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Homework 1

Exercise 2.10.10 We show there is some operator that is not bounded. Take some arbitrary hamel basis of X. Note that in general this basis will be infinitely dimensional, but we can take a subset of this basis that is countable.

Since Y is not all zero. choose some nonzero vector $y_0 \in Y$

Given this subset e_k. define $Te_k = ky_0$. We show that this operator is unbounded. take $||Te_k|| = |k|||y_0||$ clearly this bound can grow to be as large as we want so this operator is unbounded. It doesn't matter what we define the operator on the rest of the hamel basis to be (it could even be zero, in fact we set it to zero and then extend to the rest of x), since it is unbounded on this subset the linear operator is unbounded. As a result there always exists an unbounded linear operator.

Exercise 3.2.4 To show this recall that the inner product is continuous namely if we have two convergent sequences $x_n \to x, y_n \to y$ then $\langle x_n, y_n \rangle \to \langle x, y \rangle$

Take x_n as given in the problem and define $y_n = (y, y, y, \dots)$. From here note that $\langle x_n, y_n \rangle = \langle x_n, y \rangle \to \langle x, y \rangle$. However

$$\langle x_n, y \rangle = 0 \tag{1}$$

So since it is a constant sequence of zero it must approach 0 thus by continuity $\langle x_n, y \rangle \to 0 = \langle x, y \rangle$

Exercise 3.2.8 assume that $x \perp y$ so $\langle x, y \rangle = 0$ then take:

$$||x + \alpha y||^2 = ||x||^2 - \overline{\alpha}\langle x, y \rangle - \alpha(\langle y, x \rangle - \overline{\alpha}\langle y, y \rangle) \tag{2}$$
$$= ||x||^2 + |\alpha|^2 ||y||^2 \geq ||x||^2 \tag{3}$$
$$\implies \tag{4}$$
$$||x + \alpha y||^2 \geq ||x||^2 \tag{5}$$
$$||x + \alpha y|| \geq ||x|| \tag{6}$$
$$\tag{7}$$

For the other direction assume that $||x + \alpha y|| \geq ||x||$ for all scalars, then for this we take:

$$||x||^2 \leq ||x + \alpha y||^2 = ||x||^2 + \overline{\alpha}\langle x, y \rangle + \alpha(\langle y, x \rangle + \overline{\alpha}\langle y, y \rangle) \tag{8}$$
$$\tag{9}$$

choosing $\overline{\alpha} = \frac{-\langle y, x \rangle}{\langle y, y \rangle}$ then:

$$= ||x||^2 - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \tag{10}$$

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so :

$$0 \leq -\frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \tag{11}$$

$$0 \geq \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \tag{12}$$

$$0 \geq (|\langle x, y \rangle|^2) \tag{13}$$

$$0 \geq (|\langle x, y \rangle|^2) \tag{14}$$

$$0 \geq |\langle x, y \rangle| \tag{15}$$

So we must have that $\langle x, y \rangle = 0$

Exercise 3.3.10

first note that if $A \subset B$ then $B^\perp \subset A \perp$. To prove this assume that $b \in B \perp$ so $b \perp d \in B$ in particular $b \perp a \in A$ so $b \in A^\perp$. From here note that if $M \subset Y$ then $Y^\perp \subset M^\perp$. But now we can do this agian to obtain that $(M^\perp)^\perp \subset (Y^\perp)^\perp$. However since Y is closed we know that $Y = (Y^\perp)^\perp$ so we have that:

$$(M^\perp)^\perp \subset Y \tag{16}$$

Exercise 3.4.6 To do this take:

$$||x - y||^2 \tag{17}$$

Note that this has the same minimizer as $||x - y||$ then:

$$= ||x||^2 - \langle x, y \rangle - \langle y, x \rangle + ||y||^2 \tag{18}$$

note that in minimizing this we only need to minimize the β so we can actually ignore the first term x since it does not depend on beta:

$$-\langle x, y \rangle - \langle y, x \rangle + ||y||^2 \tag{19}$$

$$= -\langle x, y \rangle - \langle y, x \rangle + \left\langle \sum_k \beta_k e_k, \sum_k \beta_k e_k \right\rangle \tag{20}$$

$$= -\langle x, y \rangle - \langle y, x \rangle + \sum_k \langle \beta_k, \beta_k \rangle \tag{21}$$

$$= -\langle x, y \rangle - \langle y, x \rangle + \sum_k |\beta_k|^2 \tag{22}$$

$$= -\langle x, \sum_k \beta_k e_k \rangle - \left\langle \sum_k \beta_k e_k, x \right\rangle + \sum_k |\beta_k|^2 \tag{23}$$

$$= -\sum_k \overline{\beta_k} \langle x, e_k \rangle - \sum_k \beta_k \langle e_k, x \rangle + \sum_k |\beta_k|^2 \tag{24}$$

$$= \sum_k -2\mathrm{re}(\beta_k \langle x, e_k \rangle) + |\beta_k|^2 \tag{25}$$

$$\tag{26}$$

From here we can complete the square by adding $|\langle x, e_k \rangle|^2$ and subtracting it:

$$= \sum_k |\langle x, e_k \rangle - \beta_k|^2 - |\langle x, e_k \rangle|^2 \tag{27}$$

now note that this last term also has no dependence on β_k so we can ignore it again:

$$\sum_k |\langle x, e_k \rangle - \beta_k| \tag{28}$$

which is obviously minimized when $\beta_k = \langle x, e_k \rangle$. So if we assume that this norm is minimized then $\beta_k = \langle x, e_k \rangle$ converselly if we assume that $\beta_k = \langle x, e_k \rangle$ then clearly it minimizes this norm

Exercise 3.5.6

take

$$\langle x, y \rangle \tag{29}$$

$$= \langle \lim_{n \to \infty} \sum_{j=1}^{n} (\alpha_j) e_j, \lim_{m \to \infty} \sum_{j=1}^{m} (\beta_j) e_j \rangle \tag{30}$$

$$\tag{31}$$

by continuity of inner product we can pull out the limits

$$= \lim_{m \to \infty} \lim_{n \to \infty} \langle \sum_{j=1}^{n} (\alpha_j) e_j, \sum_{j=1}^{n} (\beta_j) e_j \rangle \tag{32}$$

$$= \lim_{m \to \infty} \lim_{n \to \infty} \sum_{j}^{n} \sum_{k}^{m} \langle (\alpha_j) e_j, (\beta_k) e_k \rangle \tag{33}$$

$$= \lim_{m \to \infty} \lim_{n \to \infty} \sum_{j}^{n} \sum_{k}^{m} \alpha_j \overline{\beta_k} \langle e_j, e_k \rangle \tag{34}$$

$$= \lim_{m \to \infty} \lim_{n \to \infty} \sum_{k}^{\min(m,n)} \alpha_k \overline{\beta_k} \tag{35}$$

$$= \lim_{n \to \infty} \sum_{k}^{n} \alpha_k \overline{\beta_k} \tag{36}$$

$$\tag{37}$$

We are able to get rid of the double liimit because whenever $m > n$ (or vice versa) the all of the entries dot product to zero, so it suffices to take one limit

So thus it is proven. Furthermore we can show that this is absolutely convergent. To prove this recall that $\alpha_j = \langle e_j, x \rangle$ and the squared sum of this by bessels inequality is

bounded. So this is in little l2 (because it works for any n). Similarly β_j is also in little l2. So then notice we can apply cuachy shwarts to:

$$\sum_k |\alpha_k \overline{\beta}_k| \leq ||\alpha|| ||\beta|| \tag{38}$$

so it is bounded and hence absolutely convergent.