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Homework 1

Exercise 6.1

a) To prove this remember that from the last assignment we learned that $AA^+ = UU^T$ so it is just a projection onto the collumn space of U (which is also the collumn space of A). furthermore this projection is hermitian since $(UU^T)^T = UU^T$. So as a result this is an orthogonal projection onto the range of A. Since this is an orthogonal projection this is the distance minimizer from b to the collumn space of A (by hw 4.5). So this is the unique minimizer

$$AA^+b = UU^Tb \tag{1}$$

b) To do this note that the minimum of $||Ax - b||$ is the same as its norm squared:

$$||Ax - b||^2 \tag{2}$$
$$= \langle Ax - b, Ax - b \rangle \tag{3}$$
$$= x^*A^*Ax - b^*Ax - x^*A^*b + b^*b \tag{4}$$

from here note that to minimize this we can take derivatives:

$$x^*(A^*A + A^*A) - b^*A - b^*A = 0 \tag{5}$$

to find the minimizer. From here note that we can do some simple algebra to get:

$$x^*A^*A = b^*A \tag{6}$$

Taking transposes we arrive at:

$$A^*Ax = A^*b \tag{7}$$

So any x that satisfies this equation will satisfy the first derivative test. Note that for the second derivative test we have:

$$A^*A \tag{8}$$

Now note we want this matrix to be positive definite. We note that it is already semipositive definite by construction (since its a matrix multiplied by its transpose). We now note that if $n < m$ that this matrix must have full rank and will have no null space, so it will be positive definite

c) For this part we plug in the pseudo inverse into the normal equations:

$$A^*AA^+b = A^*b \tag{9}$$

$$V\Sigma U^*U\Sigma V^*V\Sigma^{-1}U^*b = V\Sigma U^*b \tag{10}$$

$$V\Sigma^2 V^*V^*\Sigma^{-1}U^*b = V\Sigma U^*b \tag{11}$$

$$V\Sigma^2\Sigma^{-1}U^*b = V\Sigma U^*b \tag{12}$$

$$V\Sigma U^*b = V\Sigma U^*b \tag{13}$$

So we can see that A^+b satisfies the normal equations. if c is any other solution to the normal equations we need to somehow show that it has larger norm because they both satisfy the normal equations (set $x_0 = A^*b$):

$$A^*Ax_0 = A^*b = A^*Ac \tag{14}$$

$$A^*A(x_0 - c) = 0 \tag{15}$$

from here we can see that $x = x_0 - c \in N(A^*A)$ (recall that the null space of A^*A is the same as the null space of A $Ax = 0 \implies A^*Ax = A^*0 = 0, A^*Ax = 0 \implies x^*A^*Ax = 0 \implies ||Ax||^2 = 0 \implies Ax = 0$). so x is in the null space of A. However recall that $x_0 = A^+b$ lives in the collumn space of V which is perpendicular to the null space of A (by the SVD, since the SVD V can be partitioned into V_1, V_2 where V_2 is the null space of A). Sowe have then that $c = x_0 + x$ and $||c||^2 = ||x_0 + x||^2 = ||x_0||^2 + ||x||^2 \geq ||x_0||^2$ by the pythagorean theorem. Thus x_0 is the minimum normed among all solutions.

Exercise 6.2 For this problem I believe that we can prove $a \subset b \subset c \subset a$ which would imply that all three sets are equal. Set the shorthand for each set respectively to be A, B, C

proof of $A \subset B$

let $z \in A = \Lambda_\epsilon(A)$. This means that z is an eigenvalue with eigenvector v (of unit norm) of $A + \Delta A$ for some $||\Delta A|| < \epsilon$. then take:

$$||(A - zI)v|| \tag{16}$$

$$= ||(A + \Delta A - zI)v - \Delta Av|| \tag{17}$$

$$= ||(A + \Delta A)v - zv - \Delta Av|| \tag{18}$$

$$= ||zv - zv - \Delta Av|| \tag{19}$$

$$= ||\Delta Av|| \tag{20}$$

$$\leq ||\Delta A||||v|| \tag{21}$$

$$= ||\Delta A|| < \epsilon \tag{22}$$

So we thus know that $z \in B$

proof of $B \subset C$. recall that on an earlier assignment I proved that if v is unit norm that $||Av|| \geq \frac{1}{||A^{-1}||}$ for invertible A. For completeness I prove this again:

$$||A^{-1}x|| \leq ||A^{-1}||||x|| \tag{23}$$

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setting $y = A^{-1}x$, $Ay = x$:

$$||y|| \leq ||A^{-1}||||Ay|| \tag{24}$$

if $||y|| = 1$ then:

$$||Ay|| \geq \frac{1}{||A^{-1}||} \tag{25}$$

let $z \in B$ (also assume for the moment that $z \notin \Lambda A$) note then that we have that $||(A - zI)v|| < \epsilon$ for some unit vector v. Since this was unit we can apply my previous lemma:

$$\epsilon > ||(A - zI)v|| \geq \frac{1}{||(A - zI)^{-1}||} = \frac{1}{|| - \text{res}_A(z)||} = \frac{1}{||\text{res}_A(z)||} \tag{26}$$

$$\frac{1}{\epsilon} < ||\text{res}_A(z)|| \tag{27}$$

so it is proven and $z \in C$. if $z \in \Lambda(A)$ then clearly its in B additionally $R_A(z) = \infty$ so it clearly must also be in C

proof of $C \subset A$ let $z \in C$ then

$$\frac{1}{\epsilon} < ||(zI - A)^{-1}|| \tag{28}$$

$$= ||(zI - A)^{-1}x|| \tag{29}$$

$$\frac{1}{||(zI - A)^{-1}x||} < \epsilon \tag{30}$$

for some unit vector x. set $w = (zI - A)^{-1}x$. from here note that $zw - Aw = x$, $zw = Aw + x$. note note that we can set x by dividing by one to be: $x = x\frac{w^*w}{||w||^2}$ so then take:

$$B = \frac{xw^*}{||w||^2} \tag{31}$$

$$||B|| = ||x||||w^*||/||w||^2 \leq \epsilon \tag{32}$$

so this matrix B then satisfies the theorem for the pertubation matrix in a. from here rewrite:

$$zw = Aw + x = Aw + Bw = (A + B)w \tag{33}$$

So then we have proved that $z \in A$

Exercise 6.3 This seems fun assume that A is diagonalizable which means $A = V\Lambda V^{-1}$ then the resolvent is:

$$R_A(z) = (zI - A)^{-1} \tag{34}$$

$$= (zVV^{-1} - V\Lambda V^{-1}) \tag{35}$$

$$= (V(zI - \Lambda)V^{-1})^{-1} \tag{36}$$

$$= V(zI - \Lambda)^{-1}V^{-1} \tag{37}$$

For now assume that this is not one of the eigenvalues of A otherwise the result is trivial.

Now note that by the assumptions in the theorem we have that $\overline{\lambda} \in \Lambda_\epsilon(A)$ if we take the norm of this we see that:

$$\frac{1}{\epsilon} < ||R_A(\overline{\lambda})|| \leq ||V|| ||(\overline{\lambda}I - \Lambda)^{-1}|| ||V^{-1}|| \tag{38}$$

$$\leq \kappa(V)||(\overline{\lambda}I - \Lambda)^{-1}|| \tag{39}$$

so

$$\frac{1}{||(\overline{\lambda}I - \Lambda)^{-1}||} \leq \kappa(V)\epsilon \tag{40}$$

$$\text{by the fact that } \overline{\lambda}I - \Lambda \text{ is diagonal and invertible:} \overline{\lambda} \neq \lambda_i \tag{41}$$

$$\frac{1}{||\text{diag}\frac{1}{\overline{\lambda} - \lambda_i}||} \leq \kappa(V)\epsilon \tag{42}$$

$$\frac{1}{\frac{1}{|\overline{\lambda} - \lambda_i|}} \leq \kappa(V)\epsilon \text{ for some i} \tag{43}$$

$$|\overline{\lambda} - \lambda_i| \leq \kappa(V)\epsilon \tag{44}$$

if A is unitarily diagonalizable then $\kappa(V) = 1$ so $|\overline{\lambda} - \lambda_i| < \epsilon$

Exercise 6.4

To do this we essentially need to prove that any any point in the process there is a nonzero element in the collumn below $\tilde{a}_{kk}$ in that either the pivot is already nonzero or we can swap one of the rows below it to get a nonzero pivot.

to do this we start with the base case. Assume that A is invertible. Then there must be at least one nonzero entry in the first collumn (the row below a_{11} and including the original). If they were all zero then we would have a zero collumn and A would not be full collumn rank and hence not invertible. so there must be at least one nonzero entry in that collumn.

Say we have performed k steps of pivoting that means that:

$$A = P_1 L_1 P_2 L_2 \ldots P_k L_k A_k \tag{45}$$

Where A_k is upper triangular in the first k rows. Assume that our gaussian elimination has worked so far as an inductive step. We need now to show that $(A_k)_{k,k:}$ (row collumn) this subcollumn has at least one nonzero entry. then we could pivot that row and continue.

To prove this note that since A is invertible then $\det(A) = \det(P_1)\det(L_1)\ldots\det(P_k)\det(L_k)\det(A_k) \neq$ 0. Since each L_j, P_j are invertible then we get that $\det(A_k) \neq 0$ so A_k must be invertible. Decompose $A_k = \begin{bmatrix} (A_k)_{1:k-1,1:k-1} & * \\ 0 & (A_k)_{k:,k:} \end{bmatrix}$ (Note I am using numpy style annotations). From here since this is a block triangular matrix we know that $\det(A_k) = \det(B)\det(C)$ (Where B is the upper left block and C is the lower right block, this can be proved by noting that if a matrix K is upper block triangular). Since we assumed that gaussian elimination

with partial pivoting worked thus far we know that $\det(B) \neq 0$. that implies that $\det(C) \neq 0$ or in other works this lower right hand block is invertible.

By a similar argument to the base case the first collumn of this submatrix must have a nonzero entry otherwise it would be rank deficient and have zero determinant. Thus there is a pivot we can pivot on at step $k+1$. So gaussian elimination with partial pivoting succeeds!

Exercise 6.5 to do this take:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ W & I \end{bmatrix} \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \tag{46}$$

$$= \begin{bmatrix} X & Y \\ WX & WY + Z \end{bmatrix} \tag{47}$$

se we get that $X = A, B = Y, W = CX^{-1} = CA^{-1}, Z = D - CA^{-1}B$

b) to prove this take the LU decompmosition:

$$\det(M) = \det(L)\det(U) \tag{48}$$

note that L has the identity on its diagonal by construction above. so its determinant is zero so:

$$\det(M) = \det(U) \tag{49}$$

Which we can compute by reducing U to a Diagonal and unit Upper form note that since we know that A and D are invertible then:

$$U = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} I & X^{-1}Y \\ 0 & I \end{bmatrix} \tag{50}$$

The determinant of this last matrix is just 1 so we just need the determinant of this middle diagonal matrix which is given by:

$$\det(X)\det(Z) = \det(A)\det(D - CA^{-1}B) = \det(A)\det(M/A) \tag{51}$$

c) Note in the decopmosition above we showed that:

$$M = L\tilde{D}U \tag{52}$$

$$= \begin{bmatrix} I & 0 \\ W & I \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} I & X^{-1}Y \\ 0 & I \end{bmatrix} \tag{53}$$

which if we plug in values we get:

$$M = L\tilde{D}U \tag{54}$$

$$= \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} \tag{55}$$

From here note that since $A = A^*$ since M is hermitian then $A^{-*} = A^{-1}$. Thus we haev that in this decomposition :

$$L = U^* \tag{56}$$

So as a result: for any arbitrary x we have:

$$x^* L \tilde{D} U x \tag{57}$$

$$x^* U^* \tilde{D} U x \tag{58}$$

$$y^* \tilde{D} u \tag{59}$$

So but now $\tilde{D}$ is a block diagonal with A and M/A on the diagonal. So clearly M is positive definite iff the blocks are positive definite (Since we chose x to be artbirary and U is invertible so y is also arbitrary).

 d)

 for this problem consider the previous block decomposition $M = L \tilde{D} U$ once again:

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \tag{60}$$

$$\begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \tag{61}$$

$$\tag{62}$$

now note that a fun fact is that the inverse of a unit lower triangular matrix is just a unit lower triangular matrix with the lower part negated giving:

$$\begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \tag{63}$$

$$\tag{64}$$

From this we can move forward by recalling that both A is invertible and $D - CA^{-1}B$ is invertible by a previous problem this gives:

$$\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \tag{65}$$

$$\tag{66}$$

similarly for an upper triangular matrix one can compute the inverse by just negating the upper bllock:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & (M/A)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \tag{67}$$

$$\tag{68}$$