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Homework 2

Exercise 2.1

Note for this problem we want to find the Lipshitz bound L. This is the same as bounding the norm of the derivative with respect to mu. Then take:

$$\frac{d}{d\mu}\frac{dx}{dt} \tag{1}$$

$$= \frac{d}{dt}\frac{dx}{d\mu} \text{ since we are in } C^1 \tag{2}$$

$$\int_0^t \frac{d}{d\mu}\frac{dx}{dt} ds = \int_0^t \frac{d}{dt}\frac{dx}{d\mu} ds \tag{3}$$

$$\frac{dx}{d\mu} = \int_0^t \frac{d}{d\mu}\frac{dx}{dt} ds \tag{4}$$

$$\frac{dx}{d\mu} = \int_0^t \frac{d}{d\mu} f(s, x_\mu(s), \mu) ds \tag{5}$$

$$\frac{dx}{d\mu} = \int_0^t f_x(s, x_\mu(s), \mu)\frac{dx}{d\mu} + f_\mu(s, x_\mu(s), \mu) ds \tag{6}$$

$$|\frac{dx}{d\mu}| = \int_0^t |f_x(s, x_\mu(s), \mu)\frac{dx}{d\mu}| + |f_\mu(s, x_\mu(s), \mu)| ds \tag{7}$$

$$|\frac{dx}{d\mu}| = Mt + \int_0^t L|\frac{dx}{d\mu}| ds \tag{8}$$

So then by the advanced gronwalls inequality.

$$|\frac{dx}{d\mu}| \leq Mt + \int_0^t LMse^{\int_s^t L du} ds \tag{9}$$

$$= M(t + L\int_0^t se^{L(t-s)} ds \tag{10}$$

$$= M(t + Le^{Lt}\int_0^t se^{-Ls} ds \tag{11}$$

$$= M(t + Le^{Lt}(\frac{1 - e^{-Lt}(Lt - 1)}{L^2})) \tag{12}$$

$$= M(t + e^{Lt}(\frac{1 - e^{-Lt}(Lt - 1)}{L})) \tag{13}$$

$$= M(t - t + \frac{1}{L}(e^{Lt} - 1)) \tag{14}$$

$$= \frac{M}{L}(e^{Lt} - 1) \tag{15}$$

$$\tag{16}$$

So the bound on the lipshitz constant is $\frac{M}{L}(e^{Lt}-1)$ so:

$$||x_\mu - x_\nu|| \leq \frac{M}{L}(e^{Lt}-1)|\mu - \mu| \tag{17}$$

uniformly in t. That is the answer.

Alternative method:

Alternatively you can do a different method with derivative tricks, note this is not my main answer but is included for completeness.

For this problem note that we can write the integral form:

$$x_\mu(t) = x_0 + \int_0^t f(s, x_\mu(s), \mu)ds \tag{18}$$

Now take:

$$x_\mu(t) - x_\nu(t) == \int_0^t f(s, x_\mu, \mu) - f(s, x_\nu, \nu) \tag{19}$$

$$= \int_0^t f(s, x_\mu, \mu) - f(s, x_\nu, \mu) + f(s, x_\nu, \mu) - f(s, x_\nu, \nu) \tag{20}$$

$$= \int_0^t f(s, x_\mu, \mu) - f(s, x_\nu, \mu) + f(s, x_\nu, \mu) - f(s, x_\nu, \nu) \tag{21}$$

$$\tag{22}$$

Taking norms and applying the triangle inequality we obtain:

$$\|x_\mu - x_\nu\| \tag{23}$$

$$\leq \int_0^t ||f(s, x_\mu, \mu) - f(s, x_\nu, \mu)||ds + \int_0^t ||f(s, x_\nu, \mu) - f(s, x_\nu, \nu)||ds \tag{24}$$

Now the goal is to bound each of those norms Take

$$\|f(s, x_\mu, \mu) - f(s, x_\nu, \mu)\| = \left\| \int_0^1 f_x(s, x_\nu + \tau(x_\mu - x_\nu), \mu)(x_\mu - x_\nu)d\tau \right\| \tag{25}$$

$$\leq \int_0^1 \|f_x\| \, |x_\mu - x_\nu| d\tau \tag{26}$$

$$\leq L|x_\mu - x_\nu| \tag{27}$$

Now to bound the other one:

$$\|f(s, x_\nu, \mu) - f(s, x_\nu, \nu)\| \tag{28}$$

$$\leq \left\| \int_0^1 f_\mu(s, x_\nu, \nu + \tau(\mu - \nu))(\mu - \nu)d\tau \right\| \tag{29}$$

$$\leq \int_0^1 M|\mu - \nu|d\tau \tag{30}$$

$$= M|\mu - \nu| \tag{31}$$

Thus in total we have that:

$$\|x_\mu - x_\nu\| \tag{32}$$

$$\leq \int_0^t ||f(s, x_\mu, \mu) - f(s, x_\nu, \mu)||ds + \int_0^t ||f(s, x_\nu, \mu) - f(s, x_\nu, \nu)||ds \tag{33}$$

$$\leq \int_0^t L|x_\mu - x_\nu|ds + \int_0^t M|\mu - \nu|ds \tag{34}$$

$$\leq L\int_0^t |x_\mu - x_\nu|ds + tM|\mu - \nu| \tag{35}$$

$$\tag{36}$$

Using the more general gronwall inequality we derived earlier we have:

$$a(t) = tM|\mu - \nu| \tag{37}$$

$$b(t) = L \tag{38}$$

$$c(s) = 1 \tag{39}$$

Then:

$$\|x_\mu - x_\nu\| \leq tM|\mu - \nu| + L(\int_0^t sM|\mu - \nu|e^{\int_s^t Ldu}ds) \tag{40}$$

$$= M|\mu - \nu|(t + L\int_0^t se^{(t-s)L}ds) \tag{41}$$

$$= M|\mu - \nu|(t + L\frac{-Lt + e^{Lt} - 1}{L^2}) \tag{42}$$

$$= M|\mu - \nu|(t + \frac{-Lt + e^{Lt} - 1}{L}) \tag{43}$$

$$= M|\mu - \nu|(t - t + \frac{e^{Lt} - 1}{L}) \tag{44}$$

$$= M|\mu - \nu|(\frac{e^{Lt} - 1}{L}) \tag{45}$$

$$= \frac{M}{L}(e^{Lt} - 1)|\mu - \nu| \tag{46}$$

$$\tag{47}$$

So it is lipshitz. and the lipshitz bound is:

$$\frac{M}{L}(e^{Lt} - 1) \tag{48}$$

Which is dependent on t.

Exercise 2.2

a) To show its unique we will use the standard argument take:

$$x(t) = x_0 + \int_0^t f(x(s), s)ds \tag{49}$$

as the integral form of hte IVP. then assuming we have two different solutions

$$x(t) - y(t) = x_0 - y_0 + \int_0^t f(x(s), s) - f(y(s), s)ds \tag{50}$$

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_0^t |f(x(s), s) - f(y(s), s)|ds \tag{51}$$

$$\leq |x_0 - y_0| + \int_0^t p(|x(s) - y(s)|)ds \tag{52}$$

$$\tag{53}$$

At this point we would really like to use gronwalls inequality, but we cannot here. We will now prove a variant.

Assume that $\phi(t) \leq A + \int_0^t p(\phi(s))ds$ set $u(t) = A + \int_0^t p(\phi(s))ds$ where A is positive and p is monotonically increasing and nonnegative. then since p is continuous by the FTC we can take derivatives:

$$u'(t) = p(\phi(t)) \leq p(u(t)) \tag{54}$$

That follows since p is monotonically increasing. then:

$$\frac{u'}{p(u(t))} \leq 1 \tag{55}$$

$$\int_0^t \frac{u'(s)}{p(u(s))}ds \leq t \tag{56}$$

$$\int_{u(0)}^{u(t)} \frac{1}{p(s)}ds \leq t \tag{57}$$

$$\tag{58}$$

Now note that becaue p is increasing $\frac{1}{p}$ is decreasing. As a result $\int_0^{\frac{h}{2}} \frac{1}{p(s)} \geq \int_{\frac{h}{2}}^h \frac{1}{p(s)}$. From this we can gather that

$$2\int_0^{\frac{1}{2}} \frac{1}{p(s)} \geq \int\limits_0^{1/2} \frac{1}{p(s)}ds + \int\limits_{1/2}^{1} \frac{1}{p(s)}ds = \infty \text{ by assumption} \tag{59}$$

So $\int_0^{1/2} \frac{1}{p(s)}ds = \infty$. Similarly $\int_0^{1/4} \frac{1}{p(s)}ds = \infty$ and by induction $\int\limits_0^{1/2^k} 1/p(s)ds = \infty$. As a result of this if the upper limit is anything other than zero, the integral is infinity.

From here note that the inequality we have derived

$$\int_{u(0)}^{u(t)} \frac{1}{p(s)}ds \leq t \tag{60}$$

Lets plug in things that we derived earlier in our quest for finding a unique solution. We would set $A = |x_0 - y_0|$ and $\phi(s) = |x(s) - y(s)|$. From here we are ready to prove uniqueness. Assume that with these solutions $y_0 = x_0$

from this we gather that $A = 0 = u(0)$. from this take a closer look at our previous inequality

$$\int_{u(0)}^{u(t)} \frac{1}{p(s)} ds \leq t \tag{61}$$

$$\int_{0}^{u(t)} \frac{1}{p(s)} ds \leq t \tag{62}$$

However we know by the osgood condition that the integral blows up, specifically if the upper integrand is nonzero. So the only way this inequality holds is if $u(t) = 0$ for all time t. Which in turn means:

$$\phi(t) \leq u(t) = 0 \tag{63}$$

$$\phi(t) = ||x(t) - y(t)|| = 0 \tag{64}$$

So $x(t) = y(t)$

b) For this part we jump straight to:

$$\frac{u'(t)}{p(u(t))} \leq 1 \tag{65}$$

$$\frac{u'(t)}{Lu(t)(1 + |\log u(t)|)} \leq 1 \tag{66}$$

$$\tag{67}$$

Note that for small $u(t)$ $|\log(u(t))| = -\log(u(t))$ for $|u(t)| < 1$ while for $u(t) > 1$ it is $\log(u(t))$. Furtherome not that since ρ is positive and increasing we have that $u(t)$ is strictly increasing. (It is a positive number plus the integral of a positive number)

As a result we can split up our integral as thus:

$$\int_{u(0)}^{u(t)} \frac{dv}{Lv(1-\log(v))} \tag{68}$$

$$\frac{1}{L}\int_{u(0)}^{1} \frac{dv}{v(1-log(v)} + \frac{1}{L}\int_{1}^{u(t)} \frac{dv}{v(1+\log(v))} \leq t \tag{69}$$

$$\int_{0}^{1} \frac{dv}{v(1-log(v))} + \int_{1}^{u(t)} \frac{dv}{v(1+\log(v))} \leq Lt \tag{70}$$

$$-\log(1-\log(v))|_{v=u(0)}^{v=1} + \log(1+\log(v))|_{v=1}^{v=u(t)} \leq Lt \tag{71}$$

$$-\log(1-\log(1)) + \log(1-\log(u(0))) + \log(1+\log(u(t))) - \log(1+\log(1)) \leq Lt \tag{72}$$

$$\log(1-\log(u(0))) + \log(1+\log(u(t))) \leq Lt \tag{73}$$

$$\log(1+\log(u(t))) \leq Lt - \log(1-\log(u(0))) \tag{74}$$

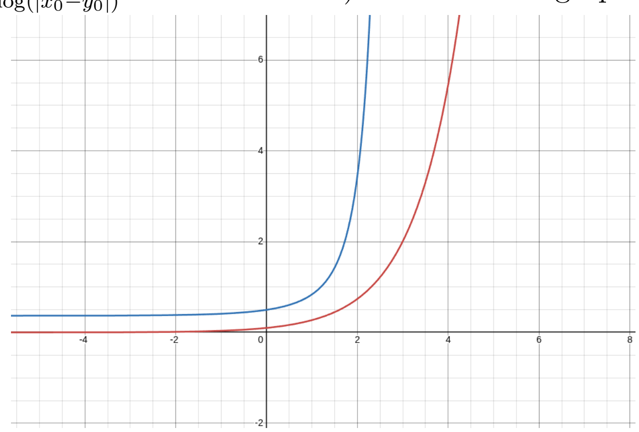
$$1+\log(u(t)) \leq e^{Lt-\log(1-\log(|x_0-y_0|))} \tag{75}$$

$$\log(u(t)) \leq e^{Lt}\frac{1}{1-\log(|x_0-y_0|))} - 1 \tag{76}$$

$$u(t) \leq e^{e^{Lt}\frac{1}{1-\log(|x_0-y_0|))}-1} \tag{77}$$

$$\tag{78}$$

Here is a graph of the normal lipshitz bound $e^{Lt}|x_0 - y_0|$ in red verses the new bound $e^{e^{Lt}\frac{1}{1-\log(|x_0-y_0|))}-1}$ in blue. You can see that the new bound grows way faster! This can be seen in the equation just because we will have an exponential to a positive exponential (since $\frac{1}{1-\log(|x_0-y_0|)} > 0$ in this case). Note for the graph I set $x_0 = 0, y_0 = 0.1, L = 1$



in the second case for $|x_0 - y_0| > 1$ then $|\log(u(t))| = \log(u(t))$ so we don't have to split up the integral (Since u(t) is increasing it will always be greater than 1)

$$\frac{1}{L}\int_0^t \frac{u'(t)}{u(t)(1+log(u(t)))} \leq t \tag{79}$$

$$\log(1+\log(u(t)))|_0^t \leq Lt \tag{80}$$

$$\log(1+\log(u(t)) - \log(1+\log(u(0))) \leq Lt \tag{81}$$

$$\log(1+\log(u(t))) \leq \log(1+\log(|x_0-y_0|)) + Lt \tag{82}$$

$$1+\log(u(t)) \geq (1+\log(|x_0-y_0|))e^{Lt} \tag{83}$$

$$\log(u(t)) \leq -1+(1+\log(|x_0-y_0|))e^{Lt} \tag{84}$$

$$u(t) \leq e^{-1+(1+\log(|x_0-y_0|))e^{Lt}} \tag{85}$$

$$\tag{86}$$

So in the second case this is the bound on $u(t) = |x(s)-y(s)| \leq e^{e^{Lt}(1+\log|x_0-y_0|)-1}$

Here is a graph of the normal lipshitz bound $e^{Lt}|x_0-y_0|$ in red verses the new bound $e^{e^{Lt}(1+\log|x_0-y_0|)-1}$ in blue. You can see that the new bound grows way faster! This can be seen in the equation just because we will have an exponential to a positive exponential (since $1+\log(|x_0-y_0|) > 0$ in this case). Note for the graph I set $x_0 = 0, y_0 = 1.1, L = 1$