Drake Brown
Date Apr 14, 2025

Homework 1

Exercise 2.1 For this problem note that we can write the integral form:

$$x_\mu(t) = x_0 + \int_0^t f(s, x_\mu(s), \mu) ds \tag{1}$$

Now take:

$$\begin{align}
x_\mu(t) - x_\nu(t) == \int_0^t f(s, x_\mu, \mu) - f(s, x_\nu, \nu) \tag{2}\\
= \int_0^t f(s, x_\mu, \mu) - f(s, x_\nu, \mu) + f(s, x_\nu, \mu) - f(s, x_\nu, \nu) \tag{3}\\
= \int_0^t f(s, x_\mu, \mu) - f(s, x_\nu, \mu) + f(s, x_\nu, \mu) - f(s, x_\nu, \nu) \tag{4}\\
\tag{5}
\end{align}$$

Taking norms and applying the triangle inequality we obtain:

$$\begin{align}
\|x_\mu - x_\nu\| \tag{6}\\
\leq \int_0^t ||f(s, x_\mu, \mu) - f(s, x_\nu, \mu)||ds + \int_0^t ||f(s, x_\nu, \mu) - f(s, x_\nu, \nu)||ds \tag{7}
\end{align}$$

Now the goal is to bound each of those norms Take

$$\begin{align}
\|f(s, x_\mu, \mu) - f(s, x_\nu, \mu)\| = \left\| \int_0^1 f_x(s, x_\nu + \tau(x_\mu - x_\nu), \mu)(x_\mu - x_\nu) d\tau \right\| \tag{8}\\
\leq \int_0^1 \|f_x\| \, |x_\mu - x_\nu| d\tau \tag{9}\\
\leq L|x_\mu - x_\nu| \tag{10}
\end{align}$$

Now to bound the other one:

$$\begin{align}
\|f(s, x_\nu, \mu) - f(s, x_\nu, \nu)\| \tag{11}\\
\leq \left\| \int_0^1 f_\mu(s, x_\nu, \nu + \tau(\mu - \nu))(\mu - \nu) d\tau \right\| \tag{12}\\
\leq \int_0^1 M|\mu - \nu| d\tau \tag{13}\\
= M|\mu - \nu| \tag{14}
\end{align}$$

Thus in total we have that:

$$\|x_\mu - x_\nu\| \tag{15}$$

$$\leq \int_0^t ||f(s, x_\mu, \mu) - f(s, x_\nu, \mu)||ds + \int_0^t ||f(s, x_\nu, \mu) - f(s, x_\nu, \nu)||ds \tag{16}$$

$$\leq \int_0^t L|x_\mu - x_\nu|ds + \int_0^t M|\mu - \nu|ds \tag{17}$$

$$\leq L \int_0^t |x_\mu - x_\nu|ds + tM|\mu - \nu| \tag{18}$$

$$\tag{19}$$

Using the more general gronwall inequality we derived earlier we have:

$$a(t) = tM|\mu - \nu| \tag{20}$$

$$b(t) = L \tag{21}$$

$$c(s) = 1 \tag{22}$$

Then:

$$\|x_\mu - x_\nu\| \leq tM|\mu - \nu| + L(\int_0^t sM|\mu - \nu|e^{\int_s^t Ldu}ds) \tag{23}$$

$$= M|\mu - \nu|(t + L\int_0^t se^{(t-s)L}ds) \tag{24}$$

$$= M|\mu - \nu|(t + L\frac{-Lt + e^{Lt} - 1}{L^2}) \tag{25}$$

$$= M|\mu - \nu|(t + \frac{-Lt + e^{Lt} - 1}{L}) \tag{26}$$

$$= M|\mu - \nu|(t - t + \frac{e^{Lt} - 1}{L}) \tag{27}$$

$$= M|\mu - \nu|(\frac{e^{Lt} - 1}{L}) \tag{28}$$

$$= \frac{M}{L}(e^{Lt} - 1)|\mu - \nu| \tag{29}$$

$$\tag{30}$$

So it is lipshitz. and the lipshitz bound is:

$$\frac{M}{L}(e^{Lt} - 1) \tag{31}$$

Which is dependent on t.

Exercise 2.2

a) To show its unique we will use the standard argument take:

$$x(t) = x_0 + \int_0^t f(x(s), s)ds \tag{32}$$

as the integral form of hte IVP. then assuming we have two different solutions

$$x(t) - y(t) = x_0 - y_0 + \int_0^t f(x(s), s) - f(y(s), s) ds \tag{33}$$

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_0^t |f(x(s), s) - f(y(s), s)| ds \tag{34}$$

$$\leq |x_0 - y_0| + \int_0^t p(|x(s) - y(s)|) ds \tag{35}$$

$$\tag{36}$$

At this point we would really like to use gronwalls inequality, but we cannot here. We will now prove a variant.

Assume that $\phi(t) \leq A + \int_0^t p(\phi(s)) ds$ set $u(t) = A + \int_0^t p(\phi(s)) ds$ where A is positive and p is monotonicall increasing and nonnegative. then since p is continuous by the FTC we can take derivatives:

$$u'(t) = p(\phi(t)) \leq p(u(t)) \tag{37}$$

That follows since p is monotonically increasing. then:

$$\frac{u'}{p(u(t))} \leq 1 \tag{38}$$

$$\int_0^t \frac{u'(s)}{p(u(s))} ds \leq t \int_{u(0)}^{u(t)} \frac{1}{p(s)} ds \leq t \tag{39}$$

Now note that becaue p is increasing $\frac{1}{p}$ is decreasing. and the only way $\int_0^1 \frac{1}{p(s)} = \infty$ blows up is when things are near the point $s = 0$. p(s) also has an asymptote at zero

From here note that the inequality we have derived

$$\int_{u(0)}^{u(t)} \frac{1}{p(s)} ds \leq t \tag{40}$$

Lets plug in things that we derived earlier in our quest for finding a unique solution. We would set $A = |x_0 - y_0|$ and $\phi(s) = |x(s) - y(s)|$. From here we are ready to prove uniqueness. Assume that with these solutions $y_0 = x_0$

from this we gather that $A = 0 = u(0)$. from this take a closer look at our previous inequality

$$\int_{u(0)}^{u(t)} \frac{1}{p(s)} ds \leq t \tag{41}$$

$$\int_0^{u(t)} \frac{1}{p(s)} ds \leq t \tag{42}$$

However we know by the osgood condition that the integral blows up, specifically around zero. So the only way this inequality holds is if $u(t) = 0$ for all time t. Which in turn means:

$$\phi(t) \leq u(t) = 0 \tag{43}$$
$$\phi(t) = ||x(t) - y(t)|| = 0 \tag{44}$$

So $x(t) = y(t)$

b) For this part we jump straight to:

$$\frac{u'(t)}{p(u(t))} \leq 1 \tag{45}$$
$$\frac{u'(t)}{Lu(t)(1 + |\log u(t)|)} \leq 1 \tag{46}$$
$$\tag{47}$$

Note that for small $u(t)$ $|\log(u(t))| = -\log(u(t))$

Here we make the substitution $v(t) = -\log(u(t)), u(t) = e^{-v(t)}$ then:

$$\int_{v(0)}^{v(t)} \frac{-e^{-v(s)}}{Le^{-v(s)}(1 + v(s))} ds \leq t \tag{48}$$
$$\int_{v(0)}^{v(t)} \frac{-1}{L(1 + v(s))} ds \geq t \tag{49}$$
$$\int_{-\log|x_0 - y_0|}^{v(t)} \frac{1}{L(1 + v(s))} ds \leq t \tag{50}$$
$$\int_{-\log|x_0 - y_0|}^{v(t)} \frac{-1}{(1 + v(s))} ds \leq Lt \tag{51}$$
$$-\log(1 + v(t)) + \log(1 - \log|x_0 - y_0|) \leq Lt \tag{52}$$
$$\log(1 + v(t)) \geq -Lt + \log(1 - \log|x_0 - y_0|) \tag{53}$$
$$1 + v(t) \geq e^{-Lt + \log(1 - \log|x_0 - y_0|)} \tag{54}$$
$$1 + v(t) \geq e^{-Lt}(1 - \log|x_0 - y_0|) \tag{55}$$
$$v(t) \geq e^{-Lt}(1 - \log|x_0 - y_0|) - 1 \tag{56}$$
$$\log(u(t)) \leq -e^{Lt}(1 - \log|x_0 - y_0|) - 1 \tag{57}$$
$$u(t) \leq e^{-e^{Lt}(1 - \log|x_0 - y_0|) - 1} \tag{58}$$
$$\tag{59}$$

thus we have $|x(s) - y(s)| \leq e^{-e^{Lt}(1 - \log|x_0 - y_0|) - 1}$. And that is the inequality bound