Drake Brown
Date: Apr 14, 2025

Homework 1

Exercise 4.1 a) for this first one notice that the characteristic polynomial is simply $\lambda^2 - 1 = 0$ so the eigenvalues are $-1, 1$. with corresponding eigenvectors $v_1 = [1,1]/\sqrt{2}, v_2 = [1,-1]/\sqrt{2}$. Constructing $V = [v_1, v_2]$ V is unitary so its inverse is its transpose V^T thus:

$$A = V \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} V^T \tag{1}$$

So:

$$e^{At} = V \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} V^T \tag{2}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3}$$

$$= \frac{1}{2} \begin{bmatrix} e^{-t} + e^t & e^t - e^{-t} \\ e^t - e^{-t} & e^{-t} + e^t \end{bmatrix} \tag{4}$$

b) For this one we take the characteristic polynomial:

$$(1-\lambda)(-\lambda) + \frac{1}{4} = \lambda^2 - \lambda + \frac{1}{4} \tag{5}$$

The roots of this are $\lambda = \frac{1}{2}$. I believe we will need to compute thee generalized eigenvector. We compute the first one:

$$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix} \tag{6}$$

The null space of this matrix is $v_1 = [1, -1]$. We coompute the generalized eigenvector:

$$\frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{7}$$

The generalized eigenvectvor here is clearly just $v_2 = [1,1]$ setting $V = [v_1, v_2]$ we compute $V^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Then we can take the jordan normal form to be:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \frac{1}{2} \tag{8}$$

We can then compute the matrix exponential to be:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{\frac{t}{2}} & e^{\frac{t}{2}} t \\ 0 & e^{\frac{t}{2}} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \frac{1}{2} \tag{9}$$

Note that I just did the jordan form matrix exponential in my head (its just a polynomial). Combining all of these together we get:

$$\frac{e^{\frac{t}{2}}}{2}\begin{bmatrix} t+2 & t \\ -t & 2-t \end{bmatrix} \tag{10}$$

c) For this third and final problem! We compute the characterstic polynomial as:

$$(-\lambda)(-\rho-\lambda)+\omega^2 \tag{11}$$
$$= \lambda^2+\rho\lambda+\omega^2 = 0 \tag{12}$$
$$\lambda = \frac{-\rho \pm \sqrt{\rho^2-4\omega^2}}{2} \tag{13}$$

Nowe we also compute the eigenvectors:

$$\begin{bmatrix} \frac{\rho\mp\sqrt{\rho^2-4\omega^2}}{2} & 1 \\ -\omega^2 & \frac{-\rho\mp\sqrt{\rho^2-4\omega^2}}{2} \end{bmatrix} \tag{14}$$

If you squint then we can see that the eigenvectors are

$$v_1 = \begin{bmatrix} \frac{-\rho\pm\sqrt{\rho^2-4\omega^2}}{2\omega^2} \\ 1 \end{bmatrix} \tag{15}$$
$$\tag{16}$$

Because this exactly cancels out the bottom term.

Ok so now what is important is if $\rho^2 < 4\omega^2$ if that is the case then we will want to convert to the standard form for imaginary ones. Assume for the moment that $\rho^2 < 4\omega^2$

$$v_1 = \begin{bmatrix} \frac{-\rho+\sqrt{\rho^2-4\omega^2}}{2\omega^2} \\ 1 \end{bmatrix} \tag{17}$$
$$\begin{bmatrix} \frac{-\rho+i\sqrt{4\omega^2-\rho^2}}{2\omega^2} \\ 1 \end{bmatrix} \tag{18}$$
$$\tag{19}$$

for this we need to convert to the normal form by constructing $V = [\mathrm{re}(v_1), \mathrm{im}(v_1)] = [\begin{bmatrix} \frac{-\rho}{2\omega^2} \\ 1 \end{bmatrix}] \begin{bmatrix} \frac{\sqrt{4\omega^2-\rho^2}}{2\omega^2} \\ 0 \end{bmatrix}$. this matrix has inverse:

$$\begin{bmatrix} 0 & 1 \\ \frac{2\omega^2}{\sqrt{4\omega^2-\rho^2}} & \frac{\rho}{\sqrt{4\omega^2-\rho^2}} \end{bmatrix} \tag{20}$$

From here we know that the standard form then looks like:

$$V \begin{bmatrix} \frac{-\rho}{2} & -\frac{\sqrt{4\omega^2-\rho^2}}{2} \\ \frac{\sqrt{4\omega^2-\rho^2}}{2} & -\frac{\rho}{2} \end{bmatrix} V^{-1} \tag{21}$$

The exponential of this center matrix is

$$e^{-\rho t/2} \begin{bmatrix} \cos(\frac{\sqrt{4\omega^2-\rho^2}}{2}t) & -\sin(\frac{\sqrt{4\omega^2-\rho^2}}{2}t) \\ \sin(\frac{\sqrt{4\omega^2-\rho^2}}{2}t) & \cos(\frac{\sqrt{4\omega^2-\rho^2}}{2}t) \end{bmatrix} \tag{22}$$

So the general solution when $\rho^2 < 4\omega^2$ is:

$$e^{At} = e^{\frac{\rho t}{2}} V \begin{bmatrix} \cos(\frac{\sqrt{4\omega^2-\rho^2}}{2}t) & -\sin(\frac{\sqrt{4\omega^2-\rho^2}}{2}t) \\ \sin(\frac{\sqrt{4\omega^2-\rho^2}}{2}t) & \cos(\frac{\sqrt{4\omega^2-\rho^2}}{2}t) \end{bmatrix} V^{-1} \tag{23}$$

which simplifies to:

$$\begin{pmatrix} e^{-t\rho/2} \frac{\rho \sin\left(\frac{t\sqrt{-\rho^2+4\omega^2}}{2}\right) + \cos\left(\frac{t\sqrt{-\rho^2+4\omega^2}}{2}\right)\sqrt{-\rho^2+4\omega^2}}{\sqrt{-\rho^2+4\omega^2}} & e^{-\rho t/2} \frac{2\sin\left(\frac{t\sqrt{-\rho^2+4\omega^2}}{2}\right)}{\sqrt{-\rho^2+4\omega^2}} \\ e^{-t\rho/2}\left(-\frac{2\omega^2 \sin\left(\frac{t\sqrt{-\rho^2+4\omega^2}}{2}\right)}{\sqrt{-\rho^2+4\omega^2}}\right) & e^{-t\rho/2} \frac{\cos\left(\frac{t\sqrt{-\rho^2+4\omega^2}}{2}\right)\sqrt{4\omega^2-\rho^2} - \rho \sin\left(\frac{t\sqrt{-\rho^2+4\omega^2}}{2}\right)}{\sqrt{4\omega^2-\rho^2}} \end{pmatrix} \tag{24}$$

In the case where $\rho^2 > 4\omega^2$ then we will have two eigenvalues and two lineraly independet eigenvectors that are as stated before:

$$V = \begin{bmatrix} \frac{-\rho+\sqrt{\rho^2-4\omega^2}}{2\omega^2} & \frac{-\rho-\sqrt{\rho^2-4\omega^2}}{2\omega^2} \\ 1 & 1 \end{bmatrix} \tag{25}$$

This matrix has inverse:

$$V^{-1} = \frac{1}{2\sqrt{\rho^2-4\omega^2}} \begin{bmatrix} 2\omega^2 & \sqrt{\rho^2-4\omega^2}+\rho \\ -2\omega^2 & \sqrt{\rho^2-4\omega^2}-\rho \end{bmatrix} \tag{26}$$

So then the matrix exponential for this one is thus:

$$e^{At} = V \begin{bmatrix} e^{t\frac{-\rho+\sqrt{\rho^2-4\omega^2}}{2}} & 0 \\ 0 & e^{t\frac{-\rho-\sqrt{\rho^2-4\omega^2}}{2}} \end{bmatrix} V^{-1} \tag{27}$$

Which once simplified is:

$$\begin{pmatrix} \frac{e^{t\frac{-\rho+\sqrt{\rho^2-4\omega^2}}{2}}\left(-\rho+\sqrt{\rho^2-4\omega^2}\right) - e^{t\frac{-\rho-\sqrt{\rho^2-4\omega^2}}{2}}\left(-\rho-\sqrt{\rho^2-4\omega^2}\right)}{2\sqrt{\rho^2-4\omega^2}} & \frac{-e^{t\frac{-\rho+\sqrt{\rho^2-4\omega^2}}{2}} + e^{t\frac{-\rho-\sqrt{\rho^2-4\omega^2}}{2}}}{\sqrt{\rho^2-4\omega^2}} \\ \frac{\omega^2\left(-e^{t\frac{-\rho-\sqrt{\rho^2-4\omega^2}}{2}} + e^{t\frac{-\rho+\sqrt{\rho^2-4\omega^2}}{2}}\right)}{\sqrt{\rho^2-4\omega^2}} & \frac{e^{t\frac{-\rho+\sqrt{\rho^2-4\omega^2}}{2}}\left(\sqrt{\rho^2-4\omega^2}+\rho\right) + e^{t\frac{-\rho-\sqrt{\rho^2-4\omega^2}}{2}}\left(\sqrt{\rho^2-4\omega^2}-\rho\right)}{2\sqrt{\rho^2-4\omega^2}} \end{pmatrix} \tag{28}$$

Finally if $\rho^2 = 4\omega^2$. Then we will have to use the generalize eigenspace because we will have a degenerate eigenvalue at $-\rho/2$. to do this first note that one eigenvector remains the same $\begin{bmatrix} -\rho/(2\omega^2) \\ 1 \end{bmatrix}$ but from here noteice that $\rho = 4\omega^2$. so this eigenvector is just $\begin{bmatrix} -2/\rho \\ 1 \end{bmatrix}$ We need to now find the generalized eigenvector. take:

$$\begin{bmatrix} \frac{\rho}{2} & 1 \\ -\omega^2 & \frac{-\rho}{2} \end{bmatrix} v = \begin{bmatrix} -2/\rho \\ 1 \end{bmatrix} \tag{29}$$

so clearly $v_2 = \begin{bmatrix} \frac{-4}{\rho^2} \\ 0 \end{bmatrix}$. We can then almost construct the jordan decomposition. We take the matrix V:

$$V = \begin{bmatrix} -2/\rho & -4/\rho^2 \\ 1 & 0 \end{bmatrix} \tag{30}$$

From here the inverse of this is:

$$V^{-1} = \begin{bmatrix} 0 & 1 \\ -\rho^2/4 & -\rho/2 \end{bmatrix} \tag{31}$$

. So then the jodran decomposition is

$$A = V \begin{bmatrix} -\rho/2 & 1 \\ 0 & -\rho/2 \end{bmatrix} V^{-1} \tag{32}$$

The matrix exponential of the inside is just $e^{-\frac{\rho}{2}t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. So then our final answer for this one is:

$$e^{At} = V \begin{bmatrix} e^{-\frac{\rho}{2}t} & e^{-\frac{\rho}{2}t}t \\ 0 & e^{-\frac{\rho}{2}t} \end{bmatrix} V^{-1} \tag{33}$$

which multiplied out is:

$$\begin{pmatrix} \left((-2/\rho)\, e^{-\frac{\rho}{2}t}t + (-4/\rho^2)\, e^{-\frac{\rho}{2}t}\right)(-\rho^2/4) & (-2/\rho)\, e^{-\frac{\rho}{2}t} + \left((-2/\rho)\, e^{-\frac{\rho}{2}t}t + (-4/\rho^2)\, e^{-\frac{\rho}{2}t}\right)(-\rho/2) \\ e^{-\frac{\rho t}{2}}t\,(-\rho^2/4) & e^{-\frac{\rho t}{2}} + e^{-\frac{\rho t}{2}}t\,(-\rho/2) \end{pmatrix} \tag{34}$$

Exercise 4.2 writing $x_1 = u, x_2 = u'$ we get:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ -2x_2 - x_1 \end{bmatrix} \tag{35}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{36}$$

to solve this we need to compute the matrix exponential. For this I will first find the eigenvalues of A:

$$(\lambda)(\lambda + 2) + 1 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 \tag{37}$$

So we have a repeated eigenvalue at $\lambda = -1$ to find the first eigenvector take:

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \tag{38}$$

Obviously the null space of this is $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So that is our first eigenvector. We need to solve for the second generalized one

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{39}$$

Once again the solution here is clear just take $v_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So the transformation vector is $[v_1, v_2]2$ The inverse of this is:

$$\begin{bmatrix} 1 & 1/2 \\ -1 & 1/2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1 & 1 \end{bmatrix} \tag{40}$$

So our jordan normal form is:

$$\begin{bmatrix} 1 & 1/2 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1 & 1 \end{bmatrix} \tag{41}$$

From this we can apply the matrix exponential:

$$e^{At} = \begin{bmatrix} 1 & 1/2 \\ -1 & 1/2 \end{bmatrix} e^{Jt} \begin{bmatrix} 1/2 & -1/2 \\ 1 & 1 \end{bmatrix} \tag{42}$$

from here we know that $e^{Jt} = e^{\Lambda t}(I + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$ So the general solution with initial data $u(0) = u_0, u'(0) = u_1$ is (setting $x_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$)

$$e^{At} = \begin{bmatrix} 1 & 1/2 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1 & 1 \end{bmatrix} \tag{43}$$

$$x(t) = \begin{bmatrix} 1 & 1/2 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1 & 1 \end{bmatrix} x_0 \tag{44}$$

$$= \begin{bmatrix} e^{-t} + te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix} x_0 \tag{45}$$

$$= \begin{bmatrix} u_0(e^{-t} + te^{-t}) + u_1 te^{-t} \\ u_0(-te^{-t}) + u_1(e^{-t} - te^{-t}) \end{bmatrix} \tag{46}$$

so since $x_1 = u$ we have that:

$$u = u_0(e^{-t} + te^{-t}) + u_1 te^{-t} \tag{47}$$

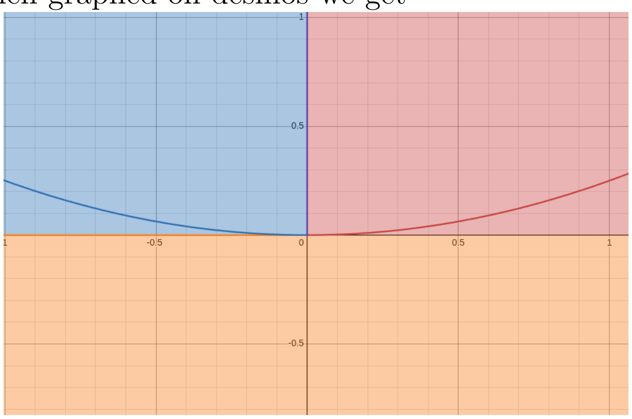
Exercise 4.3 To prove this we first solve for either eigenvalue:

$$\lambda_1 + \lambda_2 = T \tag{48}$$
$$\lambda_1 \lambda_2 = D \tag{49}$$
$$\lambda_2 = T - \lambda_1 \tag{50}$$
$$\lambda_1(T - \lambda_1) = D \tag{51}$$
$$\lambda_1^2 - T\lambda_1 + D = 0 \tag{52}$$
$$\lambda_1 = \frac{T \pm \sqrt{T^2 - 4D}}{2} \tag{53}$$

From here notice that $\lambda_2 = T - \lambda_1 = \frac{T \mp \sqrt{T^2-4D}}{2}$. So the two solutions are just given by:

$$\lambda_1 = \frac{T + \sqrt{T^2 - 4D}}{2} \lambda_2 = \frac{T - \sqrt{T^2 - 4D}}{2} \tag{54}$$

When graphed on desmos we get



Where the trace T is the x axis and the determinant D is the y axis. Here I will label each region:

I use x and y here, but just mentally remember $x =$Trace $y =$Determinant

- for $x < 0, 0 < y \leq \frac{x^2}{4}$ (the lower blue region) we have stable nodes (both eigenvalues are real and negative)

- for $x > 0, 0 < y \leq \frac{x^2}{4}$ (the lower red region) we have unstable nodes (both eigenvalues are real and positive)

- for $, y \leq 0$ and not including $(x, y) = (0, 0)$ (the orange region) we have saddle nodes (both eigenvalues are real, one is negative and one is positive and either could potentially be zero but not both)

- for $x = 0, y \geq 0$ (the purple line) we have the center nodes where both eigenvalues have zero real part.

- for $x < 0, y > \frac{x^2}{4}$ (the upper blue region) we have stable spirals (both eigenvalues have negative real part and some imaginary part)

- for $x > 0, y > \frac{x^2}{4}$ (the upper red region) we have unstable spirals (both eigenvalues have positive real part and some imaginary part)