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ON THE COMPLETENESS OF THE COHERENT STATES

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We study subsets of coherent states based on square lattices in the complex plane, namely, $\{|z_{m,n}\rangle\}$ where $z_{m,n}=\gamma(m+in)$ for $m, n=0, \pm 1, \pm 2, \dots$ Analyticity arguments suffice to establish completeness if $0<\gamma<\sqrt{\pi}$ and to disprove completeness if $\gamma>\sqrt{\pi}$. The completeness of the case $\gamma=\sqrt{\pi}$, stated without proof by von Neumann, is established by invoking square integrability along with analyticity.

1. Introduction

The coherent states,

$$|z\rangle=e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle$$

defined for all complex z, where $|n\rangle$ are the harmonic oscillator eigenstates, not only span the Hilbert space (i.e., are "total") but are overcomplete as well ([1], [3]–[6]). Here we study the totality of subsets of coherent states, especially those based on a lattice of points

$$z_{m,n}\equiv\gamma(m+n)\,; \quad m,n=0,\pm 1,\pm 2\dots$$

where $\gamma>0$. Fairly simple analyticity arguments suffice to show that if $0<\gamma<\sqrt{\pi}$ the corresponding set of states $\{|z_{m,n}\rangle\}$ is total, while if $\gamma>\sqrt{\pi}$ the set of states $\{|z_{m,n}\rangle\}$ is not total. These "fine" and "coarse" lattices are treated in Section 2. Von Neuman studied the case $\gamma=\sqrt{\pi}$ long ago ([7]) and stated, without proof, the completeness of this "von Neumann" lattice. In this case these simple analyticity arguments do not suffice and in Section 3 we exploit square integrability ([1], [5]) of the functions $\langle z|\psi\rangle$ for any state $|\psi\rangle$ to show completeness of the von Neumann lattice.

These results are physically appealing since it follows that completeness holds for a phase-space lattice of states with a density equal to or greater than one state per Planck cell, but not otherwise. The simplest way to make this connection is as follows: If $z=x+iy$, then for an oscillator with mass m and circular frequency ω the state $|z\rangle$ is — apart from a phase factor — uniquely defined by the following four conditions (we set $\hbar=1$):

$$\begin{array}{ll} \langle q\rangle=(2/\beta)^{\frac{1}{2}}x\,, & \langle p\rangle=(2\beta)^{\frac{1}{2}}y \\ (\varDelta q)^2=1/2\beta\,, & (\varDelta p)^2=\beta/2 \end{array} \qquad (\beta=m\omega)\,.$$

[Here $\langle B\rangle$ is the expectation value of the operator B, and $(\varDelta B)^2$ the expectation value of $(B-\langle B\rangle)^2$.] Note that the product $\varDelta p\cdot\varDelta q$ equals $\frac{1}{2}$, the smallest value compatible with the uncertainty relations. A square of area π in the z plane corresponds to a phase space area of $(2/\beta)^{\frac{1}{2}}(2\beta)^{\frac{1}{2}}\pi=2\pi$ which is just the area of the Planck cell in our units. Our results reconfirm the fundamental importance of this basic unit of phase-space area.

We also note that in configuration space the wave function for $|z\rangle$ is

$$\varphi(q)=(\beta/\pi)^{\frac{1}{4}}\exp\{-\tfrac{1}{2}\beta(q-\langle q\rangle)^2+i\langle p\rangle(q-\tfrac{1}{2}\langle q\rangle\}\,,$$

which we shall use in Section 3 with the special choice $\beta=2\pi$.

Our arguments of analyticity and square summability derive from several basic properties. First, for any state $|f\rangle$,

$$\langle f|z\rangle=e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{\langle f|n\rangle}{\sqrt{n!}}z^n\equiv e^{-\frac{1}{2}|z|^2}F(z)\,,$$

and it follows that

$$F(z)\equiv\sum_{n=0}^{\infty}\frac{\langle f|n\rangle}{\sqrt{n!}}z^n$$

is an entire function. Cauchy's inequality, i.e., $|\langle f|z\rangle|\leqslant\|\,|f\rangle\|$, leads to

$$|F(z)|\leqslant\|\,|f\rangle\|\,e^{\frac{1}{2}|z|^2}.$$

Thus the entire functions $F(z)$ are of order[1] ρ not exceeding 2 and, if of order 2, of type

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[1] We recall here the definition of the order ρ and type τ of an entire function $f(z)$

$$\rho=\limsup_{r\to\infty}\frac{\ln\ln M(f,r)}{\ln r}\,,$$

$$\tau=\limsup_{r\to\infty}\frac{\ln M(f,r)}{r^{\rho}}$$

where $M(f,r)$ denotes the maximum modulus of $f(z)$ for $|z|=r$.

τ not exceeding $\frac{1}{2}$; i.e., they are of "growth" $(2, \frac{1}{2})$. The second property of the coherent states we exploit is the relation ([5])

$$\pi^{-1} \int |z\rangle \langle z| \, d^2 z = 1$$

where $d^2 z = d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z)$. This relation means that

$$\begin{aligned} \langle f | f \rangle &= \pi^{-1} \int \langle f | z \rangle \langle z | f \rangle \, d^2 z \\ &= \int |F(z)|^2 d\mu(z) \end{aligned}$$

where

$$d\mu(z) \equiv \pi^{-1} e^{-|z|^2} d^2 z .$$

Thus, we can conveniently realize the abstract Hilbert space $\mathscr{H}$ as the space $\mathscr{F}$ composed of entire functions of growth $(2, \frac{1}{2})$ which are square integrable with respect to μ ([1]). An entire function of growth $(2, \tau)$ belongs to $\mathscr{F}$ if $\tau < \frac{1}{2}$.

2. Fine and coarse lattices

A. *Fine lattices*

If a vector $|f\rangle$ is orthogonal to a given state $|z\rangle$, then $F(z) = e^{\frac{1}{2}|z|^2} \langle f | z \rangle = 0$. A total set of vectors has the property that orthogonality of a vector $|f\rangle$ to all members of the set implies the vanishing of the vector $|f\rangle$. Therefore, a set of coherent states $\{|z_s\rangle\}$ is total if and only if any μ-square integrable entire function of growth $(2, \frac{1}{2})$ with zeros at the points $\{z_s\}$ vanishes identically. We call a total set of coherent states a *sufficiently dense* or *characteristic* set.

In this section we shall focus on the analytic properties of the functions $F(z)$. Two examples of characteristic sets have been previously given ([1]), but they do not cover the case of regular lattices. These examples were:

(a) Any infinite set $\{z_s\}$ which converges to a finite limit.

(b) Any diverging infinite set $\{z_s\}$, excluding $z_s = 0$, such that

$$\sum_{s=0}^{\infty} |z_s|^{-2-\varepsilon} = \infty \qquad \text{for some } \varepsilon > 0.$$

Condition (b) is a consequence of general theorems on the connection between the order of an entire function and the number of its zeros. If we take into account the type of our function, it is possible to get a more general result. To this end we may exploit the following theorem ([2], p. 152, Theorem 9.1.1): *Let $f(z)$ be an entire function of growth (ρ, τ) and $\{z_s\}$ a set of points in the complex z-plane for which either*

$$\tau < \rho^{-1} \liminf_{r \to \infty} r^{-\rho} n(r)$$

or

$$\tau < (e\rho)^{-1} \limsup_{r \to \infty} r^{-\rho} n(r)$$

where $n(r)$ *is the number of points* z_s *inside the circle* $|z| \leqslant r$ *and e is the basis of natural logs; then* $f(z)=0$, *if* $f(z_s)=0$ *for all* z_s. In our case $\rho=2$, $\tau=\frac{1}{2}$ so that we may state that a set of coherent states $\{|z_s\rangle\}$ is total if

$$\text{(c)} \liminf_{r\to\infty} \frac{n(r)}{r^2} > 1 .$$

It can easily be seen in this context that (b) supposes that the order of the function exceeds 2 (see [2], p. 17, Theorem 2.5.18).

The implication of condition (c) can be found for general lattices as easily as for the simple square lattice. Consider, in the complex plane, the lattice of the points

$$z_{n_1,n_2} = 2(n_1\omega_1 + n_2\omega_2)$$

where $2\omega_1$ and $2\omega_2$ are not colinear and n_1, n_2 run through all integers. As is customary in the theory of elliptic functions (see [8]) we assume $\operatorname{Im}(\omega_2/\omega_1)>0$. Then the area of the basis cell (the parallelogram with sides $2\omega_1$, $2\omega_2$) equals

$$a = 2i(\omega_1\omega_2^* - \omega_2\omega_1^*) .$$

It follows from elementary arguments that

$$n(r) = \pi r^2/a + O(r) .$$

Hence if $a<\pi$ the condition (c) is fulfilled and the corresponding set of coherent states is total. In particular the square lattices $\{z_{m,n}=\gamma(m+in)\}$, $0<\gamma<\sqrt{\pi}$, lead to a sufficiently dense set of coherent states. These sets are still overcomplete and deletion of any finite number of states will not invalidate condition (c).

B. *Coarse lattices*

Consider now a lattice for which $a>\pi$. In order to show that it is not sufficiently dense we construct an entire function f which vanishes at the lattice points and is square integrable with respect to μ. We start from the Weierstrass σ-function (see, e.g., Whittaker and Watson [8], p. 447)

$$\sigma(z) \equiv z \prod_{n_1,n_2}{}' \left(1 - \frac{z}{z_{n_1,n_2}}\right) \exp\left[\frac{z}{z_{n_1,n_2}} + \frac{1}{2}\left(\frac{z}{z_{n_1,n_2}}\right)^2\right]$$

where the prime denotes omission of the term $n_1=n_2=0$. Evidently, $\sigma(z_{n_1,n_2})=0$ for all lattice points. Now

$$\sigma(z+2\omega_\alpha) = -\sigma(z)\exp 2\eta_\alpha(z+\omega_\alpha) \qquad (\alpha=1,2)$$

for two constants η_1, η_2, which satisfy the "Legendre identity"

$$\eta_1\omega_2 - \eta_2\omega_1 = \tfrac{1}{2}\pi i .$$

By repeated application of the previous equations and of the Legendre identity one obtains for every complex v and every lattice vector z_{n_1,n_2}

$$\sigma(v+z_{n_1,n_2})=(-1)^{n_1+n_2+n_1 n_2}\sigma(v)\exp\psi,$$

$$\psi=(n_1\eta_1+n_2\eta_2)(2v+z_{n_1,n_2}).$$

(This equation is equivalent to that of Whittaker–Watson [8], p. 448, Example 1.)

It is now easy to obtain a sufficiently precise estimate of $|\sigma(z)|$. For any complex z there exists a lattice vector z_{n_1,n_2} such that $v=z-z_{n_1,n_2}$ belongs to the basic cell B, i.e., to the set of points of the form $v=\xi_1\omega_1+\xi_2\omega_2$, where $-1<\xi_1,\xi_2\leqslant 1$. With the help of the two equations

$$z-v=2\sum_\alpha n_\alpha\omega_\alpha, \qquad z^*-v^*=2\sum_\alpha n_\alpha\omega_\alpha^*$$

the two integers n_1, n_2 may be expressed in terms of $z-v$ and its complex conjugate, and we find that

$$\sigma(z)=(-1)^{n_1+n_2+n_1 n_2}\sigma(v)\exp\psi,$$

$$\psi=\psi_0(z)-\psi_0(v)+\psi_1(z,v)$$

where

$$\psi_0(z)=\frac{\pi}{2a}|z|^2+bz^2, \qquad \psi_1(z,v)=\frac{\pi}{2a}(z^*v-v^*z)$$

and

$$b=i(\eta_1\omega_2^*-\eta_2\omega_1^*)/a.$$

The function f defined by

$$f(z)=e^{-bz^2}\sigma(z)$$

has the required properties: (1) It vanishes at all lattice points, (2) it is an entire function of growth $(2, \pi/2a)$. In fact, since ψ_1 is imaginary, it follows that $|f(z)|\leqslant c\exp\frac{\pi}{2a}|z|^2$ where

$$c=\sup_{v\in B}|\sigma(v)\exp(-\psi_0(v))|.$$

If $a=\pi$ the function f is no longer μ-square integrable.

In the case of a square lattice ($\omega_2=i\omega_1$) the constant $b=0$. For our basic lattice $z_{m,n}=\gamma(m+in)$ with $\gamma=a^{\frac{1}{2}}>\pi^{\frac{1}{2}}$, the preceding argument directly gives a nonvanishing μ-square integrable function which nevertheless vanishes on all the lattice points. Thus such a lattice of coherent states is not sufficiently dense.

3. The von Neumann lattice

In the case of the von Neumann lattice ($\gamma=\sqrt{\pi}$) simple analyticity arguments do not suffice, and appeal must be made to the square integrability of the functions involved to establish completeness. Use of the Schrödinger (configuration) representation permits us to write

$$\psi(z)=\langle z|\psi\rangle=e^{-\frac{1}{2}ipq}\int_{-\infty}^{\infty}\varphi_0^*(x)e^{-ipx}\psi(x+q)\,dx$$

where for convenience we adopt

$$z=[(2\pi)^{\frac{1}{2}}q+i(2\pi)^{-\frac{1}{2}}p]/\sqrt{2},$$

$$\varphi_0(x)=2^{\frac{1}{4}}\exp(-\pi x^2),$$

and choose $\psi(x)$ as the Schrödinger representative of $|\psi\rangle$. We consider the lattice of points

$$z=\sqrt{\pi}(m+in),$$

or the phase-space lattice

$$(q,p)=(m,2\pi n),$$

for $m, n=0, \pm 1, \pm 2, \ldots$, and the corresponding expression [omitting a factor $(-1)^{mn}$]

$$\chi_{m,n}\equiv \int_{-\infty}^{\infty} \varphi_0^*(x)\,e^{-i2\pi nx}\psi(x+m)\,dx$$

for all m and n. Evidently, $|\psi\rangle=0$ implies that $\chi_{m,n}=0$. We wish to show the converse is also true.

We first observe that for almost all x, $0\leqslant x\leqslant 1$, we must have

$$\sum_{r=-\infty}^{\infty}|\psi(x+r)|^2<\infty$$

in order that

$$\int_0^1\sum_{r=-\infty}^{\infty}|\psi(x+r)|^2dx=\sum_{r=-\infty}^{\infty}\int_0^1|\psi(x+r)|^2dx$$

$$=\sum_{r=-\infty}^{\infty}\int_r^{r+1}|\psi(x)|^2dx$$

$$=\int_{-\infty}^{\infty}|\psi(x)|^2dx<\infty.$$

In like manner, for almost all x, $0\leqslant x\leqslant 1$,

$$\sum_{r=-\infty}^{\infty}|\varphi_0(x+r)|^2<\infty.$$

Thus there exists a set E (a subset of $[0, 1]$) with a complement having zero measure, for which the two sums converge for all $x\in E$. It follows that

$$\sum_{r=-\infty}^{\infty}|\varphi_0(x+r)\,\psi(x+m+r)|<\infty$$

for all $x\in E$ and all m.

In view of the foregoing, the expression for $\chi_{m,n}$ becomes

$$\chi_{m,n}=\sum_{r=-\infty}^{\infty}\int_r^{r+1}\varphi_0^*(x)\,e^{-i2\pi nx}\psi(x+m)\,dx$$

$$= \sum_{r=-\infty}^{\infty} \int_0^1 \varphi_0^*(x+r)\, e^{-i2\pi nx} \psi(x+m+r)\, dx$$

$$= \int_0^1 \Big\{ \sum_{r=-\infty}^{\infty} \varphi_0^*(x+r)\, \psi(x+m+r) \Big\}\, e^{-i2\pi nx} dx\,,$$

If we regard $\chi_{m,n}$ as the nth Fourier coefficient for the function $\sum \varphi_0^*(x+r)\, \psi(x+m+r)$, then the relation

$$\chi_{m,n} = 0\,, \qquad \text{for all } m \text{ and } n$$

implies that

$$\sum_{r=-\infty}^{\infty} \varphi_0^*(x+r)\, \psi(x+m+r) = 0$$

for all m and all $x \in E'$, a set (like E) whose complement in $[0, 1]$ has measure zero. For $x \in \tilde{E} \equiv E \cap E'$ and $0 \leqslant y \leqslant 1$ we define

$$M_x(y) \equiv \sum_{r=-\infty}^{\infty} \varphi_0^*(x+r)\, e^{-2\pi iry},$$

$$P_x(y) \equiv \sum_{r=-\infty}^{\infty} \psi(x+r)\, e^{2\pi iry}$$

both of which converge in the mean in view of the square summability of the coefficients for any $x \in \tilde{E}$. With $x \in \tilde{E}$

$$\int_0^1 M_x(y)\, P_x(y)\, e^{-2\pi imy} dy = \sum_{r=-\infty}^{\infty} \varphi_0^*(x+r)\, \psi(x+m+r) = 0$$

from which we deduce that

$$M_x(y)\, P_x(y) = 0$$

for $x \in \tilde{E}$ and for almost all y, $0 \leqslant y \leqslant 1$. If we assume that $M_x(y) = 0$ only at a finite number of points, then $P_x(y) = 0$ a.e. y and for all $x \in \tilde{E}$. But in that case

$$\psi(x+r) = \int_0^1 e^{-2\pi iry} P_x(y)\, dy = 0$$

for all r and for all $x \in \tilde{E}$, and therefore

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \sum_{r=-\infty}^{\infty} \int_{\tilde{E}} |\psi(x+r)|^2 dx = 0\,.$$

It remains to convince ourselves that $M_x(y) = 0$ only at a finite number of points. From the form of $\varphi_0(x)$ we find that

$$M_x(y) = 2^{\frac{1}{4}} \sum_{r=-\infty}^{\infty} e^{-\pi(x+r)^2 - 2\pi iry}$$

$$= 2^{\frac{1}{4}} e^{\pi(y^2 + 2ixy)} \sum_{r=-\infty}^{\infty} e^{-\pi(x+iy+r)^2}.$$

As a non-vanishing multiple of an entire function of $\tilde{z}=x+iy$, $M_x(y)$ can only have a finite number of zeroes in the interval $0 \leqslant x, y \leqslant 1$ and still not vanish altogether. Consequently, the appropriate conditions for $M_x(y)$ are fulfilled and the statement that

$$\langle \sqrt{\pi}(m+in) \,|\, \psi \rangle = 0$$

for all m and n implies that $|\psi\rangle = 0$ as was to be shown.

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REFERENCES

[1] Bargmann, V., Commun. Pure and Appl. Math. **14** (1961), 187.
[2] Boas, R. P., *Entire functions*, Academic Press, New York, 1954.
[3] Campagnoli, G., and G. Zambotti, Nuovo Cimento **3** (1968), 468.
[4] Glauber, R. J., Phys. Rev. **131** (1963), 2766.
[5] Klauder, J. R., Ann. Phys. N. Y. **11** (1960), 123.
[6] Klauder, J. R., and E. C. G. Sudarshan, *Fundamentals of quantum optics*, W. Benjamin, New York, 1968.
[7] von Neumann, J., *Mathematical foundations of quantum mechanics*, Princeton, 1955.
[8] Whittaker, E. T., and G. N. Watson, *A course of modern analysis*, Cambridge, 1962.