Maths - 2

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week-1

L1.1 Vectors

- often we encounter a table. A vector can be thought as a list. It could be columns or rows.

  SA : (1, 2, 3, 4, 5) row vector / matrix

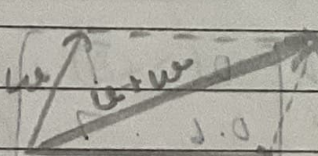
  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ column vector

- It can be used to perform arithematic opera^no on lists such as the table columns or rows.

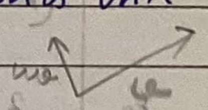
Eg. Arun 3kg rice & 2 kg dal
Neela 5kg rice & 6 kg dal } 8kg rice and 8kg dal

$(3, 2) + (5, 6) = (8, 8)$

- Add^n of vectors - it is component-wise in the list
- Scalar Multiplica^n - $x + x = 2x = 2(x)$
- Point $(a, b) \equiv$ Vector $(a, b) \equiv a\hat{i} + b\hat{j}$
  Visualization : arrow from the origin to (a, b)
- We can add 2 vectors by joining them head-to-tail or by parallelogram law.



- vectors in R^n are lists with n real entries.
  Vector with n entries $\equiv$ Vector in $R^n \equiv$ Points in R^n
- A vector has magnitude (size) and direc^n



- velocity, accelera^n, force

L1.2 Matrices

- Matrix - a rectangular array of nos., arranged in rows and columns.

$(1, 2)^{th}$ entry

Eg. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{2\times 3}$ 2×3 Matrix (2 rows and 3 columns)

- $(m \times n)$ matrix $\Rightarrow$ m rows and n column
- $(i,j)^{th}$ entry $\rightarrow$ i^{th} row & j^{th} column
- A square matrix $\Rightarrow$ no. of rows = no. of columns
- the i^{th} diagonal entry of the sq. matrix is the $(i,i)^{th}$ entry
- The diagonal of a sq. matrix is a set of diagonal entries.
- Diagonal matrix - sq. matrix in which all entries except the diagonal are 0.

$\begin{matrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{matrix}$

- Scalar matrix - A diagonal matrix in which all the entries of the diagonal are same.

$\begin{matrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{matrix}$

- Identity matrix (I) - Scalar matrix with all diagonal entries 1.

$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

- A set of linear eqns can be represented as matrices.

Eg. $3x + 4y = 5$
$4x + 6y = 10$
$\left[\begin{array}{cc|c} 3 & 4 & 5 \\ 4 & 6 & 10 \end{array}\right]$

Eg. $\begin{bmatrix} 1 & 9 \\ 0.6 & 7 \\ 4 & 1.5 \end{bmatrix}_{3\times 2} + \begin{bmatrix} 0 & 7 \\ 0.6 & 7 \\ 2.5 & 0.6 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1 & 16 \\ 1.2 & 14 \\ 6.5 & 2.1 \end{bmatrix}_{3\times 2}$

- $(A+B)_{ij} = A_{ij} + B_{ij}$

Eg. $3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$ Scalar Multiplication

- $(cA)_{ij} = c(A_{ij})$

Eg. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 7 & 10 & 13 \\ 15 & 22 & 29 \end{bmatrix}_{2\times 3}$

$7 = (1,1) \to (1\times1) + (2\times3) = 1+6 = 7$

$22 = (2,2) \to (3\times2) + (4\times4) = 6+16 = 22$

$13 = (1,2) \to (1\times3) + (2\times5) = 3+10 = 13$

Eg. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2} \times \begin{bmatrix} 5 \\ 6 \end{bmatrix}_{2\times1} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}_{2\times1}$ Matrix multiplicaⁿ

* The no. of columns in first matrix should be same as the no. of rows in second matrix.

- $A_{m\times n} \, B_{n\times p} = (AB)_{m\times p}$
  $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$] Matrix multiplicaⁿ

- $\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} c & 2c \\ 3c & 4c \\ 5c & 6c \end{bmatrix} = c \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

↳ multiplicaⁿ of a spe. (scalar/identity) matrix

$\boxed{I A_{3\times3} = A_{3\times3} = A_{3\times3} I}$ → Identity matrix acts as 1

- $(A+B) + C = A + (B+C)$ Associativity of addⁿ
- $(AB)C = A(BC)$ " of Multiplicaⁿ
- $A + B = B + A$ Commutativity of addⁿ
- $AB \neq BA$ (both should make sense)
- $\lambda(A+B) = \lambda A + \lambda B$ λ is a scalar
- $\lambda(AB) = \lambda(A)B = \lambda A(B)$
- $A(B+C) = AB + AC$
- $(A+B)C = AC + BC$

L1.3 System of Linear Equation

Eg.
$8x + 8y + 4z = 1960$ → Buyer A
$12x + 5y + 7z = 2216$ → Buyer B
$3x + 2y + 5z = 1135$ → Buyer C

solve it appropriately

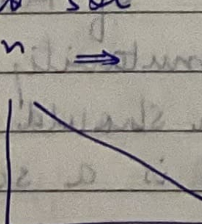
- linear eq^n is an eq^n of the form,
  $$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$
  b ↳ real no.
  x_1, x_2 are the variables
  a_1, a_2 are the coefficients
- A sys. of linear eq^n is the collec^n of more linear eq^{ns} with same set of var.
- The sol^n is the grp. of values to the variables such that all the eq^{ns} are simultaneously satisfied.
- Sys. of linear eq^n is equivalent to a matrix eq^n of form, $\boxed{Ax = b}$

  coefficient matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots \\ \vdots & & \end{bmatrix}$

  column matrix $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$

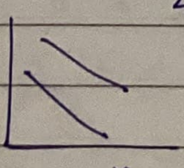
- There are 3 possibilities for the solu^{ns} of linear sys. of eq^n
  - ∞ sol^n
  - 1 unique sol^n
  - no sol^n
- ∞ sol^n ⇒ $2x + 4y = $ sth.
  $x + 2y = $ sth.

  (LHR = RHS ratio $\frac{1}{2}, \frac{1}{2}$)

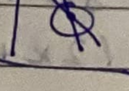


  both the equa^{ns} represent the same straight line in R^2.
- no sol^n → $(2x + y = 215) \times 2$
  $4x + 2y = 400$

  $430 \neq 400$



  both the eq^n have parallel line
- unique sol^n ⇒ $2x + y = $ sth.
  $3x + y = $ sth.



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11.4 Determinants (Part - 1)

- every sq. matrix A has an ass. no., called its determinant ⇒ det (A) or |A|
- solves sys. of linear eq^n, find inverse of matrix calculus and more.
- If $A = [a]$, a 1×1 matrix, then $\boxed{\det(A) = a}$
- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\boxed{\det(A) = ad - bc}$

Eg. $A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix}$

- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\boxed{\det(A) = a_{11} \times \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}}$$

Eg. $A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$

- det of I $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \det(I_2) = 1 - 0 = 1$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\det(I_3) = 1$

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ $AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$

$\det(AB) = [(1)(4)] - [(2)(3)]$

$$\boxed{\det(AB) = \det(A)\det(B)}$$

- Inverse of matrix,

$$(AA^{-1}) = I = A^{-1}A$$

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\bar{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ } By switching 2 rows

$$\boxed{\det(\bar{A}) = -\det(A)}$$

$\bar{A} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$ $\boxed{\det(\bar{A}) = -\det(A)}$

↳ by switching 2 columns

- Adding multiples of a row to another row

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\bar{A} = \begin{bmatrix} a + tc & b + tc \\ c & d \end{bmatrix}$

$$\boxed{\det(\bar{A}) = \det(A)}$$

- Scalar multiplicaⁿ of a row by a const. t.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\bar{A} = \begin{bmatrix} ta & tb \\ c & d \end{bmatrix}$

$$\boxed{\det(\bar{A}) = t \det(A)}$$

L1.5 Determinants (Part - 2) and lower matrices

- Upper triangular matrix. For such matrices, the determinant is the prod. of diagonal elements.
- The transpose of $A_{m \times n}$ is the $(n \times m)$ matrix with (i,j)-th entry of A_{ji}. $(A^T)_{ij} = A_{ji}$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 & 7 \\ 4 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

$$\det(A) = \det(A^T)$$

- If A is an $n \times n$ sq. with $n \leq 4$. Then the minor of the entry in the i^{th} row & j^{th} column is the determinant of the submatrix formed by deleting the i^{th} row & j^{th} column. M_{ij}

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \boxed{M_{11} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}}$$

The $(i,j)^{th}$ cofactor $\boxed{C_{ij} = (-1)^{(i+j)} M_{ij}}$

- For $A_{3\times3}$

$$\boxed{\begin{aligned} \det(A) &= a_{11} \times M_{11} - a_{12} \times M_{12} + a_{13} \times M_{13} \\ &= a_{11} \times C_{11} + a_{12} \times C_{12} + a_{13} \times C_{13} \end{aligned}}$$

- $\det(A) = \sum_{j=1}^{4} (-1)^{1+j} a_{1j} M_{1j} = \sum_{j=1}^{4} a_{1j} C_{1j}$

- $\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j} = \sum_{j=1}^{n} a_{1j} C_{1j}$

⇒ Tutorial - 1

- Geometric visualization (Geogebra)

$$\left.\begin{aligned} x + y &= 5 \\ x - y &= 5 \\ z &= 5 \end{aligned}\right\} (5, 0, 5) \Rightarrow \text{single sol}^n$$

$z = 5$ → parallel to $(x-y)$ plane

⇒ Tutorial - 2

$x + y + z = 8$

$x + y - z = 2$

$x + y = 5$

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⇒ Tutorial-3

$$x + y + z = 0$$

$$\left.\begin{matrix} x + y + z = 3 \\ x + y - z = 1 \end{matrix}\right.$$

∞ soln — no soln

$x + y = 2$

∞ soln

System of linear eqn will have no soln

⇒ Tutorial-4

$z = 5$
$z = 3$
$x = 4$

x, y, z

No solution possible

⇒ Tutorial-5

C_1	C_2	C_3	C_4 = $C_1 + C_2$
u_1	w_1	$\frac{1}{2} C_4$	$(u_1 + w_1)$
u_2	w_2	$\frac{1}{2} C_4$	$(u_2 + w_2)$
⋮	⋮	⋮	⋮

(scalar multiplica) (vector addition)

⇒ Tutorial-6

$M = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$; then find the value of λ such that $M^2 + \lambda M - 5I = 0$

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 6 \\ 4 & 7 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 0$$

$\lambda = -2$

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⇒ Tutorial - 7

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$$2x + y + 3z = 14$$
$$x + 2y + z = 17$$
$$2x + 3y + 3z = 18$$

$$A = \begin{array}{c} \begin{array}{ccc} 4 & 8 & 16 \end{array} \\ \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 3 \end{bmatrix} \end{array}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} 14 \\ 17 \\ 18 \end{bmatrix}$$

~~$2x + y + 2z = 14$~~
~~$x + 2y + 3z = 17$~~
$3x + y + 3z = 18$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 3 \end{bmatrix}$$

⇒ Tutorial - 8

$$A = \begin{bmatrix} 3x & 0 \\ 4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2y & 7 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 7 & x \\ 2x & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 3y \\ y & 0 \end{bmatrix}$$

① $A + B = C + D$, possible value of x & y

② if $\det(C) = -1$, find possible value of x

① $3x + 2y = 7$ $\qquad 3x + 2y = x + 3y$

$4 = 2x + y$ $\qquad 2x + y = 4$

$4x = 4$ $\qquad x = 1$

$y = 2$

② $7 - 2x^2$

Week-2

L2.1 Determinants (Part-2)

- 1×1 matrix $[a] \Rightarrow \det([a]) = a$
- For an $n\times n$ matrix, determinant is got by minor M_{ij}s and cofactors.
- the $(ij)^{th}$ cofactor $C_{ij} = (-1)^{i+j} M_{ij}$
- $$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \text{ for a fixed } i$$
  ↳ expansion along i^{th} row
  $$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \text{ for a fixed } j$$
  ↳ expanision along j^{th} column

Eg. for 2nd row, u get in 3×3 matrix
$= -a_{21} \times \det\begin{bmatrix} - & - \\ - & - \end{bmatrix} + a_{22} \times \det\begin{bmatrix} - & - \\ - & - \end{bmatrix} - a_{23} \times \det\begin{bmatrix} - & - \\ - & - \end{bmatrix}$

- Prop. 1 → Determinant of prod. is the prod. of det.
  $\Rightarrow \det(AB) = \det(A)\det(B) = \det(BA)$
  $\Rightarrow \det(A^n) = \det(A)^n$
  $\Rightarrow \det(A^{-1}) = \dfrac{1}{\det(A)} = \det(A)^{-1}$
  $\Rightarrow \det(P^{-1}AP) = \det(A)$
  $\Rightarrow \det(A^TA) = \det(A)^2$

  Transpose: $\det(A^T) = \det(A)$ (+ induc^n)

- Prop. 2 ⇒ Switching 2 rows/columns Δs the sign.
  $A = \begin{bmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{bmatrix}$ $\tilde{A} = \begin{bmatrix} a_{j1} & a_{j2} \\ a_{i1} & a_{i2} \end{bmatrix}$
  $\det(\tilde{A}) = -\det(A)$
- Prop. 3 ⇒ Adding multiples of a row to another row leaves the determinant unchanged.
- Prop. 4 ⇒ Scalar multiplica^n of a row by const. t, multiplies the determinant by t.
- Warning: $\det(t\,\underbrace{A_{n\times n}}_{\text{matrix}}) = t^n \det(A)$

- The det. of a matrix with row/column of 0s is 0.
- The det. of a matrix in which 1 row is the linear combination of other rows is 0.

1.2.2 Cramer's Rule

Eg.
$4x_1 - 3x_2 = 11$
$6x_1 + 5x_2 = 7$

$A = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}$ $b = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$

Unique solⁿ $\Rightarrow$ $x_1 = 2$, $x_2 = -1$

$\det(A) = 20 + 18$
$= 38$

$A_{x_1} = \begin{bmatrix} 11 & -3 \\ 7 & 5 \end{bmatrix}$ $A_{x_2} = \begin{bmatrix} 4 & 11 \\ 6 & 7 \end{bmatrix}$

$\det(A_{x_1}) = 76$ $(55-(-21))$ $\det(A_{x_2}) = -38$

$\dfrac{\det(A_{x_1})}{\det(A)} = \dfrac{-38}{38}$ $\dfrac{\det(A_{x_2})}{\det(A)} = \dfrac{76}{38} = 2$

Solutions $\rightarrow -1, 2$

- $a_{11}x_1 + a_{12}x_2 = b_1$
$a_{21}x_1 + a_{22}x_2 = b_2$

$Ax = b$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$A_{x_1} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$ $A_{x_2} = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$

$x_1 = \dfrac{\det(A_{x_1})}{\det(A)}$ $x_2 = \dfrac{\det(A_{x_2})}{\det(A)}$

The solⁿ of the sys. of eqⁿ in 2 var. → by cramer rule

- similarly goes for 3x3 matrix (invertible matrix)
goes for nxn matrix and b is a column vector with n entries.

Eg $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 4 & 3 & 1 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $b \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$\det(A) = -37$
$\det(A_{x_1}) = +12$
$\det(A_{x_2}) = -27$
$\det(A_{x_3}) = -4$

$x_1 = \dfrac{-12}{37}$ $x_2 = \dfrac{27}{37}$ $x_3 = \dfrac{4}{37}$

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L2.3 Solⁿ to a sys. of linear eqⁿ with an invertible co... matrix

- sq. matrix → is a $n \times n$ matrix ⇄ same rows / same columns
- The inverse of A is another $n \times n$ matrix B such that $AB = BA = I_{n \times n}$ & is denoted by A^{-1}
- Uniqueness of inverse

$$AB = BA = I \quad , \quad AC = CA = I$$

$$C(AB) = (CA)B$$

$$\therefore \quad B = C$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

- inverse of A exists ⇒ det (A) has to be non-zero
- $(i,j)^{th}$ minor is the det of submatrix is formed by deletion : M_{ij}

The ijth $C_{ij} = (-1)^{i+j} M_{ij}$

The adjugate matrix of A is → $\boxed{adj \cdot (A) = C^T}$

Eg. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix}$

$\det(A) = 2$

$M_{11} = -48$

$M_{21} = -18$ → C = cofactor matrix $= \begin{bmatrix} -48 & 40 & -10 \\ 18 & -15 & 4 \\ 10 & -8 & 2 \end{bmatrix}$

$M_{31} = 10$

$$adj(A) = C^T = \begin{bmatrix} -48 & 18 & 10 \\ 40 & -15 & -8 \\ -10 & 4 & 2 \end{bmatrix}$$

$$\boxed{A \frac{adj(A)}{\det(A)} = I}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix} \begin{bmatrix} -24 & 9 & 5 \\ 20 & -15/2 & -4 \\ -5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} adj(A)}$$

- $Ax = b \Rightarrow AA^{-1}x = A^{-1}b \Rightarrow \boxed{x = A^{-1}b}$

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Eg.
$8x_1 + 8x_2 + 4x_3 = 1960$
$12x_1 + 5x_2 + 7x_3 = 2215$
$3x_1 + 2x_2 + 5x_3 = 1135$

find all minors

$$C_{ij} = \begin{bmatrix} 11 & -39 & 9 \\ -32 & 28 & 8 \\ 9 & 8 & -56 \end{bmatrix}$$

$$A = \begin{bmatrix} 8 & 8 & 4 \\ 12 & 5 & 7 \\ 3 & 2 & 5 \end{bmatrix} \quad adj(A) = \begin{bmatrix} 11 & -32 & 9 \\ -39 & 28 & 8 \\ 9 & 8 & -56 \end{bmatrix}$$

$\det A = -188$

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$x = A^{-1} b = \frac{-1}{188} \begin{bmatrix} -8460 \\ -23500 \\ -28200 \end{bmatrix} = \begin{bmatrix} 45 \\ 125 \\ 150 \end{bmatrix}$$

- a sys. of linear eqn is homogeneous, if all of the constant terms is 0, ie, $b = 0$

$Ax = 0$

$x = A^{-1} 0 = 0$

→ has unique soln 0, if $\det(A) \neq 0$
→ has ∞ no. of soln, if $\det(A) = 0$

12.4 The echelon form

eg. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ $Ax = 0$

$x_1 + 2x_3 = 0$
$x_2 + 3x_3 = 0$

$x_1 = -2x_3$
$x_2 = -3x_3$

$x = \begin{bmatrix} -2c \\ -3c \\ c \end{bmatrix}$ x_1 & x_2 varies

∞ soln

- The matrix is in row echelon form if:
- (i) the first non-zero element in each row called the leading entry is 1

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(ii) each leading entry is in a column to the right of the leading entry in the previous row

(iii) rows with all 0 element, if any, are below rows having a non-0 element

for red. echelon form — (iv) for a non-zero row, the leading entry in the row is the only non-0 entry in its column

Eg. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

Eg. $A_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ⇒ yes, it's in row echelon form
⇒ its not in red echelon form

Eg. $A_2 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ⇒ yes it's row as well as reduced echelon form

$A_1 x = 0$

$A_2 x = b$

$x_1 + 2x_2 = b_1 \Rightarrow x_1 = b_1 - 2x_2$

$x_3 = b_2 \qquad \Rightarrow x_1 = b_1 - 2c$

$x_4 = b_3$

$x = \begin{bmatrix} b_1 - 2c \\ c \\ b_2 \\ b_3 \end{bmatrix}$

- A is red. row echelon form, suppose for some i, the i^{th} row of A is 0 row but $b_i \neq 0$. Then this sys. has no sol^n. $\quad 0x_1 + 0x_2 = b_i \Rightarrow$ no sol^n
- A is red. row echelon form, for i^{th} 0 row, $b_i = 0$
  - → if i^{th} column has the leading entry of some row, we call x_i ⇒ dependant var.
  - → on the other hand, x_i is independent

Eg. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$x_1, x_2 \rightarrow$ dep.

$x_3 \rightarrow$ indep. $\qquad$ only x_2 is indep.

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- for a dep. var., there is a unique solⁿ in which it occurs. All other var. in that eqⁿ are indep. var.
- All solⁿ can be obtained in this way.

1.2.5 Row Reduⁿ

- 1. Interchange 2 rows $\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

2. scalar multiplicaⁿ with t $\begin{bmatrix} & " & \end{bmatrix} \xrightarrow{R_1/3} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

3. add multiples of row to another row $\begin{bmatrix} & " & \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

- take the left most 0 column $\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

bring 1 on the top posⁿ of that column $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$ $(R_1/3)$

brings 0s under the above 1 $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix}$ $(R_2 - R_1)$

this is row-echelon form

repeat the steps for the submatrix $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 8 & 8 \end{bmatrix}$

this is row echelon form $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

now make all entries above the 1 to be 0 $\rightarrow \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $(R_2 + R_3)$ $(R_1 - \frac{1}{3} R_3)$

this is reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $(R_1 - \frac{2}{3} R_2)$

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Eg. $A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 1/3 \\ 0 & 2 & 6 \\ 0 & 0 & 8 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 1/3 \\ 0 & 1 & 3 \\ 0 & 0 & 8 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 1/3 \\ 0 & 1 & \\ 0 & 0 & \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 1 \end{bmatrix}$

Row red. form Row echelon form

$\det(A) = 2(72-42) - 4(27-35) + 1(18-40)$

$70 \to \frac{70}{2} = \frac{35}{2} = \frac{35}{2} \times \frac{2}{35} = 1$

effect on determinant:

$A \xrightarrow{R_1 \leftrightarrow R_2} B$ $\det(A) = -\det(B)$

$A \xrightarrow{R_1 / c} B$ $\det(A) = c \det(B)$

$A \xrightarrow{R_1 + cR_2}$ $\det(A) = \det(B)$

- for matrix A → it becomes upper triangular matrix after row echelon form.
- for upper triangular matrix, det = product of diagonals

L2.6 The Gaussian Elimination Method

- → Cramer's rule
- → adjugate matrix to calculate A^{-1}
- → reduced row echelon form

} Previous methods to find x in $Ax = b$

- Let $Ax = b$ be a sys. of linear eq^n where A is an $m \times n$ matrix & b is $m \times 1$ column vector.
- The augmented matrix is $m \times (n+1)$ whose first n columns are columns of A and the last column is b. → $[A \mid b]$

sol^{n}s of $Ax = b$ are the sol^{n}s of $Rx = c$

$[A|b] \to$ row red. form $\to [R|c]$

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Eg. $3x_1 + 2x_2 + x_3 + x_4 = 6$

$x_1 + x_2 = 2$

$7x_2 + x_3 + x_4 = 8$

$[A|b] \Rightarrow \left[\begin{array}{cccc|c} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_1/3} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$

1. $R_2 - R_1$
2. $3R_2$
3. $R_3 - 7R_2$
4. $R_3/8$

$\left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$

$R_1 - \frac{2}{3}R_2$

$\left[\begin{array}{cccc|c} 1 & \cancel{2/3}\,0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$

$x_1 = 1$

$x_2 = 1$

$x_3 + x_4 = 1$

$x_4 = c$

$x_3 + c = 1$

Set $\begin{bmatrix} 1 \\ 1 \\ 1-c \\ c \end{bmatrix}$

Eg. $x_1 + x_2 + x_3 = 2$

$x_2 - 3x_3 = 1$

$2x_1 + x_2 + 5x_3 = 0$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 2 & 1 & 5 & 0 \end{array}\right]$

1. $R_3 - 2R_1$
2. $R_3 - R_2$
3. $R_1 - R_2$

$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$

R   c

$\Rightarrow$ This does not have any sol^n to this sys.

- 0 is always a sol^n of homogenous sys. of linear eq^n $Ax = 0$. This sol^n is c/d trivial sol^n
- For a homogenous syst, there only 2 possibilites
  - 0 is the unique sol^n
  - there ∞ many sol^{ns} other than 0.
- In homogenous sys; if there are more var. than eq^n, then it is gauaranteed to have non-trivial sol^n

$\Rightarrow$ Tutorial 1

Eg. $2x + 3y = 5$

$6x - 5y = 1$

$[A|b] \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 6 & -5 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 6 & -5 & 1 \\ 2 & 3 & 5 \end{array}\right]$

ordering of eq^n gets interchanged

$\Rightarrow$ sol^n doesn't Δ

$R_2 + 3R_1$

$x = 1$

$y = 1$

$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 12 & 4 & 16 \end{array}\right]$

$\Downarrow$

not Δ the sol^n

$\downarrow 3R_1$

$\left[\begin{array}{cc|c} 6 & 9 & 15 \\ 6 & -5 & 1 \end{array}\right]$

multiplying the matrix with scale no Δ in sol.

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⇒ Tutorial-2

Eq. $(-x_1 + x_2 - x_3 = 0) \times 2$

$2x_1 + 2x_2 - 2x_3 = 2$

$x_2 + x_3 = -1$

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & 2 & -2 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

↓ Row echelon

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & -3/4 \end{array}\right]$$

$\cancel{-2x_1 + 2x_2 - 2x_3 = 0}$

$\cancel{2x_1 + 2x_2 - 2x_3 = 2}$

$-4x_1 = -2$

$\boxed{x_1 = \frac{1}{2}}$

$x_2 - x_3 = +\frac{1}{2}$

$x_2 + x_3 = -1$

$2x_2 = -\frac{2}{4}$

$-\frac{1}{2} + \frac{1}{1}$

$-\frac{3}{2}$

$\boxed{x_3 = \frac{-3}{4}}$

$\boxed{x_2 = \frac{-1}{4}}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -3/4 \end{array}\right]$$

⇒ Tutorial-3

Eq. $x_1 - x_3 = 0$

$-x_1 + x_2 + x_3 = -1$

$x_1 - x_2 - x_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

↓

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

No solⁿ

→ Tutorial-4

$x_2 - x_3 = 1$

$x_1 + 2x_3 = -1$

$x_1 + x_2 + x_3 = 0$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = c$

$x_1 + x_3 = 1 \Rightarrow x_1 = 1 - c$

$x_2 - x_3 = 1 \quad x_2 = 1 + c$

Sol^n $\begin{bmatrix} 1 - c \\ 1 + c \\ c \end{bmatrix}$

→ Tutorial - 5

- computing the inverse of an invertible matrix A is eq. to finding solⁿ of $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $Ay = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $Az = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{[A|I] \xrightarrow[\text{red. row echelon form}]{\text{reduce to}} [I|A^{-1}]}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 4 & 8 & 0 & 1 & 0 \\ 3 & 9 & 27 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 6 & -2 & 1 & 0 \\ 0 & 6 & 24 & -3 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\text{go on} \longleftarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/6 \end{array}\right]$$

Eg. $A = \begin{bmatrix} 2 & 4 & 6 \\ -1 & 3 & -3 \\ 0 & 4 & 2 \end{bmatrix} \Rightarrow A^{-1} = ?$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/2 & 0 & 0 \\ 0 & 5 & 0 & 1/2 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/10 & 1/5 & 0 \\ 0 & 0 & 1 & -2/10 & -4/10 & 1/2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/10 & 1/5 & 0 \\ 0 & 0 & 1 & -2/10 & -2/5 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} 9/10 & 8/10 & -3/2 \\ 1/10 & 1/5 & 0 \\ -2/10 & -4/10 & 1/2 \end{bmatrix}}_{A^{-1}}$$

→ Tutorial - 6

$$2x + y - 2z = -1$$
$$3x - 3y - z = 5$$
$$x - 2y + 3z = 6$$

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$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 3 & -3 & -1 & 5 \\ 1 & -2 & 3 & 6 \end{array}\right] \xrightarrow{\text{go on}} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 1 & -\frac{10}{3} & -\frac{13}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$z = 1$

$y - \frac{10}{3} = \frac{-13}{3}$

$x - 2 + 3 = 6$

$x + 1 = 6$

$x = 5$

$y = \frac{10}{3} - \frac{13}{3} = \frac{-3}{3} = -1$

⇒ Tutorial - 7

$2x + y - 3z = 0$

$4x + 2y - 6z = 0$

$x - y + z = 0$

$$\begin{array}{r} 2x + y - 3z = 0 \\ x - y + z = 0 \\ \hline 3x - 2z = 0 \end{array}$$

$3x = 2z \to x = \frac{2}{3}c$

$6y = 10z$

$y = z$

$z = c$

$$\begin{array}{r} 4x + 2y - 6z = 0 \\ 4x - 4y + 4z = 0 \\ \hline 6y - 10z = 0 \end{array}$$

$$\begin{bmatrix} \frac{2}{3}c \\ \frac{5}{3}c \\ c \end{bmatrix}$$

⇒ Tutorial - 8

$x - 3y + z = 4$

$-x + 2y - 5z = 3$

$5x - 13y + 13z = 8$

$-y - 3z = 7$

$$\begin{array}{r} 5x - 13y + 13z = 8 \\ 5x - 15y + 3z = 20 \\ \hline 2y + 10z = 28 \end{array}$$

$$\begin{array}{r} 2y + 6z = -14 \\ \hline 4z = 42 \end{array}$$

absurd

no sol^n

~~Introd~~ Week 3

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3.1 Introduction to Vector Spaces

- vectors in R^n
- $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$
- $c(x_1, x_2) = (cx_1, cx_2)$
- Prop of addition & scalar multiplication

(i) $v + w = w + v$

(ii) $(v + w) + v' = v + (w + v')$

(iii) $v + 0 = 0 + v = v$

(iv) $v + (-v) = 0$

(v) $1v = v$ (vi) $(ab)v = a(bv)$

(vii) $a(v + w) = av + aw$

(viii) $(a+b)v = av + bv$

- A vector space is the set with 2 operations (addition & multiplication). It has 2 functions,
  - it follows the above (i) → (viii) rules
  - $+ : V \times V \to V$ and $\cdot : R \times V \to V$

Eg. Matrices, let $M_{m\times n}(R)$ be the set of all $m \times n$ matrices with real nos.

$(A + B)_{ij} = A_{ij} + B_{ij}$

$(cA)_{ij} = c(A)_{ij}$

In this way, $M_{m\times n}$ along with addition and scalar multiplications form a vector space.

Eg. Solutions of homogenous system $(Ax = 0)$

$A(v + w) = Av + Aw = 0 + 0 = 0$

$A(cv) = c(Av) = c(0) = 0$

It is a vector space.

this is an eg. of subspace of vector space

So addition & multiplication on R^2 restricts to the solution set.

Non eg. $(x_1, x_2) + (y_1, y_2) \Rightarrow \{(x_1 + y_1), (x_1 - y_1)\}$, $c(x_1, x_2) = (cx_1, cx_2)$ — Rule i, ii, and viii fails

Non eg. $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, 0)$, $c(x_1, x_2) = (cx_1, 0)$ — Rule (iii), iv, and v fails

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L3.2 Some prop. of vectors spaces

- cancellation law of vector addition
  - $u_1 + u_3 = u_2 + u_3 \Rightarrow u_1 = u_2$
  - vector 0 in (iii) is unique
  - $-u$ in (iv) is also unique
- $0v = 0$
- $-c(v) = -(cv) = c(-v)$] all are true
- $c0 = 0$

Eg. stock taking

	stock	A	B	C	New
Rice	150	8	12	3	100
dal	50	8	5	2	75
Oil	35	4	7	5	30
biscuit	70	10	10	5	80
Soap	25	4	2	1	30

vector space {quantity rice, dal, oil, biscuit, soap}

$-u \Rightarrow$ is the demand

$+u \Rightarrow$ supply

Eg. Affine flats

Let V be a plane || to the XY-plane. Project P onto the XY plane, scale the resulting vector by c & project the result by P

L3.3 Linear Dependance

- vector addition $\Rightarrow u + w$
- scalar multiplication $\Rightarrow c.u$
- The linear combination of $u_1, u_2 \ldots u_n$ with coeff. $a_1, a_2 \ldots a_n$ is the vector $\sum_{a=1}^{n} a_i u_i \in V$

(vector space) $V \ni u_1, u_2, \ldots u_n$

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Eg. $2(1,2)+(2,1) \Rightarrow (2,4)+(2,1) = (4,5)$

$(4,5)$ is the linear combination of vectors $(1,2)$ & $(2,1)$. Moreover, each of these vectors in the expression is a linear combination of the other 2 vectors.

$$2(1,2)+(2,1)-(4,5) = (0,0) \Rightarrow 0 \text{ vector}$$

Eg. $3(2,1) + 2(-2,3) \Rightarrow (6,3)+(-4,6) = (2,9)$

$$3(2,1)+2(-2,3)-(2,9) = (0,0)$$

Eg. $2(0,2,1) + \frac{3}{2}(2,2,0) \Rightarrow (0,4,2)+(3,3,0)$
$= (3,7,2)$

- The plane of the 2 vectors $(0,2,1)$ & $(2,2,0)$ can be expressed in the eq^n $2x-2y+4z=0$. But, $(1,2,0)$ is not on this plane, so it can't expressed in the linear combination of those vectors
- The only way the 0 vectors of these 3 combina^n is if the coefficients are 0.
- A set of vectors $u_1, u_2 \dots u_n$ is said to be linearly dependant, if there exist scalars $a_1 \dots a_n$, not all 0, such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0$$

Eg. $(2,3,7)$ & $\left(\frac{5}{3}, \frac{5}{2}, \frac{35}{6}\right)$
$\times 5 \qquad \times 6$
$(0,0,0)$

$\Rightarrow$ They are linearly dependant

Eg. $(2,1,2)$, $(3,0,1)$ & $(10,-4,-2)$ $\Rightarrow + (2,3,7)$
$\times 2 \qquad \times -3 \qquad \times \frac{1}{2} \qquad \times 0$
$= (0,0,0)$

$\Rightarrow$ They are linearly dependant Then, fine

- If a set is linearly dependent, then the superset is also dependant.

L3.4 Linear Independance - Part 1

- They are linearly independent, if they are dependant,

$$a_1 v_1 + \dots a_n v_n = 0$$

when $a_i = 0$, $i = 1, 2 \dots n$

Eg. $(-1, 3)$ & $(2, 0)$

$$a(-1, 3) + b(2, 0) = (0, 0)$$

$-a + 2b = 0$ $\Rightarrow$ $a = 0$

$3a = 0$ $b = 0$

$\rightarrow$ linearly independent

- Let $v_1 \dots v_n$, be a set of vectors containing the 0 vector. Then, the linear combina^n is 0, but not all coefficients are 0.
  $\Rightarrow$ A vector set with 0 vector is linearly dependant
- if v_1 & v_2 are multiples of each other then they are linearly dependant. ($v_1, v_2 \in V$)
- 2 non-0 vectors are linearly independent when they are not multiples of each other.
- v_1, v_2 & v_3 are linearly dependant exactly when one of the vectors is a linear combination of the others.
- If 3 vectors are linearly independent then none of these vectors is a linear combina^n of the other 2.

Eg. $a(1, 1, 2) + b(1, 2, 0) + c(0, 2, 1) = 0$

$a + b = 0$ $\Rightarrow$ $b = -a$ $b = 0$

$a + 2b + 2c = 0$ $a - 2a - 4a = 0$ $\Rightarrow$ $a = 0$

$2a + c = 0$ $\Rightarrow$ $c = -2a$ $c = 0$

$\Longrightarrow$ linearly independent

6.3.5 Linear ~~Def~~ Independence - Part - 2

- $a_1 u_1 + a_2 u_2 + \dots + a_n u_n = 0$

  $u_{11} a_1 + u_{12} a_2 + \dots + u_{1n} a_n = 0$

  $\vdots$

  $u_{m1} a_1 + u_{m2} a_2 + \dots + u_{mn} a_n = 0$

- for linear independence, only choice of a_i is $a_i = 0$

Eg. $(5,2)$ & $(1,3)$ in R^2

$$x_1(5,2) + x_2(1,3) = (0,0)$$

$5x_1 + x_2 = 0$ $\qquad x_2 = -5x_1$

$2x_1 + 3x_2 = 0$

$2x_1 + 3(-5x_1) = 0 \Rightarrow 2x_1 - 15x_1 = -13x_1 = 0$

$\therefore x_1 = 0$, $x_2 = 0$

$\Longrightarrow$ linearly independent

Eg. $(1,2,0)$ & $(3,3,5)$ $\qquad$ Eg. (3×2)

$x_1 + 3x_2 = 0$

$2x_1 + 3x_2 = 0$ $\qquad x_2 = 0$, $x_1 = 0$ $\Big\} \Rightarrow$ linearly independent

$5x_2 = 0$

Eg. $(1,2)$ $(1,3)$ & $(3,4)$

$x_1 + x_2 + 3x_3 = 0$

$2x_1 + 3x_2 + 4x_3 = 0$ $\qquad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 3 & 4 & 0 \end{array}\right]$

∞ sol^n $\qquad x_1 = -5c$, $x_2 = 2c$, $x_3 = c$

$\Longrightarrow$ linearly dependant

- We have n vectors in R^2 $(n \geq 3) \Rightarrow \infty$ sol^{ns}

  $\Longrightarrow$ linearly dependant

- Any set of α vectors in R^n with $\alpha > n$ are linearly dependant.
- V is a sq. matrix, unique sol^n $x = 0$, V is invertible if $\det(V) \neq 0$

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Eg $(1,4,2)\ (0,4,3)\ \&\ (1,1,0)$

$x_1 + x_3 = 0$ $\quad -x_1 = +x_3$

$4x_1 + 4x_2 + x_3 = 0$ $\quad -\frac{2}{3}x_1 = x_2$

$2x_1 + 3x_2 = 0$

$\cancel{2}x_1 - \frac{8x_1}{3\ 2}\ \cancel{-x_1} = 0$

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$ } Independent

⇒ Tutorial-1

- subspace
① $0 \in W$
② $x \in W, y \in W \Rightarrow x+y \in W$
③ $c \in R, y \in W \Rightarrow cy \in W$

Eg. Let $V = R^2$
$W = \{(x,y) \mid x+y=0, x,y \in R\}$
① $0 \in W$ $\quad 0+0=0$
② $w_1 = (x_1, y_1) \in W$ & $w_2 = (x_2, y_2)$
$w_1 + w_2 = (x_1 + x_2, y_1 + y_2) = 0$
③ $c \in R$, $w_1 = (x_1, y_1) \in W$, $cw_1 = (cx_1, cy_1)$
$= c(x_1 + y_1) = 0$

⇒ Tutorial-2

Eg. $V = R^3$
$W = \{(x,y,z) \mid x - 4y + 2z = 0, x,y,z \in R\}$
① $0 \in R$ $\quad 0-0+0=0$ $\quad (x,y,z)+(0,0,0) = (x,y,z)$
② $w_1 = (x_1, y_1, z_1)$
$w_2 = (x_2, y_2, z_2)$ $\quad (x_1 + x_2, y_1 + y_2, z_1 + z_2)$
③ cw_1, $w_1 = (x_1, y_1, z_1) \Rightarrow (cx_1, cy_1, cz_1)$
$cx_1 - 4cy_1 + 2cz_1 = c(0) = 0$

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→ Tutorial -3

$V = M_{3\times3}(\mathbb{R})$

q. $W = \left\{ \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \;\middle|\; \begin{matrix} a_{11}+a_{12}+a_{13}=0 \\ a_{21}+a_{22}+a_{23}=0 \\ a_{31}+a_{32}+a_{33}=0 \end{matrix} \right\}$

① $\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$ $\quad 0 \in W$

② $W_1 = \left\{ \begin{matrix} - & - & - \\ - & - & - \\ - & - & - \end{matrix} \right\} \begin{matrix} =0 \\ 0 \\ 0 \end{matrix} \quad + \quad W_2 = \left\{ \begin{matrix} - & - & - \\ - & - & - \\ - & - & - \end{matrix} \right\} \begin{matrix} =0 \\ 0 \\ =0 \end{matrix}$

0

③ $cW_1 = 0$

⇒ Tutorial -4

q. $x(1,1,-3)\; y(-1,1,-1),\; z(-1,-1,3)$

$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ -3 & -1 & 3 \end{bmatrix}$ $\quad \det(A) = 0$

$x - y - z = 0$

$x + y - z = 0$

$-3x - y + 3z = 0$

$-2y = 0$

$y = 0 \qquad -6x = 0 \qquad x = 0$

⇒ linearly independent

⇒ Tutorial -5

- linearly independent vectors

$(-1,1,0)\; (1,0,1)\; (0,1,-1)$ $\quad A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$a = 0$ independent

$b = 0$ $\quad \det(A) \neq 0$

$c = 0$

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⇒ Tutorial - 6

Ex. W_1 & W_2 of subspace of V

$W_1 \cap W_2 \subseteq W_1 \subseteq V$
$\subseteq W_2 \subseteq V$

① $0 \in W_1$, $0 \in W_2$
$W_1 \cap W_2 \to 0$

② $a, b \in W_1 \cap W_2$ ⇒ $a, b \in W_1$ &
$a+b \to$
$a+b \to$ $a, b \in W_2$
$a+b \to$

③ $a \in W_1 \cap W_2$, $c \in R$ $a \in W_1$ & $a \in W_2$
$ca \in W_1 \cap W_2$

⇒ $W_1 \cap W_2$ is the subspace of V
⇒ $W_1 \cup W_2$ may not always be the subspace

Week-4

(4.1) What is the basis for a vector space?

- Let $u_1, u_2 \dots$ be vectors of vector space V.

linear dependant $\Rightarrow$ $\underbrace{a_1 u_1 + a_2 u_2 \dots + a_n u_n = 0}_{a \to \text{not all } 0}$

linear independent $\Rightarrow$ $\overbrace{a_1 u_1 + a_2 u_2 + \dots = 0}^{\text{all } a_i = 0}$

- The span of a set S is defined as set of all finite linear combina^ns^ of elements of $S \to \text{span}(S)$.

$$\text{span}(S) = \{\textstyle\sum_{i=1}^{n} a_i u_i \in V \mid a_1, a_2 \dots a_n \in R\}$$

Eg. $S = \{(1,0)\} \subset R^2$

$\text{Span}(S) = \{(a,0) \mid a \in R\}$

$\Rightarrow$ X-axis of R^2

Eg. $S = \{(1,1)\} \subset R^2$

$\text{span}(S) = \{(a,a) \mid a \in R\}$

Eg. $S = \{(1,0,0), (0,1,0)\} \subset R^3$

$\text{Span}(S) = \{a(1,0,0) + b(0,1,0)\} = \{(a,b,0) \mid a,b \in R\}$

$\Rightarrow$ The XY plane

- The set $S \subseteq V$ is a spanning set for V if $\text{span}(S) = V$

Eg. $S = \{(1,0), (0,1)\}$ $\text{span}(S) = R^2$

$S = \{(\text{"}), (\text{"}), (1,2)\}$ $\text{span}(S) = R^2$

$S = \{(1,1), (0,1)\}$ $\text{span}(S) = R^2$

$S = \{(1,0,0), (0,1,0), (0,0,1)\}$ $\text{span}(S) = R^3$

- S_0 $\text{Span}(S_0) = \text{Span}(\phi) = \{(0,0,0)\}$

$S_1 = S_0 \cup \{(0,2,1)\}$

$S_2 = S_1 \cup \{(2,2,0)\}$

$S_3 = S_2 \cup \{(0,0,5)\}$

$$(x,y,z) = \frac{y-x}{2}(0,2,1) + \frac{x}{2}(2,2,0) + \frac{x-y+2z}{10}(0,0,5)$$

$$\text{Span}(S_3) = R^3$$

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- Span $(S_0) = \{(0,0,0)\}$ $S_1 = S_0 \cup (3,0,0)$

  Span (S_1) = x-axis

  $S_2 = S_1 \cup (2,2,1)$ Span (S_2) = a plane

  $S_3 = S_2 \cup (1,3,3)$

  $$(x,y,z) = \frac{3x-5y+4z}{9}(3,0,0) + (y-z)(2,2,1) + \frac{2z-y}{3}(1,3,3)$$

  Span$(S_3) = R^3$

- A **basis** B of a vector space V is a linearly independent subset of V that spans V.
- Let $e_i \in R^n$ be the vector with i^{th} coordinate 1 and all other coordinates 0. $e_1 = (1,0,0..0)$

$\Rightarrow$ The set $e = \{e_1, e_2, ..., e_n\} \subseteq R^n$ is a basis for R^n

L4.2 Finding basis of vector spaces

- S is a spanning set for V if Span (S) = V
- Conditions are equivalent to a subset $B \subseteq V$ being a basis —

  (i) B is linearly indep. & Span (B) = V

  (ii) B is maximal linearly indep. set

  (iii) B is minimal spanning set
- Maximal $\Rightarrow$ it is linearly independent
  $\Rightarrow$ appending any vector makes it dependent
- Minimal $\Rightarrow$ it is spanning
  $\Rightarrow$ deleting a vector, no longer a spanning set
- (i) Start with ϕ and keep appending vectors which are not in the span of set, until we obtain a spanning set

  (ii) Take a spanning set, remove until rem. vectors are not a linear combinaⁿ of the others

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Eg. (i) $V = R^2$ $\{U.(1,2), U(2,3)\}$

span $\{(1,2), (2,3)\} = R^2$

Hence, this forms a basis for R^2

(size = 2)

Eg. (ii) $V = R^3$

$S = \{(1,0,0), (1,2,0), (1,1,3), (0,1,3), (0,4,1)\}$

Span $(S) = R^3$

delete delete

S_2 is the basis for R^3

(size = 3)

4.3 What is the rank / dim. of vector space?

- The dimension of vector space is the size of a basis of the vector space.
  - If B is a basis of V, then rank is the number of elements in B.
- For every vector there exists a basis, and all basis of a vector space have the same no. of elements. dim(V) / rank(V)
  (for now)
- The dimension of R^n is n.

Eg. dim? of (1,0,0), (0,1,0), (3,5,0)

delete (3,5,0)

⇒ The set $\{(1,0,0), (0,1,0)\}$ forms a basis of the subspace W

The dim(W) = 2

Eg. Write the vectors which span W as rows of $m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix}$

apply row red^n

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

no of non 0 rows = dim(W)
= 2

this forms the basis of subspace W

- Let A be $m \times n$ matrix. The column space of A is subspace of R^m spanned by the column vectors of A.

$$\boxed{\text{Column-rank} = \text{row rank} = \text{rank of } A}$$

Eg. Find rank of $\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow[\text{row echelon}]{\text{go to}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

There are 2 non-0 rows. Rank of (m) = 2

⇒ The row operations do not change the rank (A)

L4.4 Rank & Dim. using Gaussian Elimination

- find dim & basis with a given spanning set

set S = {(1, 0, 1), (-2, -3, 1), (3, 3, 0)}

→ form a matrix in the rows $\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$

→ reduce it to row echelon form (row reduced echelon form isn't necessary) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

→ the no. of non-zero rows forms the dim of vector = 2

→ vectors corresponding to matrix is the basis itself {(1, 0, 1), (0, 1, -1)}

Eg: (1, -2, 0, 4), (3, 1, 1, 0), (-1, -5, -1, 8), (3, 8, 2, -12)

$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1/7 & 12/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Dim (A) = 2

Basis ⇒ (1, -2, 0, 4), (0, 1, 1/7, 12/7)

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- The row based method produces a basis from a spanning set, but may not contain the vectors in the spanning set. Can we get a basis consisting of vectors in the spanning set?

Eg. same eq. in new way —
now keep them as columns

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 3 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rows}} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot elements

This matrix is in row echelon form and columns with the pivot entries (leading 1s) are the 1^{st} & 2^{nd} columns.

$\Rightarrow \therefore$, $(1,0,1)$, $(-2,-3,1)$ which are the 1^{st} and 2^{nd} vectors in S forms the basis.

Eg. same R^4 eq.

$$\begin{bmatrix} 1 & 3 & -1 & 3 \\ -2 & 1 & -5 & 8 \\ 0 & 1 & -1 & 2 \\ 4 & 0 & 8 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1st, 2nd

$\therefore$, $(1,-2,0,4)$ $(3,1,1,0)$ are the basis for W.

⇒ Tutorial -1

$V = M_{2\times 2}$ R
(2×2 Matrix)

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \cdots$$

$$\alpha\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \gamma\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \delta\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \alpha = \beta = \gamma = \delta = 0$$

→ linearly independent Matrix

basis for $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

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Tutorial-2

$$W = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \;\middle|\; \begin{matrix} a_{11} + a_{12} + a_{13} = 0 \\ a_{21} + a_{22} + a_{23} = 0 \\ a_{31} + a_{32} + a_{33} = 0 \end{matrix} \right\} \subseteq M_{3\times 3}\mathbb{R}$$

$a_{13} = -a_{11} - a_{12}$

$a_{23} = -a_{21} - a_{22}$ $\quad$ $a_{33} = -a_{31} - a_{32}$

Basis $\Rightarrow$ $\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \right\}$

$\Rightarrow$ linear independent

$\Rightarrow$ Tutorial-3

Eg. Find basis

$V = \{(x, y, z) \mid x = y + z\}$

$y = 1, \; z = 0, \; x = 1$ $\quad$ $a(1, 1, 0) + b(1, 0, 1)$

$y = 0, \; z = 1, \; x = 1$ $\quad$ $u_1 = \quad u_2 =$

$a + b = 0$

$a = 0$ $\quad$ $a = b = 0$

$b = 0$

$u = (y + z, y, z) \in V$ $\quad$ linearly independent

$u = y u_1 + z u_2$ $\quad$ $u \in \text{span}\{u_1, u_2\}$

$\therefore \{u_1, u_2\}$ basis of V

$\Rightarrow$ Tutorial-4

$S =$

Eg. If $\{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$, then $\{u_1, u_1 + u_2, u_1 - u_3\}$ is a basis of $\mathbb{R}^3$

$a u_1 + b(u_1 + u_2) + c(u_1 - u_3) = 0$ $\quad$ $a + b + c = 0 \Rightarrow a = 0$

$(a + b + c) u_1 + b u_2 - c u_3 = 0$ $\quad$ $b = 0$

$-c = 0$

S is linearly independent

$u = a u_1 + b u_2 + c u_3$

$(a - b - c) u_1 + b(u_1 + u_2) + c(u_1 - u_3)$

span (S)

→ Tutorial - 5

Basis ⇒ Maximal linearly independent

⇒ Minimal spanning set

(V) $\neq 0$

→ Tutorial - 6

- $V_1 = \{A \in M_{3\times3}(\mathbb{R}) : A = A^T\}$ ⇒ symm

2×2 $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ $\dim(V_1) = 6$

3×3 $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ basis = $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots 6 \text{ in all} \right\}$

- $V_2 = \{A \in M_{3\times3}(\mathbb{R}) : A$ is a scalar matrix $\}$

3×3 $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ $\dim(V_2) = 1$ basis = $\{I\}$

- V_3 = " : A is a diagonal }

$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ $\dim(V_3) = 3$ basis = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- V_4 = " : A is upper Δ

$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ $\dim(V_4) = 6$ basis (as same)

- V_5 = " : A is lower Δ matrix

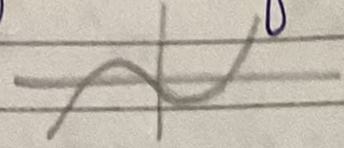
$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ $\dim(V_5) = 6$ basis (as same)

⇒ One Variable Calculus (Recall)

- $f : X \to Y$

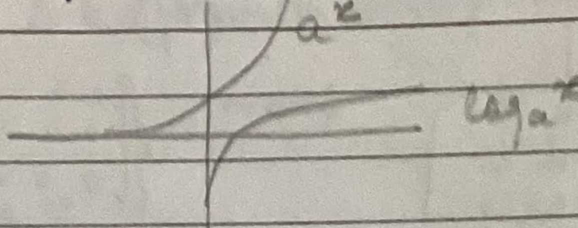
Domain : X ; $Y \to$ Codomain Range : $\{f(x) \mid x \in X\}$

- Linear funcⁿ → $f(x) = mx + c$
- Quadratic funcⁿ → $f(x) = a(x-b)^2 + c$
- $x^3 - 4x$



- $f(x) = a^x$ ⟶ ($a > 0$) exponential funcⁿ ⇒ fastest

$g(x) = \log_a x$ ⇒ : logarithmic funcⁿ ⇒ slowest



- A funcⁿ is monotone increasing, if $x_1 \leq x_2$ ⇒ $f(x_1) \leq f(x_2)$ & vice versa

Week-5

L5.1 The null space of a matrix : finding nullity and a basis : Part 1

- we can find dim & basis by Gaussian method
- The subspace $W = \{x \in \mathbb{R} \mid Ax = 0\}$ of $\mathbb{R}^n$ is the solⁿ space of the homogeneous sysᵐ of linear eqⁿ $Ax = 0$ or the null space of A.
- The null space is a subspace of $\mathbb{R}^n$. The dimension of the null space is c/d the nullity of A.
- We use row reducⁿ to find the nullity and a basis for null space of A.

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→ $[A|b]$
→ Reach to $[R|c]$ row red. echelon form
→ i^{th} column has leading entry of some row ⟹ dependent var.

$\boxed{\text{nullity}(A) = \text{no. of independent var.}}$

→ assign t_i value to the i^{th} indep. var.
→ compute the value of each dep. var. in terms of t_is from unique row.

- Vectors obtained by subsituting $t_i = 1$ & $t_j = 0 \ \forall j \neq i$ as i varies constitutes the basis of the null space of A. (i.e sol^n space of $Ax = 0$

Eg. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ $[A|b]$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \xrightarrow{RRE} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x_1 dep

nullity (A) = 2 x_1 → dep
$(x_2), x_3$ → indep.

$x_2 = t_1$, $x_3 = t_2$

$x_1 = -t_1 - t_2$

null space is $\{(-t_1 - t_2, t_1, t_2) \mid t_1, t_2 \in R\}$

$t_1 = 0, t_2 = 1$

basis is $(-1, 1, 0), (-1, 0, 1)$

- The augmented 0 vector remains unchanged during the row red^n process. → not needed

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L5.2 Part 2

Eg. $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ $Ax = 0$

$Ax = b$
$[A|b]$
$[R|C]$

$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

Nullity(A) = 1

$x_4 = t_1$ $x_1 - 3t_1 = 0$ $x_1 = 3t_1$
$x_2 = -3t_1$
$x_3 = -2t_1$

$\{(3t_1, -3t_1, -2t_1, t_1) \mid t_1 \in \mathbb{R}\}$ ⇒ Null space

$t_1 = 1$ ⇒ basis = $(3, -3, -2, 1)$

- Rank-Nullity Theorem
→ rank(A) is the no. of non-0 rows of matrix R in red. row echelon form.
→ matrix R (in row form) the no. of non-0 rows = no. of dependent var. for $Rx = 0$

rank(A) = no. of non-0 rows of R = no. of dep. var.
nullity(A) = no. of indep. var.

For $n \times n$ matrix A, $\boxed{\text{rank}(A) + \text{nullity}(A) = n}$

- n vectors of $\mathbb{R}^n$ ⇒ write them as columns of a matrix, → $n \times n$ matrix
⇒ If det = 0, then the given vectors is not a basis, otherwise it is basis

Eg. $(1,2,3), (0,1,2), (1,3,0)$ ⇒ basis? or not?

$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 0 \end{bmatrix}$ ⇒ $\det(A) = -5 \neq 0$

⇒ yes, it is a basis

Q. $(1,2,3,0)$, $(0,1,2,1)$, $(1,3,0,2)$, $(2,6,5,3)$ → basis?

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 6 \\ 3 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

det = 0

⇒ so, not a basis of R^4

What is Linear Mapping – Part 1

	Rice	Dal	Oil
Shop A	45	125	150

cost of 1 kg rice, 2 kg dal & 1 l oil

⇒ $1 \times 45 + 2 \times 125 + 1 \times 150 = ₹445$

⇒ $45x_1 + 125x_2 + 150x_3$ ⇒ This is an expression

C_A func^n is LC of x_1, x_2, x_3, it is an eq. of linear func^n.

$$C_A(x_1, x_2, x_3) = 45x_1 + 125x_2 + 150x_3 = \begin{bmatrix} 45 & 125 & 150 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Q. 30 tiffins by 20 kg rice, 10 kg dal, 4 l of oil for office 1 they got a shop A. For office 2, 40 tiffins, 30 kg of rice, 12 kg of dal & 2 l of oil. Next day, both office wants 15 & 50 tiffins. What is the cost?

1 ⇒ 10 kg rice, 5 kg dal, 2 l oil

2 ⇒ 37.5 kg rice, 15 kg dal, 2.5 l oil

Total = 47.5 kg, 20 kg, 4.5 l

= ₹5312.5

Sol

Cost of 1st office (initial) = ₹2750

Cost of office 2 (initial) = ₹3150

Cost (final) $= \frac{1}{2} \times 2750 + \frac{5}{4} \times 3150 = ₹5312.5$

	Rice	Dal	Oil	C_A
M	20	10	4	2750
T	30	12	2	3150
W	47.5	20	4.5	5312.5

L5.4 Part 2

Sam q. we have shop B & C

	Rice	Dal	Oil
A	45	125	150
B	40	120	170
C	50	130	160

$\Rightarrow$ how to decide which is cheaper?

$$C_A = \begin{bmatrix} 45 & 125 & 150 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$C_B = \begin{bmatrix} 40 & 120 & 170 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$C_C = \begin{bmatrix} 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$C_C > C_A$ (always)

but, for C_B

$$C(x_1, x_2, x_3) = (C_A(x_1, x_2, x_3), C_B(x_1, x_2, x_3), C_C(x_1, x_2, x_3))$$

$$C(x_1, x_2, x_3) = \begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

for $x_1 = 2$, $x_2 = 1$, $x_3 = 2$

- The prop. of linearity of costs can be extracted from the matrix form for the cost func^n C

$$C(\alpha(x_1, x_2, x_3) + (y_1, y_2, y_3)) = \alpha\, C(x_1; x_2, x_3) + C(y_1, y_2, y_3)$$

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- A linear mapping f from R^n to R^m

$$f(x_1, x_2 \dots x_n) = \left(\sum_{j=1}^{n} a_{1j} x_j , \dots \sum_{j=1}^{n} a_{mj} x_j \right)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & \dots & & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- A linear mapping follows linearity.

$$f(x_1 + cy_1 , x_2 + cy_2) = f(x_1, x_2) + cf(y_1, y_2)$$

15.5 What is linear transformation?

- A function $f: V \to W$ is said to be a linear transformation if for any 2 vectors v_1 & v_2 in V and $c \in R$.
  - $\to f(v_1 + v_2) = f(v_1) + f(v_2)$ → linear mappings are linear transformation
  - $\to f(cv_1) = cf(v_1)$

Eg. $f: R^2 \to R^2$ $\quad f(x, y) = (2x, y)$

$\quad f(x, y) = (2x, 0)$

$f: R^3 \to R^3$ $\quad f(x, y, z) = (x/2, 3y, 5z)$

$f: R^2 \to R$ $\quad f(x, y) = x$

- $f: V \to W$ is 1-1 (injective) if $f(v_1) = f(v_2) \Rightarrow v_1 = v_2$ and onto (surjective) if for every $w \in W$ there exists $v \in W$, such that $f(v) = w$
- For linear transformation, being 1-1 is equivalent to $f(v) = 0 \Rightarrow v = 0$
- $f: V \to W$ is bijective is it's 1-1 & onto.
  $\Leftrightarrow$ for any $w \in W$ there exists a unique $v \in ~~W~~$ such that $f(v) = w$

- A linear transformaⁿ $f: V \to W$ btw. 2 vector spaces V & W is said to be an isomorphism if it's a bijecⁿ.

Eg. $f(x, y) = (2x, y)$

1-1: $(2x, y) = (0, 0)$
$2x = 0, y = 0$
$x = 0, y = 0$
$(x, y) = (0, 0)$

onto: $(2x, y) = (u, v)$

$\Rightarrow$ bijection

$f(x, y) = (2x, 0)$ $\Rightarrow$ not bijection
$(0, 1)$ has no pre-image.

- Let V be a vector with basis $(u_1, u_2 \dots u_n)$. Let $f: V \to W$ be a linear transformaⁿ. Then the ordered vectors $f(u_1), f(u_2) \dots$ uniquely determine f.

$$f(u_k) = w_k$$

Eg. $\{(1,0), (0,1)\}$ of $R^2 \to$ std. basis

$f: R^2 \to R^2$

$f((1,0)) = (2,0) = 2(1,0)$
$f((0,1)) = (0,1)$

- If we choose to Δ the basis, we get a diff. linear transformaⁿ.

If $\{(1,0), (1,1)\}$ is basis for R^2
then $(2x - 2y, y)$ is its linear transformaⁿ.

$\Rightarrow$ Tutorial - 1

Eg. $A = \begin{bmatrix} 2 & 4 & 0 & 2 \\ 1 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}_{3\times 4} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_{4\times 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{3\times 1}$

$\rightarrow$ homogenous sys. of linear eqⁿ

$$\begin{bmatrix} 2 & 4 & 0 & 2 \\ 1 & 0 & 3 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \xrightarrow[\text{form}]{\text{Red. row echelon}} \begin{bmatrix} 1 & 0 & 0 & -4/5 \\ 0 & 1 & 0 & 9/10 \\ 0 & 0 & 1 & 3/5 \end{bmatrix}$$

Rank = 3

Nullity = 1

$x_4 = t$

$x_1 - \frac{4}{5}t = 0 \quad x_1 = \frac{4t}{5}$

$x_2 = -\frac{9t}{10}, \quad x_3 = -\frac{3t}{5}$

Null space : $\left(\left\{ \frac{4t}{5}, -\frac{9t}{10}, -\frac{3t}{5}, t \right\} \middle| t \in R\right)$

Basis of Nullspace : $\{(4/5, -9/10, -3/5, 1)\}$

Rank Nullity Th. $\Rightarrow$ A is $m \times n$ matrix

Rank (A) + Nullity (A) = n

$\Rightarrow$ Tutorial - 2

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 1 & 2 & 5 \end{pmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1, x_2 \rightarrow$ dep. $\quad x_3 = t \Rightarrow$ indep.

$x_1 = -t \quad x_2 = -2t$

Rank (A) = 2

Nullity = 1

Null space: $\{(-t, -2t, t) \mid t \in R\}$

Basis : $\{(-1, -2, 1)\}$

$\Rightarrow$ Rank - nullity theorem verified

$\rightarrow$ Tutorial - 3

$S = \{(1,2,0), (0,3,1), (3,3,-1), (3,0,-2)\} \leq R^3$

$V = \text{span}\{\underbrace{(1,2,0), (0,3,1)}_{\text{basis}}\}$

dim (V)

dim (V) $\leq$ 2

$a(1,2,0) + b(0,3,1) = 0$

$a = 2 \Rightarrow a = 0$

$2b \quad b = 0$

dim (V) = 2) $\Rightarrow$ independent

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \begin{matrix} 3 & 3 \\ 3 & 0 \\ -1 & -2 \end{matrix} \right\} \xrightarrow[\text{row echelon form}]{\text{Red.}} \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $(A) = 2$, nullity$(A) = 2$

→ independent vectors

$W = \text{span}(S) \Rightarrow \dim(W) = 2$

⇒ Tutorial - 4

Rank of $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -2 \\ 1 & -3 & a \end{bmatrix}$ is 2. Find the value of a?

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -2 \\ 1 & -3 & a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3/2 & 2 \\ 0 & 1 & -2 \\ 0 & -3/2 & a-2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3/2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 - \frac{2a}{3} \end{bmatrix}$$

$$\cancel{2} = \frac{2}{3}(a-2) \qquad \underline{a = 5}$$

⇒ Tutorial - 5

$T : R^2 \to R^3$ is a linear trans.

$T(x, y) = (2x + 3y,\ 5x - y,\ x + 6y)$

T is 1-1 / ~~onto~~ ?

Very Impt

⇒ $T : R^n \to R^m$, rank (T) + nullity $(T) = n$

rank$(T) = m$, then T is onto | rank $T \le m$

nullity$(T) = 0$, then T is 1-1 | nullity $T \le n$

$n > m \Rightarrow$ if T is one-one, rank$(T) = n \Rightarrow$ T cannot be 1-1

$m > n \Rightarrow$ if T is onto, nullity $= n - m < 0 \Rightarrow$ T can't be onto

$n = m \Rightarrow$ if T is onto, nullity$(T) = 0$

$\Rightarrow$ T is 1-1

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$T(x,y)=0 \Rightarrow (x,y)=0$

$$\left.\begin{array}{l} 2x+3y=0 \\ 5x-y=0 \\ x+6y=0 \end{array}\right\} \quad \left.\begin{array}{l} 5x=y \\ x=-6y \end{array}\right\} \quad \begin{array}{l} 5(-6y)=y \\ \Rightarrow -30y=y \\ \Rightarrow y=0 \\ \Rightarrow x=0 \end{array}$$

T is 1-1

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Week-6

L6.1 Linear Transformation, ordered bases and matrices

- Let V be a vector space. Choose a basis $\{v_1, v_2, \dots v_n\}$

$f: V \to R^n$ for $c_i \in R^n$ for each i.

$\Rightarrow$ f is isomorphism

$v = \sum c_i v_i \quad f(v) = \sum c_i e_i \Rightarrow f(v_i) = e_i$

Eg. $W = \{(x, y, z) \mid x + y + z = 0\} \Rightarrow$ basis is $(-1,0,1), (-1,1,0)$

$f(-1,1,0) = (1,0)$ & $f(-1,0,1) = (0,1)$ is an isomorphism

$(x, y, z) = y(-1,1,0) + z(-1,0,1)$

$f: W \to R^2$ is $f(x,y,z) = y(1,0) + z(0,1) = (y,z)$

Eg. $f: R^2 \to R^2 \quad f(x,y) = (2x, y)$

$f(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$f(1,0) = (2,0) = 2(1,0) + 0(0,1)$

$f(0,1) = (0,1) = 0(1,0) + 1(0,1)$

- $f: V \to W$ be a linear transformation. Let $\beta = v_1, v_2, \dots v_n$ be an ordered basis of V and $\gamma = w_1, w_2 \dots$ ordered basis of W.

Matrix corresp. to the linear transformation f w.r.t. ordered bases of β & γ $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & & \dots & a_{mn} \end{bmatrix}$

Eg. $f(1,0) = (2,0) = 2(1,0) + 0(0,1)$

$f(1,1) = (2,1) = 1(1,0) + 1(1,1)$

$\{(1,0), (1,1)\} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

A is $m \times n$ matrix.

$$f(u) = \sum_{j=1}^{n} c_j \sum_{i=1}^{m} A_{ij} w_i$$

$$f(v_k) = A_{1k} w_1 + A_{2k} w_2 + \dots + A_{mk} w_m$$

Let β & γ be ordered basis for V & W
↳ $n = \dim(V)$ and $m = \dim(W)$

There is a bijection ⇒

$\left\{ \begin{array}{l} \text{Linear Transformation} \\ \text{from } V \to W \end{array} \right\} \longleftrightarrow \{ m \times n \text{ matrices} \}$

Eg. $W = \{(x, y, z) \mid x + y + z = 0\}$ $V = \mathbb{R}^2$
$\beta = (-1, 1, 0), (-1, 0, 1)$ $\gamma = (1,0)\ (0,1)$

$$\text{Matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If $\gamma = (1, 0), (1, 1)$

$$f(-1, 1, 0) = 1(1,0) + 0(1,1)$$
$$f(-1, 0, 1) = \cancel{+}1(1,0) + 1(1,1)$$

$$\text{Matrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

- Thus, Δing the ordered basis gives us diff matrices for same linear transformations.

6.2 Image and Kernel of Linear Transformation

$f: V \to W$ be a linear Transformation

$$\ker(f) = \{ v \in V \mid f(v) = 0 \}$$

$$\operatorname{Im}(f) = \{ w \in W \mid \exists\, v \in V \text{ for which } f(v) = w \}$$

- $Im(f)$ is another name of 'range of func^n'

Eg. $f: R^2 \to R^2$ $f(x,y) = (2x, y)$
$ker(f) = \{(0,0)\}$ $Im(f) = R^2$

Eg. $f: R^2 \to R^2$ $f(x,y) = (2x, 0)$
$ker(f) = \{(0,y) \mid y \in R\}$ $Im(f) = \{(x,0) \mid x \in R\}$ = X-axis
= Y-axis

- $f: V \to W$ is 1-1 if $f(v_1) = f(v_2) \Rightarrow v_1 = v_2$
For linear transforma^n, $f(v) = 0 \Rightarrow v = 0$
$\Rightarrow$ A linear transforma^n f is 1-1 iff $ker(f) = 0$
- $f: V \to W$ is onto if $fV = W$
$\Rightarrow$ A linear transformation $f: V \to W$ is onto iff if $Im(f) = W$
- $V = \sum_{j=1}^{n} c_j v_j \in ker(f)$

$$\Longleftrightarrow \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$ is the null space of A

A is matrix corres. to wrt β & γ

- $W = \sum_{i=1}^{m} d_i w_i \in W$
$w \in Im(f)$,

Such that, $\sum_{j=1}^{n} A_{ij} c_j = d_i$

$$\Leftrightarrow \quad d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$$ is the column space of A

- The relation btw. kernal and null spaces yields an isomorphism between them

The vectors $\begin{bmatrix} c_{k1} \\ c_{k2} \\ \vdots \\ c_{kn} \end{bmatrix}$ form a basis when $u_1' \dots u_k' \in \ker(f)$

$u_i' = \sum_{j=1}^{n} c_{ij} v_j$ forms a basis of $\ker(f)$

- Similarly,
  $u_i' = \sum_{j=1}^{m} d_{ij} w_j$ form a basis for $\mathrm{Im}(f)$.

11.3 Eg. of finding bases for the kernel and img. of linear transforma^n

- basis of nullspace of (A) will be the basis of $\ker(f)$.
- basis of column space (A) yields basis of $\mathrm{Im}(f)$

Q. $T: \mathbb{R}^4 \to \mathbb{R}^3$ $\quad T(x_1, x_2, x_3, x_4) =$

$(2x_1 + 4x_2 + 6x_3 + 8x_4,\ x_1 + 3x_2 + 5x_4,\ x_1 + x_2 + 6x_3 + 3x_4)$

$\beta \Rightarrow \mathbb{R}^4 \quad \gamma = \mathbb{R}^3$

$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}$

$x_3 = t_1, \quad x_4 = t_2$

$x_2 = 3t_1 - t_2$

Null space of (A) $\Rightarrow (-9t_1 - 2t_2,\ 3t_1 - t_2,\ t_1,\ t_2)$

Basis of (A) $= \{(-9, 3, 1, 0), (-2, -1, 0, 1)\}$

$\boxed{\ker(T) = \text{basis of (A)}}$

For column space

$\{(2,1,1), (4,3,1)\}$ is basis of column space

$\boxed{\mathrm{Im}(T) = \text{Basis of col. space}}$

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Eg. $V = R^2$, $W = \{(x, y, z) \mid x + y + z = 0\}$

$\beta = (1,1), (1,-1)$ $\gamma = (-1,1,0), (-1,0,1)$

$T(x,y) = (0, x+2y, -x-2y)$ $V \to W$

$A_T = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$

$x_1 = 1/3\, t$

$(t/3, -t/3)$

$(1/3, -1/3)$

Basis of (A) =

$\ker(T) = 1/3(1,1) + 1(1,-1)$

$= \left(\frac{4}{3}, \frac{-2}{3}\right)$

col space of $(A) = (3, -3)$

$Im(T) = 3(-1,1,0) + (-3)(-1,0,1)$

$= (0, 3, -3)$

- The rank-nullity thm. for linear transforma^n

  $\text{rank}(T) = \dim(Im(T))$

  $\text{nullity}(T) = \dim(\ker(T))$

  $\text{rank}(T) + \text{nullity}(T) = \dim(V)$

⇒ Tutorial - 1

Eg. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ T $\beta = \{(1,1,0), (0,1,1), (1,0,1)\}$

Find matrix representa^n of T wrt std. ordered basis of R^3

$\{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,1,0) = 0(1,1,0) + 0(0,1,1) + 0(1,0,1) = (0,0,0)$

$T(0,1,1) = 0(1,1,0) + 0(0,1,1) + 0(1,0,1) = (0,0,0)$

$T(1,0,1) = 0(1,1,0) + 0(0,1,1) + 1(1,0,1) = (1,0,1)$

$(1,0,0) = a(1,1,0) + b(0,1,1) + c(1,0,1)$

$= (a+c,\ a+b,\ b+c)$

$a = c = 1/2, \quad b = -1/2$

$(0,1,0) = \frac{1}{2}(1,1,0) + \frac{1}{2}(0,1,1) - \frac{1}{2}(1,0,1)$

$(0,0,1) = -\frac{1}{2}(1,1,0) + \frac{1}{2}(0,1,1) + \frac{1}{2}(1,0,1)$

$T(1,0,0) = \frac{1}{2}T(1,1,0) - \frac{1}{2}T(0,1,1) + \frac{1}{2}T(1,0,1)$

$= \frac{1}{2}(1,0,1) = \frac{1}{2}(1,0,0) + 0(0,1,0) + \frac{1}{2}(0,0,1)$

$T(0,1,0) = -\frac{1}{2}(1,0,1)$

$T(0,0,1) = \frac{1}{2}(1,0,1)$

$T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$

$= \frac{1}{2}(x-y+z,\ 0,\ x-y+z)$

$$[T] = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

$\ker(T) = \{(x,y,z) \mid y = x+z,\ x,z \in \mathbb{R}\}$

$\ker(T) = \text{Span} = \{(1,1,0),(0,1,1)\}$

→ Tutorial 2

$T: \mathbb{R}^3 \to \mathbb{R}^3 \qquad T(x,y,z) = (x+y,\ y+z,\ 2y+2z)$

$\beta = \{(1,0,0),(1,1,0),(1,1,1)\}$

$\gamma = \mathbb{R}^3$ standard

1. Is linear transformation injective (Not)

$T(x_1) = T(x_2) \Rightarrow x_1 = x_2 \qquad T(0,0,0) = 0$

$T(x,y,z) = 0 \Rightarrow (x,y,z) = (0,0,0)$

$\Rightarrow (x+y,\ y+z,\ 2y+2z) = (0,0,0)$

$x+y=0 \qquad x = z \qquad W = \{(z,-z,z) \mid z \in \mathbb{R}\}$

$y+z=0 \qquad y = -z \qquad z=1 \Rightarrow (1,-1,1)$

$2y+2z=0 \qquad T(1,-1,1) = (1-1,\ -1+1,\ -2+2) = 0$

2. Find basis of ker(T)

basis of ker(T) = (1, -1, 1)

3. Is linear transformaⁿ surjective ? α (Not)

dim (V) = 3

$\underbrace{\text{Rank}(T)}_{\text{Im}} + \underbrace{\text{Nullity}(T)}_{\text{ker}} = \text{dim}(V)$

$2 \quad 1 \Rightarrow 3$

Im(T) = dim(W) Rank (T) = 2

(2 = 3) α

4. Find matrix representaⁿ of T

T(1, 0, 0) = (1, 0, 0) = 1(1, 0, 0) + 0(0, 1, 0) + 0(0, 0, 1)

T(1, 1, 0) = (2, 1, 2) = 2(1, 0, 0) + 1(0, 1, 0) + 2(0, 0, 1)

T(1, 1, 1) = (2, 2, 4) = 2(1, 0, 0) + 2(0, 1, 0) + 4(0, 0, 1)

$$[T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

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Week 7

Equivalence and similarity of matrices

- Let A & B be matrices be 2 matrices of order $m\times n$. A is **equivalent** to B if $B = QAP$

$n\times n \rightarrow$ matrix P

$m\times m \rightarrow$ matrix Q (invertible)

(i) $A \rightsquigarrow B$ by a combinaⁿ of elementary operations

(ii) rank (A) = rank (B)

- Equivalence of matrices → equivalence relaⁿ

(i) A is equivalent to itself ⇒ $A = I_{m\times m} A I_{n\times n}$

(ii) A is equivalent B ⇒ B is equivalent to A

(iii) A is equi. to B & B to C ⇒ A to C

Eg $f: R^3 \to R^2$ $\quad f(x,y,z) = (x+y, y+z)$

R^3 $\beta_1 \Rightarrow$ std. $\quad \beta_2 = (1,1,0), (0,1,1), (0,0,1)$

R^2 γ_1 = std. $\quad \gamma_2 = (1,0), (1,1)$

$$[A]_{\beta_1}^{\gamma_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad [A]_{\beta_2}^{\gamma_2} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

how to get P & Q

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow \underline{\underline{B = QAP}}$$

A & B are equivalent

- $T: V \to W \Rightarrow \beta_1$ & β_2 for V & γ_1 & γ_2 for W

$[A]_{\beta_1}^{\gamma_1}$ & $[B]_{\beta_2}^{\gamma_2}$

$\Rightarrow$ $\boxed{\text{A is equivalent to B}}$

P → express the basis β_2 in terms of β_1

Q → express the basis γ_2 in terms of γ_1

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- An $n \times n$ matrix A is similar to an $n \times n$ matrix B if $\boxed{B = P^{-1}AP}$ $\Rightarrow$ P matrix $(n \times n)$
- Prop. of A & B similar matrices —
  (i) A and B are equivalent
  (ii) A & B have same rank
  (iii) $\det(B) = \det(A)$

Eg. $f: \mathbb{R}^3 \to \mathbb{R}^3$ $f(x, y, z) = (-x + y + z, \; x - y + z, \; x + y - z)$
$\beta = \gamma$ are std. $\mathbb{R}^3$ basis

$\beta' = \gamma' = \{(1,1,1), (-1,1,0), (1,0,1)\}$

$$[A]_{\beta}^{\gamma} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$[B]_{\beta'}^{\gamma'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\boxed{P^{-1}AP = B}$

$\Rightarrow$ A & B are similar

how to get P

$P \to$ Express γ in terms of β

- Under some basis, we hope that the corresp. matrix is a diag. matrix which gives an easy understanding of linear transformation.

L7.2 <u>Affine Subspaces and Affine Mappings</u>

- Let V be a vector space. An <u>affine subspace</u> of V is a subset L that there exists $v \in V$ and $U \subseteq V$

$$L = v + U := \{v + u \mid u \in U\}$$

- an affine subspace L is n-dim. if corres. U is n-dim.

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- The subspace U corres. to an affine subspace is unique
- The vector v isn't unique and infact can be any vector in L.
- Affine subspaces are thus translates of a vector space of V.
- Affine subspaces in R^2 $\longrightarrow$ Points, lines, the entire plane R^2

$y = mx + c$ $y = mx$

$L = \{(x,y)\} = (x,y) + (0,b)$

↳ not an subspace but subset ⇒ parabola ($y = x^2 + 1$)

curve ($y^2 = x^3$)

- Affine subspaces in R^3 → Points, lines, planes, entire space R^3

↳ $L = v + \{\lambda_1 u_1 + \lambda_2 u_2\}$ w

- $Ax = b$
  - → $b = 0$, sol^n set of subspace of R^n, null space (A)
  - → $b \notin$ column space (A) : no sol^n / empty set
  - → $b \in$ column space (A) : → The sol^n set L is an affine subspace of R^n.

$L = v +$ null space (A)

↳ any sol^n of eq^n $Ax = b$

- Let L & L' be affine subspaces of V & W respec.

$f : L \to L'$ ⇒ $L = v + U$

$L' = f(v) + U'$

⇒ f is an affine mapping from L to L' if the func^n $g : U \to U'$ is defined by $g(u) = f(u+v) - f(v)$ is a linear transforma^n.

Eg. $T(x,y,z) = (2x + 3y + z, 4x - 5y + 3)$. This an affine mapping from $R^3 \to R^2$.

- $T : V \to W$ be an linear Trans. $T' : V \to W$

$T'(u) = w + T(u)$

is an affine mapping from $V \to W$

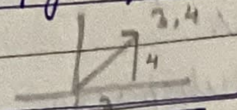
L7.3 Lengths & angles

- $(3,4)$ and $(2,7)$ in R^2

dot Prod. → $(3\times2)+(4\times7) = 6+28 = 34$

Gen. ⇒ $\boxed{(x_1, y_1)\cdot(x_2, y_2) = x_1x_2 + y_1y_2}$

- Length of vector $(3,4)$ ⇒ 5



$$\boxed{\text{Length of } (x,y) = \sqrt{y^2 + x^2}}$$

- The length of (x,y) is the square root of the dot product of the vector with itself

$(x,y)\in R^2$ is $\boxed{\sqrt{x^2+y^2} = (x,y)\cdot(x,y)}$

- The ∠ btw. the vectors u and v and measures how far the direcⁿ is of v from u.

↳ degrees (0 & 360) & radians (0 & 2π)

- The angles are measured from trigonometry

- angle btw. u & v

$$\boxed{\cos(\theta) = \frac{u\cdot v}{\sqrt{(v\cdot v)\times(u\cdot u)}}}$$

- for R_3

⇒ $(x_1, y_1, z_1)\cdot(x_2, y_2, z_2) = x_1x_2 + y_1y_2 + z_1z_2$

⇒ length $= \sqrt{x^2+y^2+z^2}$

⇒ same $\cos\theta$ formula to calculate angle

Eg. Angle btw. $(1,0,0)$ & $(1,0,1)$

$\cos\theta = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$ $\theta = 45°$

[General]

⇒ $u\cdot v = u_1v_1 + u_2v_2 + \dots u_nv_n$ → dot prod.

⇒ $||u|| = \sqrt{u\cdot u}$ ⇒ length

⇒ $\cos\theta = \frac{u\cdot v}{||u||\times||v||}$ ⇒ angles

Inner products and norms

- An **inner prod.** on a V **vector space** is a func^n $\langle ?, \cdot \rangle$: $V \times V \to R$
  - $\to \langle v, v \rangle > 0$ for all $v \in V \setminus \{0\}$
  - $\to \langle u_1 + u_2, u_3 \rangle = \langle u_1, u_3 \rangle + \langle u_2, u_3 \rangle$
  - $\to \langle u_1, u_2 \rangle = \langle u_2, u_1 \rangle$
  - $\to \langle c u_1, u_2 \rangle = c \langle u_1, u_2 \rangle = \langle u_1, c u_2 \rangle$
- A vector space V together with an inner prod. $\langle \cdot, \cdot \rangle$ is c/d an inner prod. space.
- The dot prod. is an eg. of an inner prod.

$$\langle \cdot, \cdot \rangle : V \times V \to R \; ; \; \langle u, v \rangle = u \cdot v$$

Eg. $u = (x_1, x_2)$ $\quad v = (y_1, y_2)$

$$\langle u, v \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2 x_2 y_2$$

- A **norm** on a V is func^n

$$\| \cdot \| : V \to R \qquad x \mapsto \|x\|$$

  - $\to \|x + y\| \leq \|x\| + \|y\|$
  - $\to \|cx\| = \|c\| \|x\|$
  - $\to \|x\| \geq 0$
- Length is an eg. of norm

$$\|u\| = \sqrt{(x_1^2 + x_2^2 + \dots x_n^2}$$

Eg. $\|u_1\| = |x_1| + |x_2| + \dots |x_n|$

$\|u\| = 0 \iff |x_i| = 0 \Rightarrow x_i = 0 \; \forall i \Rightarrow u = 0$

$\|cu\| \Rightarrow |cx_1| + |cx_2| \dots + |cx_n|$

$= |c| (|x_1| + |x_2| + \dots + |x_n|) = |c| \|u\|$

- $\| \cdot \| : V \to R$ by $\boxed{\|v\| = \sqrt{\langle v, v \rangle}}$ is a norm on V.

→ Tutorial-1

Similar matrices

$T(x,y,z) = (x, x+y, x+z)$.

$\beta_1 \Rightarrow$ std. $\mathbb{R}^3$ $\quad \beta_2 = \{(1,0,1),(0,1,1),(1,1,0)\}$

$T(1,0,0) = (1,1,1)$ $\quad [T]_{\beta_1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

A

$T(0,1,0) = (0,1,0)$

$T(0,0,1) = (0,0,1)$

$T(1,0,1) = (1,1,2) = 1(1,0,1) + 1(0,1,1) + 0(1,1,0)$

$T(0,1,1) = (0,1,1) =$

$T(1,1,0) = (1,2,1) =$

$$B = [T]_{\beta_2} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$A = PBP^{-1}$

T $(1,0,1) = 1 \quad 0$ std ordered matrix

$(0,1,1) = 0 \quad 1$

$(1,1,0) = 1 \quad 0 \quad 0$

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$PB = AP = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

$PB = AP$

$A = P^{-1}BP$

→ Tutorial-2

sol^n of non-homo. sys. of eq^n and affine space

$Ax = b \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

Null space (A) $\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $x_3 = t$
$x_1 = -t$
$x_2 = -t$

$\{-t, -t, t \mid t \in R\}$
Span $\{(-1, -1, 1)\}$

$(A|b) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 \end{array}\right]$ $x_1 + x_3 = 2$
$x_2 + x_3 = 2$

sol^n $(2-x_3, 2-x_3, x_3) \mid x \in R$

$\underbrace{(2, 2, 0)}_{u} + \underbrace{\{(-x_3, -x_3, x_3) \mid x_3 \in R\}}$

$Au = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$ $N(A) \rightarrow$ subspace of R^3

$u +$ subspace $=$ Affine space

→ Tutorial - 3

$\beta = \{(1,1,0), (0,1,1)\}$ and a vector $(3,3,3) \in R^3$

$L = u + W \Rightarrow$ affine subspace of W

$W = \{x(1,1,0) + y(0,1,1) \mid x, y \in R\}$
$= \{(x, x, 0) + (0, y, y) \mid x, y \in R\}$
$= \{(\underset{X}{x}, \underset{Y}{x+y}, \underset{Z}{y}) \mid x, y \in R\}$

$\Rightarrow Y = X + Z$

$W = \{X, Y, Z \mid Y = X + Z, (X, Y, Z) \in R\}$
$W = \{x, y, z \mid y = x + z, (x, y, z) \in R\}$

$L = 3,3,3 + W = \{(3,3,3) + (x, y, z) \mid y = x + z\}$
$= \{(x+3, y+3, z+3) \mid y = x + z\}$

$L = \{(X, Y, Z) \mid Y = X + Z - 3, (X, Y, Z) \in R\}$
$= \{(x, y, z) \mid y = x + z - 3, (x, y, z) \in R\}$

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→ Tutorial-4

$L = \{x, y, z \mid x + y + z = 2\}$. Find $\dim(L)$

$L = u + W$

$u \in L$

$u = (\alpha, \beta, \gamma) \in L$

$(-u) + u + W = W$

$-u = (-\alpha, -\beta, -\gamma)$

$-u + u + W = (-\alpha, -\beta, -\gamma) + \{(x, y, z) \mid x + y + z = 2\}$

$= \{(\underbrace{x - \alpha}_{X}, \underbrace{y - \beta}_{Y}, \underbrace{z - \gamma}_{Z}) \mid x + y + z = 2\}$

$X + Y + Z = 2 - (\alpha + \beta + \gamma)$

$= 2 - 2 = 0$

$W = \{(X, Y, Z) \mid X + Y + Z = 0\}$

$\{(-1, 1, 0), (-1, 0, 1)\}$

$\dim(L) = 2$

⇒ Tutorial-5

$V = \mathbb{R}^2$ $\quad \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$

$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 x_2 + 2 y_1 y_2$

Is it inner product?

$u, u_1, u_2, u_3 \in V$

① $f(u, u) \geq 0$ ② $f(u_1 + u_2, u_3) = f(u_1, u_3) + f(u_2, u_3)$

③ $f(u_1, u_2) = f(u_2, u_1)$ ✓

④ $f(cu_1, u_2) = c f(u_1, u_2)$ ✓

$u_1 = (x_1, y_1)$ ....

→ $\langle u_1, u_1 \rangle = \langle (x_1, y_1), (x_1, y_1) \rangle = x_1^2 + 2y_1^2$

$= x_1^2 + 2y_1^2 \geq 0$

⇒ $f(u_1 + u_2, u_3) = \langle (x_1, y_1) + (x_2, y_2), (x_3, y_3) \rangle$

$\langle (x_1 + x_2), (y_1 + y_2), (x_3, y_3) \rangle$

$= x_1 \cdot x_3 + x_2 \cdot x_3 + 2y_1 y_3 + 2y_2 y_3$ ✓

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Tutorial - 6

$\|(x, y)\| = |x+y|$

Is the given func a norm on V.

Three condi^n:

✓ ① $f(u+u) \leq f(u) + f(u)$ $u, u \in V$

✓ ② $f(cv) = |c| f(v)$

✓ ③ $f(u) \geq 0$

$u_1 = (x_1, y_1)$ $u_2 = (x_2, y_2)$

$\Rightarrow \|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$

$\|(x_1, y_1) + (x_2, y_2)\| \leq |x_1 + y_1| + |x_2 + y_2|$

$\leq \|u_1\| + \|u_2\|$

Week-8

L8.1 Orthogonality and Linear Independence

- $\boxed{\cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|}} \Rightarrow \theta$ is the $\angle$ btw. u & v
- inner prod. $\Rightarrow$ dot prod.
  norm $\Rightarrow$ length
- $\theta = 90° \Rightarrow \cos(\theta) = 0 \Rightarrow u \cdot v = 0$
  eg. $(1,2,3)$ & $(2,2,-2)$ are orthogonal
  they are at $90°$ $\angle$
- Two vectors u & v of an inner prod. space V are said to be orthogonal, $\langle u, v\rangle = 0$
  $\langle u, v\rangle = x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2$
  $u = (x_1, x_2)$ & $v = (y_1, y_2)$
  Then the vectors $(1,1)$ & $(1,0)$ are orthogonal
- Orthogonality depends on particular inner product.
- An orthogonal set of vectors whose elements are mutually orthogonal.
  $S = \{(4,3,-2), (-3,2,-3), (-5,18,17)\}$
  0 0
  0
- Let $\{u_1, u_2 \ldots u_k\}$ is an orthogonal set of vectors in V, then they are linearly indep. set of vectors.
- A basis consisting of mutually orthogonal vectors is cld orthogonal basis.
- Since, an ortho. set is already linearly ind.
  If $\dim(V) = n$,
  $\boxed{\text{ortho. basis} \equiv \text{ortho. set of } n \text{ vectors}}$
  Eg. → the std. basis
  → $\{(4,3,-2), (-3,2,-3), (-5,18,17)\} \subseteq \mathbb{R}^3$

Q. what is an orthonormal basis?

- A maximal orthogonal set is a basis and is called an orthogonal basis.
- A orthonormal set of vectors of an V is an orthogonal set of vectors such that norm of each vector is 1.

$$\left.\begin{array}{l} \rightarrow \langle u_i, u_j \rangle = 0 \\ \rightarrow \|u_i\| = 1 \end{array}\right\} \text{for orthonormal condition}$$

- An orthonormal basis is an orthonormal set of vectors which forms a basis.
- Orthonormal $\rightarrow$ orthogonal basis + each vector with norm 1.
- orthonormal basis a maximal orthonormal set.
- $\gamma = \{u_1, u_2 \dots u_k\}$ is the orthogonal set of vectors

$$\boxed{\underset{\text{ortho normal vectors}}{\beta} = \left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \dots, \frac{u_k}{\|u_k\|} \right\}}$$

- $\gamma = \{u_1, u_2 \dots u_n\}$ is an orthonormal basis of inner prod. space V

$$u = c_1 u_1 + c_2 u_2 + \dots c_n u_n$$

$$\boxed{c_i = \langle u, u_i \rangle}$$

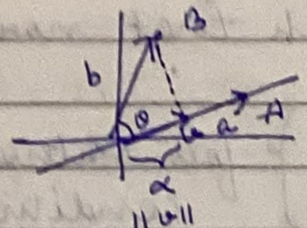
Q. $\left\{ \frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1) \right\}$. $(2,5) \Rightarrow u$ Find c_i?

$$c_1 = \langle (2,5), \frac{1}{\sqrt{10}}(1,3) \rangle = \frac{17}{\sqrt{10}}$$

$$c_2 = \langle (2,5), \left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right) \rangle = \frac{-1}{\sqrt{10}}$$

L8.3 Projections using inner products

A & B are pts. in $\mathbb{R}^2$ and we want to find the nearest pt. from B on the lines passing thru A and the origin



$$v = \alpha a$$

$$\|v\| = \alpha \|a\|$$

$$\alpha = \frac{\|v\|}{\|a\|}$$

$$v = \|v\| \frac{a}{\|a\|}$$

$$\|v\| = \|b\| \cos\theta$$

$$= \|b\| \frac{\langle a, b\rangle}{\|a\| \|b\|}$$

$$v = \frac{\langle a, b\rangle a}{\|a\|^2}$$

$$\boxed{v = \frac{\langle a, b\rangle}{\langle a, a\rangle} a}$$

- Let V be an inner prod space, $v \in W$ & $W \subseteq V$ be a subspace. Then the projectⁿ of v onto W

Orthonormal basis $\{u_1, u_2, \dots, u_n\}$ for W

$$\boxed{\text{proj}_W(v) = \sum_{i=1}^{n} \langle v, u_i \rangle u_i}$$

↳ indep. of the chosen orthonormal basis.

↳ proj of v onto W is the vector in W closest to v

Eg. $V = \mathbb{R}^2$, $W = \langle (3,1) \rangle$, $v = (1,3)$ ⇒ $\text{proj}_W(v) = (1.8, 0.6)$

$\frac{1}{\sqrt{10}}(3,1)$ orthonormal

$$\text{proj}_W(v) = \left\langle (1,3), \frac{1}{\sqrt{10}}(3,1) \right\rangle \frac{1}{\sqrt{10}}(3,1)$$

$$= \frac{6}{10}(3,1) = (1.8, 0.6)$$

ON basis

Eg. $V = \mathbb{R}^3$, $W = \langle (1,0,0), (0,1,0) \rangle$, $v = (2,3,5)$

$$\text{proj}_W(V) = \langle (2,3,5), (1,0,0) \rangle (1,0,0) + \langle (2,3,5), (0,1,0) \rangle (0,1,0)$$

$$= 2(1,0,0) + 3(0,1,0)$$

$$= (2,3,0)$$

$proj_w(v) = proj_{\langle w \rangle}(v) \qquad \langle w \rangle = \frac{w}{\|w\|}$

- For the orthogonal basis,

$$\boxed{proj_w(v) = \frac{\langle v, w\rangle}{\langle w, w\rangle} \cancel{\times}\, w}$$

Eg. W = 2d subspace of $V = R^3$ orthogonal → $u_1 = (1,2,1)$, $u_2 = (1,-1,1)$
$v = (-2, 2, 2)$ on w?

$proj_{u_1}(v) = \frac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle} u_1 = \frac{4}{6}(1,2,1) = \frac{2}{3}(1,2,1)$

$proj_{u_2}(v) = -\frac{2}{3}(1,-1,1) \qquad proj_W(v) = proj_{u_1}(v) + proj_{u_2}(v)$

$= (0, 2, 0)$

- V is inner prod. space & W is subspace. Then the proj. of vectors in $V \to W$ is a linear transforman from $V \to V$ with image W. ⟶ P_W –
- Prop. of P_W –
  (i) $P_W(v) = v \;\forall\; v \in W$
  (ii) Range of $P_W = W$
  (iii) $W^\perp = \{ v \mid v \in V, \langle v, w\rangle = 0 \;\forall\; w \in W\}$ is the null space of P_W
  (iv) $P_W^2 = P_W$
  (v) $\|P_W(v)\| \leq \|v\|$

6.4 The Gram – Schmidt process

- In an inner prod. space,
  Any basis $x_1, x_2 \ldots x_n$ $\xrightarrow{\text{process}}$ Orthonormal basis $u_1, u_2 \ldots u_n$

Eg. $\beta = \{(1,2,2), (-1,0,2), (0,0,1)\}$

$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$ not 0 det ⟶ orthonormal basis

$v_1 = (1,2,2) \qquad v_2 = (-1,0,2) - P_{v_1}((-1,0,2)) \qquad \langle v_2, v_1\rangle = 0$

$= (-1,0,2) - \frac{\langle (-1,0,2), (1,2,2)\rangle}{\langle (1,2,2), (1,2,2)\rangle} (1,2,2) = \left(-\frac{4}{3}, \frac{-2}{3}, \frac{4}{3}\right)$

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Again, a orthogonal to both u_1 & u_2, a vector in $\text{Span}(\{u_1, u_2\})^\perp$

$$u_3 = (0,0,1) - P_{u_1}((0,0,1)) - P_{u_2}((0,0,1))$$
$$= \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right)$$

$\Rightarrow \{u_1, u_2, u_3\}$ is an orthogonal basis and divide by their norms to get orthonormal basis.

- $u_1 = x_1$; $w_1 = \dfrac{u_1}{\|u_1\|}$

  $u_2 = x_2 - \langle x_2, w_1\rangle w_1$; $w_2 = \dfrac{u_2}{\|u_2\|}$

  ⋮ go on like this

  basis $= \{x_1, x_2 \dots x_n\}$
  og. basis $= \{u_1, u_2 \dots u_n\}$
  on. basis $= \{w_1, w_2 \dots w_n\}$

  $\Rightarrow$ The Gram-Schmidt process

- Theorem - Any finite dimensional vector space with an inner prod. has an on. basis.
  ↳ Any basis can be Δ to on. basis using the Gram Schmidt process.

L8.5 Orthogonal Transforma^n and rotations

- Let V be an inner prod-space & T be a linear transforma^n from $V \to V$. T is said to be orthogonal transforma^n $\Leftrightarrow \langle T_v, T_w\rangle = \langle v, w\rangle \; \forall \; v, w \in V$
- When $V = \mathbb{R}^n$ with the usual inner prod., a linear transforma^n is orthogonal if it preserves angles and lengths.

Week 9

L9.1 Multivariable func^{ns} → Visualization

- Func^{ns} of 1 variables → linear func^{ns}, polynomials, rational, trigonometric, exponential, logarithmic, compositions
- A scalar-value multivar. func^n is a $f: D \to R$ where D is a domain in R^n; $n > 1$
  Eg. Linear transformation^{ns} $R^n \to R$
  Polynomial func^n $x_1^2 + x_2^2 \dots + x_n^2$
  Combina^n / Composition
- A vector-value multivar. func^n is a $f: D \to R^m$ where D is a domain in R^n, $n, m > 1$ / Vector of scalar-valued multivariable func^{ns}.
- Multivar. func^n is either scalar or vector-valued
  $f: D \to R^m$ on R^n $m > 1$ and $m = 1$ (also)

Eg. $f(x,y) = 2.5x - 3.9y$ $\qquad f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$
$f(x,y) = \sin(x^2+y^2)$
$f(x,y) = \frac{xy}{x^2+y^2}$ $\qquad f(x,y,z) = x^2 + y^2 + z^2$
$f(x,y,z) = (2x, 2y, 2z)$ $\qquad f(x,y) = \begin{cases} 1 & 0 \leq (x,y) \leq 1 \\ 0 & \text{o.w.} \end{cases}$
(Piecewise → set on plane of R^2)

- $f: D \to R^m$, $g: D \to R^m$, $D \subset R^n$
  (i) $(f+g)x = f(x) + g(x)$, $x \in D$
  (ii) $(cf)(x) = c \times f(x)$
  (iii) $m = 1 \Rightarrow fg(x) = f(x) \times g(x)$
  (iv) $m = 1$, $g(x) \neq 0 \Rightarrow \left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}$

- $g \circ f: D \to R^p = g(f(x))$, $x \in D$
  Eg. $f(x,y) = x^2 + y^2$ $\quad g(x) = \sqrt{x}$ $\Rightarrow g(f(x)) = \sqrt{x^2+y^2}$
- A curve in R^m refers to range of a func^n $f: D \to R^m$ where D is a domain in R. Eg. Lines in R^m, conics in R^2, helix in R^3, etc.

Partial derivatives

rate of Δ → instantaneous speed → time int. → an infinitesimal time $= \lim_{\Delta t \to 0} \frac{\text{dist. in } \Delta t}{\Delta t}$

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

- Func^n → scalar-valued multivariable func^n
- $\underset{\sim}{a}$ is a pt. R^n, open ball of r around $\underset{\sim}{a}$

$$\{ x \in R^n \mid \|x - \underset{\sim}{a}\| < r \}$$

Let $f(x_1 \dots x_n)$ be a func^n on D in R^n a pt. $\underset{\sim}{a}$ and an open ball around it.

rate of Δ of f w.r.t x_i $\boxed{= \lim_{h \to 0} \frac{f(\underset{\sim}{a} + h e_i) - f(\underset{\sim}{a})}{h}}$

1. $f(x,y) = (x+y)$ at $(0,0)$ w.r.t. x

$\lim_{h \to 0} \frac{f((0,0) + h(1,0)) - f(0,0)}{h} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$

$= \lim_{h \to 0} \frac{h - 0}{h} = \frac{h}{h} = 1$

2. $f(x,y,z) = (xy + yz + zx)$ at $(1,2,3)$ w.r.t. y

$\lim_{h \to 0} \frac{f((1,2,3) + h(0,1,0)) - f(1,2,3)}{h} = \lim_{h \to 0} \frac{f(1,2+h,3) - f(1,2,3)}{h}$

$= \lim_{h \to 0} \frac{h + 3h}{h} = 4$

3. $f(x,y) = \sin(xy)$ at $(1,0)$ w.r.t. x

$\lim_{h \to 0} \frac{f((1,0) + h(1,0)) - f(1,0)}{h} = \lim_{h \to 0} \frac{f(1+h,0) - f(1,0)}{h}$

$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$

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$$f_{x_i}(\underset{\sim}{x}) \text{ or } \frac{\delta f}{\delta x_i}(\underset{\sim}{x}) = \lim_{h \to 0} \frac{f(\underset{\sim}{x} + h e_i) - f(\underset{\sim}{x})}{h}$$

↳ partial derivative of func^n f w.r.t x_i

Eg. $f(x,y,z) = xy + yz + zx$

$\frac{\delta f}{\delta x}(x,y,z) = y + 0 + z = y + z$

Eg. $f(x,y,z) = \sin(xy)$

$\frac{\delta f}{\delta x}(x,y) = \cos(xy) \times y$; $\frac{\delta f}{\delta y}(x,y) = x \cos(xy)$

Eg. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$\boxed{(x,y) \neq (0,0)}$

$\frac{\delta f}{dx}(x,y) = \frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}$

$\frac{\delta f}{\delta y}(x,y) = \frac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2} = \frac{x^3 - y^2x}{(x^2+y^2)^2}$

$\boxed{(x,y) = (0,0)}$

$\frac{\delta f}{\delta x} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$; $\frac{\delta f}{dy} = 0$