18 Optimal Control and Pontryagin's Maximum Principle

Control your life through insanity
—Cliff Burton

Control what you can control. Don't lose sleep worrying about things that you don't have control over because, at the end of the day, you still won't have any control over them
—Cam Newton

Time management is an oxymoron. Time is beyond our control, and the clock keeps ticking regardless of how we lead our lives. Priority management is the answer to maximizing the time we have.
—John C. Maxwell

The most difficult thing is the decision to act, the rest is merely tenacity. The fears are paper tigers. You can do anything you decide to do. You can act to change and control your life; and the procedure, the process is its own reward
—Amelia Earhart

18.1 Introduction to optimal control

From a variational perspective, optimal control can be thought of as the minimization (or equivalently maximization) of a cost functional relative to some given constraints that can often be highly nonlinear and fairly complicated. The system under consideration consists of primarily two different variables, the state space variable and the control variable, both of which are likely vector valued functions of time t. The state space variable provides the evolution of some interesting state, while the control variable is a quantity that can be chosen to provide the optimal solution to the minimization (equivalently maximization) problem. For example, we may consider the minimization of the cost functional

$$J[\mathbf{u}] = \int_a^b L(t, \mathbf{x}, \mathbf{u})dt,$$

subject to the state space constraint

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \mathbf{u}), \tag{18.1}$$

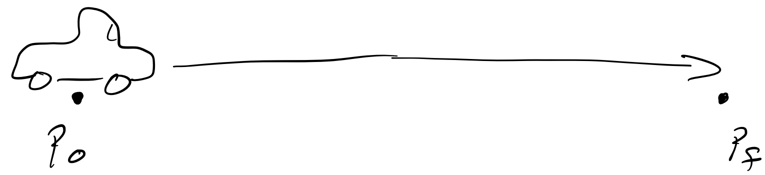

Figure 18.1: Driving a car in a straight line between two points p_0 and p_f as described in Example 18.1.2.

where $L(t, \mathbf{x}, \mathbf{u})$ may depend on time derivatives of $\mathbf{x}(t)$ and $\mathbf{u}(t)$ as well as the functions themselves. The optimal choice of the control $\mathbf{u}(t)$ will yield the best possible minimum of $J[\mathbf{u}]$ under the state space constraint (18.1). Additional constraints are common in control problems, and often make the problem more tractable. Generically we will denote this as the additional constraint on our control variable that $\mathbf{u}(t) \in U$ where U is some set that holds whatever properties we desire of our control.

Remark 18.1.1. Typically although the Lagrangian $L(t, \mathbf{x}, \mathbf{u})$ may depend on derivatives of $\mathbf{x}(t)$ and $\mathbf{u}(t)$, in practice this is avoided by instead rewriting each of these derivatives as a new variable, much as was done to reduce a 2nd order n-dimensional ODE to a 1st order 2n-dimensional ODE. This also ensures that (18.1) is first order. Such a distinction is somewhat particular to the field of optimal control, although the results are equivalent to keeping the higher derivative terms.

Something to note is that we will be discussing optimal control which in essence assumes that the state evolution is known a priori and there are no adjustments or alterations to the originally chosen control once the optimal path is found. This means that we set the problem up, solve for the optimal control and then let the system evolve accordingly. Often times in practice, one will instead see a more practical approach wherein the control variable may be adapted to changing conditions and respond to various inputs. One version of this occurs in model predictive control where the optimal control is updated at discrete intervals.

Example 18.1.2. A self-driving car is at rest at time t_0 and point y_0. The passenger wants to get to point y_f at time t_f. Suppose that the car is a point mass, i.e. we will neglect the distribution of mass over the car's chassis as well as the possible effective mass of the passenger. The car can be controlled by accelerating or braking. This is the 'control' $u(t)$, i.e. $u(t) \geq 0$ is pushing on the gas and $u(t) \leq 0$ is pushing on the brake. We will neglect the frictional forces beyond the fact that the braking provides a negative acceleration on the vehicle. Physically we know that $U_l \leq u(t) \leq U_u$ where U_l and U_u depend on the design and make of the vehicle (and the passenger's comfort). We also want the car to stop at y_f. We are interested in the position and velocity, so we are interested in

$$\mathbf{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}, \mathbf{x}(t_0) = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}, \mathbf{x}(t_f) = \begin{pmatrix} y_f \\ 0 \end{pmatrix}$$

Note that $y''(t) = u(t)$ so that

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

If we want to optimize the time spent driving, we minimize time

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt,$$

where we are optimizing $J[u]$ and have the additional constraint that we have a finite amount of gas which we claim is proportional to the velocity and/or acceleration of the vehicle which we represent as

$$K = \int_{t_0}^{t_f} [k_1 u(t) + k_2 y'(t)] dt,$$

where $K \leq G$ for some constant G. We are trying to find the relationship between $\mathbf{x}(t)$ and $u(t)$.

There are of course other instances where optimal control is important and useful. For instance you don't have to be into designing cars or pursuing autonomous vehicles. You may instead just be interested in gardening. Even then, you should take full advantage of your opportunity to model the optimal garden production with optimal control.

Example 18.1.3. Suppose that a certain type of pesticide $u(t)$ (it may be completely organic and non-chemical...this problem isn't that specific yet, no judgments please) works remarkably well at controlling pests to the point that the growth of watermelon satisfies the differential equation

$$x' = f(x, u),$$

where $f(x, u)$ is non-decreasing with u, i.e. the more pesticide is used, the better the watermelon grows. Without a cost on the pesticide the solution would clearly be to use an infinite amount on the watermelon since $f(x, u)$ is non-decreasing in u. This is not reality though, and we would expect that instead there is some cost to using more of the pesticide. At the same time we want to maximize the growth of the watermelon plants. This could be accomplished by minimizing a cost functional of the type

$$J[u] = \int_0^{t_f} \left[u^2 - x^2\right] dt,$$

with some definitive end time t_f that defines when we expect the first frost (and hence when we will have to harvest the last of the watermelon).

Remark 18.1.4. Note that neither of these two examples were very specific. Nor were they all that complete. There are also a lot of variations we can place on these problems. For instance, we are really interested in large, juicy watermelons, not necessarily the biggest watermelon plants (these don't always correlate) so maybe we need to change our model. We may also want the optimal watermelon for the county fair which would give us an endpoint condition i.e we really want to maximize $x(t_f)$, and don't care that much about what happens for $t < t_f$ to get there.

In addition we point out that most of the examples and exercises that we will consider in this course are going to be simplified in order to create a tractable problem. In particular, we will typically have an explicit state equation (evolution equation for the state variable $x(t)$) when in reality these evolution equations are only approximately known, i.e. everything is really only done numerically in practice. Even so, the primary ideas and concepts that we explore in the following carry over into the fully nonlinear and realistic setting.

18.1.1 A few definitions

To formalize these control problems, we will use the following definitions.

Definition 18.1.5. *For a given system described by the state space variable $\mathbf{x}(t) \in \mathbb{R}^n$ and control variable $\mathbf{u}(t) \in \mathbb{R}^m$ $(m \neq n)$ there is a physical restriction $\mathbf{u}(t) \in U \subseteq \mathbb{R}^m$ where U is called the control region.*

Definition 18.1.6. *A piecewise continuous control $\mathbf{u}(t)$ defined on the interval $t_0 \leq t \leq t_f$ where $\mathbf{u}(t) \in U$, the control region, for all $t \in [t_0, t_f]$ is called an 'admissible control'.*

Remark 18.1.7. As already described above, the physical system is described by a dynamical system typically written as (18.1). Solutions $\mathbf{x}(t; \mathbf{x}_0), \mathbf{u}(t)$ of (18.1) are called responses to the system corresponding to control $\mathbf{u}(t)$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. If $\mathbf{f}$ is Lipschitz in $\mathbf{x}$ (and depends nicely on $\mathbf{u}(t)$) then solutions $\mathbf{x}(t)$ exist, and are continuously dependent on $\mathbf{u}(t)$.

Definition 18.1.8. *A cost functional associated with the physical system has the following 3 standard forms:*

- *Lagrange form* $J[\mathbf{u}] = \int_{t_0}^{t^f} L(t, \mathbf{x}(t), \mathbf{u}(t))dt$.
- *Mayer form* $J[\mathbf{u}] = \phi(t_0, \mathbf{x}(t_0); t_f, \mathbf{x}(t_f))$.
- *Bolza form* $J[\mathbf{u}] = \int_{t_0}^{t^f} L(t, \mathbf{x}(t), \mathbf{u}(t))dt + \phi(t_0, \mathbf{x}(t_0); t_f, \mathbf{x}(t_f))$.

Remark 18.1.9. It turns out that all 3 forms of the cost functional are equivalent. This can be shown by introducing a new state variable x_l and a new state vector. For example to go from Lagrange to Mayer form one may use

$$\tilde{\mathbf{x}} = \begin{pmatrix} x_l \\ x_1 \\ \vdots \\ x_n \end{pmatrix}, x_l' = L(t, \mathbf{x}, \mathbf{u}), x_l(t_0) = 0.$$

Remark 18.1.10. Constraints may be pointwise, integral or even in the form of a differential equation, and they need not be an equality constraint, i.e. inequality constraints are acceptable, such as the one illustrated in the example above. For simplicity, we will most frequently only be concerned with equality constraints here, but the reader should be aware that it is a relatively straightforward generalization of the theory presented here to include inequality constraints.

18.1.2 Formal approach and the mother functional

Suppose that we are interested in minimizing the cost functional

$$J[\mathbf{u}] = \int_{t_0}^{t_f} L(t, \mathbf{x}, \mathbf{u}) dt, \tag{18.2}$$

subject to the state space constraint

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \mathbf{u}). \tag{18.3}$$

From the Calculus of Variations we expect that we can proceed simply by including a Lagrange multiplier to enforce the state space equation constraint. This would indicate that we want to minimize the modified cost functional (which is actually referred to as the mother functional in some circles)

$$J^*[\mathbf{u}] = \int_{t_0}^{t_f} \left\{ L(t, \mathbf{x}, \mathbf{u}) + \mathbf{p} \cdot \left[\mathbf{x}' - \mathbf{f}(t, \mathbf{x}, \mathbf{u}) \right] \right\} dt, \tag{18.4}$$

where $\mathbf{p}(t)$ is the Lagrange multiplier of the system. Using the definition of the Hamiltonian $H(t; \mathbf{x}, \mathbf{u}, \mathbf{p}) = \mathbf{p}(t) \cdot \mathbf{f}(t; \mathbf{x}, \mathbf{u}) - L(t; \mathbf{x}, \mathbf{u})$ this can be rewritten as

$$J^*[\mathbf{u}] = \int_{t_0}^{t_f} \left[\mathbf{p} \cdot \mathbf{x}' - H(t; \mathbf{x}, \mathbf{u}, \mathbf{p}) \right] dt. \tag{18.5}$$

The Euler-Lagrange equations for this modified cost functional (where we are now supposing that we are optimizing over both the control variable $\mathbf{u}(t)$ and the state space variable $\mathbf{x}(t)$) yield exactly the same equations derived in the next section.

If we are a bit more careful then we will see in the following section that it turns out that the optimal control $\mathbf{u}(t)$ is not only a stationary point for the Hamiltonian, but is actually a maximizer for H considered as a function of $\mathbf{u}$. This remarkable result can be illustrated in what follows, but a rigorous proof we will avoid at all peril as it is not instructive and would take too long to work through for our present purposes. The heuristic derivation in the next section is sufficient to understand how to generalize the statement of what has become known as the celebrated Pontryagin's Maximum Principle.

Remark 18.1.11. It is worth pausing here and stating a strong note of caution. As with most fields of Mathematics (and the hard sciences in general), there is little to no agreement as to the notational approach to take with optimal control. Several textbooks will take the same approach that we have here, but others will define the Hamiltonian as $L - \mathbf{p} \cdot \mathbf{f}$ and then solve a minimization problem by minimizing the Hamiltonian. Other resources will use this different form of the Hamiltonian on a maximization problem, and correspondingly maximize the Hamiltonian. Even other books will consider maximization problems with the form of the Hamiltonian used here and hence desire to minimize the Hamiltonian.

Basically be very wary of consulting other resources. They are very informative and useful, but nobody has a universal statement of Pontraygin's principle (whether maximum or minimum). Just make sure if you look at some other resources that you are correctly maximizing your minimization problem (or was it the other way around?).

18.2 Pontryagin's maximum principle

The goal of this section is to derive the necessary conditions for the minimizing control and the corresponding state, and justify the fact that the optimal control $\mathbf{u}(t)$ is a maximizer of the Hamiltonian H. With this in mind, we recognize that we were being a bit cavalier in the previous section where we pretended to be able to optimize over $\mathbf{x}(t)$ and $\mathbf{u}(t)$ simultaneously as if they were independent of each other, when in fact this is not legitimate. In reality we are interested in variations of $\mathbf{u}(t)$ which induce a variation on $\mathbf{x}(t)$, but such variations are not truly independent, i.e. if we change $\mathbf{u}(t) \to \mathbf{u}(t) + \varepsilon\boldsymbol{\eta}(t)$ then what does this imply will happen to $\mathbf{x}(t)$ through the state space equation?

To see how this works, first we ensure that $\boldsymbol{\eta}(t)$ is chosen so that $\mathbf{u}(t) + \varepsilon\boldsymbol{\eta}(t) \in U$ the admissible control region. We expect that varying $\mathbf{u}(t)$ in this way will lead to $\mathbf{x}(t) \to \mathbf{x}(t) + \varepsilon\mathbf{h}(t) + O\left(\varepsilon^2\right)$. Both $\mathbf{h}(t)$ and $\boldsymbol{\eta}(t)$ are assumed to be piece-wise continuously differentiable (corner-points are on a set of measure zero). In addition, $\mathbf{x}(t_0) = \mathbf{x}_0$, so $\mathbf{h}(t_0) = 0$ (different boundary conditions are of course possible and we will address them later). For the time being we will allow the end point $\mathbf{x}(t_f)$ to be unknown, i.e. $\mathbf{h}(t_f)$ is free as well. The physical system is described by the state space equation

$$\mathbf{x}' = \mathbf{f}(t; \mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{18.6}$$

Variations of $\mathbf{u}(t)$ make (18.6) become

$$\mathbf{x}'(t;\varepsilon) = \mathbf{f}(t; \mathbf{x}(t;\varepsilon), \mathbf{u}(t) + \varepsilon\boldsymbol{\eta}(t)), \mathbf{x}(t_0) = \mathbf{x}_0,$$

which has solutions $\mathbf{x}(t;\varepsilon) = \mathbf{x}(t) + \varepsilon\mathbf{h}(t) + O\left(\varepsilon^2\right)$ where

$$\frac{\partial \mathbf{x}}{\partial \varepsilon}(t; \varepsilon = 0) = \mathbf{h}(t),$$

for all t. Now we want to determine what, if any relationship $\mathbf{h}(t)$ has to $\boldsymbol{\eta}(t)$. Differentiating both sides of this equation with respect to time we arrive at

$$\mathbf{h}' = \frac{d}{dt}\left(\frac{\partial \mathbf{x}}{\partial \varepsilon}(t; \varepsilon = 0)\right) = \mathbf{x}_{\varepsilon t}\big|_{\varepsilon=0} = \mathbf{x}_{t\varepsilon}\big|_{\varepsilon=0}.$$

Thus assuming that $\mathbf{x}$ is sufficiently smooth in both t and ε,

$$\begin{aligned}
\mathbf{h}' &= \frac{d}{d\varepsilon}\left(\mathbf{x}'\right)\big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\left[\mathbf{f}(t; \mathbf{x}(t;\varepsilon), \mathbf{u} + \varepsilon\boldsymbol{\eta})\right]|_{\varepsilon=0} \\
&= \frac{D\mathbf{f}}{D\mathbf{x}}(t; \mathbf{x}_\varepsilon(t;0), \mathbf{u}(t)) \cdot \mathbf{x}_\varepsilon(t;0) + \frac{D\mathbf{f}}{D\mathbf{u}}(t; \mathbf{x}_\varepsilon(t;0), \mathbf{u}(t))\boldsymbol{\eta}(t) \\
&= \frac{D\mathbf{f}}{D\mathbf{x}}(t; \mathbf{x}, \mathbf{u})\mathbf{h}(t) + \frac{D\mathbf{f}}{D\mathbf{u}}(t; \mathbf{x}, \mathbf{u})\boldsymbol{\eta}(t)
\end{aligned}$$

Let $A(t) = \frac{D\mathbf{f}}{D\mathbf{x}}(t; \mathbf{x}, \mathbf{u})$ and $B(t) = \frac{D\mathbf{f}}{D\mathbf{u}}(t; \mathbf{x}, \mathbf{u})$. We have

$$\mathbf{h}' = A(t)\mathbf{h}(t) + B(t)\boldsymbol{\eta}(t),$$

which is a linear system describing the evolution of $\mathbf{h}(t)$ in terms of $\boldsymbol{\eta}(t)$.

Remark 18.2.1. Notice that we could have just as easily considered variations of $\mathbf{x}(t)$, and tried to determine how this would effect variations of the control $\mathbf{u}(t)$. That isn't completely true however, because actually determining how the variations of $\mathbf{u}(t)$ depend on variations of $\mathbf{x}(t)$, i.e. trying to write out the evolution of $\boldsymbol{\eta}(t)$ in terms of $\mathbf{h}(t)$ is a much more difficult problem than what is addressed above. This is why we consider variations of the control, and how those affect variations of the state space variable. In addition, from a more philosophical perspective, we are claiming that we can choose the control variable $\mathbf{u}(t)$ which means that we can select how $\mathbf{u}(t)$ varies, and then we must determine how that effects variations in $\mathbf{x}(t)$ and not the other way around.

Now that we know how these variations are related, returning to (18.5) we want to compute the first Gateaux differential of $J^*[\mathbf{u}]$, that is we are interested in first computing

$$\frac{d}{d\varepsilon}\, J^*[\mathbf{u}+\varepsilon\boldsymbol{\eta}]\big|_{\varepsilon=0}$$
$$= \frac{d}{d\varepsilon}\left\{\int_{t_0}^{t_f}\left[\mathbf{p}\cdot\left(\mathbf{x}'+\varepsilon\mathbf{h}'+O\left(\varepsilon^2\right)\right)\right]dt - \int_{t_0}^{t_f}\left[H(t;\mathbf{x}+\varepsilon\mathbf{h}+O\left(\varepsilon^2\right),\mathbf{u}+\varepsilon\boldsymbol{\eta},\mathbf{p})\right]dt\right\}_{\varepsilon=0}$$

Recalling that $\mathbf{h}(t_0)=0$ (because at least for now we are supposing that $\mathbf{x}(t_0)$ is fixed) then

$$\begin{aligned}\int_{t_0}^{t_f}\left[\mathbf{p}\cdot\mathbf{h}'\right]dt &= -\int_{t_0}^{t_f}\left[\mathbf{p}'\cdot\mathbf{h}\right]dt + \left[\mathbf{p}(t)\cdot\mathbf{h}(t)\right]\big|_{t_0}^{t_f}\\ &= -\int_{t_0}^{t_f}\left[\mathbf{p}'\cdot\mathbf{h}\right]dt + \mathbf{p}(t_f)\cdot\mathbf{h}(t_f),\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{d\varepsilon}\int_{t_0}^{t_f} H\left(t;\mathbf{x}+\varepsilon\mathbf{h}+O\left(\varepsilon^2\right),\mathbf{u}+\varepsilon\boldsymbol{\eta},\mathbf{p}\right)dt\bigg|_{\varepsilon=0}\\ &=\int_{t_0}^{t_f}\left[\frac{DH}{D\mathbf{x}}\cdot\mathbf{h}(t)+\frac{DH}{D\mathbf{u}}\cdot\boldsymbol{\eta}(t)\right]dt.\end{aligned}$$

Putting these together, we reach

$$\delta J^*[\mathbf{u};\boldsymbol{\eta}] = -\int_{t_0}^{t_f}\left[\left(\mathbf{p}'+\frac{DH}{D\mathbf{x}}\right)\cdot\mathbf{h}+\frac{DH}{d\mathbf{u}}\cdot\boldsymbol{\eta}\right]dt+\mathbf{p}(t_f)\cdot\mathbf{h}(t_f). \tag{18.7}$$

For a minimizer (or maximizer) of $J[\mathbf{u}]$ we need $\delta J^*=0$ for all 'admissible' $\boldsymbol{\eta}(t)$ and corresponding $\mathbf{h}(t)$. We may still specify $\mathbf{p}(t)$ to be whatever we like (at least at this point). So, choose $\mathbf{p}(t)$ so that

$$\mathbf{p}'(t) = -\frac{DH}{D\mathbf{x}}(t;\mathbf{x},\mathbf{u},\mathbf{p}),$$

and $\mathbf{p}(t_f)=0$. Thus $\delta J^*[\mathbf{u};\boldsymbol{\eta}]=0$ for all 'admissible' $\boldsymbol{\eta}(t)$ only if $\frac{DH}{D\mathbf{u}}=0$ for the optimal control $\mathbf{u}(t)$. Thus for a control $\tilde{\mathbf{u}}(t)$ to yield a minimum (or maximum) of the original cost functional, then the Hamiltonian must have a stationary point at $\mathbf{u}(t)=\tilde{\mathbf{u}}(t)$. Now recalling that $H(t;\mathbf{x},\mathbf{u},\mathbf{p})=\mathbf{p}\cdot\mathbf{f}-L(t,\mathbf{x},\mathbf{u})$ and $\mathbf{x}'=\mathbf{f}$, then along the optimal path we have

$$\mathbf{x}(t_0)=\mathbf{x}_0,\quad \mathbf{p}(t_f)=0,\quad \mathbf{x}'=\frac{DH}{D\mathbf{p}},\quad \mathbf{p}'=-\frac{DH}{D\mathbf{x}}. \tag{18.8}$$

As noted earlier, these are Hamilton's canonical equations, or more specifically we refer to $\mathbf{p}(t)$ as the adjoint of $\mathbf{x}(t)$ and these equations as the adjoint equation of $\mathbf{x}'=\mathbf{f}$.

Note that

$$\begin{aligned}\mathbf{p}' &= -\frac{DH}{\mathbf{x}}\\ &= -\frac{D}{D\mathbf{x}}(\mathbf{p}\cdot f - L)\\ &= -\mathbf{p}\cdot\frac{D\mathbf{f}}{D\mathbf{x}}+\frac{DL}{D\mathbf{x}}\\ &= -\left(\frac{D\mathbf{f}}{D\mathbf{x}}\right)^T\mathbf{p}+\frac{DL}{D\mathbf{x}}\\ &= -A(t)^T\mathbf{p}+\frac{DL}{D\mathbf{x}},\end{aligned}$$

where this is the same $A(t)$ that appeared in

$$\mathbf{h}' = A(t)\mathbf{h} + B(t)\boldsymbol{\eta},$$

i.e. the operator $-A(t)^T$ is the adjoint of $A(t)$, so this is the reason that $\mathbf{p}(t)$ is often called the costate or adjoint vector.

For a minimum of the cost functional

$$J[\mathbf{u}] = \int_{t_0}^{t_f} [\mathbf{p}\cdot\mathbf{x}' - H(t;\mathbf{x},\mathbf{u},\mathbf{p})]dt,$$

a variationally-motivated necessary condition is that $\delta^2 J$ is positive definite, along the optimal path, i.e.

$$\delta^2 J[\mathbf{u},\boldsymbol{\eta}] = -\frac{1}{2}\int_{t_0}^{t_f} \begin{pmatrix} \mathbf{h}_1^T & \boldsymbol{\eta}^T \end{pmatrix} \mathscr{H} \begin{pmatrix} \mathbf{h}_1 \\ \boldsymbol{\eta} \end{pmatrix} dt,$$

where

$$\mathscr{H} = \begin{pmatrix} \frac{D^2H}{D\mathbf{x}^2} & \frac{D^2H}{D\mathbf{x}D\mathbf{u}} \\ \frac{D^2H}{D\mathbf{x}D\mathbf{u}} & \frac{D^2H}{D\mathbf{u}^2} \end{pmatrix}.$$

It turns out that $\delta^2 J \geq 0$ if

$$\boldsymbol{\eta}^T \frac{D^2H}{D\mathbf{u}^2}\boldsymbol{\eta} \leq 0$$

for all $t \in [t_0, t_f]$ impliying $\frac{D^2H}{D\mathbf{u}^2}$ is negative definite. Recalling optimization in multivariable calculus, this condition combined with the fact that the optimal control $\mathbf{u}(t)$ is a stationary point of H indicates that $\mathbf{u}(t)$ is a maximum of H.

Remark 18.2.2. Everything we have done so far relies on expanding the Lagrangian and Hamiltonian as if everything in the discussion were infinitely differentiable (or at least as differentiable as we need it to be). Surprisingly this same result holds true regardless of how differentiable our solutions are. In other words, the following summary of Pontryagin's Maximum Principle applies not only to continuously differentiable controls, but also to piecewise continuously differentiable controls that yield continuous state space variables $\mathbf{x}(t)$. This is what Pontryagin's Maximum Principle states, the proof of which we will not attempt to scale at this time.

In summary we can state the following version of the celebrated Pontryagin Maximum Principle.

Theorem 18.2.3. *Consider the optimal control problem where* $\mathbf{x}' = \mathbf{f}(t;\mathbf{x},\mathbf{u})$ *with* $\mathbf{x}(t_0) = \mathbf{x}_0$ *that is seeking to minimize the cost functional*

$$J[\mathbf{u}] = \int_{t_0}^{t_f} L(t;\mathbf{x},\mathbf{u})dt,$$

via the control $\mathbf{u}(t) \in U$ *where* U *is the admissible control region. If* $\mathbf{f}$, $\frac{D\mathbf{f}}{D\mathbf{x}}$, L, *and* $\frac{DL}{D\mathbf{x}}$ *are continuous and if* $\tilde{\mathbf{u}}(t) : [t_0, t_f] \to U$ *is a global optimal control and* $\tilde{\mathbf{x}}(t) : [t_0, t_f] \to \mathbb{R}^n$ *is the corresponding optimal state trajectory, then there exists a function* $\mathbf{p}(t) : [t_0, t_f] \to \mathbb{R}^n$ *so that:*

(i)

$$\tilde{\mathbf{x}}' = \frac{DH}{D\mathbf{p}}, \qquad \tilde{\mathbf{x}}(t_0) = \mathbf{x}_0,$$
$$\tilde{\mathbf{p}}' = -\frac{DH}{D\mathbf{x}}, \qquad \tilde{\mathbf{p}}(t_f) = 0,$$

(*ii*) *The Hamiltonian* $H = \mathbf{p} \cdot \mathbf{f} - L$ *has a global maximum for each* $t \in [t_0, t_f]$ *at the optimal control* $\tilde{\mathbf{u}}(t)$*, i.e.* $H(\tilde{\mathbf{u}}) \geq H(\mathbf{u})$ *for all* $\mathbf{u}(t) \in U$*.*

Remark 18.2.4. It is valuable to point out at this point that while the derivation presented in this Chapter is formal (we have stealthily avoided making any of this rigorous so far), it is instructive in that generalizations of this 'simplest problem' to other cases such as when an endpoint is not specified, can be derived in a very similar fashion.

18.2.1 A brief foray into generalizations

In a later chapter we will delve into the generalizations of this simple derivation, but for now we will also consider the influence/impact of a few very specific generalizations that we may run into. Everything that follows is an extension of Theorem 18.2.3 i.e. unless stated explicitly all of the following generalizations include the same hypotheses in addition to those stated above.

Autonomous systems

Now suppose that we are considering an autonomous system with autonomous cost function, meaning that there is no explicit dependence on the time variable, i.e.

$$\mathbf{f} = \mathbf{f}(\mathbf{x}, \mathbf{u}),$$
$$L = L(\mathbf{x}, \mathbf{u}).$$

In this case we can see that the Hamiltonian will also have no explicit t dependence, i.e. $H = H(\mathbf{x}, \mathbf{u}, \mathbf{p})$ so that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{DH}{D\mathbf{x}} \cdot \mathbf{x}' + \frac{DH}{D\mathbf{u}} \cdot \mathbf{u}' + \frac{DH}{D\mathbf{p}} \cdot \mathbf{p}',$$

but since H is independent of t then $\frac{\partial H}{\partial t} = 0$, and for the optimizer $\tilde{\mathbf{u}}$ then $\frac{DH}{D\mathbf{u}} = 0$. Finally, we resort to our choice of the co-state $\mathbf{p}$, and replacing $\frac{DH}{D\mathbf{x}} = -\mathbf{p}'$ and $\frac{DH}{D\mathbf{p}} = \mathbf{x}'$ to see that $\frac{dH}{dt} = 0$ so that the Hamiltonian is constant along the optimal trajectory with the optimally chosen control $\tilde{\mathbf{u}}$.

In summary, while the Hamiltonian is maximized in $\mathbf{u}$ for the optimal setting, it is also constant in time for a completely autonomous system. This should feel familiar and comparable to our earlier usage of the Hamiltonian for physical systems.

Endpoint condition

If in addition to $\mathbf{x}(t_0) = \mathbf{x}_0$ being prescribed, we also have $\mathbf{x}(t_f) = \mathbf{x}_1$ being given as part of the problem such as in the self-driving car problem from the previous chapter. In this case, one can check the integration by parts and find that $\mathbf{p}(t_f)$ is no longer necessarily 0. Instead it turns out that fixing $\mathbf{x}$ at both t_0 and t_f is enough, and that $\mathbf{p}$ is free at the two end points.

Optimization over the final endpoint

Now suppose that we are interested in the system described by the hypotheses of Theorem 18.2.3 except that now we are interested in optimizing over both the control variable $\mathbf{u}(t)$ and the final time t_f, i.e. we seek the solution to

$$\min_{\mathbf{u}, t_f} J[\mathbf{u}] = \int_{t_0}^{t_f} L(t; \mathbf{x}, \mathbf{u}) dt. \tag{18.9}$$

In this case we must consider not only variations in $\mathbf{u}(t)$ and the corresponding changes to the state variable $\mathbf{x}(t)$, but we must directly consider variations to the final time t_f, i.e. we consider $t_f \to t_f + \delta$ and note that under this new variation, the derivative with respect to δ must vanish as well. In particular circumstances we may have some additional conditions that will relate the δ here to the variations in the control variable $\boldsymbol{\eta}$, but in general such conditions are rare and avoidable. Hence we arrive at:

$$\begin{aligned}
&\frac{d}{d\delta} J^*[\mathbf{u}, t_f + \delta] = 0 \\
&\qquad\qquad \Rightarrow \frac{d}{d\delta} \int_{t_0}^{t_f+\delta} [\mathbf{p} \cdot \mathbf{x}' - H] dt \\
&\qquad\qquad = \mathbf{p}(t_f) \cdot \mathbf{x}'(t_f) - H(t_f) = 0.
\end{aligned}$$

This provides an endpoint condition on the Hamiltonian. In the case where $\mathbf{x}(t_0) = \mathbf{x}_0$ is fixed but $\mathbf{x}(t_f)$ is not then we already found that $\mathbf{p}(t_f) = 0$ so that this would imply that $H(t_f) = 0$.

Remark 18.2.5. It turns out that this condition ($H(t_f) = 0$) works even if $\mathbf{x}(t_f)$ is fixed but we are optimizing over t_f, something that is not easily seen from the current derivation.

If we are optimizing over the final time for a completely autonomous system, then this discussion combined with that above implies that the Hamiltonian is in fact zero for the optimizing trajectory, i.e. $H(t) = 0$ for all t and the optimal control and corresponding state variable.

18.3 Examples in gardening

The more relevant and interesting examples will come in the following chapters. Here we lay out a few simple mathematical examples that are meant only to illustrate the use of the maximum principle. After working through a few of these, we will outline the general approach to such problems. In the following chapters we will address the issues of modeling with optimal control. For the present we assume that the modeling steps have already been taken care of for us.

Example 18.3.1. Samwise is an eco-conscious gardener and is trying to minimize the amount of wizardry pesticides he uses on Frodo's garden. At the same time, he would prefer not to have the grasshoppers completely take over everything. To start off with, he chooses to minimize

$$J[u] = \int_0^1 u^2 dt,$$

subject to

$$x' = x - u, \quad x(0) = 1,$$

where $x(t)$ is the grasshopper population at time t with an original total saturation of the garden with grasshoppers i.e. the grasshopper population as a percentage of the carrying capacity of the garden is at the maximum of $x(0) = 1$. $u(t)$ is the control parameter that represents the amount of magic that Sam must use to keep the grasshoppers at bay (much better than pesticides). Sam is interested in working over the interval $[0, 1]$ because Gandalf is visiting in 1 month (one time unit) and he wants the garden to be in good shape when Gandalf arrives.

We will proceed by identifying what form of the maximum principle we will apply, including any potential other conditions, then set up the problem by computing the Hamiltonian, maximizing it in the control variable u, and then computing the evolution of the state and co-state with the appropriate initial/endpoint conditions.

(i) First we observe that this system is completely autonomous, but we are not optimizing over the end time as the end point is fixed at $t_f = 1$, hence we know that the Hamiltonian will be constant, but we don't automatically have an endpoint condition on the Hamiltonian.

(ii) The Hamiltonian in this problem is defined as

$$H = pf - L = p(x-u) - u^2.$$

(iii) The optimal control will maximize the Hamiltonian, meaning that the optimal control $\tilde{u}$ will satisfy $\frac{\partial H}{\partial u} = 0$ implying that $\tilde{u} = -\frac{p}{2}$. Thus once we have found the co-state, we will know the optimal control.

(iv) The state and co-state will evolve according to

$$\begin{aligned} x' &= \frac{DH}{Dp} = x + \frac{p}{2}, \quad x(0) = 1, \\ p' &= -\frac{DH}{Dx} = -p, \quad p(1) = 0. \end{aligned}$$

Solving the co-state equation first, we see that $p(t) = ce^{-t}$ and inserting the endpoint condition shows that $p(t) = 0$, and hence the optimal solution for this setup is to do nothing, i.e. $\tilde{u} = 0$. Inserting this back into the state equation we arrive at the differential equation

$$x' = x, \quad x(0) = 1.$$

This ODE has solution $x(t) = e^t$.

(v) Now, although we have the final solution to the optimization problem, we still want to check that the Hamiltonian is constant for this solution.

$$\begin{aligned} H &= p(x-u) - u^2 \\ &= 0 \end{aligned}$$

which is trivially constant.

There are a variety of potential steps that we may have taken in this system. For instance, once we found that the optimal control was satisfied by $u(t) = \frac{p(t)}{2}$ and we solved for the exact functional dependence of $p(t)$ then we could have employed the $H =$constant condition to determine the resulting constants of integration for the state variable. While this would have eventually gotten us to the right solution, it wouldn't have been pretty. Basically at this stage we note that there may be a choice as to which conditions to employ first in determining the solution. It is worthwhile, but not necessary to check the additional conditions as we did above, unless you are violently opposed to additional algebraic manipulations.

There are also several variations of this problem that we could have made. For instance, what if we decided that we wanted to minimize

$$\int_0^1 u(t)dt,$$

rather than $\int_0^1 u^2 dt$? How does this change the derivation we performed above? Is there anything physically different about these two approaches, or is it just a matter of choosing the one that makes the solution easier to compute?

We could also consider imposing an endpoint cost on the final number of grasshoppers, i.e. Samwise really wants to eliminate the grasshoppers at the end of the month $x(1)$ so maybe we should consider the cost functional given by:

$$\int_0^1 u^2 dt + \alpha x(1)^2.$$

This setting will yield a more interesting (and realistic) solution that should indicate a strategy for removing the grasshopper population by the end of the month. What would the parameter α mean in this case, and what would be the reasons for Sam picking different values of it?

Remark 18.3.2. Just as we have discussed throughout this book, this example yields an ideal example of how *NOT* to model the problem. Clearly Samwise would not last long as Frodo's gardener if he decided that the grasshoppers could take over everything, and there was no need to worry about them. This doesn't indicate that optimal control failed, but instead shows us that our modeling has failed. Hence, we must try again...

Example 18.3.3. Consider the same set up as before, except, now we want to eliminate all of the grasshoppers by time $t = 1$ then we would have the additional endpoint condition that $x(1) = 0$. In this setting the Hamiltonian is the same as the previous example, and in fact, up to the multiplicative constant, the co-state and optimal choice of control are the same as well, that is

$$H = p(x - u) - u^2,$$

and

$$\tilde{u} = -\frac{p}{2} = -\frac{c}{2}e^{-t}.$$

Now this means that

$$x' = x + \frac{c}{2}e^{-t}, \quad x(0) = 1, \quad x(1) = 0.$$

This has solution

$$\begin{aligned} x(t) &= e^t + \frac{c}{2}e^t \int_0^t e^{-2s} ds \\ &= \left(1 + \frac{c}{4}\right) e^t - \frac{c}{4}e^{-t}. \end{aligned}$$

The endpoint condition requires that

$$\begin{aligned} 0 = x(1) &= \left(1 + \frac{c}{4}\right) e^1 - \frac{c}{4}e^{-1} \\ \Rightarrow c &= -\frac{2e^1}{\sinh 1}, \end{aligned}$$

so that the optimal control is actually given by

$$\tilde{u}(t) = \frac{e^{1-t}}{\sinh(1)},$$

with corresponding state and co-state as described above.

Remark 18.3.4. The previous example is likely not even the best solution to Sam's problem. Note that this is assuming that at time $t = 1$ that all the grasshoppers are eliminated, but ignores what happens afterward. Sam had better assume in this case that he has a different job for $t > 1$ (such as helping Frodo in some other ornerous task far more important than caring for flowers), because otherwise, he knows nothing about what the grasshopper population will be for $t > 1$.

A perhaps better approach to this problem might be to minimize both the pesticides used, and the number of grasshoppers, i.e. minimize a cost functional of the form

$$\int_0^1 (u^2 + \alpha x^2)dt,$$

where α is a constant that Sam can choose dependent on how much he abhors the pesticides or grasshoppers. In this case, he may or may not choose to have a fixed end value for $x(t)$ (likely not).

Exercises

Note to the student: Each section of this chapter has several corresponding exercises, all collected here at the end of the chapter. The exercises between the first and second line are for Section 1, the exercises between the second and third lines are for Section 2, and so forth.

You should **work every exercise** (your instructor may choose to let you skip some of the advanced exercises marked with *). We have carefully selected them, and each is important for your ability to understand subsequent material. Many of the examples and results proved in the exercises are used again later in the text. Exercises marked with ⚠ are especially important and are likely to be used later in this book and beyond. Those marked with † are harder than average, but should still be done.

Although they are gathered together at the end of the chapter, we strongly recommend you do the exercises for each section as soon as you have completed the section, rather than saving them until you have finished the entire chapter.

18.1. Show that the Mayer form of a cost functional can be transformed into a Lagrange form. Consider introducing a new variable x_l so that $x_l' = 0$ and $x_l(t_0) = \frac{1}{t_f - t_0}\phi(t_0, \mathbf{x}(t_0); t_0, \mathbf{x}(t_f))$.

18.2. Show that the Lagrange form of a cost functional can be transformed into a Mayer form. Consider introducing a new variable $x_l(t_0) = 0$ and $x_l' = L(t, \mathbf{x}, \mathbf{u})$.

18.3. Come up with an alternative constraint that would represent that the self-driving vehicle has a finite amount of gas, and the passenger's are not comfortable with instantaneous acceleration.

18.4. Suppose that the self-driving car's goal is to minimize the gas, with a constraint of arriving at a certain time t_f after beginning. Re-phrase the example as a minimization problem that would work in this case.

18.5. Using the formal derivation and the 'mother functional' concept, show that one can formally arrive at the same set of evolution equations for the state and co-state as was done in the second section. Consider the specific case where the initial and final endpoint of the state are specified.

18.6. Verify the formula for the second variation of the functional

$$J[u] = \int_{t_0}^{t_f} \left[\mathbf{p}(t) \cdot \mathbf{x}'(t) - H(t; \mathbf{x}, \mathbf{u}, \mathbf{p})\right] dt,$$

as shown in class, that is:

$$\delta^2 J[\mathbf{u}; \boldsymbol{\eta}] = -\frac{1}{2} \int_{t_0}^{t_f} (\mathbf{h}_1^T \ \boldsymbol{\eta}^T) \mathscr{H} \begin{pmatrix} \mathbf{h}_1 \\ \boldsymbol{\eta} \end{pmatrix} dt$$

where

$$\mathscr{H} = \begin{pmatrix} \frac{D^2 H}{D\mathbf{x}^2} & \frac{D^2 H}{D\mathbf{x} D\mathbf{u}} \\ \frac{D^2 H}{D\mathbf{x} D\mathbf{u}} & \frac{D^2 H}{D\mathbf{u}^2} \end{pmatrix}.$$

Recall that $\mathbf{u} \to \mathbf{u} + \varepsilon \boldsymbol{\eta}$ corresponds to $\mathbf{x} \to \mathbf{x} + \varepsilon \mathbf{h}_1 + \varepsilon^2 \mathbf{h}_2 + o(\varepsilon^2)$. In addition, Taylor's Theorem for multidimensional vector valued functions can be expressed in this context as $H(\mathbf{x} + \varepsilon \mathbf{h}, \mathbf{u} + \varepsilon \boldsymbol{\eta}) = H(\mathbf{x}, \mathbf{u}) + \varepsilon \left(\frac{DH}{D\mathbf{x}} \mathbf{h} + \frac{DH}{D\mathbf{u}} \boldsymbol{\eta}\right) + \frac{\varepsilon^2}{2} \left(\mathbf{h}^T \frac{D^2 H}{D\mathbf{x}^2} \mathbf{h} + 2\mathbf{h}^T \frac{D^2 H}{D\mathbf{x} D\mathbf{u}} \boldsymbol{\eta} + \boldsymbol{\eta}^T \frac{D^2 H}{D\mathbf{u}^2} \boldsymbol{\eta}\right) + o(\varepsilon^2)$.

18.7. Set up the Hamiltonian, and find the optimal control (in terms of the state and co-state) for the problem of minimizing

$$J[u] = \frac{1}{2} \int_0^{t_f} u^2 dt,$$

subject to

$$x' = -x + u, \quad x(0) = 1, \quad x(t_f) = 2.$$

Are you optimizing over the final time t_f in this problem?

18.8. Continue the previous problem by solving the state and co-state evolution equations with the appropriate initial and endpoint conditions. You may end up with some rather ugly looking relations here. Don't stress about simplifying the algebraic equations that come as a result.

18.9. Finish up the problem from the previous 2 exercises by employing the appropriate conditions to find the value of the final time t_f. This should be a rather clean calculation relative to some of what you already did above.

18.10. Set up the optimal control problem (that is determine the co-state evolution equation and relevant boundary conditions, as well as the optimal control in terms of the co-state and state) discussed in the remark at the end of this Chapter, that is when Samwise minimizes both the pesticides and number of grasshoppers over the entire time interval $[0, 1]$.

Notes