21 Various modifications to the 'standard' problem

Insanity: doing the same thing over and over again and expecting different results
—Albert Einstein

Up to this point we have defined optimal control problems and then made a formal derivation of Pontryagin's maximum principle. Then we demonstrated particular solutions for two very different setups, LQR and bang-bang or singular control problems. Although we would prefer to remain in the land of quadratic cost functionals and linear evolution equations for the rest of our time, reality is a very different universe, and so we return to the formal derivation here to extend our version of Pontryagin's famous result.

Although we showed previously that the different forms of the cost functional are equivalent, it is sometimes convenient to consider the Bolza or Mayer form for the problem setup. The earlier focus on the Lagrange form was purely because the form of Pontraygin's Maximum Principle is more straightforward in that case. This section is dedicated to extensions of these results. First we will consider the Bolza cost function when the final time is fixed, then when it is free, and finally we will consider the Mayer form of the cost functional.

21.1 Bolza cost functional with a fixed end time

We begin by motivating this setup with the modification of an example that we have seen before.

Example 21.1.1 (Chemotherapy). Let $x(t)$ be the number of cancerous tumor cells at time t (with an exponential growth rate) and let $u(t)$ be the chemotherapy drug concentration. The desired outcome is to minimize both the end state of $x(t)$ and the potential damage caused by $u(t)$. This leads to the optimization problem.

$$\min_u J[u] = x(t_f) + \int_0^{t_f} u(t)^2 dt,$$

subject to

$$x' = \alpha x - u, \quad x(0) = x_0 > 0,$$

which describes the exponential growth rate of the cancer and the negative effects of chemotherapy. For the current setting we are supposing that t_f is fixed, i.e. we need the chemotherapy to end at a definitive time, say 6 months from the beginning of treatment.

Remark 21.1.2. Before proceeding to the formal derivation, we comment that this model clearly has some issues. For example, most cancer growth is not truly exponential in the absence of treatment. The body has natural ways of fighting cancer that do not rely on chemotherapy intervention, and will inhibit the growth rate. The current model does not account for this. In addition, there may be several different drug types that are more useful depending on the stage of the cancer, and at the same time will have different costs to the patient. One can see that this optimal control problem can quickly become more complicated, and less and less tractable. The same principles and concepts apply even in these more complicated settings, and are illustrated in what follows.

To determine the optimal solution for this type of control problem we will consider a generic Bolza cost functional.

$$J[\mathbf{u}] = \int_0^{t_f} L(t; \mathbf{x}(t), \mathbf{u}(t))dt + \phi(\mathbf{x}(0), \mathbf{x}(t_f)). \tag{21.1}$$

As before, we consider necessary conditions that arise from setting the first Gateaux differential to zero for all admissible variations of the control $\mathbf{u}$. To incorporate the dynamics of the system: $\mathbf{x}' = f(t; \mathbf{x}, \mathbf{u})$, we need to use the costate vector (Lagrange Multiplier) just as was done previously, i.e. we consider the modified cost functional (recalling that $H = \mathbf{p} \cdot \mathbf{f} - L$):

$$J^*[\mathbf{u}] = \int_0^{t_f} [\mathbf{p} \cdot \mathbf{x}' - H(t; \mathbf{x}, \mathbf{u}, \mathbf{p})]dt + \phi(\mathbf{x}(0), \mathbf{x}(t_f)).$$

Just as we did previously, we note that $\mathbf{u} \to \mathbf{u} + \varepsilon\boldsymbol{\eta}$ maps $\mathbf{x} \to \mathbf{x} + \varepsilon\mathbf{h} + O(\varepsilon)$. Then

$$\begin{aligned}
\delta J^*[\mathbf{u}; \boldsymbol{\eta}] &= \frac{d}{d\varepsilon}\left\{\int_0^{t_f} \left[\mathbf{p} \cdot \left(\mathbf{x}' + \varepsilon\mathbf{h}' + O\left(\varepsilon^2\right)\right) - H\left(t; \mathbf{x} + \varepsilon\mathbf{h} + O\left(\varepsilon^2\right), \mathbf{u} + \varepsilon\boldsymbol{\eta}, \mathbf{p}\right)\right] dt\right. \\
&\quad \left. + \, \phi\left(\mathbf{x}(0) + \varepsilon\mathbf{h}(0) + O\left(\varepsilon^2\right), \mathbf{x}(t_f) + \varepsilon\mathbf{h}(t_f) + O\left(\varepsilon^2\right)\right)\right\}_{\varepsilon=0} \\
&= \int_0^{t_f} \left[\mathbf{p} \cdot \mathbf{h}' - \frac{DH}{D\mathbf{x}} \cdot \mathbf{h} - \frac{DH}{D\mathbf{u}} \cdot \boldsymbol{\eta}\right] dt + \frac{D\phi}{D\mathbf{x}(0)} \cdot \mathbf{h}(0) + \frac{D\phi}{D\mathbf{x}(t_f)} \cdot \mathbf{h}(t_f) \\
&= -\int_0^{t_f} \left\{\left[\mathbf{p}' + \frac{DH}{D\mathbf{x}}\right] \cdot \mathbf{h} + \frac{DH}{D\mathbf{u}} \cdot \eta\right\} dt \\
&\quad + \left[-\mathbf{p}(0) + \frac{D\phi}{D\mathbf{x}(0)}\right] \cdot \mathbf{h}(0) + \left[\mathbf{p}(t_f) + \frac{D\phi}{D\mathbf{x}(t_f)}\right] \cdot \mathbf{h}(t_f) = 0.
\end{aligned}$$

For this to be true for any admissible $\boldsymbol{\eta}$ (and hence $\mathbf{h}(t)$) then we choose $\mathbf{p}(t)$ so that

$$\mathbf{p}' = -\frac{DH}{D\mathbf{x}}.$$

If both of the endpoints for the state variable $\mathbf{x}(t)$ are free, then we further select the co-state $\mathbf{p}(t)$ to satisfy the endpoint conditions

$$\mathbf{p}(0) = \frac{D\phi}{D\mathbf{x}(0)}, \qquad \mathbf{p}(t_f) = -\frac{D\phi}{D\mathbf{x}(t_f)}. \tag{21.2}$$

If $\mathbf{x}(0) = \mathbf{x}_0$ is specified as part of the problem then $\mathbf{h}(0) = 0$ and $\mathbf{p}(0)$ is left unspecified. This holds similarly for $\mathbf{x}(t_f)$ being specified. With this choice of $\mathbf{p}(t)$ the first Gateaux differential becomes

$$\delta J^*[\mathbf{u}; \boldsymbol{\eta}] = -\int_0^{t_f} \frac{DH}{D\mathbf{u}} \cdot \boldsymbol{\eta} dt = 0,$$

which leads to the same maximization condition on the Hamiltonian for the optimal control that we saw earlier. Further consideration of the second Gateaux differential for this setting further confirms the same principle that we saw for the purely Lagrange form of $J[\mathbf{u}]$.

In summary, for an optimization problem of the form (21.1) where the endpoints of $\mathbf{x}(t)$ are both free at $t = 0$ and $t = t_f$, but t_f is fixed itself, the optimal solution will maximize the Hamiltonian as a function of $\mathbf{u}$, i.e. $H = H(\mathbf{u})$ will be maximized. In addition the state and co-state will evolve according to:

$$\mathbf{x}' = \frac{DH}{D\mathbf{p}}, \tag{21.3}$$

$$\mathbf{p}' = -\frac{DH}{D\mathbf{x}}, \qquad \mathbf{p}(0) = \frac{D\phi}{D\mathbf{x}(0)}, \qquad \mathbf{p}(t_f) = -\frac{D\phi}{D\mathbf{x}(t_f)}. \tag{21.4}$$

If $\mathbf{x}(0) = \mathbf{x}_0$ is fixed (similarly for $\mathbf{x}(t_f)$) then $\mathbf{p}(0)$ is free and only the second condition on $\mathbf{p}(t_f)$ above will apply.

Remark 21.1.3. We comment at this point that if $\mathbf{x}(0)$ (the final endpoint can be considered equivalently) is fixed in the problem then it doesn't make any sense for the cost function to depend on $\mathbf{x}(0)$, i.e. we would expect that $\phi = \phi(\mathbf{x}(t_f))$ but for no dependence on $\mathbf{x}(0)$ which is a fixed constant. Thus the endpoint conditions on the co-state derived above will only apply when the corresponding endpoint is not fixed, which is consistent with our earlier observations balancing the necessary number of endpoints.

Remark 21.1.4. Just because I like the formatting for remarks, we'll put in another one here. As mentioned in the previous remark, we can observe that in every example of an optimal control problem that we have considered (and indeed for those we haven't yet considered) we necessarily need to have the 'correct' number of endpoint conditions on the state and co-state. The correct number it turns out are exactly one at the initial time and one at the end time for each component of $\mathbf{x}(t)$. This means that if $\mathbf{x}(t) \in \mathbb{R}^n$ then we will need n initial conditions and n final conditions, one each for each component of $\mathbf{x}(t)$, although we can equally assign these conditions to either $\mathbf{x}(t)$ or the co-state $\mathbf{p}(t)$ depending on the nature of the problem.

Now we are finally prepared to resolve the type of treatment that our cancer patient requires.

Example 21.1.5 (Return to the chemotherapy problem). Recall that we want to minimize

$$J[u] = mx(t_f) + \int_0^{t_f} u(t)^2 dt$$

subject to

$$x' = \alpha x - u, \quad x(0) = x_0 > 0.$$

The Hamiltonian is

$$H = \alpha p x - p u - u^2,$$

with the adjoint (co-state evolution) equation

$$p' = -\frac{\partial H}{\partial x} = -\alpha p$$

with $p(t_f) = -m$. The optimality condition gives

$$\frac{\partial H}{\partial u} = -p - 2u = 0,$$

and thus $\tilde{u}(t) = -\frac{p}{2}$. Solving the co-state equation, we see that

$$\begin{aligned}\tilde{p}(t) &= -me^{\alpha(t_f - t)}\\ \Rightarrow \tilde{u}(t) &= \frac{m}{2}e^{\alpha(t_f-t)}.\end{aligned}$$

Thus

$$x' = \alpha x - \tilde{u} = \alpha x - \frac{m}{2}e^{\alpha(t_f-t)}, \quad x(0) = x_0.$$

To solve this equation we consider the change of variables $y = e^{-\alpha t}x$ so that $y' = -\alpha e^{-\alpha t}x + e^{-\alpha t}x'$. This further tells us that

$$\begin{aligned}y' &= -\frac{m}{2}e^{\alpha(t_f-2t)}\\ y(t) &= \frac{m}{4\alpha}e^{\alpha(t_f-2t)} + c_0\\ x(t) &= c_0e^{\alpha t} + \frac{me^{\alpha(t_f-t)}}{4\alpha}.\end{aligned}$$

The initial condition further leads to

$$\begin{aligned}x(0) &= x_0\\ &\Rightarrow x_0 = c_0 + \frac{me^{\alpha t_f}}{4\alpha}\\ &\Rightarrow c_0 = x_0 - \frac{me^{\alpha t_f}}{4\alpha},\end{aligned}$$

so that the final solution is given by

$$\tilde{x}(t) = x_0e^{\alpha t} + me^{\alpha t_f}\frac{e^{-\alpha t} - e^{\alpha t}}{4\alpha}$$

so that $\tilde{x}(t_f) = x_0e^{\alpha t_f} + m\frac{1-e^{2\alpha t_f}}{4\alpha}$ is the final concentration of the cancer cells.

Remark 21.1.6. We may notice that the problem defined in the previous example is actually an LQR problem, and hence we may have worked out the solution using the most optimal of optimal control methods, but we slogged through the current setup just to make a point, i.e. how do we treat the endpoint costs for a fixed endpoint time? For instance the same approach employed here would work if we used a more realistic model of the cancer cell's growth with a fully nonlinear model.

21.2 Other considerations with endpoint costs

21.2.1 Bolza cost functional while optimizing over the final time

One may legitimately remark that the key example of the previous section was clearly flawed. If at all possible, the physicians (and patients for that matter) would be delighted to extend the treatment by some short time period if it produced significantly better results both in terms of the cancer treatment and the potential damage of the drug. In other words, although there may be mitigating circumstances for which the final time is set, this is likely not the norm in such a situation. As such, we need to consider the impact of varying the final end time for the problem.

Consider the general problem of minimizing over the control $\mathbf{u}(t)$ and the final time t_f (similar conditions can be derived for the variations in the starting time t_0 if such was the case).

$$\min_{t_f, \mathbf{u}(t)} \left\{ \phi(t_f, \mathbf{x}(t_f)) + \int_0^{t_f} L(t, \mathbf{x}(t), \mathbf{u}(t)) dt \right\},$$

subject to

$$\mathbf{x}' = \mathbf{f}(t; \mathbf{x}(t), \mathbf{u}(t)), \quad x(0) = x_0.$$

Then setting $\delta J^*[\mathbf{u}; \boldsymbol{\eta}] = 0$ gives the exact same necessary conditions as shown above when we consider variations in $\mathbf{u}$ (and hence $\mathbf{x}$) only. To see how minimizing over t_f affects things, we consider variations in t_f, i.e. $t_f \to t_f + \varepsilon$ just as we did previously. Now, we formalize this a bit more, by clarifying that we need to continue $\mathbf{x}(t)$, $\mathbf{p}(t)$ and $\mathbf{u}(t)$ past time t_f, i.e.

$$\tilde{\mathbf{u}}(t) \to \begin{cases} \tilde{\mathbf{u}}(t) & : t < t_f \\ \tilde{\mathbf{u}}(t_f) & : t \geq t_f \end{cases},$$

and the same for $\mathbf{x}(t)$ and $\mathbf{p}(t)$. This gives us a way of defining the relevant variables even if the original interval over which they were defined is extended. Let $\tilde{t_f}$ be the optimal end time, then

$$\begin{aligned} 0 &= \frac{\partial}{\partial \varepsilon} \left\{ \int_0^{\tilde{t}_f + \varepsilon} \left[\mathbf{p} \cdot \mathbf{x}' - H \right] dt + \phi(\tilde{t}_f + \varepsilon, \mathbf{x}(\tilde{t}_f + \varepsilon)) \right\}_{\varepsilon = 0} \\ &= \mathbf{p}(\tilde{t}_f) \cdot \mathbf{x}'(\tilde{t}_f) - H(\tilde{t}_f) + \frac{\partial \phi}{\partial t}(\tilde{t}_f) + \frac{D\phi}{D\mathbf{x}} \mathbf{x}'(t_f) \\ &= -H(\tilde{t}_f) + \frac{\partial \phi}{\partial t}(\tilde{t}_f) \end{aligned}$$

where we once again used Leibniz' rule and the fact that $\mathbf{p}(\tilde{t}_f) = -\frac{D\phi}{D\mathbf{x}}(\tilde{t}_f)$. So this gives the additional condition that

$$H(\tilde{t}_f; \tilde{\mathbf{x}}(\tilde{t}_f), \tilde{\mathbf{u}}(\tilde{t}_f), \tilde{\mathbf{p}}(\tilde{t}_f)) = \frac{\partial \phi}{\partial t}(\tilde{t}_f)$$

Example 21.2.1 (Chemotherapy treatment yet again). Going back to the cancer problem where we want to

$$\min_{t_f, u(t)} \left\{ m x(t_f) + \int_0^{t_f} u(t)^2 dt \right\},$$

subject to

$$x' = \alpha x - u, \quad x(0) = x_0 > 0,$$

where $\alpha > 0$. Just as before $H = \alpha px - pu - u^2$ with the adjoint equation

$$p' = -\frac{\partial H}{\partial x} = -\alpha p,$$

and $p(t_f) = -m$. Thus $\tilde{p}(t) = -me^{\alpha(t_f - t)}$ so

$$\tilde{u}(t) = \frac{m}{2}e^{\alpha(t_f-t)},$$

and thus

$$\tilde{x}(t) = x_0 e^{\alpha t} + me^{\alpha t_f}\frac{e^{-\alpha t} - e^{\alpha t}}{4\alpha}.$$

Now we can find the optimal $\tilde{t_f}$, using the endpoint condition we just derived:

$$H(\tilde{t}_f) = \frac{\partial \phi}{\partial t}(\tilde{t}_f).$$

For this problem, this condition becomes

$$\alpha\tilde{p}(\tilde{t}_f)\tilde{x}(\tilde{t}_f) - \tilde{p}(\tilde{t}_f)\tilde{u}(\tilde{t}_f) - (\tilde{u}(\tilde{t}_f))^2 = 0.$$

This implies that $\tilde{t}_f = \frac{1}{\alpha}\log\left(\frac{4\alpha x_0}{m}\right)$. But this does not work if $4\alpha x_0 < m$ because then $\tilde{t}_f < 0$. This gives a condition that relates the cancer's growth with it's initial state, determining at which level it is cost effective to start chemotherapy, i.e. if the cancer cells are regenerating at a sufficiently slow rate then the chemotherapy will do more harm than good.

We can further compare with our earlier derivation by finding that in this case $x(t_f) = \frac{m}{4\alpha}$ which is independent of the initial state x_0, something that we would expect in this case.

There are a few things that may seem odd about the final solution we achieved in the example above. First off, it appears that the cancer concentration is proportional to m, the weight we are putting on reducing the final cancer population itself. This is counter intuitive, as we would expect a higher value of m to lower the final cancer population. the reason is a little subtle, but it is because the term $mx(t_f)$ doesn't explicitly depend on t_f, and hence minimizing with respect to t_f will focus on the integral costs which is purely minimizing the cost (amount of chemo) $u(t)^2$. We would need to include $x(t)$ somehow in the integral costs if we want the final cancer population to decrease when minimizing over t_f. This is an issue that is considered in more detail in the exercises.

21.2.2 Mayer form of the cost functional and absolute control

The car (train) driving example that we started our discussion of optimal control with was quickly converted into a problem in Lagrange form, even though it was in reality a Mayer cost functional. We will now return to an example where the Lagrange form of the cost functional is computable, but would have far less meaning.

Example 21.2.2 (Cash Balance). You have recently inherited the family bakery and have been tasked with managing some of the financial setup for the business. You know that there is a known demand for cash on hand, as some function of time (for instance ski season brings in more expenses with heating the building, but more customers also indicate more expenses in preparing the baked items). However, the more cash on hand restricts how much can be invested at a given time (it turns out that the family business includes some gambling enterprises in the back room). If the cash runs out, investments must be sold off, incurring a fee from the broker (and you lose some very important customers in the back room). The goal is to maintain a proper balance between investments and cash on hand.

To this end we will define the following quantities:

t_f : Final time (finite horizon)
$x(t)$: Money on hand
$y(t)$: Money in investments
$d(t)$: The demand for money ($d(t)$ can be positive or negative)
$u(t)$: Rate of selling (if positive) or buying (if negative) investments
$r_1(t)$: Interest rate for cash
$r_2(t)$: Interest rate for investiments. We have $r_2 \geq r_1$
α : Broker's fee as a percentage ($0 < \alpha < 1$)

The state evolution is

$$x' = r_1 x - d + u - \alpha|u|, \quad x(0) = x_0$$
$$y' = r_2 y - u, \quad y(0) = y_0$$

Where we have $-U_2 \leq u \leq U_1$ where $U_1, U_2 > 0$.

We want to maximize the final, total amount of money at time $t = t_f$ when you plan to split with all the proceeds and flee the country, hopefully staying ahead of both the IRS, and your former customers from the back room of the bakery. This leads to the optimization problem:

$$\max_{u(t)} J[u] = \max_{u(t)} \{x(t_f) + y(t_f)\} \equiv \min_{u(t)} \{-x(t_f) - y(t_f)\}.$$

Thus $\phi(x(t_f), y(t_f)) = -x(t_f) - y(t_f)$. Then because the Lagrangian is $L = 0$, the Hamiltonian is given by

$$H = r_1 x p_1 - d p_1 + u p_1 - \alpha|u| p_1 + r_2 y p_2 - u p_2.$$

Hence the co-state evolution is dictated by

$$p_1' = -\frac{\partial H}{\partial x} = -r_1 p_1,$$

with boundary condition $p_1(t_f) = -\frac{\partial \phi}{\partial x(t_f)} = 1$ and

$$p_2' = -\frac{\partial H}{\partial y} = -r_2 p_2,$$

with boundary condition $p_2(t_f) = -\frac{\partial\phi}{\partial y(t_f)} = 1$. From these two equations, we have

$$p_1(t) = e^{\int_t^{t_f} r_1(\tau)d\tau},$$

and

$$p_2(t) = e^{\int_t^{t_f} r_2(\tau)d\tau}.$$

In this particular example, the co-state has a reasonable physical (economical) interpretation. $p_1(t)$ can be seen as the future value of 1 dollar held in cash from t to t_f. $p_2(t)$ is the future value of 1 dollar held in investments from t to t_f

A primary reason for looking at this example is to determine how to work with the $|u|$ term. To handle this term, we let $u = u_1 - u_2$, where $u_1 \geq 0$ and $u_2 \geq 0$, with the additional constraints that $u_1u_2 = 0$. This forces $|u| = u_1$ if $u > 0$ and $|u_2| = u_2$ if $u < 0$. The constraint $u_1u_2 = 0$ should always hold for the optimal control because buying and selling investments simultaneously will not be an optimal solution. We can also rewrite $|u| = u_1 + u_2$.

Using this decomposition we see that

$$\begin{aligned} H &= r_1xp_1 - dp_1 + r_2yp_2 + (u_1 - u_2)(p_1 - p_2) - \alpha p_1(u_1 + u_2) \\ &= r_1xp_1 - dp_1 + r_2yp_2 + u_1[p_1 - p_2 - \alpha p_1] - u_2[p_1 - p_2 + \alpha p_1]. \end{aligned}$$

Maximizing H in u_1 and u_2 will lead to a bang-bang solution, where $0 \leq u_1 \leq U_1$, $0 \leq u_2 \leq U_2$. Thus the maximum of H will be achieved for $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$ where

$$\tilde{u}_1 = \begin{cases} 0 & : (1-\alpha)p_1 - p_2 < 0 \\ U_1 & : (1-\alpha)p_1 - p_2 > 0 \end{cases}$$

and

$$\tilde{u}_2 = \begin{cases} 0 & : (1+\alpha)p_1 - p_2 > 0 \\ U_2 & : (1+\alpha)p_1 - p_2 < 0 \end{cases}$$

Notice that if $\tilde{u}_1 = U_1$ then $(1-\alpha)p_1 > p_2$ implying that $(1+\alpha)p_1 > p_2$ so $\tilde{u}_2 = 0$ and if $\tilde{u}_2 = U_2$ then $(1-\alpha)p_1 < (1+\alpha)p_1 < p_2$ so $\tilde{u}_1 = 0$. Thus $u_1u_2 = 0$ for the optimal solution for all t, which is consistent with our description above.

Remark 21.2.3. The model considered above allows for $x(t)$ and $y(t)$ to be negative which does not make a lot of financial sense, particularly if you think about the type of customers you are catering to. To rectify this, we need to enforce the inequality constraints $x(t) \geq 0$ and $y(t) \geq 0$. This isn't an easy thing to do though, and requires quite a bit of massaging relevant inequality constraints.

Remark 21.2.4. By the way, have you ever considered why mathematicians 'model' so many things? Without reference to certain fashion-conscious bullies that said scientists may have endured in their formative years, we will go with the ansatz that mathematicians are *the* model citizen so that all of their modeling is just an extension of themselves.

Unfortunately we will likely not get to cover the pure state inequality constraints like that which appears when removing over-drafts or under-drafts in banking/financial problems. These are actually quite a bit more complicated in practice. Instead, we will consider what are referred to as mixed inequality constraints in which the control variable is constrained by some inequality. The reason this is easier is that if you recall we have formally derived the maximum principle by allowing for variations in the control variable $\mathbf{u}(t)$. Variations in the state variable $\mathbf{x}(t)$ are related, but not quite the same. In this way, inequality constraints that act on $\mathbf{u}(t)$ are decently easy to implement, but capturing inequality constraints on $\mathbf{x}(t)$ only are more difficult to translate back into variations on the control itself, a problem that we stealthily avoided earlier.

Just as in nonlinear programming, this results in something that is very similar to a Lagrange multiplier, but with some additional conditions that allow for the inequality constraint to be valid.

21.3 Numerical solutions of various control problems

We have already reviewed one approach to numerical solutions for LQR, and here present solutions of optimal control problems using SciPy's built in boundary value problem solver. The basic idea is the same, you need to use the maximization of the Hamiltonian H to determine how the control u is a function of the co-state p and state x, then insert this into the coupled evolution equations for $x(t)$ and $p(t)$. Generically this will be a boundary value problem over the interval $[0, t_f]$.

To set this up we return to the same example as in the previous section.

Example 21.3.1 (Chemotherapy v3). Once again allowing the evolution equation

$$x' = \alpha x - u, \quad x(0) = x_0 > 0,$$

we want to numerically determine the optimal control and evolution to minimize the cost functional

$$J[u] = mx(t_f) + \int_0^{t_f} u(t)^2 dt.$$

As already derived in the previous section, this defines the Hamiltonian $H = \alpha px - pu - u^2$ which is maximized when $u = -\frac{p}{2}$ so that the evolution of the state and co-state are now given by:

$$x' = \alpha x + \frac{1}{2}p, \quad x(0) = x_0,$$
$$p' = -\alpha p, \quad p(t_f) = -m.$$

Hence, the optimal evolution of this problem is given simply by solving the coupled boundary value problem described above. We can then compute the optimal control $u(t)$ afterward as $u(t) = -\frac{p(t)}{2}$. This is shown in Algorithm 21.1 where the solution is calculated for $\alpha = 1$, $x_0 = 2$, $m = 3$, and $t_f = 1$.

```python
import numpy as np
from matplotlib import pyplot as plt
from scipy.integrate import solve_bvp

#First define the right hand side of the differential equation.
#Note that we are avoiding parameters in the function itself.
def chemo3(t,y):
    alpha = 1
    return np.vstack((alpha*y[0] + 0.5*y[1], -alpha*y[1]))

#Now define a function that treats the boundary conditions.
def bc(ya,yb):
    x0 = 2
    m = 3
    return np.array([ya[0]-x0, yb[1]+m])

#Now compute the solution.  We may need more than 10 points...
t=np.linspace(0,1,10)
y = np.zeros((2, t.size)) #specify an 'initial guess' for the soln y
res = solve_bvp(chemo3, bc, t, y)

#Plotting the results (if desired)
t_plot = np.linspace(0, 1, 100)
x_plot = res.sol(t_plot)[0]
p_plot = res.sol(t_plot)[1]
plt.plot(t_plot, x_plot) # the state
plt.plot(t_plot, -.5*p_plot) #the control
plt.legend(['cancer cells','chemo concentration'])
```

Algorithm 21.1: Algorithm for computing the solution of the Chemotherapy problem in Example 21.3.1.

The numerical solution of this problem is relatively simple, but the concepts this illustrates are sufficient to demonstrate how to use the boundary value problem solver. First you need to define a function that describes the right hand side of the first order ODE, and then a function that describes the boundary conditions. The boundary conditions are going to depend on both the endpoint conditions given for the state evolution, and the homogeneous conditions that may arise for the co-state. With these pieces in place, *solve_ bvp* is called to find the solution, using an initial guess for the solution (the zero solution is the initial guess in Algorithm 21.1).

The previous example illustrates how to find the optimal time to stop treatment, but it also assumes that there is no preference for ending the treatment sooner (most patients and insurance plans would prefer avoiding plans that have an indefinite treatment period). Perhaps the physician would like for the treatment to last approximately one month (we will use months as our unit of time), but is willing to extend the treatment by a few days if it makes a significant difference to the patient. We can model this in the following example.

Example 21.3.2 (Chemo in finite time). We will consider the same cost functional as before, except now we will incorporate a cost on the length of time. Hence we propose the cost functional

$$\min_{t_f, u(t)} \left\{ mx(t_f) + a(t_f - 1)^2 + \int_0^{t_f} u(t)^2 dt \right\},$$

subject to

$$x' = \alpha x - u, \quad x(0) = x_0.$$

We don't go through the derivation of the Hamiltonian, co-state etc. yet again, as nothing has changed from the pervious iteration of this problem. Suffice it to say that we also have the evolution of the co-state:

$$p' = -\alpha p, \quad p(t_f) = -m.$$

What has changed however, is that the optimality of t_f has now given us the condition:

$$\begin{aligned} H(\tilde{t}_f) &= \frac{\partial \phi}{\partial t_f}(\tilde{t}_f), \\ \Rightarrow \alpha p(t_f)x(t_f) - p(t_f)u(t_f) - u(t_f)^2 &= 2a(t_f - 1), \end{aligned}$$

where we dropped the $\tilde{}$ on t_f because I was tired of writing it out in LaTex.

While we can solve this problem analytically, we will instead demonstrate the numerical solution to show how the numerical solutions are constructed. To begin, we need to convert the variable endpoint problem into one on a fixed interval where the final time t_f is a parameter. To do this, we consider the change of variables $t = t_f \tilde{t}$ which leads to $\frac{\partial}{\partial t} = \frac{1}{t_f}\frac{\partial}{\partial \tilde{t}}$, and $\tilde{t} \in [0, 1]$. Using this, and recognizing that the optimal control is given by $u = -\frac{p}{2}$, this means that we want to find the solution of the coupled boundary value problem

$$\begin{aligned} x' &= t_f\left(\alpha x + \frac{p}{2}\right), \quad x(0) = x_0, \\ p' &= -t_f \alpha p, \quad p(1) = -m, \quad \alpha p(1)x(1) + \frac{p(1)^2}{4} = 2a(t_f - 1), \end{aligned}$$

where the final condition came from the endpoint condition on the Hamiltonian using the substitution $u = -\frac{p}{2}$.

The numerical solution for some specific values of the relevant parameters is computed in Algorithm 21.2. Note that the additional endpoint condition $H(t_f) = \frac{\partial \phi}{\partial t_f}$ is added to the boundary conditions that are used by bvp_solve. Also note that we provide $t_f = 1$ as the initial guess. This is reasonable because our design of the optimal control problem was to push t_f as close to $t_f = 1$ as possible.

Remark 21.3.3. It is worth pausing here and considering the ramifications of all of the modifications we have made to the simple chemotherapy model that we introduced previously. In the exercises you will also explore what happens if we use a more realistic model of the cancer growth and come to recognize the importance of using a realistic model (even if it is nonlinear). With all of these changes though, have we really come to a 'final model' that we are satisfied will work for chemo treatment? It is very likely that we can still consider other possibilities. For instance, why don't we allow the endpoint cost to be quadratic as that will guarantee convexity of the cost functional which, as we have repeatedly emphasized, is always desirable? We could also consider other modifications to the cost functional, and we also have quite a few parameters that are available to adjust and 'tune'. There is a lot of work left on this problem, and we could spend years looking for the best approach (this is often what is done in Phd programs the world over).

Often though, we must come to the final conclusion that our model needn't be perfect, but just be 'good enough' so that we get sufficiently good results. In practice, if the chemotherapy treatment prescribed by our model isn't working, we will have an oncologist monitoring the patient who can adjust the treatment accordingly. In fact, it is often important to keep in mind that we will not have perfect data when choosing our optimal control model, so it won't make sense to have a model that is more precise than the data we can collect, i.e. if we are checking in on a patient and able to adjust their treatment schedule weekly, then we shouldn't be concerned if our model isn't as reliable on a day-to-day schedule.

21.4 Inequality constraints in optimal control

Now we want to return to the problem of considering an inequality constraint. In this section we will restrict our attention to constraints that explicitly apply to the control $\mathbf{u}$. State-based constraints are possible, but overly complicate the situation. This can be readily seen by recognizing that if we have an inequality constraint on the control $\mathbf{u}$, we can incorporate it much like we did in the Calculus of Variations because we are optimizing with respect to $\mathbf{u}$. On the other hand if the constraint is applied to the state variable $\mathbf{x}(t)$ but we are optimizing with respect to $\mathbf{u}$ then implementation of the constraint is not immediate, i.e. we will need to determine first how $\mathbf{x}(t)$ depends on $\mathbf{u}(t)$ and then implement the constraint in that way.

Specifically, in this section we will consider the problem:

$$\text{minimize } J[\mathbf{u}] = \int_0^{t_f} L(t;\mathbf{x},\mathbf{u})dt + \phi(\mathbf{x}(t_f),t_f), \tag{21.5}$$

$$\text{subject to } \mathbf{x}' = \mathbf{f}(t;\mathbf{x},\mathbf{u}), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{21.6}$$

$$\mathbf{g}(\mathbf{x},\mathbf{u}) = 0 \tag{21.7}$$

$$\mathbf{h}(\mathbf{x},\mathbf{u}) \leq 0. \tag{21.8}$$

Before considering the inequality constraint dictated by the function $\mathbf{h}$, we note that the equality constraint for $\mathbf{g}$ is easily handled by introducing a Lagrange multiplier $\boldsymbol{\lambda}(t)$ and minimizing over the modified cost functional

$$J^*[\mathbf{u}] = \int_0^{t_f} [L + \boldsymbol{\lambda}\cdot\mathbf{g}]\,dt + \phi(\mathbf{x}(t_f),t_f). \tag{21.9}$$

Just as in the case for nonlinear programming, the inequality constraint prescribed by $\mathbf{h}$ is handled similarly. The rigorous justification for this is beyond the normal scope of this textbook, particularly as the assumed smoothness of the underlying control and state variables is not valid as we have seen already. Nevertheless we will proceed as if the solution fields were as smooth as required by the following manipulations.

Before proceeding we also note that we are only going to consider full rank conditions, that is if $\mathbf{h} \in \mathbb{R}^q$ then we will restrict our attention to only those $\mathbf{h}$ so that the rank of $\frac{D\mathbf{h}}{D\mathbf{u}}$ is full, i.e.

$$rank\left\{\frac{D\mathbf{h}}{D\mathbf{u}}\ diag(\mathbf{h})\right\} = q, \tag{21.10}$$

which is equivalent to ensuring that the effects of each component of the constraint function h_k on the control, are independent, i.e. each of $\frac{Dh_k}{D\mathbf{u}}$ are linearly independent. This is very reminiscent of the definition of a regular point in nonlinear optimization for nonlinear programming.

Remark 21.4.1. Note that the discussion contained here is going to be dedicated to what we will refer to as *hard* constraints, that is inequality constraints that must be enforced exactly. Theoretically this is great, and we can pat ourselves on the back and say we have done a good job creating a new theory for such constrained optimization problems. If there is some flexibility in the constraints themselves, then we will fare much better if instead we allow for soft constraints, that is constraints that have some flexibility to them. For instance, rather than enforcing that $\mathbf{g}(\mathbf{x}, \mathbf{u}) = 0$ in the past we may have added a term proportional to $\|\mathbf{g}(\mathbf{x}, \mathbf{u})\|$ to the cost functional itself. We can do the same thing if we have soft inequality constraints, that is, rather than strictly enforcing $\mathbf{h}(\mathbf{x}, \mathbf{u}) \leq 0$ we may just modify our cost functional to

$$\int_0^{t_f} L(t; \mathbf{x}, \mathbf{u})dt \to \int_0^{t_f} L(t; \mathbf{x}, \mathbf{u}) + \boldsymbol{\gamma} \cdot \mathbf{h}dt,$$

where $\boldsymbol{\gamma}$ is a vector of positive coefficients that indicate how much we wish to penalize violating the inequality constraint we are working with.

Soft constraints are almost always preferred for computational ease of implementation, but there are some cases where the hard constraints can't be avoided and we must resort to the derivation provided below.

21.4.1 Formal derivation of the KKT conditions

Using intuition gained from the variational problem with inequality constraints, we can revisit the optimal control problem stated in (21.5)-(21.8), where for now we will omit the equality constraint, i.e. $g(\mathbf{x}, \mathbf{u}) = 0$ is ignored as this would be a straightforward addition. Defining the modified Lagrangian as

$$\mathscr{L} = H - \boldsymbol{\mu}(t) \cdot \mathbf{h}, \tag{21.11}$$

where the Hamiltonian is as defined before, that is:

$$H = \mathbf{p} \cdot \mathbf{f} - L, \tag{21.12}$$

then we still obtain the Hamiltonian maximization principle, i.e. the optimal control and state will maximize the Hamiltonian, but now the evolution of the co-state will be given by

$$\mathbf{p}' = -\frac{D\mathscr{L}}{D\mathbf{x}}, \qquad \mathbf{p}(t_f) = -\frac{D\phi}{D\mathbf{x}(t_f)}. \tag{21.13}$$

In addition

$$\mathbf{h}(\mathbf{x}, \mathbf{u}) \leq 0, \qquad \boldsymbol{\mu}(t) \geq 0, \qquad \boldsymbol{\mu}(t) \odot \mathbf{h}(\mathbf{x}, \mathbf{u}) = 0, \tag{21.14}$$

where $\odot$ denotes the Hadamard (pointwise) mutliplication, and the Lagrange multiplier $\boldsymbol{\mu}(t)$ must satisfy

$$\frac{D\mathscr{L}}{D\mathbf{u}} = \frac{DH}{D\mathbf{u}} - \boldsymbol{\mu}\frac{D\mathbf{h}}{D\mathbf{u}} = 0, \tag{21.15}$$

as well, and most importantly of all, we still must maximize the Hamiltonian H as a function of the control variable $\mathbf{u}$.

21.4.2 Some examples

We will first consider a tractable, but not physically motivated example, and then resort to one that is far more economically useful.

Example 21.4.2. We wish to minimize

$$J[u] = -\int_0^1 u(t)dt, \tag{21.16}$$

subject to

$$x' = u, \quad x(0) = 1, \tag{21.17}$$

$$0 \leq u \leq x. \tag{21.18}$$

The Hamiltonian for this problem is given by

$$H = pu + u = (p+1)u, \tag{21.19}$$

so that the optimal control is chosen as a bang-bang solution, i.e.

$$\tilde{u} = \begin{cases} 0 & \text{if } p+1 < 0 \\ x(t) & \text{if } p+1 > 0. \end{cases} \tag{21.20}$$

Note before proceeding any further that because $u \geq 0$ for all t and because $x(0) = 1$ then $x(t) > 0$ for all $t \geq 0$, i.e. there is no ambiguity in the control between $x(t)$ and 0.

To determine the evolution of $p(t)$ itself, we need to formulate the modified Lagrangian which incorporates the two constraints $-u \leq 0$ and $u - x \leq 0$ which leads to

$$\mathscr{L} = H + \mu_1 u + \mu_2(x - u) = (p + 1 + \mu_1 - \mu_2)u + \mu_2 x. \tag{21.21}$$

Hence

$$p' = -\frac{\partial \mathscr{L}}{\partial x} = -\mu_2, \tag{21.22}$$

with final endpoint condition $p(1) = 0$ so that we know that the final value of the control is $\tilde{u}(1) = x(1)$. Now, if we know $\mu_2(t)$ then we could simply integrate in time to obtain $p(t)$ and hence find the optimal control for this problem.

The additional conditions that we may use include

$$\frac{\partial \mathscr{L}}{\partial u} = p + 1 + \mu_1 - \mu_2 = 0, \tag{21.23}$$

$$\mu_1(t) \geq 0, \quad \mu_2(t) \geq 0, \tag{21.24}$$

$$\mu_1(t)u(t) = 0, \quad \mu_2(t)(u(t) - x(t)) = 0. \tag{21.25}$$

Now since we know that $\tilde{u} = x$ at the final time $t = 1$ then we need only consider the possibility that $\tilde{u}(t^*) = 0$ for some time $t^* < 1$. Note that if this is the case then the slackness condition on μ_2 indicates that $\mu_2(t^*) = 0$ and hence

$$p' = 0 \Rightarrow p = c_0, \tag{21.26}$$

i.e. $p(t)$ for all $t > t^*$ is a constant, but this would indicate that there is no possible switch to the final state $\tilde{u}(1) = x(1)$, and hence the optimal control will always be given by

$$\tilde{u}(t) = x(t), \tag{21.27}$$

for all $t \in [0, 1]$.

This of course only makes sense in order to minimize the given cost functional, something we could have readily recognized from the outset. Nevertheless we will now ensure that this optimal solution will satisfy the additional conditions.

For $\tilde{u}(t) = x(t)$ then the slackness condition indicates that $\mu_1(t) = 0$ and hence $\mu_2 = p + 1$ and the co-state evolution will be given by

$$p' = -p - 1, \quad p(1) = 0, \tag{21.28}$$

indicating that

$$p(t) = ce^{-t} - 1, \tag{21.29}$$

where $p(1) = ce^{-1} - 1 = 0$ indicating that $c = e$ and thus

$$p(t) = e^{1-t} - 1. \tag{21.30}$$

Thus $p(t) + 1 > 0$ for all $t \in [0, 1]$, justifying the choice of the optimal control $\tilde{u}(t) = x(t)$.

The example above could have been worked out perhaps without even using the inequality constraints in this way, but it does illustrate how they play a role. For a more realistic example, we consider a realistic monetary concern that arises on Bespin.

Example 21.4.3. Lando Calrissian has settled down to the mines in Bespin, but needs some help optimizing his financial outlook. He finances his investments by optimizing a combination of retained earnings and external equity. The earnings not retained are paid out as dividends to the other stockholders of Cloud City (typically meaning payouts to the Empire). Lobot has been tasked with determining the optimal investment policy, and has come up with the following model.

$y(t)$: value of Cloud City's assets or invested capital at time t

$x(t)$: the current earnings rate in credits per unit time at time t

$u(t)$: the external or new equity financing expressed as a multiple of current earnings, $u \geq 0$

$v(t)$: fraction of the current earnings retained, i.e.

$1 - v(t)$ represents the rate of dividend payout $v \in [0, 1]$

$1 - c$: proportional floatation (transaction) cost for external equity, $c \in [0, 1]$

ρ : continuous discount rate, or stockholders required rate of return

r : actual rate of return, $r \geq \rho$

g : upper bound on the growth rate of Cloud City's assets

t_f : planning horizon (Lando is planning to cash out at that time).

Using these definitions, Lobot determines that the current earnings at time t are given by $x(t) = ry(t)$ (for now we can assume that r is a constant). The rate of change in the earnings is then given by

$$x' = ry' = r(cu + v)x, \quad x(0) = x_0.$$

In addition, the upper bound on the growth of the assets leads to:

$$y'/y = \frac{(cu + v)x}{x/r} = r(cu + v) \leq g.$$

Lando wants Lobot to maximize the value of Cloud City which is described as the present value of the future dividends. To determine what this is exactly, Lobot recognizes that the present value of total dividends issued by the Cloud City enterprise can be described by

$$\int_0^{t_f} (1 - v)xe^{-\rho t} dt.$$

A proportion of these dividends go directly to new equity which (with an efficient market which is likely a poor assumption with the Empire in control) will get a rate of return exactly equal to the discount rate ρ, which is then given by

$$\int_0^{t_f} uxe^{-\rho t} dt,$$

which is the present value of the external equity raised over time.

It follows that the net present value of the total future dividends that Cloud City has is the difference between the two expressions above, i.e. Lobot needs help with the following optimal control problem:

$$\begin{aligned} \text{maximize} \int_0^{t_f} e^{-\rho t} (1 - v - u)\, x dt, \\ x' = r(cu + v)x, \quad x(0) = x_0, \\ cu + v \leq g/r, \quad u \geq 0, \quad v \in [0, 1]. \end{aligned}$$

The Hamiltonian for this problem can be written as

$$H = pr(cu + v)x + e^{-\rho t}(1 - v - u)x.$$

There are a variety of ways that we can solve this problem. It turns out that the use of the Karush-Kuhn-Tucker conditions may not be the best way to do so, but it is nevertheless illustrative of what one could do to solve such a problem where the inequality constraints are reliant on the control variables u and v.

In reality this problem is a very non-trivial exercise with close to a dozen cases that focus on different situations where the control variables can change in time. The solution will be a combination of singular and bang-bang controls, dependent on the various quantities of interest. Who knew that Lobot was really good at nonlinear programming?

Remark 21.4.4. Now this brings up a very good point. The solution to most optimal control problems is at best a coupled system of linear ordinary differential equations. When inequality constraints are enforced the actual solution is more frequently couched in the language of nonlinear (or linear if you are lucky) programming tied to the solution of a system of ODEs. Occasionally the optimal solution can be found as the solution of a partial differential equation or a nonlinear optimization procedure applied to the solution of a PDE (this would be a very bad scenario). This is the reason that so much time and energy has been spent on linear and nonlinear programming and convex optimization. The optimal solution to a far more complicated problem such as the infinite dimensional cases we have considered here, can be reduced numerically to a series of finite dimensional problems via a given discretization. Hence the solution of these finite dimensional optimization problems must be extremely efficient and scalable (infinity is best approximated by large values, not small or $O(1)$ values).

Remark 21.4.5. If the last remark seemed weird and out of place, maybe this will make more sense. What we are doing now is the infinite dimensional version of much of what happened in Volume II, i.e. nonlinear optimization. Actual solutions of these problems are typically intractable, so a discretization of the problem is required which results in a high dimensional nonlinear optimization problem. *This is hard.*

Exercises

Note to the student: Each section of this chapter has several corresponding exercises, all collected here at the end of the chapter. The exercises between the first and second line are for Section 1, the exercises between the second and third lines are for Section 2, and so forth.

You should **work every exercise** (your instructor may choose to let you skip some of the advanced exercises marked with *). We have carefully selected them, and each is important for your ability to understand subsequent material. Many of the examples and results proved in the exercises are used again later in the text. Exercises marked with ⚠ are especially important and are likely to be used later in this book and beyond. Those marked with † are harder than average, but should still be done.

Although they are gathered together at the end of the chapter, we strongly recommend you do the exercises for each section as soon as you have completed the section, rather than saving them until you have finished the entire chapter.

21.1. A more advanced model of chemotherapy treatment is to consider a cancer that has what is referred to as Gompertzian growth (a specific model of cancer growth). If we let $N(t)$ be the cancer concentration, and $u(t) > 0$ is the dosage of the drug i.e. some drugs are more potent than others, then this indicates that the cancer growth is governed by

$$N' = rN \log\left(\frac{1}{N}\right) - \delta u N,$$

where $r > 0$ is the growth rate of the cancer, and $\delta > 0$ is the potency of the drug. At time $t = 0$ we also have the initial condition $N(0) = N_0$. Set up an optimization problem that simultaneously minimizes both the harmful effects of the drug over the time interval $[0, t_f]$ and the total concentration of the cancer cells in that time interval $[0, t_f]$. Make the cost functional quadratic to make later computations more tractable.

21.2. Find the Hamiltonian for the previous problem and determine the optimal control as well as the state equation and adjoint equation (including boundary conditions if present) for the costate (*do not solve these as I expect they will be rather drawn out*).

21.3. For the situation outlined in the previous two problems, now set up an optimization problem that minimizes the final number of cancerous cells, but does not regard the cancer concentration at some intermediate point $t < t_f$.

21.4. Find the Hamiltonian, state and costate equations for the previous problem as well as any new boundary conditions that may arise. Do you expect this optimal control to be fundamentally different from that discussed in the previous version of this problem?

21.5. Modify the optimization problem considered in the previous two exercises so that the final time t_f is variable i.e. t_f is a variable in the optimization itself. How does this change the setup? What additional conditions arise?

21.6. Find the optimal policies (solution) for the simple cash balance model from Example 21.2.2 for $x_0 = 2,\ y_0 = 2,\ U_1 = U_2 = 5,\ t_f = 1,\ \alpha = 0.01$ and $r_1 = 1/3,\ r_2 = 1/2$ with $d = 1.5$.

21.7. For the same problem as above, find the optimal solution for $r_1(t) = 1/3$ and $r_2(t) = t/2$.

21.8. Find a numerical solution of the optimal control problem from exercises 21.1-21.2. Use $r = 1,\ N_0 = 2,\ \delta = 0.5$, and $t_f = 5$. Plot both $N(t)$ and $u(t)$ for this problem. Does the solution make sense? If not, can you adjust your cost functional to make this work?

21.9. Solve problem 21.5 numerically and plot $N(t)$ and $u(t)$. Use $r = 1,\ N_0 = 2,\ \delta = 0.5$. What is the optimal value of t_f?

21.10. Modify the previous problem to include an endpoint cost, i.e. a term in the cost functional that penalizes large values of t_f. Compute the numerical solution, and comment on the differences in the solution.

21.11. (adapted from Sethi & Thomson): Consider the motion of a rocket that is dictated by acceleration or deceleration at the rate $u \in [-1, 1]$ and consumes fuel at a rate proportional to that acceleration. We desire to have the rocket start from an initial velocity x_0 and end with zero velocity (neglecting all other potential effects including friction). At the same time, the fuel is expensive and you desire to minimize the total cost. Set up an optimal control problem, including the state equation.

21.12. Find the optimal control and corresponding state for the previous problem.

21.13. Set up the optimal control solution (define all the necessary conditions that must be satisfied) for the following problem:

$$
\begin{aligned}
\text{minimize } & -\int_0^1 x(t)dt, \\
x' &= x + u, \quad x(0) = 0, \\
-1 \le u &\le 1, \quad x + u \le 2.
\end{aligned}
$$

21.14. Set up the optimization problem (including specifying the modified Lagrangian, and all relevant endpoint and initial conditions) to solve the following:

$$\begin{aligned}
&\text{minimize } x_1(2),\\
&\qquad x_1' = u_1 - u_2, \quad x_1(0) = 2,\\
&\qquad x_2' = u_2, \quad x_2(0) = 1,\\
&\qquad 0 \leq u_1(t) \leq x_2(t), \quad 0 \leq u_2(t) \leq 2, \quad t \in [0, 2].
\end{aligned}$$

21.15. Find the optimal control and state for the previous problem.

21.16. Set up the final example of this Chapter so it can be formulated with the Karush-Kuhn-Tucker conditions for the inequality constraints. This includes rewriting the inequality constraints so they are in the form discussed in this Chapter, formulating the modified Lagrangian and the complimentary slackness conditions.

21.17. This problem is an example of what is called a 'current value' problem meaning that the current value of something is typically more valued than its future value (we're all greedy and impatient in the end). The discount rate ρ indicates how impatient we are. If we are trying to minimize the cost function

$$\begin{aligned}
&\int_0^{t_f} L(\mathbf{x}, \mathbf{u}) e^{-\rho t} dt + \phi_s(\mathbf{x}(t_f)) e^{-\rho t_f},\\
&\mathbf{x}' = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x}(0) = \mathbf{x}_0,\\
&h(\mathbf{x}, \mathbf{u}) \leq 0,
\end{aligned}$$

which will have the standard Hamiltonian

$$H_s = \mathbf{p}_s \cdot \mathbf{f} - L e^{-\rho t},$$

with standard modified Lagrangian

$$\mathscr{L}_s = H_s - \mu_s \cdot \mathbf{h},$$

and the resultant co-state equation is exactly as we have previously considered. Defining the current value Hamiltonian as $H = \mathbf{p} \cdot \mathbf{f} - L$ and the current value modified Lagrangian $\mathscr{L} = H + \boldsymbol{\mu} \cdot \mathbf{h}$. From here derive the evolution of the current value co-state $\mathbf{p}(t)$, and the complementary slackness conditions for $\boldsymbol{\mu}(t)$.

21.18. Using the current value formulation derived in the previous problem, rewrite Lobot's optimization problem as a current value problem.

Notes