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Homework 1

Exercise 2.3.8

Take some arbitrary cauchy sequence. Note that you can write it via telescoping as:

$$x_n \tag{1}$$
$$= x_0 + \sum_{j=0}^{n-1}(x_{j+1} - x_j) \tag{2}$$

We need to show that for a cauchy sequence this is absolutely convergent, then that would mean that it converges and as a result x_n converges so:

To do this choose N large enough so that for $m, l > N$ we have that $||x_m - x_l|| < \epsilon$ for some epsilon. We can do this because the sequence is cauchy. Then for $n > N$:

$$\|x_n\| = \left\| x_0 + \sum_{j=0}^{n-1}(x_{j+1} - x_j) \right\| \tag{3}$$
$$\leq \|x_0\| + \left\| \sum_{j=0}^{N}(x_{j+1} - x_j) \right\| + \left\| \sum_{j=N+1}^{n-1}(x_{j+1} - x_j) \right\| \tag{4}$$
$$\leq \|x_0\| + \|x_N - x_0\| + \|x_n - x_{N+1}\| \tag{5}$$
$$\leq \|x_0\| + \|x_N - x_0\| + \epsilon \tag{6}$$

Now choose $\epsilon < 1$ from here:

$$\leq \|x_0\| + \|x_n - x_0\| + 1 < \infty \tag{7}$$

This is less than infinity because the other two norms are positive. so This series converges absolutely! Namely the series $x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) = x_n$ converges absolutely. Because it converges absolutely we know that $x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) = x_n$ converges. So

$$x_n \tag{8}$$

must also converge. And since x_n was cauchy then the space is complete since every cauchy sequence converges.

Exercise 2.3.10 To prove this assume that a normed space has a schauder basis. that means that for every $x \in X$ we have a unique sequence of scalars:

$$||x - (\alpha_1 e_1 + \cdots + \alpha_n e_n)|| \to 0 \tag{9}$$

or that the partial sums converge to x. Define the space P to be the set of all finite series in the schauder basis $\sum_{k=1}^{n} \beta_k e_k$ (n is not fixed, β is rational).

We show that any point x is also a limit point of things in P. Take $x \in X$

$$\begin{aligned} & ||x - \sum_{k=1}^{n} \beta_k e_k|| && (10)\\ & \leq ||x - \sum_{k=1}^{n} \alpha_k e_k|| + ||\sum_{k=1}^{n} \alpha_k e_k - \sum_{k=1}^{n} \beta_k e_k|| && (11)\\ & \leq ||x - \sum_{k=1}^{n} \alpha_k e_k|| + ||\sum_{k=1}^{n} (\alpha_k - \beta_k) e_k|| && (12)\\ & \leq ||x - \sum_{k=1}^{n} \alpha_k e_k|| + \sum_{k=1}^{n} |\alpha_k - \beta_k| ||e_k|| && (13)\\ & && (14) \end{aligned}$$

From here note that we can always choose n large enough so that the first term is less than $\epsilon/2$ this just follows from the definition of the shauder basis.

Given this n we can choose β_k close enough to α_k (by density of the rationals) to be $|\alpha_k - \beta_k| \leq \frac{\epsilon}{2n||e_k||}$ thus:

$$\begin{aligned} & \leq \epsilon/2 + \sum_{k=1}^{n} (\epsilon/2n) = \epsilon/2 + \epsilon/2 && (15)\\ & = \epsilon && (16) \end{aligned}$$

So we can get arbitrarily close to anything in x with a thing from P. Now note that P is countable. This follows from the fact that Q is countable and we are only taking finite sums over the countable shauder basis.

Exercise 2.7.6

To do this note first that by problem 5 the operator defined by $y = (\eta_j)$ where $y = Tx, \eta_j = \xi_j/j, x = \xi_j$ is linear and bounded in other words if:

$$x = (x_1, \dots) \in l^\infty \tag{17}$$

$$Tx = (\frac{x_1}{1}, \frac{x_2}{2}, \dots) \tag{18}$$

then this operator is linear and bounded.

To do this note that if $y \in R(T)$ then there is an x such that $y_j = \frac{x_j}{j}$ furthermore we know that x_j is in l^∞ so $\max_j x_j = \max_j y_j j < \infty$

So for y to be in the range it must be bounded even when multiplied by j.

Take the sequence $y = \frac{1}{\sqrt{j}}$ clearly this is in the closure of $R(T)$ because we can take

$x = (\sqrt{1}, \sqrt{2}, \dots, \sqrt{n}, 0, \dots)$ and have that:

$$||x - y|| = \max_j |(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}, \dots, \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n}}, -\frac{1}{\sqrt{n+1}}, \dots)| \tag{19}$$

$$= \max_j |(0, 0, \dots, 0, -\frac{1}{\sqrt{n+1}}, \dots)| \tag{20}$$

$$= \frac{1}{\sqrt{n+1}} \tag{21}$$

So we can get arbitrarily close to this vector. However we know that the vector x $x = (1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots)$ that would achieve this is unbounded. so $x \notin l^\infty$. As a result the range of T is not closed

Exercise 2.7.8 To prove this we do it by contradiction. Assume there was a bound so that $||Tx|| \leq M||x||$ for all x. then remember that $T^{-1} : R(T) \to D(T)$ is defined by :

$$T^{-1}y = (y_1 * 1, y_2 * 2, \dots, y_j * j, \dots) \tag{22}$$

Choose $K > M, K \in \mathbb{Z}$. from here take the vector $y = (1, \dots, 1, 0, \dots)$ where the ones take up K positions. Note that $y \in R(T)$ just take $x = (1, 2, \dots, K, 0, \dots)$ clearly $x \in l^\infty$ so that $Tx = y$.

from here note that $T^{-1}y = x$ from what we just showed and that $\max_j |x_j| = K$ however that means that:

$$||T^{-1}y|| = ||x|| = \max_j |x_j| = K \tag{23}$$

Furthermore note that y is unit norm so:

$$||T^{-1}y|| = ||x|| = K = K||y|| \tag{24}$$

but this is a contradiction because $K > M$ and we assumed that M was the bound. Thus T^{-1} is unbounded.

Exercise 2.9.10 So the idea for this problem is that we actually need to create a basis. Choose the basis for x so that we have a basis $\{z_1, \dots z_k\}$ for Z and a separate basis for $X - Z$ $b_{k+1}, \dots b_n$ (Note that both of these are nonepty by the fact that this is a proper subspace). Choose $b_{k+1} = x_0$ (we are allowed to do this since we can just pull other linearly independent vectors in). First notice that this is a bsis for the whole space because any thing in X is either in Z or not in Z ($X = (X \cap Z^c) \cup (X \cap Z)$). From here define the linear functional by:

$$f(z_j) = 0 \tag{25}$$

$$f(b_{k+1}) = 1 \tag{26}$$

$$f(b_j) = 1 \tag{27}$$

From this we know that linear functionals are uniquely defined by their action on the basis vectors so that:

$$x = \xi_1 z_1 + \cdots + \xi_k z_k + \xi_{k+1} b_{k+1} + \cdots + \xi_n b_n \tag{28}$$

$$f(x) = \sum_{j=1}^{k} \xi_j f(z_j) + \sum_{j=k+1}^{n} \xi_j f(b_j) \tag{29}$$

$$= \sum_{j=k+1}^{n} \xi_j f(b_j) \tag{30}$$

So if $x \in Z$ then clearly ξ_k is only nonzero for the first k values (since those are a basis). In other words the last $n-(k+1)$ elements are zero corresponding to the basis for $X - Z$. thus:

$$f(x) = \sum_{j=k+1}^{n} \xi_j f(z_j) \tag{31}$$

$$= 0 \text{ since } \xi_j = 0 \text{ for } j \geq k+1 \tag{32}$$

However if $x = x_0$ then $x = 1 * b_{k+1} = 1 * x_0 = x_0$ and $\xi_{k+1} = 1$ but zero everywhere else so

$$f(x) = \sum_{j=k+1}^{n} \xi_j f(b_j) = \xi_{k+1} f(b_{k+1}) = f(b_{k+1}) = f(x_0) = 1 \tag{33}$$

Thus it is proven.

Exercise A take :

$$|f(x)| \leq ||f||||x|| \tag{34}$$

now note that $0 \in N(f), f(0) = 0 * f(0) = 0$ so that $d(x, Y) \leq d(x, \{0\}) = ||x||$ thus:

$$\leq ||f||d(x, Y) \tag{35}$$

For this one note that $||f|| = \sup_{||x||=1} |f(x)|$. Or that it is the supremum of all vectors of unit length. Since it is a supremum we can get arbitrarily close to it with some u this means that we can get within $\epsilon||f||$ distance or $f(u) \geq ||f|| - \epsilon||f|| = (1-\epsilon)||f||$ where u is unit length by definition of norm.

now note that

$$y = x - \frac{f(x)}{f(u)} u \in Y \text{ because:} \tag{36}$$

$$f(y) = f(x) - \frac{f(x)}{f(u)} f(u) = 0 \tag{37}$$

so

$$d(x,Y) \leq ||x-y|| = ||x-(x-\frac{f(x)}{f(u)}u)|| \tag{38}$$
$$= ||\frac{f(x)}{f(u)}u|| \tag{39}$$
$$= \frac{|f(x)|}{|f(u)|}||u|| \tag{40}$$
$$= \frac{|f(x)|}{|f(u)|} \leq \frac{|f(x)|}{(1-\epsilon)||f||} \tag{41}$$

So in total:

$$(1-\epsilon)||f||d(x,Y) \leq |f(x)| \tag{42}$$

Taking the limit as $\epsilon \to 1$ we get:

$$||f||d(x,Y) \leq |f(x)| \tag{43}$$

So thus $||f||d(x,Y) \leq |f(x)|$

Exercise B (HELP, B seems to easy)

To do this take:

$$||f(x)|| = ||\int_0^1 x(t)z(t)dt|| \tag{44}$$
$$\leq \int_0^1 |x(t)||z(t)|dt \tag{45}$$
$$\leq \max|x(t)|\int_0^1 |z(t)|dt \tag{46}$$
$$= \|x\|\int_0^1 |z(t)|dt \tag{47}$$

So we know that the norm is bounded by this value. Take

$$x = \begin{cases} t/a \text{ if } t < a \\ 1 \text{ if } t > a \end{cases} \tag{48}$$

This is a continuous function and satisfies $x(0) = 0, \|x\| = 1$ for $a \in (0,1]$. and note that:

$$|f(x)| \leq \int_0^a t/a|z(t)|dt + \int_a^1 |z(t)|dt \tag{49}$$
$$< \int_0^a 1 * |z(t)|dt + \int_a^1 |z(t)|dt \tag{50}$$
$$= \int_0^1 |z(t)|dt \tag{51}$$

So we have a strict inequality. however we can choose a arbitrarily small so that the first integral can be made smaller that ϵ. In that way we can get arbitrarily choice using a unit norm x to $\int_0^1 |z(t)|dt$ The details of this are as follows:

$$\int_0^1 z(t)dt - \int_0^a t/az(t)dt - \int_a^1 z(t)dt \tag{52}$$

$$\int_0^a z(t) - \frac{t}{a}z(t)dt \tag{53}$$

$$\int_0^a (1 - \frac{t}{a})z(t)dt \tag{54}$$

$$\leq \int_0^a z(t)dt \tag{55}$$

Note that since z is bounded (by continuity on compact subset):

$$|| \int_0^1 z(t)dt - \int_0^a t/az(t)dt - \int_a^1 z(t)dt|| \leq \int_0^a |z(t)|dt \tag{56}$$

$$\leq aM \tag{57}$$

So we can choose a to make to make this quantity as small as we want. so we know then that the supremum of all such x over a is in fact $\int_0^1 |z(t)|dt$ but since $z(t) > 0$ we always have that little term in fron that makes it so that no x actually attains it (that is unit norm).

b) To show this, first we know it is bounded by the above theorem. Secondly we know it is linear because:

$$f(ax + by) = \int_0^1 (ax(t) + by(t))z(t)dt \tag{58}$$

$$= a\int_0^1 x(t)z(t)dt + b\int_0^1 y(t)z(t)dt \tag{59}$$

So then we know by corollary 2,7-10 of the book that if T is a bounded linear operator then the null space is closed. So this is a bounded linear operator, a special case as a functional. So its null space is closed

Its a proper subspace because since $z(t) > 0$ if we choose

$$x = \begin{cases} t/a \text{ if } t < a \\ 1 \text{ if } t > a \end{cases} \tag{60}$$

then as before we will have

$$f(x) = \int_0^a t/az(t)dt + \int_a^1 z(t)dt \tag{61}$$

$$> 0 \tag{62}$$

Since all of the quanities involved are strictly positive. So we know that there are things in X that are not in the null space. So it is a proper subspace.

c) To show this take f for part A to be our example here then:

$$\int_0^1 |z(t)|dt d(x,Y) = |\int_0^1 x(t)z(t)dt| \tag{63}$$

$$d(x,Y) = \frac{|\int_0^1 x(t)z(t)dt|}{\int_0^1 |z(t)|dt} \tag{64}$$

$$d(x,Y) = \frac{|f(x)|}{||f||} \tag{65}$$

However we know that for any $x \in X$ (if $||x|| = 1$)we always have $|f(x)| < ||f||$ by part a. So thus we have that:

$$d(x,Y) < 1 \tag{66}$$

So this quantity can not be greater than or equal to one.