20 Bang-bang and singular control problems

I cannot trust a man to control others who cannot control himself
—Robert E. Lee

As we noted in the previous chapter, quadratic cost functions are preferable to almost anything else, but often times reality dictates a different cost function that we can't avoid. Specifically, there are times when the corresponding Hamiltonian is linear in the control variable $\mathbf{u}$ which leads to a rather surprising result. Solutions of this type of control problem are usually referred to as bang-bang or singular control problems, and have one of the most interesting and complicated solutions you can imagine.

20.1 A straightforward bang-bang, and a straightforward singular control problem

Bang-bang control problems are those for which the optimal control is to switch between two different extremes of the control variable. Typically this happens when the control variable is constrained to live in a compact region, and the optimal choice of the control is to switch between the extreme allowed values of the control. In most cases this is not ideal, but there are times when such a control is not only natural but is the most ideal as well.

Example 20.1.1. Returning to the simplified glucose-insulin problem, recall that $x(t)$ is the glucose level in the blood stream and obeys the simplified evolution equation

$$x' = \alpha x - u,$$

where $\alpha > 0$, and insulin is injected at a rate $u(t)$. Physically, $0 \leq x(t)$ and $0 \leq u(t) \leq m$. We want $x(t) \to l$, so that $x(t_f) = l$ and $x(0) = a > 0$. There are two options to consider.

(i) Minimize t_f or the time to reach this steady state, in which case the cost functional could be written as:

$$J[u] = \int_0^{t_f} dt.$$

This problem is explored in more detail in the exercises at the end of the chapter.

(ii) Minimize the cost because insulin is expensive, i.e.

$$J[u] = \int_0^{t_f} u(t)dt.$$

We will consider this problem in more detail.

First, before proceeding, note that t_f is not chosen, i.e. we are optimizing not only over u, but over t_f as well. As discussed previously, this gives us one more very important additional condition on $H(t_f)$. To reiterate this point, we recall the argument presented previously in the Remark (below) immediately following this example.

Now, once we have reached $x(t_f) = l$, this state can be maintained by letting $u(t) = \alpha l$ which leads to $x' = 0$ (check this for yourself). This is why we are only concerned with what happens for $t \in [0, t_f]$. Assume $m \geq \alpha l$ (otherwise $x(t) < l$ forever and there wouldn't be any point to the exercise, and the desired glucose level would be unattainable). Returning to Pontraygin's maximum principle we see that the Hamiltonian is defined as

$$H = \alpha p x - p u - u = \alpha p x - u(p+1),$$

so that

$$p' = -\frac{\partial H}{\partial x} = -\alpha p,$$

implying that $p(t) = c_0 e^{-\alpha t}$. We note that because $x(0) = a$ and $x(t_f) = l$ are fixed that we have no endpoint conditions on the co-state, and hence can't identify the constant c_0 at this point.

To maximize the Hamiltonian in this case we notice that it is linear in the control variable u, which is problematic if we attempt to maximize it using the derivative as we are accustomed to doing. Instead we note that if $p+1 > 0$ then the optimal control would be to set $u(t) = 0$ (the minimal value), and if $p+1 < 0$ then we would set $u(t) = m$ (the maximal value). Hence our optimal control solution looks like

$$\tilde{u}(t) = \begin{cases} m: & \text{if } c_0 e^{-\alpha t} < -1 \\ 0: & \text{if } c_0 e^{-\alpha t} > -1 \end{cases}$$

A switch in the control occurs when/if $c_0 e^{-\alpha t} = -1$ i.e. when the insulin pump switches from one extreme to another.

Before proceeding we consider the consequences of there being a switch to this problem.

- For instance, if initially $a < l$ then there would be no need to inject insulin and thus $\tilde{u} = 0$ would be the optimal choice until the point that $x(t)$ has reached the desired level, and then the insulin injection would be maintained at $\tilde{u} = \alpha l$.
- On the other hand, if $a > l$ then the initial insulin injection should be positive, and we have already seen that the optimal rate is $\tilde{u} = m$, the maximal injection rate. Once the final state is reached then the insulin injection is switched to the maintenance stage with $\tilde{u} = \alpha l$.

In either case it does not make sense physically to have a switching time (point in time when the control switches from one extreme to another) before $t = t_f$, and we know how to maintain the optimal glucose level after $t = t_f$.

Hence, we already know what our solution is:

- If $x(0) = a < l$ then $\tilde{u} = 0$ until $x(t) = l$ at which time we switch to $\tilde{u} = \alpha l$.
- If $x(0) = a > l$ then $\tilde{u} = m$ until $x(t) = l$ at which time we switch to $\tilde{u} = \alpha l$.

The issue that we must consider is how long we have to wait until we have reached the steady state in each case. There is more than one way to identify this and we will only consider one such route here.

- If $x(0) = a < l$ so that $\tilde{u} = 0$ is the optimal control then the state equation in this case has solution $x(t) = ae^{\alpha t}$, and using $x(t_f) = l$ leads to $t_f = \frac{1}{\alpha}\log\left(\frac{l}{a}\right)$. Although we have already convinced ourselves that there isn't a switch here, we do want to make sure the mathematics tells us the same thing, so we utilize the additional constraint that because we are optimizing over t_f as well then $H(t_f) = 0$ (see the remark at the end of this example) which leads to
$$c_0 \alpha l e^{-\alpha t_f} = 0 \Rightarrow c_0 = 0,$$
meaning that the co-state $p(t) = 0$ for all values of t, and hence no switch is possible.
- If $x(0) = a > l$ so that $\tilde{u} = m$ then the state equation has solution
$$x(t) = c_1 e^{\alpha t} - \frac{m}{\alpha}.$$
Inserting the initial condition this implies that $c_1 - \frac{m}{\alpha} = a$ so that the full solution looks like
$$x(t) = ae^{\alpha t} + \frac{m}{\alpha}\left(e^{\alpha t} - 1\right).$$
The final condition that $x(t_f) = l$ leads to:
$$\begin{aligned} l &= \left(a + \frac{m}{\alpha}\right) e^{\alpha t_f} - \frac{m}{\alpha} \\ \Rightarrow \frac{\alpha a + m}{\alpha} e^{\alpha t_f} &= \frac{\alpha l + m}{\alpha} \\ \Rightarrow t_f &= \frac{1}{\alpha}\log\left(\frac{\alpha l + m}{\alpha a + m}\right). \end{aligned}$$
To ensure that no switch occurs in this case either, we again return to the endpoint condition on the Hamiltonian (using the fact that $x(t_f) = l$ in our favor):
$$\begin{aligned} &\alpha l c_0 e^{-\alpha t_f} - m(c_0 e^{-\alpha t_f} + 1) = 0 \\ &\Rightarrow c_0 e^{-\alpha t_f}(\alpha l - m) = m, \\ &\Rightarrow c_0 = \frac{m e^{\alpha t_f}}{\alpha l - m}. \end{aligned}$$

After a substantial amount of effort we come to the conclusion that for a switch to occur where $p(t^*) = -1$ then

$$t^* = \frac{1}{\alpha}\log\left(\frac{m(\alpha l + m)}{(m-\alpha l)(\alpha a + m)}\right) > t_f.$$

However this would mean that the switch would take place after the final time t_f which isn't reasonable, i.e. no switch takes place after all.

Remark 20.1.2. Consider again the problem of minimizing a cost functional $J[t_f, \mathbf{u}]$ over both the control and the final time t_f, subject to the state equation constraint $\mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \mathbf{u})$. For now we do not specify the endpoint conditions of $\mathbf{x}(t)$ although those will dictate the behavior of the co-state $\mathbf{p}(t)$. As we have already seen, optimization over the control $\mathbf{u}$ yields the evolution of the co-state $\mathbf{p}' = -\frac{DH}{D\mathbf{x}}$ where the Hamiltonian is defined as $H = \mathbf{p}\cdot\mathbf{f} - L$. In addition the Hamiltonian $H(\mathbf{u})$ is maximized as a function of $\mathbf{u}$ as well.

Optimizing over t_f leads to the following

$$\begin{aligned}
\frac{d}{dt_f}J^*[t_f, \mathbf{u}] &= 0,\\
\Rightarrow \frac{d}{dt_f}\int_0^{t_f}\left[\mathbf{p}\cdot\mathbf{x}' - H\right]dt &= 0,\\
\Rightarrow \mathbf{p}(t_f)\cdot\mathbf{x}'(t_f) - H(t_f) &= 0.
\end{aligned}$$

This gives an additional condition on the Hamiltonian, i.e. $H(t_f) = \mathbf{p}(t_f)\cdot\mathbf{x}'(t_f)$ when optimizing over the final time t_f as well. In the case as considered in the Example above where $\mathbf{x}(t_f)$ is fixed then $\mathbf{p}(t_f)$ is free, however in that specific example we are assuming that at time $t = t_f$ the solution has reached equilibrium so that $\mathbf{x}'(t_f) = 0$, and hence in this setting $H(t_f) = 0$.

Remark 20.1.3. Now, having gone through a few examples (from the past Chapter as well as the example above), we reiterate how to formulate an optimal control problem, and then apply Pontryagin's maximum principle if the setting can not be manipulated to LQR. For a given control problem:

(i) Compute / derive the Hamiltonian $H = H(t; \mathbf{x}, \mathbf{p}, \mathbf{u}) = \mathbf{p}\cdot\mathbf{f} - L$, where the state evolution is given by $\mathbf{x}' = \mathbf{f}$ and the Lagrangian cost functional is L.

(ii) Maximize H as a function of $\mathbf{u}(t)$. This gives the optimal control $\tilde{\mathbf{u}}(t)$ as a function of the co-state $\tilde{\mathbf{p}}(t)$ and state $\tilde{\mathbf{x}}(t)$. For a linear Hamiltonian the optimal control value will depend on the bounds on the admissible controls. Typically this will yield a bang-bang solution, but may produce a singular control, shown in the following example.

(iii) Return to the dynamical system for $\mathbf{x}(t)$ and the adjoint equation for $\mathbf{p}(t)$ to find out the specific form of $\tilde{\mathbf{x}}(t)$ and $\tilde{\mathbf{p}}(t)$. You will have

$$\mathbf{x}' = \mathbf{f}(t; x, u) = \frac{DH}{D\mathbf{p}}, \quad \mathbf{p}' = -\frac{DH}{D\mathbf{x}}.$$

(iv) Use the initial and endpoint conditions on the state and co-state (there should be two conditions, one at $t = 0$ and one at $t = t_f$) as well as any additional constraints given by the Hamiltonian being constant (if the system is completely autonomous) or conditions on $H(t_f)$ (if the final time t_f is being optimized over as well) . For a linear Hamiltonian this will lead to identifying the optimal switching time for the bang-bang solution if there is one.

Now we will turn to a problem that certainly isn't bang-bang, but for which the Hamiltonian is still linear with respect to the control u. Unfortunately we don't have a good motivation for studying this problem, so it is just one of those that is mathematically of interest if not necessarily of great physical interest.

Example 20.1.4. Consider minimization of the cost functional

$$J[u] = \int_0^2 \left(x - t^2\right)^2 dt,$$

subject to the state space evolution

$$x' = u, \quad x(0) = 1,$$

where the control is bounded according to $0 \leq u(t) \leq 4$. The Hamiltonian for this problem is

$$H = pu - \left(x - t^2\right)^2,$$

indicating that the optimal control will depend on the sign of the co-state $p(t)$ as this is what determines the sign of the linear term involving u. To see how this works we consider the co-state evolution as well:

$$p' = 2(x - t^2), \quad p(2) = 0.$$

Optimizing $H(u)$ will be a problem if $p(t) = 0$ on some finite interval $t \in [a, b]$ because over that interval changes in the control $u(t)$ will not affect the maximization of the Hamiltonian and hence the optimal solution will be independent of the control $u(t)$. If this is indeed the case, then on this interval $p'(t) = 0$ as well indicating that

$$0 = p'(t) = 2(x - t^2).$$

This implies that $x(t) = t^2$ on such an interval. Using this, we return to the state evolution equation to see that the optimal control on such an interval would be $u(t) = x'(t) = 2t$. Thus we can state the optimal control in terms of the co-state as

$$\tilde{u}(t) = \begin{cases} 0 & \text{when } p > 0, \\ 2t & \text{when } p = 0, \\ 4 & \text{when } p < 0. \end{cases}$$

This problem can be investigated in more detail, but it suffices to say that one cannot simply ignore the possibility that the linear Hamiltonian will yield a singular control, i.e. the bang-bang solution is not always optimal. The final solution to this particular problem would be a solution of the coupled state and co-state evolution equations:

$$\begin{aligned} x' &= u, \quad x(0) = 1, \\ p' &= 2(x - t^2), \quad p(2) = 0, \end{aligned}$$

with the control u is defined piecewise in terms of $p(t)$ as above.

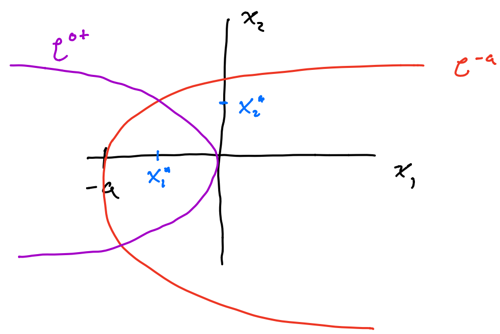

Figure 20.1: Magnificently sketched image of the position $(x_1(t))$ and velocity $(x_2(t))$ space for the bang-bang solution to the car driving problem.

20.2 Driving a car in a straight line optimally guarantees a ticket

We now turn to an example that we have been alluding to for some time, which is a classic, if not very complicated example of a bang-bang solution. We return to the introductory example of driving a car in a straight line, but trying to optimize the time it takes to get from one point to another. This is an example of a host of control problems often referred to as time-optimal control to the origin. In reality, if you instead thought of this problem as a subway train on a one-dimensional track (which is actually realistic) then the problem may make some physical sense, even though our final solution may be a bit ludicrous.

Remark 20.2.1. Another option is to consider the following example as being lifted from a society in which the primary objective is always to optimize time even at the expense of comfort and other variables. Maybe this would make a good episode of Dr. Who...

Example 20.2.2 (Driving a Car in a Straight Line). We will set the initial time as $t_0 = 0$, the initial velocity as $x_2(0) = 0$, initial position as $x_1(0) = -a$, final velocity as $x_2(t_f) = 0$ and the final position as $x_1(t_f) = 0$. As we established earlier,

$$x_1'(t) = x_2(t),$$
$$x_2'(t) = u(t),$$

where $u(t)$ is the control satisfying $U_l \leq u(t) \leq U_u$. We want to minimize the total time

$$J[u] = \int_0^{t_f} dt.$$

This means the Lagrangian is simply $L = 1$. Neglecting the constraint on fuel consumption for the time being, the co-state is

$$\mathbf{p}(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}.$$

It follows that for this problem the Hamiltonian is defined as $H = p_1x_2 + p_2u - 1$ with $x_1' = x_2$, $x_1(0) = -a$ and $x_2' = u$, $x_2(0) = 0$. We also find that

$$p_1' = -\frac{\partial H}{\partial x_1} = 0,$$
$$p_2' = -\frac{\partial H}{\partial x_2} = -p_1.$$

Then $p_1(t) = c_1$, a constant, implying that $p_2(t) = c_2 - c_1t$.

The maximum principle tells us that H should be maximized as a function of u. As noted in the previous Section, this is difficult because the Hamiltonian is linear in u, i.e. the maximum will depend on the sign of $p_2(t)$ and will be satisfied by the bounds on $u(t)$, i.e. there is a discontinuos switch so that

$$\tilde{u} = \begin{cases} U_l & \text{if } p_2(t) < 0 \\ U_u & \text{if } p_2(t) > 0 \end{cases}.$$

Since $p_2(t)$ is linear, its sign can change only once, at some time t^*, where $p_2(t^*) = 0$. We need to identify what the value of this switching time t^* is so we know when to stop stomping on the gas pedal, and slam on the brake instead (this is after all the optimal solution...just keep this in mind if you have passengers that don't like switching signs of acceleration on short notice).

There are a host of different approaches to take in determining the switching time (and eventually the final time t_f) for this problem. We have chosen to take a geometric approach that can generalize easily to other time-optimal control to the origin problems.

Let $\tilde{u}(t)$ be the optimal control we just identified (up to the sign of $p_2(t)$), then

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = \tilde{u}$$

implying that

$$\frac{dx_2}{dx_1} = \frac{\tilde{u}}{x_2},$$

Thus $x_2dx_2 = \tilde{u}dx_1$ so that $\frac{1}{2}x_2^2 = \tilde{u}x_1 + k$ (for some constant k of integration which is given by the initial/endpoint conditions) which is a family of parabolas in the $x_1 - x_2$ plane facing either right or left depending on the value of $\tilde{u}$, and hence the sign of $p_2(t)$.

Solutions will follow these parabolas as time advances, i.e. unless there is a switching time, the solution will move along one of these parabolas in $x_1 - x_2$ space. At a switching time, the motion will switch to a parabola passing through the same point, but facing the other direction.

- To see how the solution will evolve along such a parabola, note that if $p_2(t) > 0$ then $x_2' = U_u > 0$ so $x_2(t)$ will be increasing i.e. this will correspond to parabolas that are opening to the right such as $\mathscr{C}^{-a}$ in Figure 20.1.
- On the other hand if $p_2(t) < 0$ then $x_2' = U_l < 0$ so that $x_2(t)$ will be decreasing such as along the purple curve $\mathscr{C}^{0+}$ in Figure 20.1.

We note that as time increases, the nature of this problem is such that $x_1(t)$ is always increasing because the car is always moving toward the origin, even though the velocity is allowed to change.

At time $t = 0$ we are starting in the 2nd-quadrant i.e. at the point $x_1 = -a$ and $x_2 = 0$, and want to reach the origin. This is only possible if we end up on the curve $\mathscr{C}^{o+}$ as illustrated. We must start on the curve $\mathscr{C}^{-a}$ because this is the only curve that intersects our initial point, and gives an allowable trajectory. Thus, we will travel along the curve $\mathscr{C}^{-a}$ with $x_2(t)$ increasing until reaching the intersection with the curve $\mathscr{C}^0$ whereupon the switch will occur (the gas will be released and the brake depressed) and the solution will travel along $\mathscr{C}^0$ to the origin. This is illustrated in Figure 20.1 as the intersection (in the upper half plane) of the red and purple curves.

Now we need to determine the cost t_f and the time of the switch t^*. With this in mind, let, $x_1^* = x_1(t^*)$ and $x_2^* = x_2(t^*)$ as illustrated in Figure 20.1. For $0 \le t < t^*$, recall that

$$x_2' = U_u, \quad x_2(0) = 0,$$

implying that $x_2(t) = U_u t$ up to the switching time t^*. At $t = t^*$, $x_2(t^*) = U_u t^*$ which is an initial condition for the interval $t^* < t < t_f$, wherein $x_2' = U_l$. We also have the condition that $x_2(t_f) = 0$. These two conditions are permissible for such a first order ODE because neither t^* nor t_f are known at this point. This implies that

$$x_2(t) = U_l t + \tilde{c},$$

and thus

$$x_2(t^*) = U_l t^* + \tilde{c} = U_u t^*$$

so that $\tilde{c} = t^*(U_u - U_l)$. Then for $t \ge t^*$,

$$x_2(t) = U_l t + t^*(U_u - U_l).$$

At $t = t_f$, the condition $x_2(t_f) = 0$ then yields

$$U_l t_f + t^*(U_u - U_l) = 0$$

implying that

$$t_f = \frac{t^*(U_l - U_u)}{U_l} = t^*\left(1 - \frac{U_u}{U_l}\right).$$

Thus if we know the switching time t^* then we have found the optimal solution exactly, and know the time it takes to travel to the origin from the point $x_1 = -a$ and $x_2 = 0$. The problem is that t^* is still unknown.

To find t^*, we will explicitly derive the quadratic curves $\mathscr{C}^{-a}$ and $\mathscr{C}^{0+}$ and determine where they intersect, giving the point (x_1^*, x_2^*) which can be used to determine t^*. We know that

$$x_2^2 = 2\tilde{u}x_1 + \text{constant}$$

Then for $\mathscr{C}^{-a}$,

$$x_2^2 = 2U_u x_1 + \text{constant},$$

which must intersect the point $(-a, 0)$. Thus $0 = -2U_u a + \text{constant}$, implying that

$$x_2^2 = 2U_u x_1 + 2U_u a.$$

For $\mathscr{C}^{0+}$, the curve is explicitly given by

$$x_2^2 = 2U_l x_1 + \text{const},$$

which intersects the origin so that the constant is zero, i.e. $\mathscr{C}^{0+}$ is defined by

$$x_2^2 = 2U_l x_1.$$

These curves intersect at (x_1^*, x_2^*) which means

$$2U_l x_1^* = 2U_u x_1^* + 2U_u a,$$

so that

$$x_1^* = \frac{U_u a}{U_l - U_u},$$

and thus

$$x_2^* = \sqrt{\frac{2U_l U_u a}{U_l - U_u}}$$

and finally

$$t^* = \frac{x_2^*}{U_u} = \sqrt{\frac{2U_l a}{U_u(U_l - U_u)}},$$

which is the optimal switching time. The final time is given by

$$t_f = t^* \left(1 - \frac{U_u}{U_l}\right),$$

as $U_l < 0$.

An earlier switch, and the car would stop before the destination, and a later switch and the car would overrun the destination. Thus this switching time is the key quantity from this solution, from which one can calculate the cost or total amount of time to reach the destination.

Remark 20.2.3. The car driving example is a linear, time-optimal control to the origin problem. Unfortunately as you can see by working through this example, linearity in this case is not necessarily simplifying nor desirable. Although bang-bang problems are typically dictated by the Hamiltonian being linear in the control variable $\mathbf{u}(t)$ this inherently leads to a difficult problem to analyze primarily because the switching time is very difficult to find.

On the other hand, singular control problems as described in the previous Section are just as difficult to work with, and worse in some ways. The moral of the story is that linear Hamiltonian structures are not very pleasant to work with.

Exercises

Note to the student: Each section of this chapter has several corresponding exercises, all collected here at the end of the chapter. The exercises between the first and second line are for Section 1, the exercises between the second and third lines are for Section 2, and so forth.

You should **work every exercise** (your instructor may choose to let you skip some of the advanced exercises marked with *). We have carefully selected them, and each is important for your ability to understand subsequent material. Many of the examples and results proved in the exercises are used again later in the text. Exercises marked with ⚠ are especially important and are likely to be used later in this book and beyond. Those marked with † are harder than average, but should still be done.

Although they are gathered together at the end of the chapter, we strongly recommend you do the exercises for each section as soon as you have completed the section, rather than saving them until you have finished the entire chapter.

20.1. Consider the problem of minimizing

$$\int_0^2 e^t \,(u(t) - 1)\, dt,$$

subject to $x' = ux$ with $x(0) = 1,\ x(2) = 2$ and $0 \le u \le 1$. Determine the Hamiltonian, co-state evolution and endpoint conditions.

20.2. Prove that the problem considered in the previous question is not singular, i.e. the coefficient on the control variable $u(t)$ in the Hamiltonian can not vanish over a finite interval in t.

20.3. Find the optimal control for the problem considered in the previous two questions.

20.4. Recall the original glucose blood level problem in the text above, that is

$$x' = \alpha x - u(t),$$

for $\alpha > 0$, again with the constraints that $0 \le u(t) \le m$ and $x(t) \ge 0$ and boundary conditions $x(0) = a,\ x(t_f) = l$. Set up and solve the optimal control problem to minimize the time t_f to reach the desired steady state.

20.5. A water reservoir of uniform rectilinear shape, is set to discharge outtake at a rate so that the height of the reservoir $x(t)$ is known to be described by

$$x' = -0.1x + u, x(0) = 10,$$

where $u(t)$ denotes the amount of inflow that can be specified by releasing water from a dam at higher elevation (neglecting the loss of water due to evaporation etc.). For this situation, suppose that $0 \le u \le 3$. If we desire to maximize the pressure (which is proportional to the height) of the outflow, we wish to maximize the cost functional

$$J[u] = \int_0^{t_f} x(t)dt$$

where t_f is some final time. Set up and describe the optimal control problem here, including the region of admissible controls. Compute the Hamiltonian.

20.6. Using the maximum principle, find the optimal control for the previous problem.

20.7. What is the maximum $x(t)$ reached for the pressure maximization problem described above. What happens if the final time $t_f \to \infty$?

20.8. Determine the optimal control of driving the car in a straight line (just as the example in class), but this time suppose that the car is starting at an initial velocity $x_2(0) = v_0$ with initial position $x_1(0) = -a$. You do not need to find the optimal switching time and the exact state evolution yet.

20.9. Find the optimal switching time for the previous problem.

20.10. For the previous two problems there are conditions on a and v_0 so that the car can stop with velocity zero at the origin. What are those conditions?

20.11. For the car driving problem discussed in class ($v_0 = 0$ from the previous three questions) determine the exact costate vector using the initial and final conditions for the state vector $\mathbf{x}(t) = (x_1(t), x_2(t))^T$ and the fact that $H(t; \mathbf{x}, \mathbf{u}, \mathbf{p}) = 0$ for all t (the system is autonomous and $\mathbf{x}'(t_f) = 0$ is a valid assumption here).

Notes