For N samples of a signal x_n, $n = 0, 1, \ldots, N-1$ the (discrete) Fourier transform is defined as

$$\hat{x}_k = C \sum_{n=0}^{N-1} x_n e^{-i\left(\frac{2\pi k}{N}\right)n}, \quad k = 0, 1, \ldots, N-1$$

If the sampling rate is F_s Hz $\left(\frac{\text{samples}}{\text{sec}}\right)$, the frequencies matching the coefficients are

$$f_k = \frac{k}{N} F_s \text{ Hz} \qquad \left(\text{and } C = (N F_s)^{-1/2}\right)$$

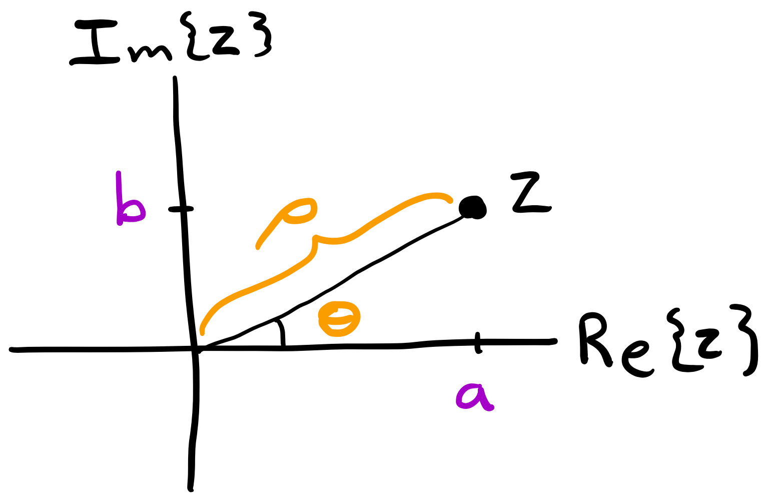
For each k, the sum is basically the correlation of the data signal with the complex exponential signal over time n with parameter k,

$$u_n^k \equiv e^{-i\left(\frac{2\pi k}{N}\right)n} \qquad \text{the "kernel"}$$

What does this definition have to do with sinusoids?

Complex numbers can be written in two equivalent ways:

$$z = \underbrace{a + bi}_{\text{Rectangular}} = \underbrace{\rho e^{i\theta}}_{\text{Polar}} \quad \text{where } i = \sqrt{-1}$$



By the Pythagorean Theorem:

$$\begin{cases} \rho^2 = a^2 + b^2 \\ \tan(\theta) = \dfrac{b}{a} \end{cases}$$

$$\begin{cases} a = \rho\cos(\theta) \\ b = \rho\sin(\theta) \end{cases}$$

The special case $\rho = 1$ is Euler's relation

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Looking back to the "kernel" signal

$$u_n^k \equiv e^{-i\left(\frac{2\pi k}{N}\right)n}, \quad \text{we set } \boxed{\theta = -\left(\frac{2\pi k}{N}\right)n}$$

and then

$\cos(-\psi) = \cos(\psi)$ $\qquad$ $\sin(-\psi) = -\sin(\psi)$

$$u_n^k = \cos\left(\frac{2\pi k}{N}n\right) - i\sin\left(\frac{2\pi k}{N}n\right)$$

and

$$\hat{x}_k = \underbrace{C\sum_{n=0}^{N-1} x_n \cos\left(\frac{2\pi k}{N}n\right)}_{\text{Correlation of } x_n \text{ with cosine of frequency } \frac{2\pi k}{N}} - \underbrace{iC\sum_{n=0}^{N-1} x_n \sin\left(\frac{2\pi k}{N}n\right)}_{\text{Correlation of } x_n \text{ with sine of frequency } \frac{2\pi k}{N}}$$

Punchline: The complex exponential form of the Fourier transform is a compact way to simultaneously write the correlations of x_n with cosine (in the real part of $\hat{x}_k$) and sine (in the imaginary part of $\hat{x}_k$).

Now each $\hat{x}_k$ can also be written in polar form

$$\hat{x}_k = \rho_k e^{i\theta_k}$$

The power at frequency k is (by definition)

$$|\hat{x}_k|^2 = \rho_k^2$$

and the phase (or angle) is $\measuredangle x_k = \theta_k$

What's up with the discrete frequencies?

If our sample rate is F_s, the time of the nth sample is $t_n = \frac{n}{F_s}$ sec.

So
$$e^{-i\left(\frac{2\pi k}{N}\right)n} = e^{-i\,\frac{2\pi k}{N/F_s}\,\frac{n}{F_s}} = e^{-i\,2\pi\left(\frac{k}{N}F_s\right)t_n}$$
$$= e^{-i\,2\pi f_k t_n}$$
$$= \cos(2\pi f_k t_n) - i\sin(2\pi f_k t_n)$$

The 2π just converts physical to radian frequency:

f_k	$\text{Hz} = \frac{\text{cycles}}{\text{sec}}$	
		$\times 2\pi\ \frac{\text{radians}}{\text{cycle}}$
$2\pi f_k$	$\frac{\text{radians}}{\text{sec}}$	
		$\div F_s\ \frac{\text{samples}}{\text{sec}}$ (also Hz...)
$\frac{2\pi k}{N}$	$\frac{\text{radians}}{\text{sample}}$	