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Homework 1

Exercise 5.1.10 To prove this we just use the mean value theorem take:

$$\begin{aligned} d(g(x), g(y)) &\leq g'(c)d(x,y) \text{ for some } c \in [a,b] && (1)\\ &\leq \alpha d(x,y) && (2) \end{aligned}$$

And if $\alpha < 1$ then this converges by the banach fixed point theorem.

Exercise 5.2.8 To prove this we will walk through the argument

$$\begin{aligned} d(y,w) &= d(Tx, Tz) && (3)\\ &= \sqrt{\sum_{j=1}^{n} (\eta_j - \omega_j)^2} && (4)\\ &= \sqrt{\sum_{j=1}^{n}(\sum_{k=1}^{n} c_{jk}(\xi_k - \zeta_k))^2} && (5)\\ &\leq \sum_k \sqrt{\sum_{j=1}^{n}(c_{jk}(\xi_k - \zeta_k))^2} \text{ by triangle inequality since this is a norm} && (6)\\ &\leq \sum_k \sqrt{\sum_{j=1}^{n} c_{jk}^2(\xi_k - \zeta_k)^2} && (7)\\ &\leq \sum_k |\xi_k - \zeta_k| \sqrt{\sum_{j=1}^{n} c_{jk}^2} && (8)\\ & && (9) \end{aligned}$$

We can now apply cauchy shwartz. set $a_k = |\xi_k - \zeta_k|, b_k = \sqrt{\sum_{j=1}^{n} c_{jk}^2}$

$$\begin{aligned} &\leq ||x - z||_2 \sqrt{\sum_{j,k} c_{jk}^2} && (10)\\ &\leq d(x,z) \sqrt{\sum_{j,k} c_{jk}^2} && (11) \end{aligned}$$

Thus for the contraction mapping to apply we need $\sqrt{\sum_{j,k} c_{jk}^2} < 1$ or equivalently $\sum_{j,k} c_{jk}^2 < 1$

Exercise 5.3.6 To show this let y be a limit point of $\tilde{C}$ by definition of limit poitn there exists $y_n \to y, y_n \in \tilde{C}$

$$|y_n(t) - x_0| \leq c\beta \tag{12}$$

Note that since the absolute valiue function is continuous, we can take limits here and obtain"

$$|y(t) - x_0| \leq c\beta \tag{13}$$

" This is because inequalities are preserved under limits (non strict at least). So $\tilde{C}$ contains its limit points thus it is closed

Exercise 2.2.14 We show that the metric:

$$\tilde{d}(x,x) = 0, \tilde{d}(x,y) + d(x,y) + 1 \tag{14}$$

cannot come from a norm.

Assume that it came from a norm then it should satisfy scalar preservation:

$$\begin{aligned} ||ax||_{\tilde{d}} = ||ax - a0||_{\tilde{d}} && (15)\\ = \tilde{d}(ax, 0) && (16)\\ = d(ax, 0) + 1 && (17)\\ = ||ax||_d + 1 && (18)\\ = |a|||x|| + 1 && (19) \end{aligned}$$

Note that we do not have the scalar preservation here. So it is not induced by a norm.