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Homework 1

Exercise 1.1 a) Let $x \in \text{range}(AB)$ then $\exists y$ with $x = ABy$ thus we know that $x = A(By)$ set $z = By$ thus $x = Az$ so $x \in \text{range}(A)$

b) Let $x \in \text{corange}(AB)$ then $\exists y$ with $x = (AB)^\star y$ thus we know that $x = B^\star(Ay)$ set $z = A^\star y$ thus $x = B^\star z$ so $x \in \text{corange}(B)$

c) Let $x \in \text{kernel}(B)$ then $Bx = 0$ thus $ABx = A0 = 0$ so $x \in \text{kernel}(AB)$

d) Let $x \in \text{cokernel}(A)$ then $A^\star x = 0$ thus $(AB)^\star x = B^\star A^\star x = B^\star 0 = 0$ so $x \in \text{cokernel}(AB)$

Exercise 1.2 From the definition lets compute RC

$$(RC)_{k,l} = \sum_{q=1}^{n} R_{k,q} C_{q,l} \tag{1}$$

From here note that $R_{k,:}$ is just the kth row of R and $C_{:,l}$ is merely the lth collumn of C. Thus we are just taking an inner product here between the kth row of R and the lth column of C

Thus

$$(RC)_{k,l} = r_k^T c_l \tag{2}$$

b) to do this we follow the definition note that:

$$(CR)_{k,l} = \sum_q C_{k,q} R_{q,l} \tag{3}$$

If we let l range over $1, L$ then writing this in vector form we get:

$$(CR)_{k,:} = \sum_q C_{k,q} R_{q,:} \tag{4}$$

Where here $R_{q,:}$ is the vector r_q^T:

$$(CR)_{k,:} = \sum_q C_{k,q} r_q^T \tag{5}$$

Now if we let k range from $1, K$:

$$(CR)_{:,:} = \sum_q C_{:,q} r_q^T \tag{6}$$

now we know that $C_{:,q}$ is just c_q thus:

$$(CR)_{:,:} = \sum_q c_q r_q^T (CR) = \sum_q c_q r_q^T \tag{7}$$

c) The maximal possible rank would thus have to be just n because each of these matrix are rank one and the sum of n rank one matrices can be at most rank n (Each matrix contributes one rank).

To prove this remember that the rank of a matrix is the dimension of its range. taking some arbitrary vector x we have

$$CRx = \sum_j c_j r_j^T x = \sum_j c_j (r_j^T x) = \sum_j (r_j^T x) c_j \tag{8}$$

This is just a linear combination of the c_j since there are only n c_j This can be at most n dimensional. Thus the range is at most n dimensional and the rank is at most n.

Exercise 1.3 the necessary and sufficient conditions are that A is invertible.

sufficient proof Assume A is invertible then:

$$\|x+y\|_A = \|A(x+y)\| = \|Ax+Ay\| \leq \|Ax\| + \|Ay\| = \|x\|_A + \|y\|_A \tag{9}$$

$$\|cx\|_A = \|A(cx)\| = \|cAx\| = |c|\,\|Ax\| = |c|\,\|x\|_A \tag{10}$$

note that since A is invertible $Ax = 0$ iff $x = 0$ thus we know that if $x \neq 0, Ax = y \neq 0$:

$$\|x\|_A = \|Ax\| = \|y\| > 0 \tag{11}$$

by definition of norm and if $x = 0, Ax = 0$

$$\|x\|_A = \|Ax\| = \|0\| = 0 \tag{12}$$

So all of the condition for a norm are satisfied

To prove the necessary condition assume that $\|-\|_A$ is a norm. Assume BWOC that A is singular. Since A is singular $\exists x, Ax = 0$ with $x \neq 0$

Then taking this x $\|x\|_A = \|Ax\| = \|0\| = 0$

But this is a contradiction since x is nonzero. So A must be Nonsingular **Exercise 1.4**

First note that

$$\|x\|_2^2 = \sum_i |x_i|^2 \leq \sum_i \max_k |x_k|^2 \leq n \max_k |x_k^2| = n\,\|x\|_\infty^2 \; \|x\|_2^2 = n\,\|x\|_\infty^2 \tag{13}$$

$$\|x\|_2 = \sqrt{n}\,\|x\|_\infty \tag{14}$$

$$\tag{15}$$

Now note that:

$$\|x\|_2^2 = \sum_i |x_i|^2 \geq \max_i |x_i|^2 = \|x\|_\infty^2 \tag{16}$$

So in our case $c = 1$ and $k = \sqrt{n}$

Choose $x = 1$ The ones vector for the first inequality with k and notice $\|1\|_2 = \sqrt{n}$ furthermore $\|1\|_\infty = 1$ thus $\|x\|_2 = \sqrt{n}\,\|x\|_\infty$

For the c inequality choose $x = e_1$ Note that $\|e_1\|_1 = 1$ and $\|e_1\|_2 = \sqrt{1+0+0+\dots} = 1$, thus $\|x\|_1 = \|x\|_2$ **Exercise 1.5**

$$\|Ax\|_1 = \sum_i |\sum_j A_{ij}x_j| \leq \sum_i \sum_j |A_{ij}||x_j| \tag{17}$$

$$= \sum_j \sum_i |A_{ij}||x_j| = \sum_j |x_j| \sum_i |A_{ij}| \leq \sum_j |x_j| \max_k \sum_i |A_{ik}| \tag{18}$$

$$= (\sum_j |x_j|) \max_k \sum_i |A_{ik}| \tag{19}$$

$$= \|x\|_1 \max_k \sum_i |A_{ik}| \tag{20}$$

$$\implies \tag{21}$$

$$\frac{\|Ax\|_1}{\|x\|_1} \leq \max_k \sum_i |A_{ik}| \tag{22}$$

$$\tag{23}$$

Note that since we need to take a supremum over all possible x values we can pick an x that achieves this bound namely e_k where k corresponds to the largest collumn sum thus:

$$\|A\|_1 = \sup_x \frac{\|Ax\|_1}{\|x\|_1} = \max_k \sum_i |A_{ik}| = \max_{j \in [n]} \|c_j\|_1 \tag{24}$$

$$\tag{25}$$

To prove the infinity norm statement note that:

$$\|Ax\|_\infty = \sup_i |\sum_j A_{ij}x_j| \tag{26}$$

$$\leq \sup_i \sum_j |A_{ij}||x_j| \tag{27}$$

$$\leq \sup_i \sum_j |A_{ij}|\,\|x\|_\infty \tag{28}$$

This means that:

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \sup_i \sum_j |A_{ij}| \tag{29}$$

We can actually choose an x that achieve this bound as well.

Namely choose $x = [\text{sign}(A_{i1}), \text{sign}(A_{i2}, \ldots, \text{sign}(A_{in})]$

Thus:

$$\|A\|_\infty = \sup_x \frac{\|Ax\|_\infty}{\|x\|_\infty} = \sup_i \sum_j |A_{ij}| = \max_{j \in [m]} \|r_j\|_1 \tag{30}$$

Exercise 1.6 To prove this take the definition of the induced norm (Note let $A \in \mathbb{R}^{m \times k}$, $B \in \mathbb{R}^{k \times n}$)

$$\|AB\| = \max_x \frac{\|ABx\|_m}{\|x\|_n} \tag{31}$$

$$= \max_x \frac{\|A(Bx)\|_m}{\|x\|_n} \tag{32}$$

$$= \max_x \frac{\|A(Bx)\|_m}{\|Bx\|_k} \frac{\|Bx\|_k}{\|x\|_n} \tag{33}$$

$$\tag{34}$$

Setting $y = Bx$

$$= \max_x \frac{\|A(y)\|_m}{\|y\|_k} \frac{\|Bx\|_k}{\|x\|_n} \tag{35}$$

$$\leq \max_x \max_y \frac{\|A(y)\|_m}{\|y\|_k} \frac{\|Bx\|_k}{\|x\|_n} \tag{36}$$

$$= \max_y \frac{\|A(y)\|_m}{\|y\|_k} \max_x \frac{\|Bx\|_k}{\|x\|_n} \tag{37}$$

$$= \|A\| \, \|B\| \tag{38}$$

For the frobenius norm note that:

$$\|AB\|_F^2 = \sum_i \sum_j (AB)_{ij} \tag{39}$$

$$= \sum_i \sum_j \sum_k |A_{ik} B_{kj}|^2 \tag{40}$$

$$= \sum_i \sum_j |A_{i,:} B_{:,j}|^2 \tag{41}$$

$$\leq \sum_i \sum_j \|A_{i,:}\|^2 \, \|B_{:,j}\|^2 \tag{42}$$

$$\leq (\sum_i \|A_{i,:}\|^2)(\sum_j \|B_{:,j}\|^2) \tag{43}$$

$$= (\sum_i \sum_k A_{i,k}^2)(\sum_i \sum_k B_{k,i}^2) \tag{44}$$

$$= \|A\|_F^2 \, \|B\|_F^2 \tag{45}$$

In this we used the cauchy-schwarts inequality. Taking square roots of both sides yields the theorem.