(P)ISCES: A Spectral Element Code to Tackle Staircases in Double-Diffusive Convection

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The Problem
- Double-Diffusive Convection
- ODDC in Massive Stars
- Setup
- Model

(P)ISCES

Future

Double-diffusive convection defines instabilities in nominally stable fluids with two opposing buoyant fields.

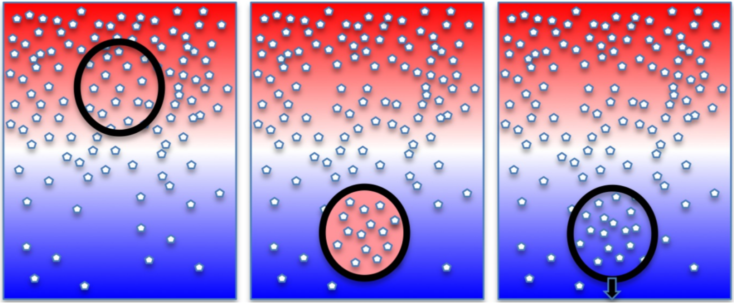

[1]

Figure: Fingering (FC; also Thermohaline, Thermocompositional) Convection

[1] (Garaud, 2014)

Double-diffusive convection defines instabilities in nominally stable fluids with two opposing buoyant fields.

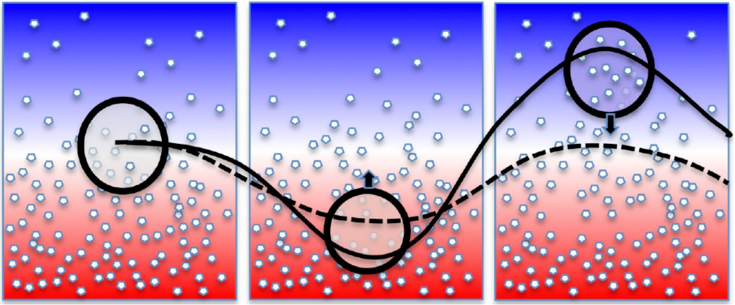
[1]

Figure: Oscillatory Double-Diffusive Convection (ODDC; also sometimes Semi-Convection)

[1](Garaud, 2014)

Double-diffusive convection occurs in several physical scenarios.

- FC
    - Thermocline of the ocean
    - Red Giant Branch of Sun-like stars
- ODDC
    - Beneath the arctic ice cap (also intrusions)
    - Throughout massive star evolution

In this talk (because I'm an astrophysics student), I will focus on the massive star applications.

ODDC is modeled in stars using the Langer et al. (1983) prescription by deriving a diffusion coefficient, D.

MESA (Langer et al., 1983)

- Only mixes composition
- One free parameter, α

$$D = \frac{\alpha \kappa_T}{6} \frac{\frac{dT}{dz} - \frac{\partial T}{\partial z}\big|_{\mathrm{ad}}}{\frac{\phi T}{\delta \mu}\frac{d\mu}{dz} + \frac{\partial T}{\partial z}\big|_{\mathrm{ad}} - \frac{dT}{dz}} \tag{1}$$

$$\delta \equiv -\frac{\partial \ln \rho}{\partial \ln T}, \phi \equiv \frac{\partial \ln \rho}{\partial \ln \mu} \tag{2}$$

KEPLER (Woosley & Weaver, 1988)

- Only mixes composition
- One free parameter, F

$$D = \frac{F \kappa_T D_b}{F \kappa_T + D_b} \tag{3}$$

$$D_b \propto \left(\frac{dT}{dz} - \frac{\partial T}{\partial z}\bigg|_{\mathrm{ad}} \right)^{\frac{1}{2}} \tag{4}$$

Governing equations (Spiegel & Veronis, 1960)

$$\rho_0 \frac{D}{Dt}\mathbf{u} = -\nabla p + (-\alpha T + \beta\mu)\,\rho_0 \mathbf{g} + \nu\rho_0\nabla^2\mathbf{u} \quad (5)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (6)$$

$$\frac{D}{Dt}T + w\frac{d}{dz}T_0 = \kappa_T \nabla^2 T \quad (7)$$

$$\frac{D}{Dt}\mu + w\frac{d}{dz}\mu_0 = \kappa_\mu \nabla^2 \mu \quad (8)$$

Governing equations (Spiegel & Veronis, 1960)

$$\frac{1}{\mathrm{Pr}}\frac{D}{Dt}\mathbf{u} = -\nabla p + (-T + \mu)\,\hat{\mathbf{z}} + \nabla^2\mathbf{u}, \mathrm{Pr} \equiv \frac{\nu}{\kappa_{\mathrm{T}}} \tag{5}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{6}$$

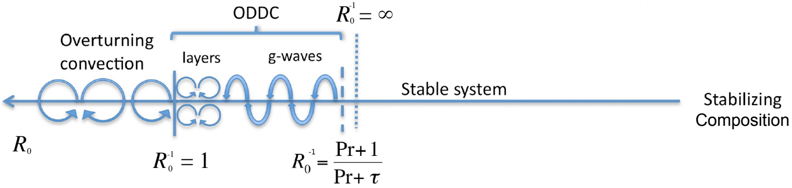
$$\frac{D}{Dt}T \pm w = \nabla^2 T \tag{7}$$

$$\frac{D}{Dt}\mu \pm \frac{w}{R_0} = \tau\nabla^2\mu, R_0 \equiv \frac{\alpha\,(T_{0z} - T_{\mathrm{ad,z}})}{\beta\mu_{0z}}, \tau \equiv \frac{\kappa_\mu}{\kappa_T} \tag{8}$$

The $+$ corresponds to FC and the $-$, ODDC.

Depending on the strength of the stabilizing composition, semi-convection can be in various forms.

$$1 < R_0^{-1} < \frac{\mathrm{Pr}+1}{\mathrm{Pr}+\tau} \tag{9}$$



[2] (Garaud, 2014)

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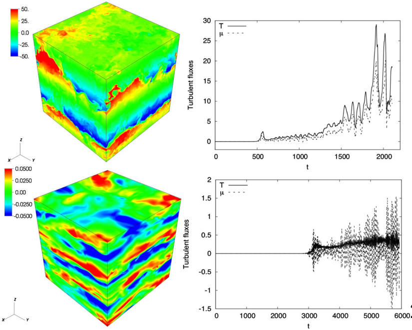
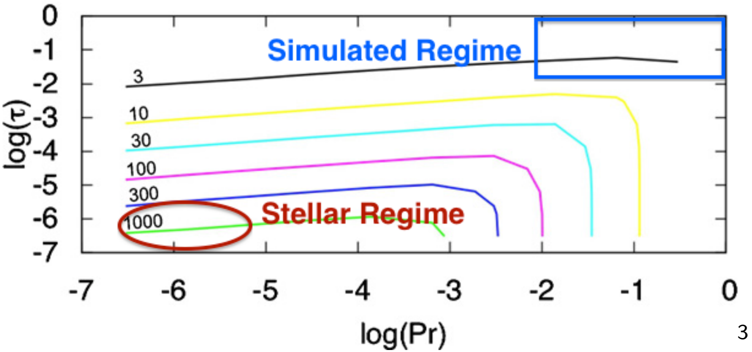


[2]

Figure: $\mathrm{Pr} = \tau = 0.03,\ R_0^{-1} = 1.5\ (\text{top}), 5.0\ (\text{bottom})$

[2](Mirouh et al., 2012)

To determine which regime is relevant to stars, we look at the condition for layer formation from Mirouh et al. (2012).



3

Figure: R_L^{-1}, the maximum R_0^{-1} for which layers exist

3 (Mirouh et al., 2012)

We find in KEPLER simulations that semi-convection regions exist preferentially in the layer-forming regime.

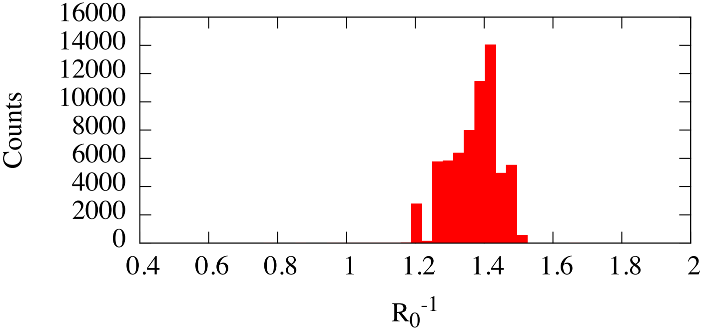
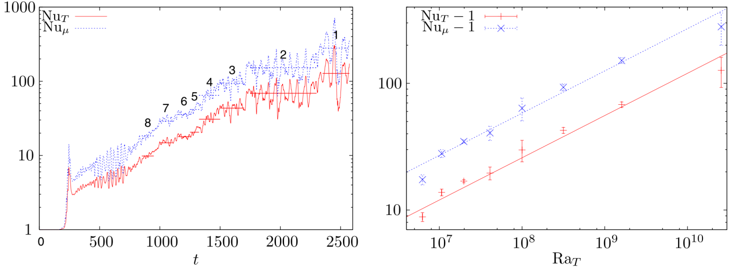


Figure: For semi-convection to be in the layered regime, $1 < R_0^{-1} < R_L^{-1} \sim 1000$

Wood et al. (2013) determined an empirical model for the fluxes in layers with one free parameter, l_{semi}.



[4]

$$Q_{\mathrm{semi}} = -0.1 \left(\frac{g\delta \left| \frac{dT}{dr} - \left.\frac{dT}{dr}\right|_{\mathrm{ad}} \right| (l_{\mathrm{semi}})^4}{T\kappa_T^2} \right)^{1/3} \rho c_p \kappa_T \left(\frac{dT}{dr} - \left.\frac{dT}{dr}\right|_{\mathrm{ad}} \right) \tag{9}$$

[4](Wood et al., 2013)

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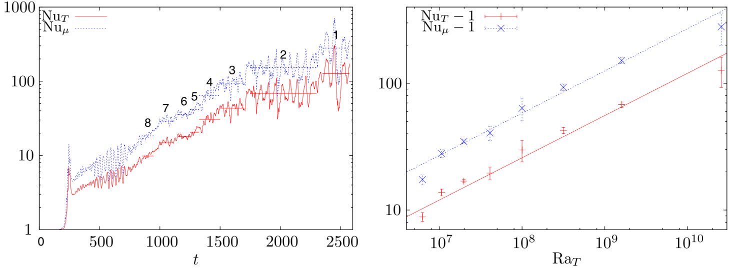
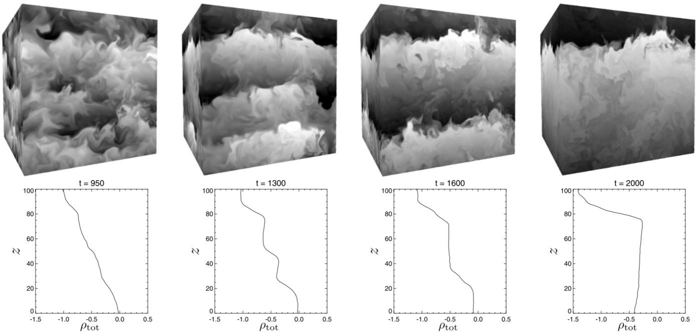


[4]

$$\kappa_{\mu,\mathrm{semi}} = 0.03 \left(\nu \kappa_T^3\right)^{1/4} \left(\frac{g\delta \left| \frac{dT}{dz} - \frac{dT}{dz}\right|_{\mathrm{ad}} \left| (l_{\mathrm{semi}})^4 \right.}{\nu T \kappa_T} \right)^{0.37} \quad (9)$$

[4] (Wood et al., 2013)

The Boussinesq approximation does not constrain the layer height; layers merge until they fill the domain.



[5]

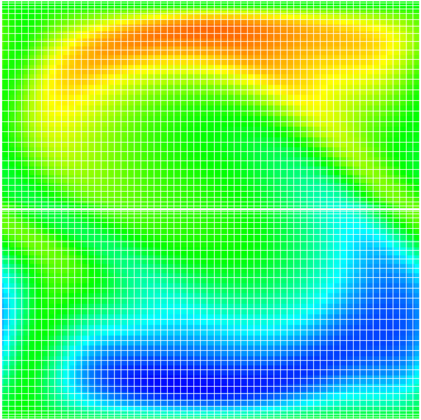
[5] (Wood et al., 2013)

We'd like to perform a new series of simulations to determine the maximum height of these layers.

- Cover new physics not present in PADDI
  - Non-linear diffusion
  - Pseudo-incompressibility
- Attempt larger domains
  - Move to 2D for this to be feasible
- Finely resolve only layer interfaces
  - Not possible in a traditional spectral code!

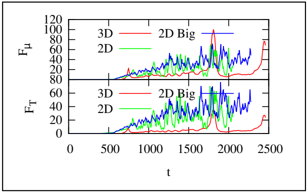
A simple solution to this is a spectral element code using Chebyshev polynomials in the vertical direction.

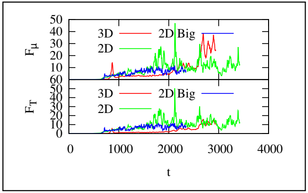
- Solves by Chebyshev collocation
  - Trivial to solve space-dependent diffusion
  - Fine resolution at element boundaries
  - Coarse resolution at element centers



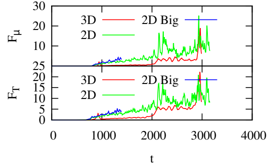
We've run simulations in 2D and 3D with PADDI, and find similar results—except in particular cases with strong shear.

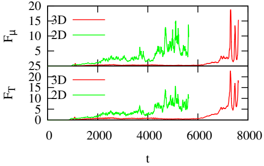
$\mathrm{Pr} = 0.03$
$\tau = 0.3$
$R_0^{-1} = 1.1, 1.2$





$\mathrm{Pr} = 0.01$
$\tau = 0.01$
$R_0^{-1} = 1.5, 2$





At early stages, the simulations have similar behavior; however, at late stages, large shear dominates the 2D runs.

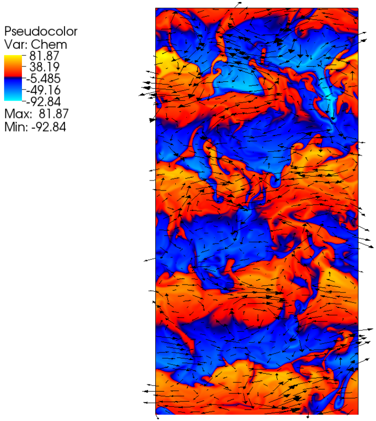


Figure: $\mathrm{Pr} = 0.3$, $\tau = 0.3$, $R_0^{-1} = 1.15$

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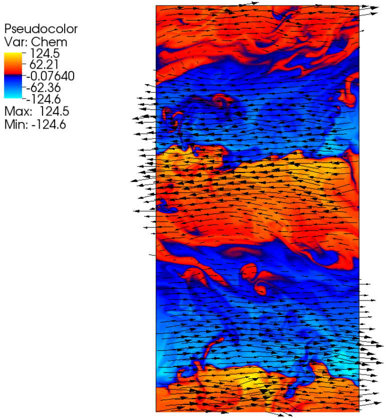
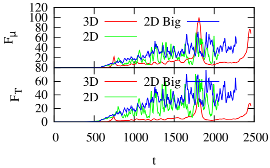
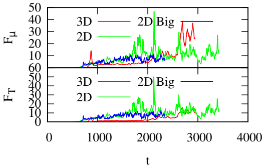


Figure: $\mathrm{Pr} = 0.3,\ \tau = 0.3,\ R_0^{-1} = 1.15$

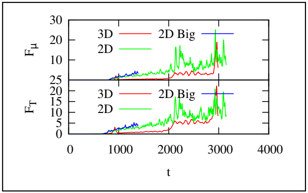
Fortunately, this problem appears absent in the low viscosity and compositional diffusivity runs at low R_0^{-1}.

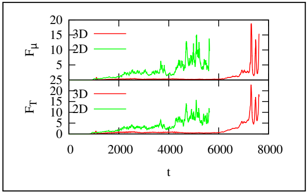
$\mathrm{Pr} = 0.03$
$\tau = 0.3$
$R_0^{-1} = 1.1, 1.2$





$\mathrm{Pr} = 0.01$
$\tau = 0.01$
$R_0^{-1} = 1.5, 2$





These simulations show no sign of the strong horizontal shear apparent in the simulations at higher viscosity.

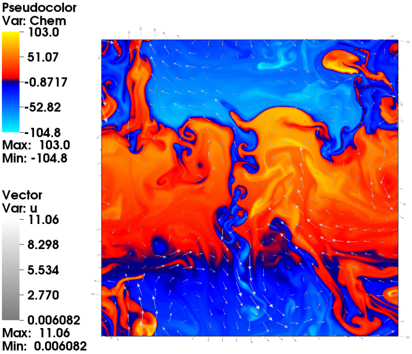


Figure: $\mathrm{Pr} = 0.03$, $\tau = 0.03$, $R_0^{-1} = 1.5$

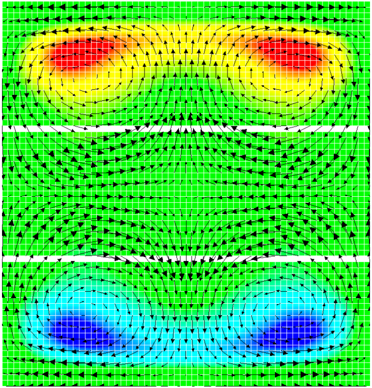
The Problem

(P)ISCES
- (Pseudo-)Incompressible
- Spectral-Chebyshev
- Element Solver
- Results

Future

Boundary conditions

- Fixed outer vertical boundaries
- Periodic horizontal boundaries
- Cross-element boundaries must be continuous and smooth



The code currently constrains the velocity using a predictor-corrector method according to the incompressible velocity constraint (Chorin, 1997).

$$\nabla \cdot \mathbf{u}^{\mathrm{new}} = 0 \tag{10}$$

Update the velocity according to the momentum equation, ignoring the pressure term.

$$\rho_0 \frac{D}{Dt}\mathbf{u}^{\mathrm{mid}} = \left(-\alpha T + \beta\mu\right)\rho_0 \mathbf{g} + \nu\rho_0 \nabla^2 \mathbf{u}^{\mathrm{mid}} \tag{11}$$

$$\frac{\mathbf{u}^{\mathrm{new}} - \mathbf{u}^{\mathrm{mid}}}{\Delta t} = -\frac{\nabla p}{\rho_0} \tag{12}$$

$$\frac{-\nabla \cdot \mathbf{u}^{\mathrm{mid}}}{\Delta t} = -\nabla \cdot \frac{\nabla p}{\rho_0} \tag{13}$$

The code is pseudo-spectral, calculating the non-linear terms in Cartesian space and implicit terms in spectral space.

- Fourier series in the horizontal to enforce periodicity
- Chebyshev polynomials in the vertical for fine boundary resolution

$$F\left(x,z\right)=\sum_{j=-N_x/2}^{N_x/2}\sum_{k=0}^{N_z}f_{i,j}e^{\frac{2\pi ij\left(x-x_0\right)}{L_x}}T_k\left(\frac{z-z_0}{L_z/2}\right) \tag{14}$$

The code solves the diffusion implicitly by alternating directions, solving by collocation in the vertical and by spectral method in the horizontal.

$$\sum_k \left(T_k \left(\frac{z_l - z_0}{L_z/2} \right) - \kappa \left(z_l \right) \frac{\Delta t}{L_z^2/4} T_k'' \left(\frac{z_l - z_0}{L_z/2} \right) \right) f_{j,k}^{\mathrm{new}} = F_j^{\mathrm{old}} \left(z_l \right) + \Delta t RHS_j \left(z_l \right) \tag{15}$$

We can solve equations with z-dependent coefficients implicitly

$$\left(1 - \kappa \left(z_l \right) \Delta t \left(\frac{2\pi ij}{L_x} \right)^2 \right) F_j^{\mathrm{new}} \left(z_l \right) = F_j^{\mathrm{old}} \left(z_l \right) + \Delta t RHS_j \left(z_l \right) \tag{16}$$

The collocation method matches the boundaries of the elements with spectral accuracy; the matrix can be solved as follows (Beaume, private communication).

$$\begin{pmatrix} A_{0,0} & A_{1,0} & 0 & 0 & \dots & 0 & 0 \\ A_{0,1} & A_{1,1} & A_{2,1} & 0 & \dots & 0 & 0 \\ 0 & A_{1,2} & A_{2,2} & A_{3,2} & \dots & 0 & 0 \\ 0 & 0 & A_{2,3} & A_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & A_{N-2,N-3} & 0 \\ 0 & 0 & 0 & 0 & A_{N-3,N-2} & A_{N-2,N-2} & A_{N-1,N-2} \\ 0 & 0 & 0 & 0 & 0 & A_{N-2,N-1} & A_{N-1,N-1} \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{N-2} \\ X_{N-1} \end{pmatrix} = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_{N-2} \\ B_{N-1} \end{pmatrix}$$

The collocation method matches the boundaries of the elements with spectral accuracy; the matrix can be solved as follows (Beaume, private communication).

$$\left(\begin{array}{ccccc|cccc}
A_{0,0} & 0 & \dots & \dots & 0 & A_{1,0} & 0 & \dots & 0 \\
0 & A_{2,2} & \dots & \dots & 0 & A_{1,2} & A_{3,2} & \dots & 0 \\
\vdots & \vdots & \ddots & \dots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \dots & \ddots & 0 & 0 & 0 & \ddots & A_{N-2,N-3} \\
0 & 0 & \dots & \dots & A_{N-1,N-1} & 0 & 0 & \dots & A_{N-2,N-1} \\
\hline
A_{0,1} & A_{2,1} & \dots & 0 & 0 & A_{1,1} & 0 & \dots & 0 \\
0 & A_{2,3} & \ddots & 0 & 0 & 0 & A_{3,3} & \dots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & A_{N-3,N-2} & A_{N-1,N-2} & 0 & 0 & \dots & A_{N-2,N-2}
\end{array}\right)$$

The collocation method matches the boundaries of the elements with spectral accuracy; the matrix can be solved as follows (Beaume, private communication).

$$\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \begin{pmatrix} X_L \\ X_R \end{pmatrix} = \begin{pmatrix} B_T \\ B_B \end{pmatrix} \tag{17}$$

$$A_{TL} X_L + A_{TR} X_R = B_T \tag{18}$$

$$A_{BL} X_L + A_{BL} A_{TL}^{-1} A_{TR} X_R = A_{BL} A_{TL}^{-1} B_T \tag{19}$$

$$\begin{pmatrix} A_{TL} & 0 \\ 0 & A_{BR} - A_{BL} A_{TL}^{-1} A_{TR} \end{pmatrix} \begin{pmatrix} X_L + A_{TR} X_R \\ X_R \end{pmatrix} = \tag{20}$$

$$\begin{pmatrix} B_T \\ B_B - A_{BL} A_{TL}^{-1} B_T \end{pmatrix} \tag{21}$$

The extents of these elements are dynamic, and can be rezoned using simulated annealing.

We define $P(E(s), E(s'), T)$, the probability to accept a new state, s', to maximize E, given a temperature, T, where $P > 0$ even if $E(s') < E(s)$ but that $P(E(s') < E(s)) \to 0$ as $T \to 0$.[6]

- Let $s = s_0$
- For k in 0 to k_{max}:
  - $T =$ temperature $\left(\frac{k}{k_{\mathrm{max}}}\right)$
  - Pick a random neighbor, $s' =$ neighbor (s)
  - If $P(E(s), E(s'), T) >$ random(0, 1):
    - $s = s_{\mathrm{new}}$
- Return: s

[6] Pseudocode from Wikipedia

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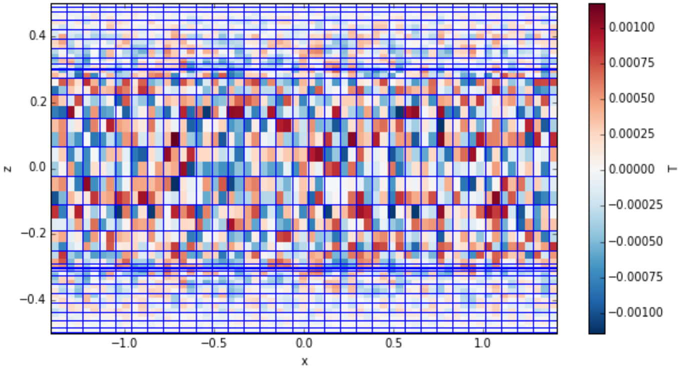
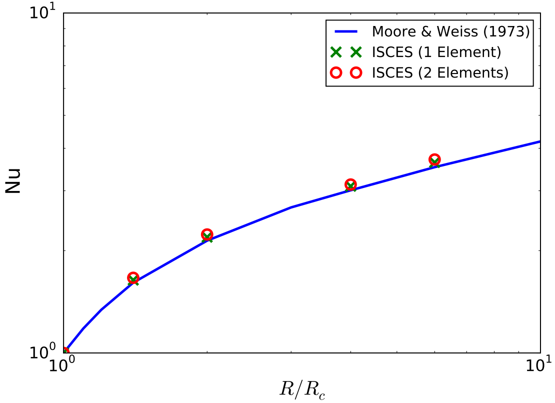


To verify the code, we ran the standard Rayleigh Bénard Convection problem and compared to Moore & Weiss (1973).



We can also compare to the results of the same equations with the same parameters as PADDI.

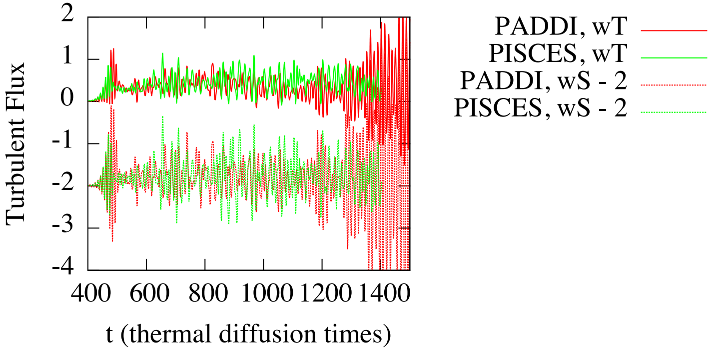


Figure: The late time behavior is dominated by the boundary conditions, which differ between the codes

The Problem

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Future

Layered Semi-Convection

Additional Slides

Future Code Development

- Implement nonlinear diffusion (working in 1D) in 2D
  - Examine layer properties with μ-dependent diffusion
- Upgrade from Boussinesq to Pseudo-Incompressible
  - Determine layer height across density scale heights
- Update KEPLER and MESA with layer height

The Problem

(P)ISCES

Future
- Layered Semi-Convection
- Additional Slides
  - Pseudo-Incompressibility
  - Two-Component Ideal Mixture

Governing equations

Fully Compressible

$$\rho \frac{D}{Dt}\mathbf{u} = -\nabla p + \rho \mathbf{g} + \nabla \cdot \mathbf{\Pi} \tag{22}$$

$$\frac{D}{Dt}\rho = -\rho \nabla \cdot \mathbf{u} \tag{23}$$

$$\rho T \frac{D}{Dt}s = \nabla \mathbf{u} : \mathbf{\Pi} + \sum_i \mu_i \nabla \cdot \mathbf{C}_i - \nabla \cdot \mathbf{H} \tag{24}$$

$$\rho \frac{D}{Dt}\xi_i = -\nabla \cdot \mathbf{C}_i \tag{25}$$

Governing equations

Incompressible, Current Equations

$$\rho_0 \frac{D}{Dt}\mathbf{u} = -\nabla p + \left(-\alpha T + \beta\mu\right)\rho_0 \mathbf{g} + \nu\rho_0\nabla^2\mathbf{u} \tag{22}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{23}$$

$$\frac{D}{Dt}T + w\frac{d}{dz}T_0 = \kappa_T\nabla^2 T \tag{24}$$

$$\frac{D}{Dt}\xi_i + w\frac{d}{dz}\xi_{i,0} = \kappa_i\nabla^2\xi_i \tag{25}$$

Governing equations

Pseudo-Incompressible

$$\rho\frac{D}{Dt}\mathbf{u} = -\nabla\left(p_0 + p_1\right) + \frac{p_1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)^{-1}\nabla p_0 - \rho\nabla\Phi + \nabla\cdot\mathbf{\Pi} \tag{22}$$

$$p_0\left(z\right) = p\left(\rho, s, \xi_1, \xi_2, \dots\right) \tag{23}$$

$$\frac{D}{Dt}\rho = -\rho\nabla\cdot\mathbf{u} \tag{24}$$

$$\rho T\frac{D}{Dt}s = \nabla\mathbf{u} : \mathbf{\Pi} + \sum_i \mu_i\nabla\cdot\mathbf{C}_i - \nabla\cdot\mathbf{H} \tag{25}$$

$$\rho\frac{D}{Dt}\xi_i = -\nabla\cdot\mathbf{C}_i, \tag{26}$$

Governing equations

Pseudo-Incompressible, Energy Conserving (Wood, private communication)

$$\rho \frac{D}{Dt}\mathbf{u} = -\nabla\left(p_0 + p_1\right) + \frac{p_1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)^{-1}\nabla p_0 - \rho\nabla\Phi + \nabla\cdot\mathbf{\Pi} \tag{22}$$

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$$\rho T\frac{D}{Dt}s = \nabla\mathbf{u} : \mathbf{\Pi} + \sum_i \mu_i \nabla\cdot\mathbf{C}_i - \nabla\cdot\mathbf{H} \tag{25}$$

$$\rho\frac{D}{Dt}\xi_i = -\nabla\cdot\mathbf{C}_i,\ \left(T, \mu_i\right) \equiv \left(\frac{\partial e}{\partial s}, \frac{\partial e}{\partial \xi_i}\right) + p_1\frac{\partial\left(\frac{\partial e}{\partial s}, \frac{\partial e}{\partial \xi_i}\right)/\partial\rho}{\partial p/\partial\rho} \tag{26}$$

To solve these equations, we choose the equation of state of an ideal mixture.

$$\rho\frac{D}{Dt}\mathbf{u} = -\nabla\left(p_0 + p_1\right) + \frac{p_1}{\gamma p_0}\nabla p_0 - \rho\nabla\Phi + \nabla\cdot\mathbf{\Pi} \tag{27}$$

$$p_0\left(z\right) = \frac{k\rho\tilde{T}}{\overline{m}} \tag{28}$$

$$\frac{D}{Dt}\rho = -\rho\nabla\cdot\mathbf{u} \tag{29}$$

$$\rho T\frac{D}{Dt}s = \nabla\mathbf{u}:\mathbf{\Pi} + \sum_i \mu_i\nabla\cdot\mathbf{C}_i - \nabla\cdot\mathbf{H} \tag{30}$$

$$\rho\frac{D}{Dt}\xi_i = -\nabla\cdot\mathbf{C}_i, \left(T,\mu_i\right) \equiv \frac{c_p p_0 + k p_1}{c_p p_0}\left(\tilde{T},\tilde{\mu_i}\right) \tag{31}$$

We can simplify this greatly by assuming a perfect gas and converting the entropy equation into a pseudo-temperature equation.

$$\frac{De}{Dt} = \frac{\partial e}{\partial \rho}\frac{D\rho}{Dt} + \frac{\partial e}{\partial s}\frac{Ds}{Dt} + \sum_i \frac{\partial e}{\partial \xi_i}\frac{D\xi_i}{Dt} = c_v \frac{D}{Dt}\frac{\partial e}{\partial s} \qquad (32)$$

$$(33)$$

We draw a distinction between $\frac{\partial e}{\partial s} \equiv \tilde{T}$ and T, as there's no guarantee they are the same in the pseudo-incompressible equations.

We can simplify this greatly by assuming a perfect gas and converting the entropy equation into a pseudo-temperature equation.

$$\frac{De}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt} + \tilde{T}\frac{Ds}{Dt} + \sum_i \tilde{\mu}_i \frac{D\xi_i}{Dt} = c_v \frac{D}{Dt}\tilde{T} \tag{32}$$

$$\rho T \frac{D}{Dt}s = \rho\frac{T}{\tilde{T}}\left(c_v \frac{D}{Dt}\tilde{T} - \frac{p}{\rho^2}\frac{D\rho}{Dt} - \sum_i \tilde{\mu}_i \frac{D\xi_i}{Dt}\right) \tag{33}$$

$$= \nabla \mathbf{u} : \mathbf{\Pi} + \sum_i \mu_i \nabla \cdot \mathbf{C}_i - \nabla \cdot \mathbf{H}(T) \tag{34}$$

We can simplify this greatly by assuming a perfect gas and converting the entropy equation into a pseudo-temperature equation.

After some algebra,

$$\rho c_v \frac{D}{Dt}\tilde{T} = \frac{c_p p}{c_p p + k p_1}\left(\nabla \mathbf{u} : \mathbf{\Pi} - \nabla \cdot \mathbf{H}\left(\frac{c_p p + k p_1}{c_p p}\tilde{T}\right)\right) - p\nabla \cdot \mathbf{u} \tag{32}$$

Governing Equations

$$\rho \frac{D}{Dt}\mathbf{u} = -\nabla\left(p_0 + p_1\right) + \frac{p_1}{\gamma p_0}\nabla p_0 - \rho\nabla\Phi + \nabla\cdot\mathbf{\Pi} \quad (33)$$

$$\nabla\cdot p_0^{\frac{1}{\gamma}}\mathbf{u} = \frac{1}{\gamma p_0^{\frac{\gamma-1}{\gamma}}}\left(\frac{\partial p}{\partial s}\frac{D}{Dt}s + \sum_i \frac{\partial p}{\partial \xi_i}\frac{D}{Dt}\xi_i\right) \quad (34)$$

$$p_0\left(z\right) = \frac{k\rho\tilde{T}}{\overline{m}} \quad (35)$$

$$\rho\frac{k}{\gamma-1}\frac{D}{Dt}\tilde{T} = \frac{p_0}{p_0 + \frac{\gamma-1}{\gamma}p_1}\left(\nabla\mathbf{u} : \mathbf{\Pi} - \nabla\cdot\mathbf{H}\right) - p\nabla\cdot\mathbf{u} \quad (36)$$

$$\rho\frac{D}{Dt}\xi_i = -\nabla\cdot\mathbf{C}_i \quad (37)$$

Derivation of the Equation of State for a two-component mixture.

Start with the EOS for an ideal gas:

$$e_i\left(\rho_i, s_i\right) = \frac{c_V}{m_i}\left(\frac{\rho_i}{m_i}\phi_i e^{\frac{m_i s_i}{k}}\right)^{\frac{k}{c_V}} \tag{38}$$

Each component of the gas must follow this, and the total energy is $\overline{m}e = m_1 e_1 + m_2 e_2$.

Derivation of the Equation of State for a two-component mixture.

Note the entropy of the mixture is defined as $\overline{m}s = m_1\xi_1 s_1 +_2 \xi_2 s_2 - k\xi_1 \ln \xi_1 - k\xi_2 \ln \xi_2$. We assume the gases are thermally coupled:

$$T = \left(\frac{\rho\xi_1\phi_1}{\overline{m}} e^{\frac{m_1 s_1}{k}} \right)^{\frac{k}{c_V}} = \left(\frac{\rho\xi_2\phi_2}{\overline{m}} e^{\frac{m_2 s_2}{k}} \right)^{\frac{k}{c_V}} \tag{38}$$

$$e^{\frac{m_1 s_1}{k}} = \left(\frac{\xi_2\phi_2}{\xi_1\phi_1} \right)^{\xi_2} e^{\frac{\overline{m}s}{k} + \xi_1 \ln \xi_1 + \xi_2 \ln \xi_2} \tag{39}$$

And likewise for s_2

Derivation of the Equation of State for a two-component mixture.

Thus, the total energy is

$$e_1\left(\rho, s, \xi_1, \xi_2\right) = c_V \left(\frac{\rho \xi_1 \phi_1}{m_1} \left[\frac{\xi_2 \phi_2}{\xi_1 \phi 1} \right]^{\xi_2} e^{\frac{\overline{m} s}{k} + \xi_1 \ln \xi_1 + \xi_2 \ln \xi_2} \right)^{\frac{k}{c_V}}, \tag{38}$$

$$e_2\left(\rho, s, \xi_1, \xi_2\right) = c_V \left(\frac{\rho \xi_2 \phi_2}{m_2} \left[\frac{\xi_1 \phi_1}{\xi_2 \phi 2} \right]^{\xi_1} e^{\frac{\overline{m} s}{k} + \xi_1 \ln \xi_1 + \xi_2 \ln \xi_2} \right)^{\frac{k}{c_V}}, \tag{39}$$

$$\overline{m} e\left(\rho, s, \xi_1\right) = \sum_i m_i \xi_i e_i\left(\rho, s, \xi_1, 1 - \xi_1\right). \tag{40}$$

Derivation of the Equation of State for a two-component mixture.

		$\frac{\partial}{\partial \rho}$	$\frac{\partial/\partial\rho}{\partial p/\partial\rho}$
p	$\frac{k}{c_v}\rho e$	$\frac{c_p}{c_v}\frac{p}{\rho}$	1
T	$\frac{e}{c_v}$	$\frac{kT}{c_v\rho}$	$\frac{kT}{c_p p}$
μ_1	$\frac{\frac{1}{\xi_1}(m_2-\overline{m})(c_p-\overline{m}s)+k\ln\frac{\phi_1}{\phi_2}}{c_v}e$	$\frac{k\mu_1}{c_v\rho}$	$\frac{k\mu_1}{c_p p}$

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