement.

method A named sequence of statements that can be called by other code to perform a computation.

method call An expression that executes a method and returns a value.

modular programming A style of programming that emphasizes using separate, independent modules to address a task.

module (software) An independent program, such as a Java class, that implements an API.

Moore's law The observation, by Gordon Moore, that both processor power and memory capacity have doubled every two years since the introduction of integrated circuits in the 1960s.

mutable data type A data type for which the data-type value of an instance can change, such as Counter, Picture, or arrays.

mutable object An object whose data-type value can change.

null reference The special literal null that represents a reference to no object.

object An in-computer-memory representation of a value from a particular data type, characterized by its state (data-type value), behavior (data-type operations), and identity (location in memory).

object-oriented programming A style of programming that emphasizes modeling real-world or abstract entities using data types and objects.

object reference A concrete representation of an object's identity (typically, the memory address where the object is stored).

operand A value on which an operator operates.

operating system The program on your computer that manages resources and provides common services for programs and applications.

operator A special symbol (or sequence of symbols) that represents a built-in data-type operation, such as +, -, *, or [].

operator associativity Rules that determine the order in which to apply operators that have the same precedence, such as 1 - 2 - 3.

operator precedence Rules that determine the order in which to apply the operators in an expression, such as 1 + 2 * 3.

order of evaluation The order in which subexpressions, such as `f1() + f2() * f5(f3(), f4())`, are evaluated. Regardless of operator precedence or operator associativity, Java evaluates subexpressions from left to right. Java evaluates method arguments from left to right, prior to calling the method.

overflow When the value of the result of an arithmetic operation exceeds the maximum possible value.

overloading a method Defining two or more methods with the same name (but different parameter lists).

overloading an operator Defining the behavior of an operator—such as +, *, <=, and []—for a data type. Java does not support operator overloading.

overriding a method Redefining an inherited method, such as `equals()` or `hashCode()`.

package A collection of related classes and interfaces that share a common namespace. The package `java.lang` contains the most fundamental classes and interfaces and is imported automatically; the package `java.util` contains Java’s Collections Framework.

parameter variable A variable specified in the definition of a method. It is initialized to the corresponding argument when the method is called.

parsing Converting a string to an internal representation.

pass by value Java’s style of passing arguments to methods—either as a data-type value (for primitive types) or as an object reference (for reference types).

polymorphism Using the same API (or partial API) for different types of data.

polynomial-time algorithm An algorithm that is guaranteed to run in time bounded by some polynomial function of the input size.

primitive data type One of the eight data types defined by Java, which include `boolean`, `char`, `double`, and `int`. A variable of a primitive type stores the data-type value itself.

private Data-type implementation code that is not to be referenced by clients.

program A sequence of instructions to be executed on a computer.

pure function A function that, given the same arguments, always returns the same value, without producing any observable side effect.

reference type A class type, interface type, or array type, such as `String`, `Charge`, `Comparable`, or `int[]`. A variable of a reference type stores an object reference, not the data-type value itself.

resizing array A data structure that ensures that a constant fraction of an array's elements are used.

return value The value provided to the caller as the result of a method call.

run-time error An error that occurs while the program is executing.

scope of a variable The part of a program that can refer to a particular variable by name.

side effect A change in state, such as printing output, reading input, throwing an exception, or modifying the value of some persistent object (instance variable, parameter variable, or global variable).

source code A program or program fragment in a high-level programming language, such as Java.

standard input, output, drawing, and audio Our input/output modules for Java.

statement An instruction that Java can execute, such as an assignment statement, an `if` statement, a `while` statement, or a `return` statement.

static method The implementation of a function in a Java class, such as `Math.abs()`, `Euclid.gcd()`, or `StdIn.readInt()`.

static variable A variable associated with a class.

string A finite sequence of alphabet symbols.

terminal window An application for your operating system that accepts commands.

this Within an instance method or constructor, a keyword that refers to the object whose method or constructor is being called.

throw an exception Signal a compile-time or run-time error.

trace Step-by-step description of the operation of a program.

type parameter A placeholder in a generic class for some concrete type that is specified by the client.

Unicode An international standard for encoding text.

unit testing The practice of including code in every module that tests the code in that module.

variable An entity that holds a value. Each Java variable has a name, type, and scope.

wrapper type A reference type corresponding to one of the primitive types, such as `Integer`, `Double`, `Boolean`, or `Character`.

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APIs

```
public class Math
```

double abs(double a)	<i>absolute value of a</i>
double max(double a, double b)	<i>maximum of a and b</i>
double min(double a, double b)	<i>minimum of a and b</i>

Note 1: abs(), max(), and min() are defined also for int, long, and float.

double sin(double theta)	<i>sine of theta</i>
double cos(double theta)	<i>cosine of theta</i>
double tan(double theta)	<i>tangent of theta</i>

Note 2: Angles are expressed in radians. Use toDegrees() and toRadians() to convert.

Note 3: Use asin(), acos(), and atan() for inverse functions.

double exp(double a)	<i>exponential (e^a)</i>
double log(double a)	<i>natural log ($\log_e a$, or $\ln a$)</i>
double pow(double a, double b)	<i>raise a to the bth power (a^b)</i>
long round(double a)	<i>round a to the nearest integer</i>
double random()	<i>random number in [0, 1)</i>
double sqrt(double a)	<i>square root of a</i>
double E	<i>value of e (constant)</i>
double PI	<i>value of π (constant)</i>

<code>public class String</code>	
<code>String(String s)</code>	<i>create a string with the same value as s</i>
<code>String(char[] a)</code>	<i>create a string that represents the same sequence of characters as a[]</i>
<code>int length()</code>	<i>string length</i>
<code>char charAt(int i)</code>	<i>ith character</i>
<code>String substring(int i, int j)</code>	<i>ith through (j-1)st characters</i>
<code>boolean contains(String sub)</code>	<i>does string contain sub as a substring?</i>
<code>boolean startsWith(String pre)</code>	<i>does string start with pre?</i>
<code>boolean endsWith(String post)</code>	<i>does string end with post?</i>
<code>int indexOf(String p)</code>	<i>index of first occurrence of p</i>
<code>int indexOf(String p, int i)</code>	<i>index of first occurrence of p after i</i>
<code>String concat(String t)</code>	<i>this string with t appended</i>
<code>int compareTo(String t)</code>	<i>string comparison</i>
<code>String replaceAll(String a, String b)</code>	<i>result of changing as to bs</i>
<code>String[] split(String delim)</code>	<i>strings between occurrences of delim</i>
<code>boolean equals(String t)</code>	<i>is this string's value the same as t's?</i>

<code>public class System.out/StdOut/Out</code>	
<code>Out(String name)</code>	<i>create output stream from name</i>
<code>void print(String s)</code>	<i>print s</i>
<code>void println(String s)</code>	<i>print s, followed by newline</i>
<code>void println()</code>	<i>print a newline</i>
<code>void printf(String format, ...)</code>	<i>print the arguments to standard output, as specified by the format string format</i>

Note: For System.out/StdOut, methods are static and constructor does not apply.

```
public class StdIn/In
```

In(String name)	<i>create input stream from name</i>
<i>methods for reading individual tokens</i>	
boolean isEmpty()	<i>is input stream empty (or only whitespace)?</i>
int readInt()	<i>read a token, convert it to an int, and return it</i>
double readDouble()	<i>read a token, convert it to a double, and return it</i>
boolean readBoolean()	<i>read a token, convert it to a boolean, and return it</i>
String readString()	<i>read a token and return it as a String</i>
<i>methods for reading characters</i>	
boolean hasNextChar()	<i>does input stream have any remaining characters?</i>
char readChar()	<i>read a character from input stream and return it</i>
<i>methods for reading lines from standard input</i>	
boolean hasNextLine()	<i>does input stream have a next line?</i>
String readLine()	<i>read the rest of the line and return it as a String</i>
<i>methods for reading the rest of standard input</i>	
int[] readAllInts()	<i>read all remaining tokens and return them as an int array</i>
double[] readAllDoubles()	<i>read all remaining tokens and return them as a double array</i>
boolean[] readAllBooleans()	<i>read all remaining tokens and return them as a boolean array</i>
String[] readAllStrings()	<i>read all remaining tokens and return them as a String array</i>
String[] readAllLines()	<i>read all remaining lines and return them as a String array</i>
String readAll()	<i>read the rest of the input and return it as a String</i>

Note 1: For StdIn, methods are static and constructor does not apply.

Note 2: A token is a maximal sequence of non-whitespace characters.

Note 3: Before reading a token, any leading whitespace is discarded.

Note 4: Analogous methods are available for reading values of type byte, short, long, and float.

Note 5: Each method that reads input throws a run-time exception if it cannot read in the next value, either because there is no more input or because the input does not match the expected type.

```

public class StdDraw/Draw

    Draw()                                create a new Draw object
    drawing commands

        void line(double x0, double y0, double x1, double y1)
        void point(double x, double y)
        void circle(double x, double y, double radius)
        void filledCircle(double x, double y, double radius)
        void square(double x, double y, double radius)
        void filledSquare(double x, double y, double radius)
        void rectangle(double x, double y, double r1, double r2)
        void filledRectangle(double x, double y, double r1, double r2)
        void polygon(double[] x, double[] y)
        void filledPolygon(double[] x, double[] y)
        void text(double x, double y, String s)

    control commands

        void setXscale(double x0, double x1)      reset x-scale to (x0, x1)
        void setYscale(double y0, double y1)      reset y-scale to (y0, y1)
        void setPenRadius(double radius)          set pen radius to radius
        void setPenColor(Color color)            set pen color to color
        voidsetFont(Font font)                  set text font to font
        void setCanvasSize(int w, int h)          set canvas size to w-by-h
        void enableDoubleBuffering()             enable double buffering
        void disableDoubleBuffering()            disable double buffering
        void show()                            copy the offscreen canvas to the onscreen canvas
        void clear(Color color)                 clear the canvas to color color
        void pause(int dt)                     pause dt milliseconds
        void save(String filename)              save to a .jpg or .png file

```

Note 1: For StdDraw, the methods are static and the constructor does not apply.

Note 2: Methods with the same names but no arguments reset to the default values.

<code>public class StdAudio</code>	
<code>void play(String filename)</code>	<i>play the given .wav file</i>
<code>void play(double[] a)</code>	<i>play the given sound wave</i>
<code>void play(double x)</code>	<i>play sample for 1/44,100 second</i>
<code>void save(String filename, double[] a)</code>	<i>save to a .wav file</i>
<code>double[] read(String filename)</code>	<i>read from a .wav file</i>

<code>public class Stopwatch</code>	
<code>Stopwatch()</code>	<i>create a new stopwatch and start it running</i>
<code>double elapsedTime()</code>	<i>return the elapsed time since creation, in seconds</i>

<code>public class Picture</code>	
<code>Picture(String filename)</code>	<i>create a picture from a file</i>
<code>Picture(int w, int h)</code>	<i>create a blank w-by-h picture</i>
<code>int width()</code>	<i>return the width of the picture</i>
<code>int height()</code>	<i>return the height of the picture</i>
<code>Color get(int col, int row)</code>	<i>return the color of pixel (col, row)</i>
<code>void set(int col, int row, Color c)</code>	<i>set the color of pixel (col, row) to c</i>
<code>void show()</code>	<i>display the picture in a window</i>
<code>void save(String filename)</code>	<i>save the picture to a file</i>

<code>public class StdRandom</code>	
<code>void setSeed(long seed)</code>	<i>set the seed for reproducible results</i>
<code>int uniform(int n)</code>	<i>integer between 0 and n-1</i>
<code>double uniform(double lo, double hi)</code>	<i>floating-point number between lo and hi</i>
<code>boolean bernoulli(double p)</code>	<i>true with probability p, false otherwise</i>
<code>double gaussian()</code>	<i>Gaussian, mean 0, standard deviation 1</i>
<code>double gaussian(double mu, double sigma)</code>	<i>Gaussian, mean mu, standard deviation sigma</i>
<code>int discrete(double[] p)</code>	<i>i with probability p[i]</i>
<code>void shuffle(double[] a)</code>	<i>randomly shuffle the array a[]</i>

`public class StdArrayIO`

<code>double[] readDouble1D()</code>	<i>read a one-dimensional array of double values</i>
<code>double[][] readDouble2D()</code>	<i>read a two-dimensional array of double values</i>
<code>void print(double[] a)</code>	<i>print a one-dimensional array of double values</i>
<code>void print(double[][] a)</code>	<i>print a two-dimensional array of double values</i>

*Note 1. 1D format is an integer n followed by n values.**Note 2. 2D format is two integers m and n followed by m × n values in row-major order.**Note 3. Methods for int and boolean are also included.*`public class StdStats`

<code>double max(double[] a)</code>	<i>largest value</i>
<code>double min(double[] a)</code>	<i>smallest value</i>
<code>double mean(double[] a)</code>	<i>average</i>
<code>double var(double[] a)</code>	<i>sample variance</i>
<code>double stddev(double[] a)</code>	<i>sample standard deviation</i>
<code>double median(double[] a)</code>	<i>median</i>
<code>void plotPoints(double[] a)</code>	<i>plot points at (i, a[i])</i>
<code>void plotLines(double[] a)</code>	<i>plot lines connecting points at (i, a[i])</i>
<code>void plotBars(double[] a)</code>	<i>plot bars to points at (i, a[i])</i>

Note: Overloaded implementations are included for all numeric types.

```
public class Stack<Item> implements Iterable<Item>
```

Stack()	<i>create an empty stack</i>
boolean isEmpty()	<i>is the stack empty?</i>
int size()	<i>number of items in the stack</i>
void push(Item item)	<i>insert an item onto the stack</i>
Item pop()	<i>return and remove the item that was inserted most recently</i>

```
public class Queue<Item> implements Iterable<Item>
```

Queue()	<i>create an empty queue</i>
boolean isEmpty()	<i>is the queue empty?</i>
int size()	<i>number of items in the queue</i>
void enqueue(Item item)	<i>insert an item into the queue</i>
Item dequeue()	<i>return and remove the item that was inserted least recently</i>

```
public class SET<Key extends Comparable<Key>> implements Iterable<Key>
```

SET()	<i>create an empty set</i>
boolean isEmpty()	<i>is the set empty?</i>
int size()	<i>number of elements in the set</i>
void add(Key key)	<i>add key to the set</i>
void remove(Key key)	<i>remove key from set</i>
boolean contains(Key key)	<i>is key in the set?</i>

```
public class ST<Key extends Comparable<Key>, Value>
```

ST()	<i>create an empty symbol table</i>
void put(Key key, Value val)	<i>associate val with key</i>
Value get(Key key)	<i>value associated with key</i>
void remove(Key key)	<i>remove key (and its associated value)</i>
boolean contains(Key key)	<i>is there a value paired with key?</i>
int size()	<i>number of key-value pairs</i>
Iterable<Key> keys()	<i>all keys in sorted order</i>
Key min()	<i>minimum key</i>
Key max()	<i>maximum key</i>
int rank(Key key)	<i>number of keys less than key</i>
Key select(int k)	<i>kth smallest key in symbol table</i>
Key floor(Key key)	<i>largest key less than or equal to key</i>
Key ceiling(Key key)	<i>smallest key greater than or equal to key</i>

```
public class Graph
```

Graph()	<i>create an empty graph</i>
Graph(String filename, String delimiter)	<i>create graph from a file</i>
void addEdge(String v, String w)	<i>add edge v-w</i>
int V()	<i>number of vertices</i>
int E()	<i>number of edges</i>
Iterable<String> vertices()	<i>vertices in the graph</i>
Iterable<String> adjacentTo(String v)	<i>neighbors of v</i>
int degree(String v)	<i>number of neighbors of v</i>
boolean hasVertex(String v)	<i>is v a vertex in the graph?</i>
boolean hasEdge(String v, String w)	<i>is v-w an edge in the graph?</i>