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Homework 1

Exercise 1.8 We need to show that this does not define a metric. There are two ways of doing this. One way is to show that positivity is not satisfied. Take Two sets in the metric space A, B that have a nonempty intersection or $A \cap B \neq$ but also are not the same (Say take $A = [0, 1], B = [.5, 1.5]$)

From this note that $D(A, B) = \inf_{\substack{a \in A \\ b \in B}} d(a, b) = 0$. If this were a metric that would mean that $A = B$, but that is clearly not the case by construction. So this is not a metric.

Alternate proof:

We show that the triangle inequality does not hold. Take three sets A, B, C such that $A \cap B =$ but $A \cap C \neq, B \cap C \neq$ Where $A \neq B \neq C$ (This could be $A = [0, 1], B = [2, 3], C = [.5, 2.5]$) From this it is easy to see that $D(A, B) > 0$ but $D(A, C) = 0, D(C, B) = 0$ so $D(A, B) > D(A, C) + D(C, B)$. So the triangle inequality breaks. (In our example $D(A, B) = 1 > D(A, C) + D(C, B) = 0 + 0 = 0$)

Exercise 2.10 Before I prove this i prove a lemma that will make it easier. x is a limit point of A iff $\exists x_n \in A, x_n \to x$.

pf: Assume x is a limit point. From this we know that every neighborhood contains at least one point $y \in A$. Choose a sequence of neighborhoods that are open balls $B(x, \frac{1}{n})$. Take each x_n from each neighborhood in succession. Clearly $x_n \in A$ also $d(x, x_n) < \frac{1}{n}$. So $x_n \to x$.

in the other direction assume that $\exists x_n \in A, x_n \to x$. Take some arbitrary neighborhood of x, call it K. Since K is a neighborhood it contains an epsilon neighborhood of x of radius some ϵ.

Since $x_n \to x$ we can choose an N such that $d(x, x_n) < \frac{\epsilon}{2}$ for $n > N$. Thus $x_n \in K$ since it is within this epsilon neighborhood. and since $x_n \in A$ by assumption then x is a limit point.

Now we prove the main theorem that $x \in \overline{A} \iff d(x, A) = 0$

pf:

Assume that $x \in \overline{A}$. From this we know that either $x \in A$ or x is a limit point of A.

Case 1: $x \in A$. From this $D(x, A) = 0$ since x is a part of A.

Case 2: x is a limit point of A. since x is a limit point of A, we can construct a sequence of elements $x_n \in A$ such that $x_n \to x$. Because $x_n \to x$. we can choose a N such that for $m > N$ $d(x_m, x) < \epsilon$. from this we can make the distance from any point in A as small as we want. Written formally:

$$D(x, A) = \inf_{y \in A} d(x, y) \leq d(x, x_m) < \epsilon \tag{1}$$

Since epsilon was arbitrary. we then know that we can send this to zero so $D(x, A) = 0$

Now we prove the other direction. Assume that $D(x, A) = 0$. This means that for every $\epsilon, \exists y \in A, d(x, y) < \epsilon$ Generate a sequence by choosing x_n such that $d(x, x_n) < \frac{1}{n}$. From this we generate a convergent sequence. So x is a limit point of A and this a part of $\overline{A}$

Exercise 4.4 By the the definition of a cauchy sequence we know that for any ϵ we can choose an N so that for $n, m > N, d(x_n, x_m) < \epsilon$. In particular choose $\epsilon = 1$.

Now take some element in the sequence x_L. Now for $d(x_0, x_L)$. if $L > N$ then:

$$d(x_0, x_n) \leq \sum_{k=1}^{N} (d(x_{k-1}, x_k)) + d(x_{N+1}, x_L) \tag{2}$$

$$\leq \sum_{k=1}^{N} (d(x_{k-1}, x_k)) + 1 \tag{3}$$

This follows from the triangle inequality and what we chose for ϵ. Similarly if $L \leq N$ then:

$$d(x_0, x_n) \leq \sum_{k=1}^{L} (d(x_{k-1}, x_k)) \tag{4}$$

$$\leq \sum_{k=1}^{N} (d(x_{k-1}, x_k)) + 1 \tag{5}$$

Either way $d(x_0, x_L) \leq M$ where $M = \sum_{k=1}^{N} (d(x_{k-1}, x_k)) + 1$.

Thus if we take the $\overline{B(x_0, M)}$ then we know that the entire sequence is contained within this ball. since $d(x_0, x_L) \leq M$

So every cauchy sequence is bounded.

Exercise 5.8 I am assuming that we are using the supremum metric.

There are a couple of ways to do this. In theorem 1.4-7 we proved that if M si a subset of a complete metric space it itself is complete if and only if M is closed. So we just need to show that M is closed or that it contains all of its accumulation points because we already know that $C[a, b]$ is complete by 1.5-5

Let x_n be a convergetn sequence in $C[a, b], x(a) = x(b)$. We just need to show that its limit is also in this space.

First note that

$$d(x(b), x(a)) \leq d(x(b), x_n(b)) + d(x_n(b), x_n(a)) + d(x_n(a), x(a)) \tag{6}$$

$$= d(x(b), x_n(b)) + d(x_n(a), x(a)) \tag{7}$$

$$= \max_t d(x(x), x_n(t)) + d(x_n(a), x(a)) \tag{8}$$

$$\leq 2 \max_t d(x(t), x_n(t)) = 2d(x_n, x) \tag{9}$$

Note we were able to express the distance between $x(b), x(a)$ in terms of the distance between the two functions themselves. Since $x_n \to x$ we can choose N such that for $n > N, d(x_n, x) < \epsilon/2$

$$= 2d(x_n, x) < \frac{2\epsilon}{2} = \epsilon \tag{10}$$

So rthe distance between $d(x(a), x(b)) < \epsilon$ for arbitary epsilon. Thus $x(a) = x(b)$.

furthermore we know that x is continuous, as a proof for some $c \in (a, b)$ take some sequence x_n that converges to x. Take t in a delta neighborhood of x

$$\begin{aligned}
d(x(c), x(t)) &\leq d(x(c), x_n(c)) + d(x_n(c), x_n(t)) + d(x_n(t), x(t)) && (11)\\
&\leq d(x(c), x_n(c)) + d(x_n(c), x_n(t)) + \max_s d(x_n(s), x(s)) && (12)\\
&\leq \max_s d(x(s), x_n(s)) + d(x_n(c), x_n(t)) + \max_s d(x_n(s), x(s)) && (13)\\
&\leq 2 \max_s d(x(s), x_n(s)) + d(x_n(c), x_n(t)) && (14)\\
&\leq 2 \max_s d(x(s), x_n(s)) + d(x_n(c), x_n(t)) && (15)\\
&\leq 2d(x, x_n) + d(x_n(c), x_n(t)) && (16)\\
& && (17)
\end{aligned}$$

From here choose N such that for $n > N$ $d(x, x_n) < \frac{\epsilon}{4}$. We can do this since $x_n \to x$. Furthermore we will choose the delta of $d(c, t) < \delta$ such that $d(x_n(c), x_n(t)) < \frac{\epsilon}{2}$

Thus we have:

$$< \frac{2}{4}\epsilon + \frac{\epsilon}{2} = \epsilon \tag{18}$$

Thus we see that x is continuous.

Thus any convergent sequence converges to something within this new space. So it is closed (it contains all of its limit points). Thus by 1.4-7 this space is complete.

Exercise 6.6 To do this we need to come up with a mapping $T : C[0, 1] \to C[a, b]$ that is an isometry and bijective.

Take $f \in C[0, 1]$ then define T as

$$(Tf)(t) = f((b - a)t + a) \tag{19}$$

First of all this function is bijective. This can be seen because the inverse is:

$$\begin{aligned}
(T^{-1}f)(t) &= f(\frac{t-a}{b-a}) && (20)\\
(T^{-1}(Tf))(t) &= T^{-1}f((b-a)t + a) && (21)\\
&= f((((b-a)t + a) - a)/(b - a)) = f(t) && (22)
\end{aligned}$$

So this function bijective. between the two spaces we now show that it is an isometry.

Take two continuous function on $[0,1]$

$$\tilde{d}(Tx,Ty) = \sup_{t\in[a,b]} |Tx(t) - Ty(t)| \tag{23}$$

$$= \sup_{t\in[a,b]} |x((b-a)t+a) - y((b-a)t+a)| \tag{24}$$

if we perform an s substitution this leaves us with:

$$= \sup_{t\in[0,1]} |x(t) - y(t)| = d(x,y) \tag{25}$$

Thus this is an isometry

Brief description. this is how it is proved

$$a^2 + b^2 = c^2 \tag{26}$$

□