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Homework 1

Exercise 1

So to prove this take the schur decomposition and assume that A is normal:

$$A = UTU^* \tag{1}$$
$$\tag{2}$$

Then:

$$\begin{aligned} A^*A &= AA^* && (3)\\ UT^*U^*UTU^* &= UTU^*UT^*U^* && (4)\\ UT^*TU^* &= UTT^*U^* && (5)\\ T^*T &= TT^* && (6) \end{aligned}$$

So T must be normal. We now prove that any normal upper triangular matrix must also be diagonal... take e_1:

$$\begin{aligned} &T^*Te_1 && (7)\\ &= T^*[T_{11}, 0, \dots].T \text{ since T is upper trinagular} && (8)\\ &= [|T_{11}|^2, 0, \dots].T && (9) \end{aligned}$$

but in addition:

$$\begin{aligned} &TT^*e_1 && (10)\\ &= T[\overline{T_{11}}, \overline{T_{12}}, \dots].T && (11)\\ &= [\sum_j |T_{1j}|^2, \dots].T && (12) \end{aligned}$$

Note I left off the last n-1 entries of the vector because we will not be using them.

Now note that since these two must be equal we know that $\sum |T_{1j}|^2 = |T_{11}|^2$. Thus $T_{1j} = 0$ for $j \neq 1$. From this we immidiately get equality above.

Now that we have established that as a base case we prove assume that T is normal and diagonal up to k and prove it is diagonal on $k+1$ or that $T_{k+1,j} = 0$ for $j > k+1$ to do this take the vector e_{k+1} then:

$$\begin{aligned} &T^*Te_{k+1} && (13)\\ &= T^*[0, \dots, T_{k+1,k+1}, 0, \dots].T \text{ because of inductive hypothesis all rows above are zero} && (14)\\ &= [0, \dots, |T_{k+1,k+1}|^2, 0, \dots].T && (15) \end{aligned}$$

Furthermore

$$TT^* e_{k+1} \tag{16}$$
$$= T[0, \ldots \overline{T_{k+1,k+1}}, \overline{T_{k+1,k+2}}, \ldots].T \tag{17}$$
$$= [0, \ldots, \sum_{j=k+1}^{n} |T_{k+1,j}|^2, \ldots].T \tag{18}$$

(Once again we ignore the last $n-k-1$ entries because we do not use them)

From here we then know that $\sum_{j=k+1}^{n} |T_{k+1,j}|^2 = |T_{k+1,k+1}|^2$. So that must mean that $T_{k+1,j} = 0$ for $j > k+1$.

Carry out this induction to n and we are left with the fact that T is Diagonal.

From this we can see that $A = UTU^*$ where T is diagonal so A is unitarily diagonalizable.

In the opposite direction assume that A is unitarily diagonalizable then:

$$AA^* = UDU^*UD^*U \tag{19}$$
$$= UDD^*U^* \tag{20}$$

Now remember that all diagonal matrices commute so:

$$= UD^*DU^* \tag{21}$$
$$= UD^*U^*UDU^* \tag{22}$$
$$= A^*A \tag{23}$$

So A is normal. and it is proven

Exercise 2 Assume that A is hermitian and positive semi definite. We know that all of the eigenvalues are nonnegative because of semidefiniteness $x^*Ax \geq 0 \implies W(A) \geq 0$. Since it is also hermitian all of these valeus are also strictly real

Furthermore take its eigenvalue decomposition (since it is hermitian this exists):

$$A = V\Lambda V^* \tag{24}$$

Then note that:

$$A^*A = V\Lambda^* V^* V\Lambda V^* \tag{25}$$
$$= V\Lambda^2 V^* \tag{26}$$

from here note that this is the eigenvalue decomposition of A^*A and $\Lambda * = \Lambda$ because all of the eigenvalues are real. Then note that the singular values are:

$$\sigma_i = \sqrt{\lambda_i^2} = \lambda_i \tag{27}$$

So since all of the eigenvalues are real and positive this exists and all of the singualr values equal the eigenvalues (at least for the nonzero eigenvalues.)

If the matrix is normal then we get that:

$$A^*A = V|\Lambda|^2V^* \tag{28}$$

so $\sigma_i = \sqrt{|\lambda_i|^2} = |\lambda_i|$

So the singular values are actually the magnitudes of the eigenvalues

Exercise 3

To do this take the frobenius norm of a matrix:

$$\|A\|_{fr}^2 = \operatorname{tr}(AA^*) \tag{29}$$
$$= \operatorname{tr}(U\Sigma V^*V\Sigma^*U^*) \tag{30}$$

Because every matrix has an singular value decomposition. Note that $\Sigma^* = \Sigma$ since they are all real and positive. Also note that $V^*V = I$ so:

$$= \operatorname{tr}(U\Sigma^2U^*) \tag{31}$$
$$= \operatorname{tr}(U^*U\Sigma^2) \text{ cyclic property of trace} \tag{32}$$
$$= \operatorname{tr}(\Sigma^2) \tag{33}$$
$$= \sum_{k=1}^{r}(\sigma^2) \tag{34}$$
$$= \|\sigma\|_2^2 \tag{35}$$

Exercise 4 To do this take:

$$\|A - B\|_2 \tag{36}$$
$$\geq \|(A - B)v\| \tag{37}$$

For some particular v. Assume rank of B is strictly less than A or else we just choose $B = A$ and the proof is trivial. Since rank of B is strictly less than A then the space spanned by the $k+1$ right singular vectors of A must include a vector in the null space of B. Take this right singular vector v_j then:

$$= \|(A - B)v_j\| = \|Av_j\| \tag{38}$$
$$= \left\|\sum_{i=1}^{r}\sigma_i u_i v_i^* v_j\right\| \tag{39}$$
$$= \|\sigma_j u_j\| = \sigma_j \geq \sigma_{k+1} \tag{40}$$

So we know that the norm of $A - B$ is bounded below by the $k+1$ singular value of A. if

$A = \sum_{i=1}^{r} \sigma_i u_i v_i^*$ then set $B = \sum_{i=1}^{k} \sigma_i u_i v_i^*$ and we get:

$$\|A - B\| = \tag{41}$$

$$= \left\| \sum_{j=1}^{k} (\sigma_j - \sigma_j) u_j v_j^* + \sum_{j=k+1}^{r} (\sigma_j) u_j v_j^* \right\| \tag{42}$$

$$= \left\| \sum_{j=k+1}^{r} (\sigma_j) u_j v_j^* \right\| \tag{43}$$

And remember that the two norm of a matrix is the largest singular value (I think we proved this but if not note that since the two norm is invariant under unitary transformation $\|A\| = \|\Sigma\|$ and the norm of that diagonal (even if rectangular) matrix is just the largest entry on the diagonal). so

$$= \sigma_{k+1} \tag{44}$$

So the lower bound that we set up is achieved specifically when B the rank k approximation of A. So that is the answer.

Exercise 5 To prove this take some random vector in V and take P to be the orthogonal projector onto V then:

$$\|x - v\|^2 = \|x - Px + Px - v\|^2 \tag{45}$$

$$= \|(I - P)x + Px - Pv\|^2 \text{ since v is in V} \tag{46}$$

$$= \|(I - P)x + P(x - v)\|^2 \tag{47}$$

Remember now that $I - P$ projects onto a space orthogonal to P so the two things in our norm are orthogonal. We can then apply pythagoras:

$$= \|(I - P)x\|^2 + \|Px - Pv\|^2 = \|x - Px\|^2 + \|Px - v\|^2 \geq \|x - Px\|^2 \tag{48}$$

so as a result

$$\|x - v\| \geq \|x - Px\| \tag{49}$$

for any arbitrary $v \in V$. We now show that this minimum can be attain. Well trivially by choosing $v = Px$ since $Px \in V$. $\square$

Exercise 6 a) to prove this part remember that we can construct projections like:

$$P_{U_k} = \sum_{j=1}^{k} \left(u_j u_j^*\right) \tag{50}$$

So :

$$P_{U_k}A = \sum_{l=1}^{k}(u_l u_l^*)\sum_{k=1}^{n}(\sigma_k u_k v_k^*) \tag{51}$$

$$= \sum_{j=1}^{k}\sum_{l=1}^{n}\left(u_j u_j^*\right)(\sigma_l u_l v_l^*) \tag{52}$$

$$= \sum_{j=1}^{k}\sigma_j u_j u_j^* u_j v_j^* \tag{53}$$

$$= \sum_{j=1}^{k}\sigma_j u_j v_j^* \tag{54}$$

$$\tag{55}$$

And it is proven

b) To prove this part we just take remember that the B that minimizes:

$$||A - B||_F^2 \tag{56}$$

Is literally just (by smidth theorem) $\sum_{j=1}^{k}\sigma_j u_j v_j^* = P_{U_k}A$ by part A. thus if U were any other projection it would not be equal to B. so this is proven by the smidth theorem.

c) To prove this last part remember that one form of the forbenius norm is $||D||_F^2 = \sum_i ||D_i||_2^2$ where D_i are the collumns of D. now take:

$$P_V A = [P_V a_1, \ldots, P_V a_n] \tag{57}$$

$$||P_V A||_F^2 = \sum_j ||P_V a_j||_2^2 = \text{var}_V(B) \tag{58}$$

So we just need to maximize this projection. Intuitively choosing $V = U_k$ should maximize it because that is the thing that mimizeds the norm thing. But how to show it

Here can use a cheap trick note:

$$\text{var}_V(B) = ||P_V A||_F^2 \tag{59}$$

$$||A||_F^2 = ||(I - P_V)A + P_V A||_F^2 \tag{60}$$

Note that if we convert this into the $\sum ||(I - P_V)a_j + P_V a_j||_2^2$ we can apply pythagoras to each element in turn and then convert back to get:

$$||A||_F^2 = ||(I - P_V)A + P_V A||_F^2 \tag{61}$$

$$= ||(I - P_V)A||_F^2 + ||P_V A||_F^2 \tag{62}$$

$$||A||_F^2 - ||A - P_V A||_F^2 = ||P_V A||_F^2 \tag{63}$$

So if we maximize the left hand side it is the same as maximizing right hand side over V. Maximizing the left hand side is thus equivalent to minimizing:

$$||A - P_V A||_F^2 \tag{64}$$

since there was a negative. And the thing that minimizes this is P_{U_k} by the above proof (part b). So the maximal subspace is the space U_k