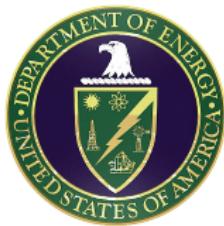


(P)ISCES: A Spectral Element Code to Tackle Staircases in Double-Diffusive Convection

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The Problem

Double-Diffusive Convection

ODDC in Massive Stars

Setup

Model

(P)ISCES

Future

Double-diffusive convection defines instabilities in nominally stable fluids with two opposing buoyant fields.

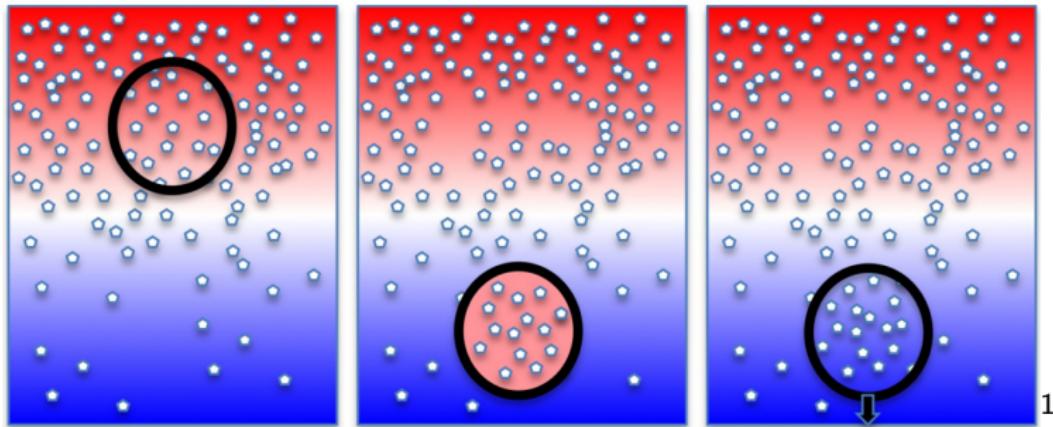


Figure: Fingering (FC; also Thermohaline, Thermocompositional) Convection

¹(Garaud, 2014)

Double-diffusive convection defines instabilities in nominally stable fluids with two opposing buoyant fields.

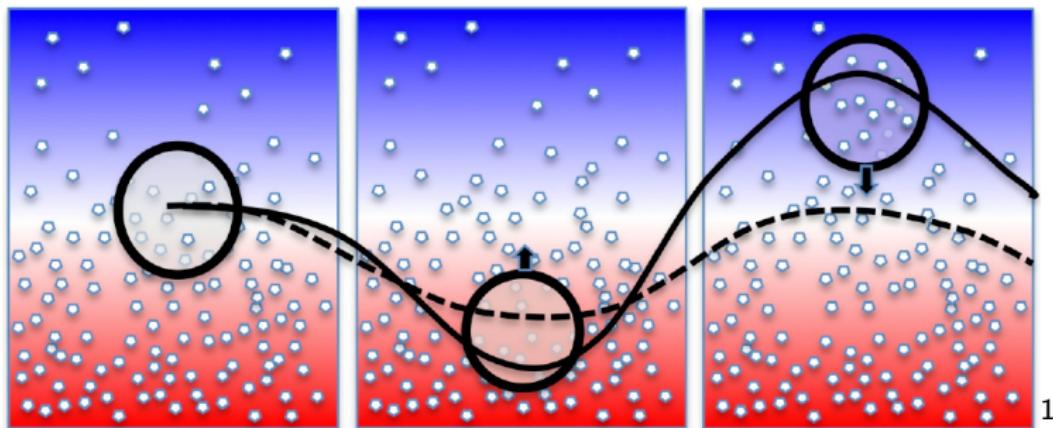


Figure: Oscillatory Double-Diffusive Convection (ODDC; also sometimes Semi-Convection)

¹(Garaud, 2014)

Double-diffusive convection occurs in several physical scenarios.

- ▶ FC
 - ▶ Thermocline of the ocean
 - ▶ Red Giant Branch of Sun-like stars
- ▶ ODDC
 - ▶ Beneath the arctic ice cap (also intrusions)
 - ▶ Throughout massive star evolution

In this talk (because I'm an astrophysics student), I will focus on the massive star applications.

ODDC is modeled in stars using the Langer et al. (1983) prescription by deriving a diffusion coefficient, D .

MESA (Langer et al., 1983)

- ▶ Only mixes composition
- ▶ One free parameter, α

$$D = \frac{\alpha \kappa_T}{6} \frac{\frac{dT}{dz} - \left. \frac{\partial T}{\partial z} \right|_{\text{ad}}}{\frac{\phi T}{\delta \mu} \frac{d\mu}{dz} + \left. \frac{\partial T}{\partial z} \right|_{\text{ad}} - \frac{dT}{dz}} \quad (1)$$

$$\delta \equiv -\frac{\partial \ln \rho}{\partial \ln T}, \phi \equiv \frac{\partial \ln \rho}{\partial \ln \mu} \quad (2)$$

KEPLER (Woosley & Weaver, 1988)

- ▶ Only mixes composition
- ▶ One free parameter, F

$$D = \frac{F \kappa_T D_b}{F \kappa_T + D_b} \quad (3)$$

$$D_b \propto \left(\frac{dT}{dz} - \left. \frac{\partial T}{\partial z} \right|_{\text{ad}} \right)^{\frac{1}{2}} \quad (4)$$

Governing equations (Spiegel & Veronis, 1960)

$$\rho_0 \frac{D}{Dt} \mathbf{u} = -\nabla p + (-\alpha T + \beta \mu) \rho_0 \mathbf{g} + \nu \rho_0 \nabla^2 \mathbf{u} \quad (5)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (6)$$

$$\frac{D}{Dt} T + w \frac{d}{dz} T_0 = \kappa_T \nabla^2 T \quad (7)$$

$$\frac{D}{Dt} \mu + w \frac{d}{dz} \mu_0 = \kappa_\mu \nabla^2 \mu \quad (8)$$

Governing equations (Spiegel & Veronis, 1960)

$$\frac{1}{\text{Pr}} \frac{D}{Dt} \mathbf{u} = -\nabla p + (-T + \mu) \hat{\mathbf{z}} + \nabla^2 \mathbf{u}, \text{Pr} \equiv \frac{\nu}{\kappa_T} \quad (5)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (6)$$

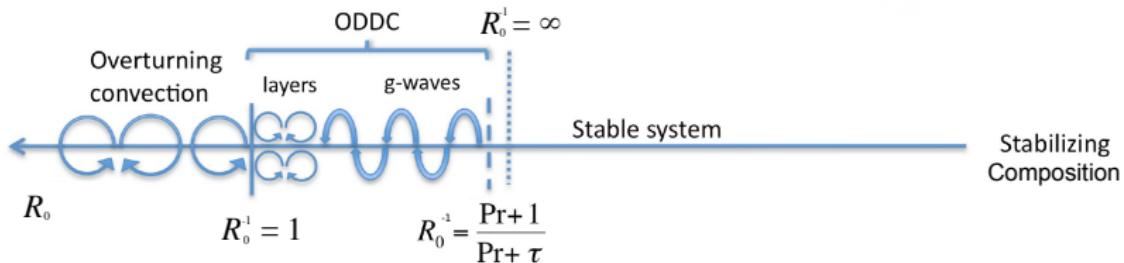
$$\frac{D}{Dt} T \pm w = \nabla^2 T \quad (7)$$

$$\frac{D}{Dt} \mu \pm \frac{w}{R_0} = \tau \nabla^2 \mu, R_0 \equiv \frac{\alpha (T_{0z} - T_{\text{ad},z})}{\beta \mu_{0z}}, \tau \equiv \frac{\kappa_\mu}{\kappa_T} \quad (8)$$

The + corresponds to FC and the -, ODDC.

Depending on the strength of the stabilizing composition, semi-convection can be in various forms.

$$1 < R_0^{-1} < \frac{\text{Pr} + 1}{\text{Pr} + \tau} \quad (9)$$



²(Garaud, 2014)

Depending on the strength of the stabilizing composition, semi-convection can be in various forms.

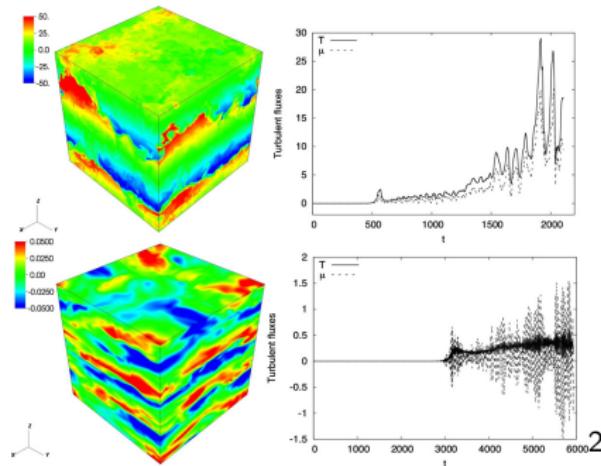


Figure: $\text{Pr} = \tau = 0.03$, $R_0^{-1} = 1.5$ (top), 5.0 (bottom)

To determine which regime is relevant to stars, we look at the condition for layer formation from Mirouh et al. (2012).

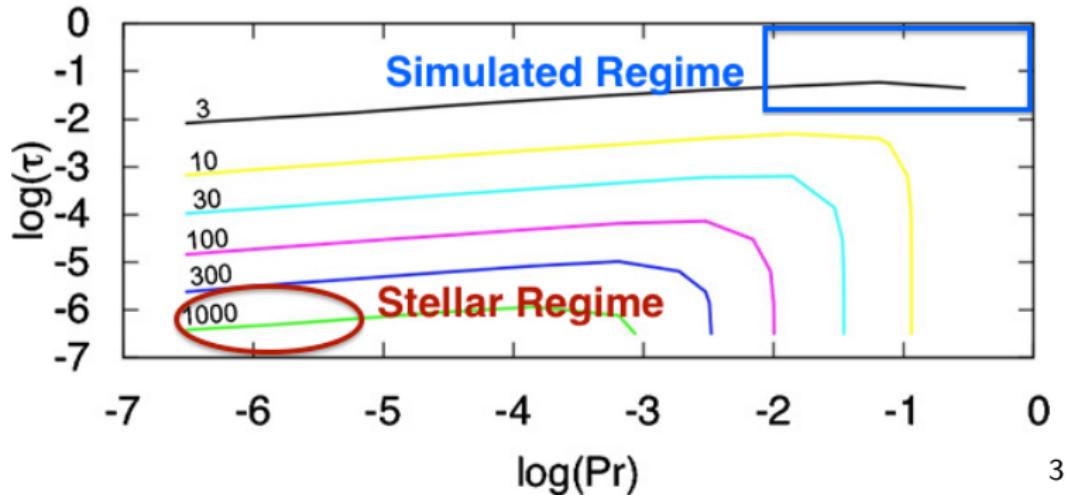


Figure: R_L^{-1} , the maximum R_0^{-1} for which layers exist

³(Mirouh et al., 2012)

We find in KEPLER simulations that semi-convection regions exist preferentially in the layer-forming regime.

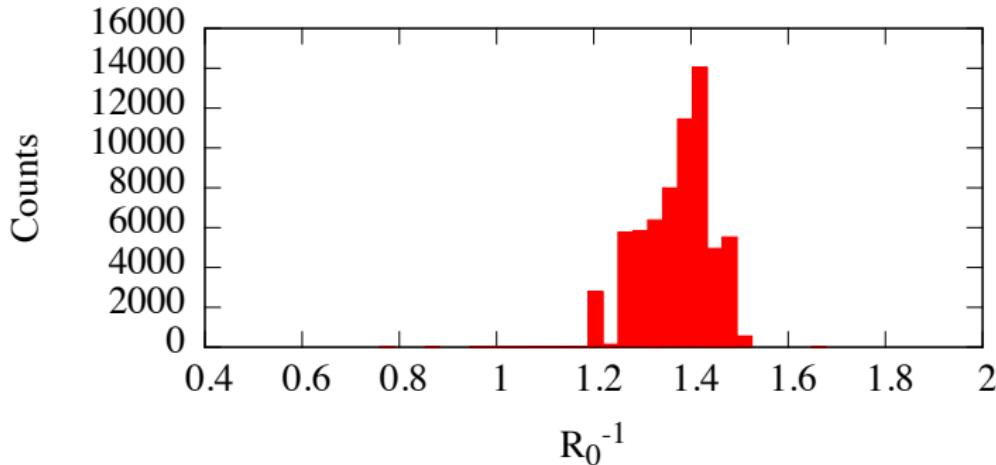
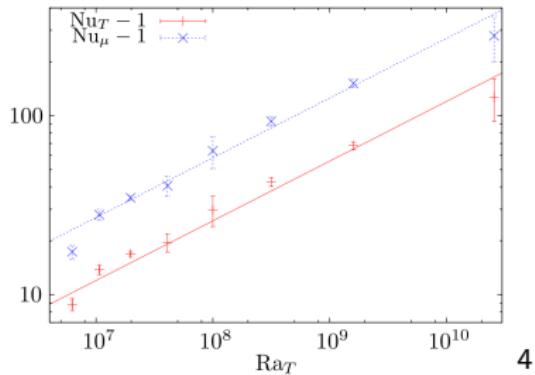
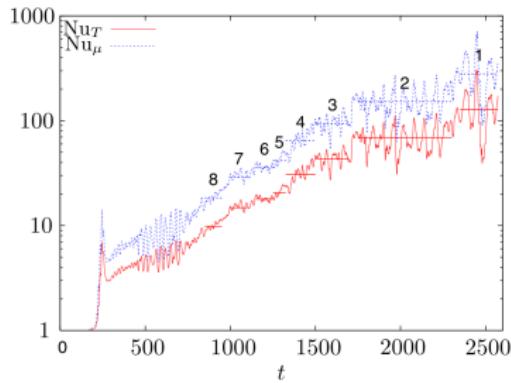


Figure: For semi-convection to be in the layered regime,
 $1 < R_0^{-1} < R_L^{-1} \sim 1000$

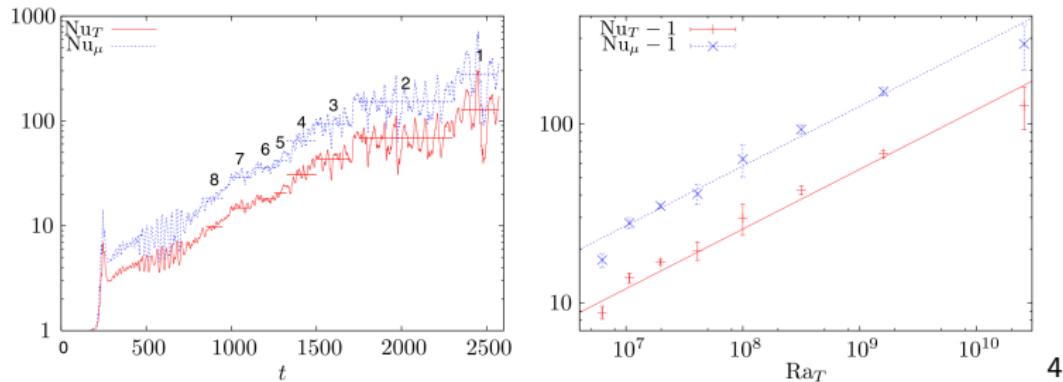
Wood et al. (2013) determined an empirical model for the fluxes in layers with one free parameter, I_{semi} .



$$Q_{\text{semi}} = -0.1 \left(\frac{g \delta \left| \frac{dT}{dr} - \left. \frac{dT}{dr} \right|_{\text{ad}} \right| (I_{\text{semi}})^4}{T \kappa_T^2} \right)^{1/3} \rho c_p \kappa_T \left(\left. \frac{dT}{dr} - \left. \frac{dT}{dr} \right|_{\text{ad}} \right) \right) \quad (9)$$

⁴(Wood et al., 2013)

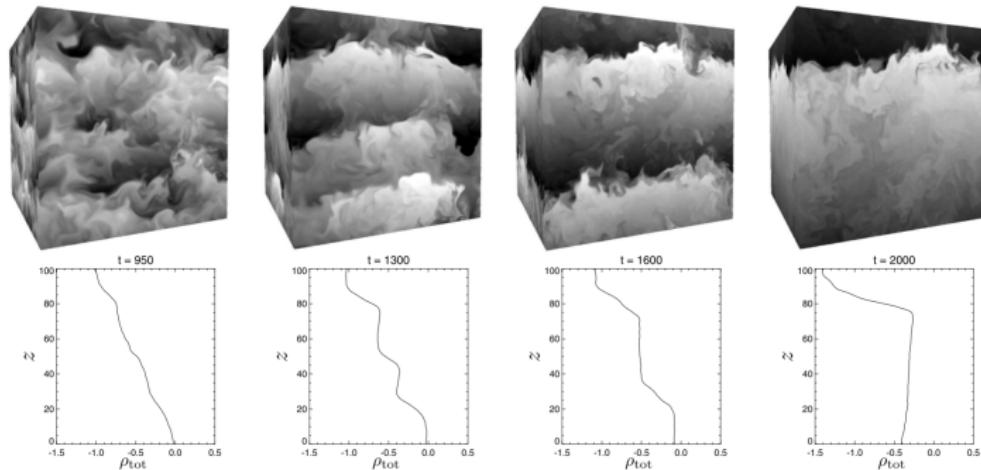
Wood et al. (2013) determined an empirical model for the fluxes in layers with one free parameter, I_{semi} .



$$\kappa_{\mu,\text{semi}} = 0.03 \left(\nu \kappa_T^3 \right)^{1/4} \left(\frac{g \delta | \frac{dT}{dz} - \frac{dT}{dz} |_{\text{ad}} | (I_{\text{semi}})^4 }{ \nu T \kappa_T } \right)^{0.37} \quad (9)$$

⁴(Wood et al., 2013)

The Boussinesq approximation does not constrain the layer height; layers merge until they fill the domain.



5

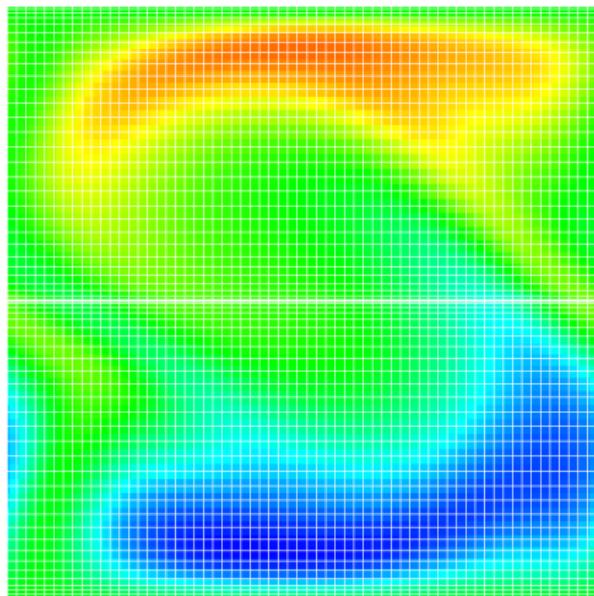
⁵(Wood et al., 2013)

We'd like to perform a new series of simulations to determine the maximum height of these layers.

- ▶ Cover new physics not present in PADDI
 - ▶ Non-linear diffusion
 - ▶ Pseudo-incompressibility
- ▶ Attempt larger domains
 - ▶ Move to 2D for this to be feasible
- ▶ Finely resolve only layer interfaces
 - ▶ Not possible in a traditional spectral code!

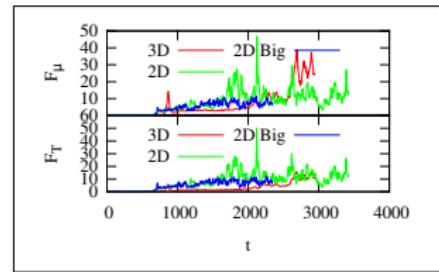
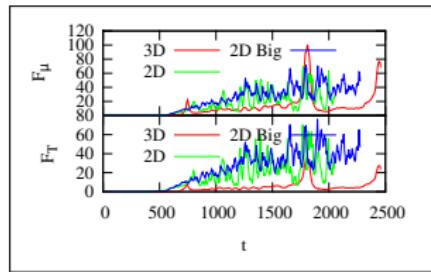
A simple solution to this is a spectral element code using Chebyshev polynomials in the vertical direction.

- ▶ Solves by Chebyshev collocation
 - ▶ Trivial to solve space-dependent diffusion
 - ▶ Fine resolution at element boundaries
 - ▶ Coarse resolution at element centers

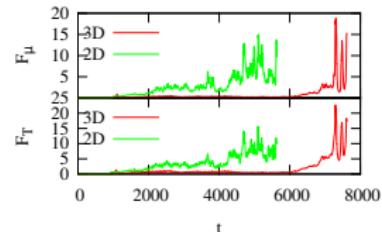
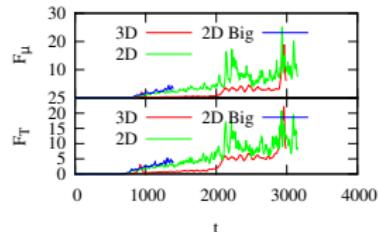


We've run simulations in 2D and 3D with PADDI, and find similar results—except in particular cases with strong shear.

$$\Pr = 0.03 \\ \tau = 0.3 \\ R_0^{-1} = 1.1, 1.2$$



$$\Pr = 0.01 \\ \tau = 0.01 \\ R_0^{-1} = 1.5, 2$$



At early stages, the simulations have similar behavior; however, at late stages, large shear dominates the 2D runs.

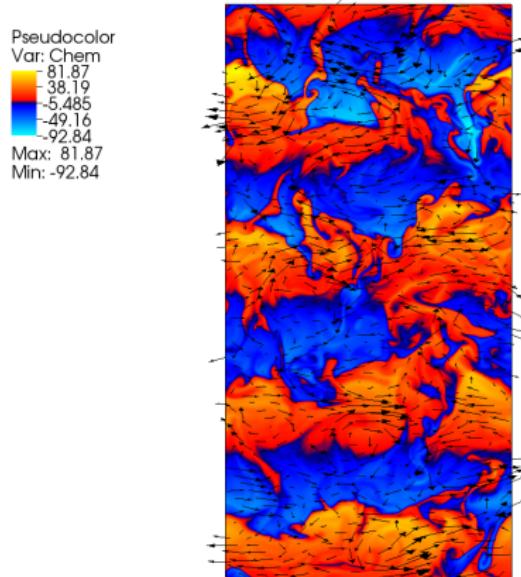


Figure: $\text{Pr} = 0.3$, $\tau = 0.3$, $R_0^{-1} = 1.15$

At early stages, the simulations have similar behavior; however, at late stages, large shear dominates the 2D runs.

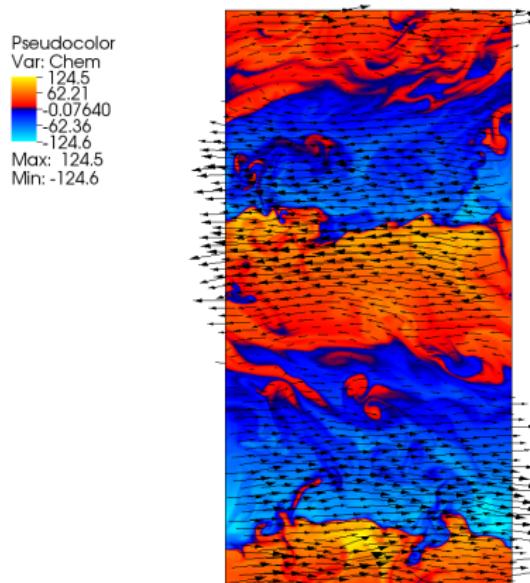
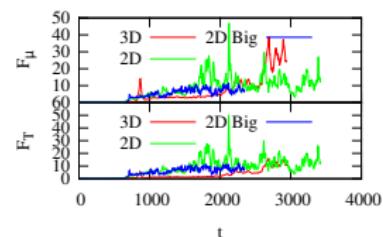
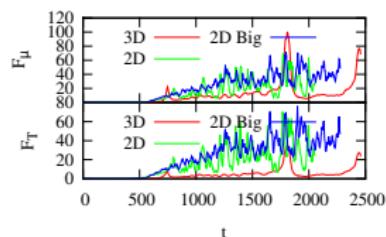


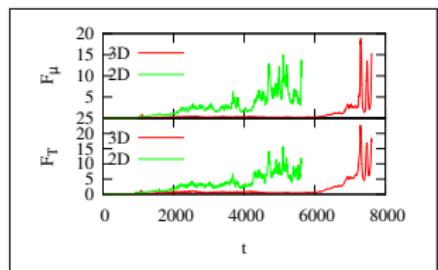
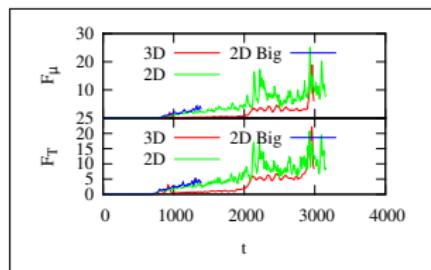
Figure: $\text{Pr} = 0.3$, $\tau = 0.3$, $R_0^{-1} = 1.15$

Fortunately, this problem appears absent in the low viscosity and compositional diffusivity runs at low R_0^{-1} .

$$\text{Pr} = 0.03 \\ \tau = 0.3 \\ R_0^{-1} = 1.1, 1.2$$



$$\text{Pr} = 0.01 \\ \tau = 0.01 \\ R_0^{-1} = 1.5, 2$$



These simulations show no sign of the strong horizontal shear apparent in the simulations at higher viscosity.

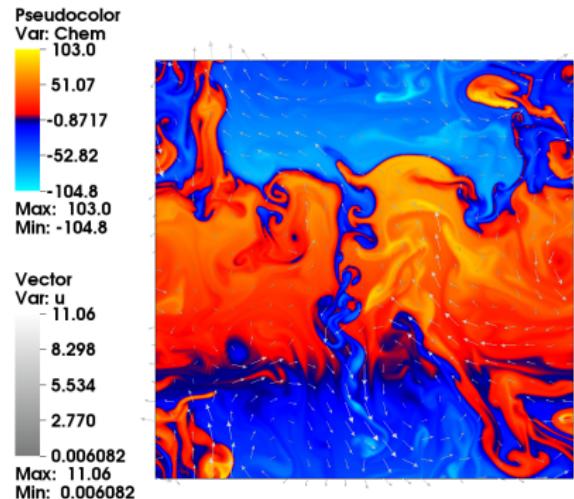


Figure: $\text{Pr} = 0.03$, $\tau = 0.03$, $R_0^{-1} = 1.5$

The Problem

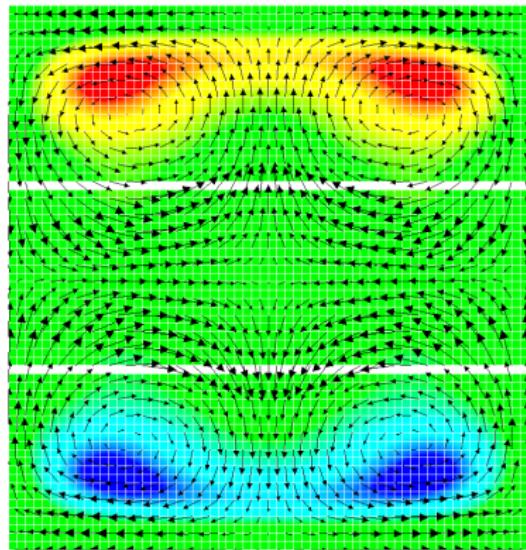
(P)ISCES

(Pseudo-)Incompressible
Spectral-Chebyshev
Element Solver
Results

Future

Boundary conditions

- ▶ Fixed outer vertical boundaries
- ▶ Periodic horizontal boundaries
- ▶ Cross-element boundaries must be continuous and smooth



The code currently constrains the velocity using a predictor-corrector method according to the incompressible velocity constraint (Chorin, 1997).

$$\nabla \cdot \mathbf{u}^{\text{new}} = 0 \quad (10)$$

Update the velocity according to the momentum equation, ignoring the pressure term.

$$\rho_0 \frac{D}{Dt} \mathbf{u}^{\text{mid}} = (-\alpha T + \beta \mu) \rho_0 \mathbf{g} + \nu \rho_0 \nabla^2 \mathbf{u}^{\text{mid}} \quad (11)$$

$$\frac{\mathbf{u}^{\text{new}} - \mathbf{u}^{\text{mid}}}{\Delta t} = - \frac{\nabla p}{\rho_0} \quad (12)$$

$$\frac{-\nabla \cdot \mathbf{u}^{\text{mid}}}{\Delta t} = - \nabla \cdot \frac{\nabla p}{\rho_0} \quad (13)$$

The code is pseudo-spectral, calculating the non-linear terms in Cartesian space and implicit terms in spectral space.

- ▶ Fourier series in the horizontal to enforce periodicity
- ▶ Chebyshev polynomials in the vertical for fine boundary resolution

$$F(x, z) = \sum_{j=-N_x/2}^{N_x/2} \sum_{k=0}^{N_z} f_{i,j} e^{\frac{2\pi i j (x - x_0)}{L_x}} T_k \left(\frac{z - z_0}{L_z/2} \right) \quad (14)$$

The code solves the diffusion implicitly by alternating directions, solving by collocation in the vertical and by spectral method in the horizontal.

$$\sum_k \left(T_k \left(\frac{z_l - z_0}{L_z/2} \right) - \kappa(z_l) \frac{\Delta t}{L_z^2/4} T_k'' \left(\frac{z_l - z_0}{L_z/2} \right) \right) f_{j,k}^{\text{new}} = F_j^{\text{old}}(z_l) + \Delta t RHS_j(z_l) \quad (15)$$

We can solve equations with z -dependent coefficients implicitly

$$\left(1 - \kappa(z_l) \Delta t \left(\frac{2\pi ij}{L_x} \right)^2 \right) F_j^{\text{new}}(z_l) = F_j^{\text{old}}(z_l) + \Delta t RHS_j(z_l) \quad (16)$$

The collocation method matches the boundaries of the elements with spectral accuracy; the matrix can be solved as follows (Beaume, private communication).

$$\begin{pmatrix} A_{0,0} & A_{1,0} & 0 & 0 & \dots & 0 & 0 \\ A_{0,1} & A_{1,1} & A_{2,1} & 0 & \dots & 0 & 0 \\ 0 & A_{1,2} & A_{2,2} & A_{3,2} & \dots & 0 & 0 \\ 0 & 0 & A_{2,3} & A_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & A_{N-2,N-3} & 0 \\ 0 & 0 & 0 & 0 & A_{N-3,N-2} & A_{N-2,N-2} & A_{N-1,N-2} \\ 0 & 0 & 0 & 0 & 0 & A_{N-2,N-1} & A_{N-1,N-1} \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{N-2} \\ X_{N-1} \end{pmatrix} = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_{N-2} \\ B_{N-1} \end{pmatrix}$$

The collocation method matches the boundaries of the elements with spectral accuracy; the matrix can be solved as follows (Beaume, private communication).

$$\left(\begin{array}{cc|cc} A_{0,0} & 0 & \dots & \dots & 0 & A_{1,0} & 0 & \dots & 0 \\ 0 & A_{2,2} & \dots & \dots & 0 & A_{1,2} & A_{3,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \ddots & 0 & 0 & 0 & \ddots & A_{N-2,N-3} \\ 0 & 0 & \dots & \dots & A_{N-1,N-1} & 0 & 0 & \dots & A_{N-2,N-1} \\ \hline A_{0,1} & A_{2,1} & \dots & 0 & 0 & A_{1,1} & 0 & \dots & 0 \\ 0 & A_{2,3} & \ddots & 0 & 0 & 0 & A_{3,3} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{N-3,N-2} & A_{N-1,N-2} & 0 & 0 & \dots & A_{N-2,N-2} \end{array} \right)$$

The collocation method matches the boundaries of the elements with spectral accuracy; the matrix can be solved as follows (Beaume, private communication).

$$\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \begin{pmatrix} X_L \\ X_R \end{pmatrix} = \begin{pmatrix} B_T \\ B_B \end{pmatrix} \quad (17)$$

$$A_{TL}X_L + A_{TR}X_R = B_T \quad (18)$$

$$A_{BL}X_L + A_{BL}A_{TL}^{-1}A_{TR}X_R = A_{BL}A_{TL}^{-1}B_T \quad (19)$$

$$\begin{pmatrix} A_{TL} & 0 \\ 0 & A_{BR} - A_{BL}A_{TL}^{-1}A_{TR} \end{pmatrix} \begin{pmatrix} X_L + A_{TR}X_R \\ X_R \end{pmatrix} = \quad (20)$$

$$\begin{pmatrix} B_T \\ B_B - A_{BL}A_{TL}^{-1}B_T \end{pmatrix} \quad (21)$$

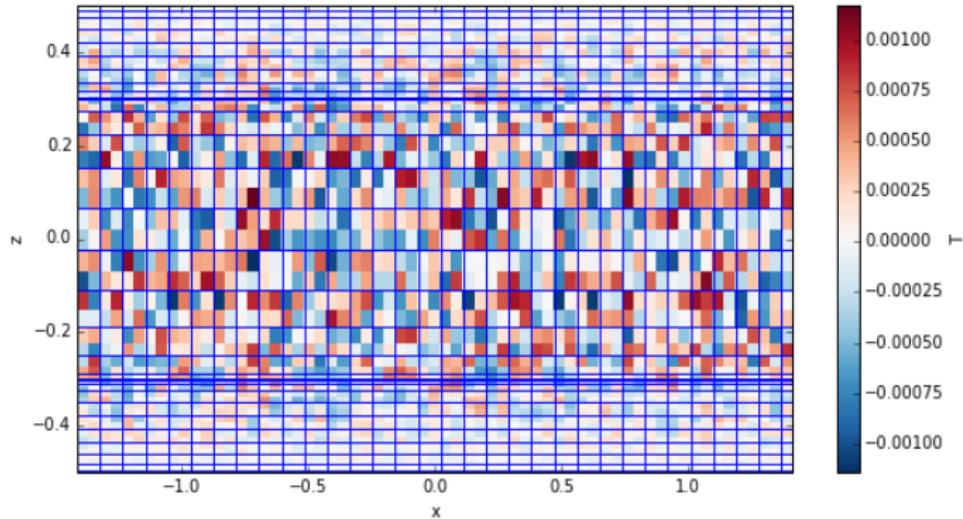
The extents of these elements are dynamic, and can be rezoned using simulated annealing.

We define $P(E(s), E(s'), T)$, the probability to accept a new state, s' , to maximize E , given a temperature, T , where $P > 0$ even if $E(s') < E(s)$ but that $P(E(s') < E(s)) \rightarrow 0$ as $T \rightarrow 0$.⁶

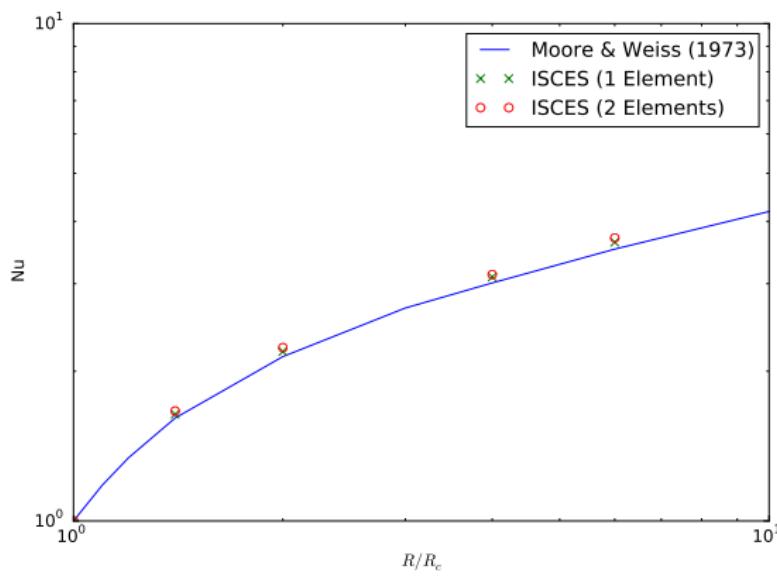
- ▶ Let $s = s_0$
- ▶ For k in 0 to k_{\max} :
 - ▶ $T = \text{temperature}(\frac{k}{k_{\max}})$
 - ▶ Pick a random neighbor, $s' = \text{neighbor}(s)$
 - ▶ If $P(E(s), E(s'), T) > \text{random}(0, 1)$:
 - ▶ $s = s_{\text{new}}$
- ▶ Return: s

⁶Pseudocode from Wikipedia

The extents of these elements are dynamic, and can be rezoned using simulated annealing.



To verify the code, we ran the standard Rayleigh Bénard Convection problem and compared to Moore & Weiss (1973).



We can also compare to the results of the same equations with the same parameters as PADDI.

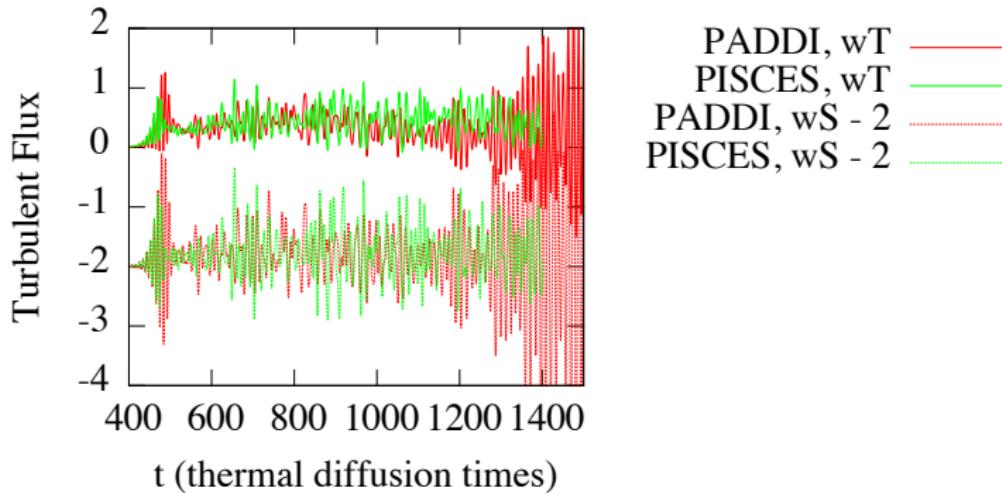


Figure: The late time behavior is dominated by the boundary conditions, which differ between the codes

The Problem

(P)ISCES

Future

Layered Semi-Convection
Additional Slides

Future Code Development

- ▶ Implement nonlinear diffusion (working in 1D) in 2D
 - ▶ Examine layer properties with μ -dependent diffusion
- ▶ Upgrade from Boussinesq to Pseudo-Incompressible
 - ▶ Determine layer height across density scale heights
- ▶ Update KEPLER and MESA with layer height

The Problem

(P)ISCES

Future

Layered Semi-Convection

Additional Slides

Pseudo-Incompressibility

Two-Component Ideal Mixture

Governing equations

Fully Compressible

$$\rho \frac{D}{Dt} \mathbf{u} = -\nabla p + \rho \mathbf{g} + \nabla \cdot \boldsymbol{\Pi} \quad (22)$$

$$\frac{D}{Dt} \rho = -\rho \nabla \cdot \mathbf{u} \quad (23)$$

$$\rho T \frac{D}{Dt} s = \nabla \mathbf{u} : \boldsymbol{\Pi} + \sum_i \mu_i \nabla \cdot \mathbf{C}_i - \nabla \cdot \mathbf{H} \quad (24)$$

$$\rho \frac{D}{Dt} \xi_i = -\nabla \cdot \mathbf{C}_i \quad (25)$$

Governing equations

Incompressible, Current Equations

$$\rho_0 \frac{D}{Dt} \mathbf{u} = -\nabla p + (-\alpha T + \beta \mu) \rho_0 \mathbf{g} + \nu \rho_0 \nabla^2 \mathbf{u} \quad (22)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (23)$$

$$\frac{D}{Dt} T + w \frac{d}{dz} T_0 = \kappa_T \nabla^2 T \quad (24)$$

$$\frac{D}{Dt} \xi_i + w \frac{d}{dz} \xi_{i,0} = \kappa_i \nabla^2 \xi_i \quad (25)$$

Governing equations

Pseudo-Incompressible

$$\rho \frac{D}{Dt} \mathbf{u} = -\nabla(p_0 + p_1) + \frac{p_1}{\rho} \left(\frac{\partial p}{\partial \rho} \right)^{-1} \nabla p_0 - \rho \nabla \Phi + \nabla \cdot \boldsymbol{\Pi} \quad (22)$$

$$p_0(z) = p(\rho, s, \xi_1, \xi_2, \dots) \quad (23)$$

$$\frac{D}{Dt} \rho = -\rho \nabla \cdot \mathbf{u} \quad (24)$$

$$\rho T \frac{D}{Dt} s = \nabla \mathbf{u} : \boldsymbol{\Pi} + \sum_i \mu_i \nabla \cdot \mathbf{C}_i - \nabla \cdot \mathbf{H} \quad (25)$$

$$\rho \frac{D}{Dt} \xi_i = -\nabla \cdot \mathbf{C}_i, \quad (26)$$

Incompressible Non-Dimensionalization

$$\frac{D}{Dt} \mathbf{u} = -\nabla p + \frac{-\text{Ra}_T T + \text{Ra}_\mu \mu}{\text{Pr}} + \nabla^2 \mathbf{u} \quad (27)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (28)$$

$$\frac{D}{Dt} T = \nabla \cdot \left(\text{Pr}^{-1} + \frac{\kappa_{T,1}}{\nu} (z) \right) \nabla T \quad (29)$$

$$\frac{D}{Dt} \xi_i = \frac{\tau}{\text{Pr}} \nabla^2 \xi_i \quad (30)$$

To solve these equations, we choose the equation of state of an ideal mixture.

$$\rho \frac{D}{Dt} \mathbf{u} = -\nabla(p_0 + p_1) + \frac{p_1}{\gamma p_0} \nabla p_0 - \rho \nabla \Phi + \nabla \cdot \boldsymbol{\Pi} \quad (31)$$

$$p_0(z) = \frac{k\rho\tilde{T}}{\bar{m}} \quad (32)$$

$$\frac{D}{Dt} \rho = -\rho \nabla \cdot \mathbf{u} \quad (33)$$

$$\rho T \frac{D}{Dt} s = \nabla \mathbf{u} : \boldsymbol{\Pi} + \sum_i \mu_i \nabla \cdot \mathbf{C}_i - \nabla \cdot \mathbf{H} \quad (34)$$

$$\rho \frac{D}{Dt} \xi_i = -\nabla \cdot \mathbf{C}_i, (T, \mu_i) \equiv \frac{c_p p_0 + k p_1}{c_p p_0} \left(\tilde{T}, \tilde{\mu}_i \right) \quad (35)$$

We can simplify this by expressing the entropy equation as a temperature equation.

$$\frac{D}{Dt} T = \frac{\partial T}{\partial \rho} \frac{D}{Dt} \rho + \frac{\partial T}{\partial s} \frac{D}{Dt} s + \frac{\partial T}{\partial \xi_i} \frac{D}{Dt} \xi_i. \quad (36)$$

We can use the remainder of the governing equations and the equation of state to get the full equation.

We can simplify this by expressing the entropy equation as a temperature equation.

$$\rho \frac{D}{Dt} T = -(\gamma - 1) \rho T \nabla \cdot u + (\gamma - 1) \frac{\overline{m}}{k} (\nabla \mathbf{u} : \boldsymbol{\Pi} - \nabla \cdot \mathbf{H}). \quad (36)$$

Governing Equations

$$p_0 = \frac{\rho T}{\bar{m}} \quad (37)$$

$$\rho \frac{D}{Dt} \mathbf{u} = -\nabla (p_0 + p_1) + \frac{p_1}{\gamma p_0} \nabla p_0 - \rho \hat{y} + \nabla \cdot \boldsymbol{\Pi} \quad (38)$$

$$\rho \frac{D}{Dt} \xi = \nabla \cdot \rho \tau \nabla \xi \quad (39)$$

$$\frac{1}{\gamma - 1} \rho \frac{D}{Dt} T = -\rho T \nabla \cdot \mathbf{u} + \bar{m} \left(\nabla \mathbf{u} : \boldsymbol{\Pi} + \nabla \cdot \frac{\rho}{\bar{m}} \nabla T \right) (1 + Q) \quad (40)$$

$$\nabla \cdot p_0^{\frac{1}{\gamma}} = \frac{\gamma - 1}{\gamma} p_0^{\frac{1}{\gamma} - 1} \left(\nabla \mathbf{u} : \boldsymbol{\Pi} - \nabla \cdot \frac{\rho}{\bar{m}} \nabla T \right) (1 + Q) \quad (41)$$

$$\boldsymbol{\Pi}_{ij} = \rho \text{Pr} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right) \quad (42)$$

Derivation of the Equation of State for a two-component mixture.

Start with the EOS for an ideal gas:

$$e_i(\rho_i, s_i) = \frac{c_V}{m_i} \left(\frac{\rho_i}{m_i} \phi_i e^{\frac{m_i s_i}{k}} \right)^{\frac{k}{c_V}} \quad (43)$$

Each component of the gas must follow this, and the total energy is $\bar{m}e = m_1e_1 + m_2e_2$.

Derivation of the Equation of State for a two-component mixture.

Note the entropy of the mixture is defined as

$\bar{ms} = m_1\xi_1s_1 + 2\xi_2s_2 - k\xi_1 \ln \xi_1 - k\xi_2 \ln \xi_2$. We assume the gases are thermally coupled:

$$T = \left(\frac{\rho \xi_1 \phi_1}{\bar{m}} e^{\frac{m_1 s_1}{k}} \right)^{\frac{k}{c_v}} = \left(\frac{\rho \xi_2 \phi_2}{\bar{m}} e^{\frac{m_2 s_2}{k}} \right)^{\frac{k}{c_v}} \quad (43)$$

$$e^{\frac{m_1 s_1}{k}} = \left(\frac{\xi_2 \phi_2}{\xi_1 \phi_1} \right)^{\xi_2} e^{\frac{\bar{ms}}{k} + \xi_1 \ln \xi_1 + \xi_2 \ln \xi_2} \quad (44)$$

And likewise for s_2

Derivation of the Equation of State for a two-component mixture.

Thus, the total energy is

$$e_1(\rho, s, \xi_1, \xi_2) = c_V \left(\frac{\rho \xi_1 \phi_1}{m_1} \left[\frac{\xi_2 \phi_2}{\xi_1 \phi_1} \right]^{\xi_2} e^{\frac{ms}{k} + \xi_1 \ln \xi_1 + \xi_2 \ln \xi_2} \right)^{\frac{k}{c_V}}, \quad (43)$$

$$e_2(\rho, s, \xi_1, \xi_2) = c_V \left(\frac{\rho \xi_2 \phi_2}{m_2} \left[\frac{\xi_1 \phi_1}{\xi_2 \phi_2} \right]^{\xi_1} e^{\frac{ms}{k} + \xi_1 \ln \xi_1 + \xi_2 \ln \xi_2} \right)^{\frac{k}{c_V}}, \quad (44)$$

$$\overline{m}e(\rho, s, \xi_1) = \sum_i m_i \xi_i e_i(\rho, s, \xi_1, 1 - \xi_1). \quad (45)$$

Derivation of the Equation of State for a two-component mixture.

| | | $\frac{\partial}{\partial \rho}$ | $\frac{\partial/\partial \rho}{\partial p/\partial \rho}$ |
|---------|---|----------------------------------|---|
| p | $\frac{k}{c_v} \rho e$ | $\frac{c_p}{c_v} \frac{p}{\rho}$ | 1 |
| T | $\frac{e}{c_v}$ | $\frac{kT}{c_v \rho}$ | $\frac{kT}{c_p p}$ |
| μ_1 | $\frac{\frac{1}{\xi_1} (m_2 - \bar{m})(c_p - \bar{m}s) + k \ln \frac{\phi_1}{\phi_2}}{c_v} e$ | $\frac{k\mu_1}{c_v \rho}$ | $\frac{k\mu_1}{c_p p}$ |

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