Stats - 2

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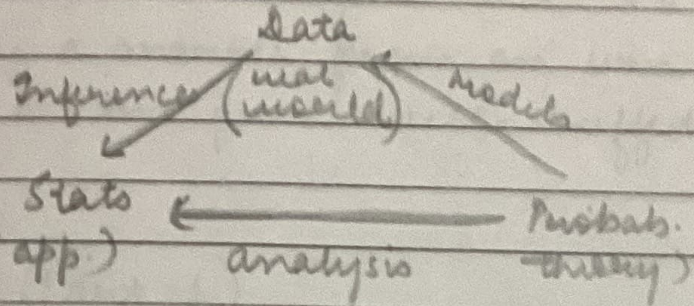
Week-0 - Part-1

→ Intro to course - Data, Stats, Probab.
- obs. (provide data) → patterns → theory → applica^n
- deterministic patterns (d=s×t) / some patterns do not have deterministic equations
- when phenomenons are too complex, we cannot expect exact models, still we can find some pattern



Act1- sites.google.com → student portfolio template → create a webpage → Pages → Classes → Stats-2 → Press it reach there → Summary of what you're learning → let it be an internal site

→ Format of Match Data in Spreadsheet
- IPL data - running ex.
- cricsheet.org → 816 matches IPL (2008 - 2020) → ZIP file → Unzip → .yaml (human readable form)
- to study this data, we need to process it.
- descrip. - useful to parse the necessary data into a spreadsheet

⇒ Summarising Powerplay Overs Data
- IPL powerplay overs - Overs 1-6 of an inning
  - Only 2 fielders outside the boundary

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- explore button in corner of google sheet → ask some reasonable questions → average for see for avg., std. devia^n, etc. for any data

Act. 2 - collect any other data (100 rows & 5 columns) → describe the rows & columns, add interesting things to it.

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Week 0 - Part 2

→ Expt., Outcome, and Sample Space

Book
- (Prob. & Stats by Siva Athreya, D. Sarkar & Steve Tanner)
- (Think Stats 2 by Allen B Downey)

- Expt. - process that we wish to study statistically
- Outcome - result of the expt.
- Sample space (S) - a set that contains all outcomes of an expt. Tossing a coin: S = {heads, tails}

Eg. → drawing a marble from an urn (pot)
→ draw a card from well-shuffled pack of 52 cards

→ Events (Part - 1)
- event - an event is a subset of the sample space.
- Toss a coin : Events : ϕ, {h}, {t}, {h, t}
- an event is said to have occurred if the actual outcome of an expt. belongs to the event.

→ Events (Part - 2)
- $A \subseteq B$, if A occurs, then B has already occurred.
- complement of A ⇒ A^c, if A occurs, A^c can't occur
- Union ($\cup$) or, Intersecⁿ ($\cap$) and
- draw 5 cards w/o replacement: no aces
  $\{1^{st}$ not ace$\} \cap \{2^{nd}$ not ace$\} \cap \{3^{rd}$ not$\} \cap \{4^{th}$ not$\} \cap \{5^{th}$ not$\}$

- Disjoint events - 2 event with an empty $\cap$
  Eg. E_1 = even nos., E_2 = odd nos.; E_1 & E_2 are disjoint,
  $A \cap A^c = \phi$ & $A \cup A^c = S$ (partition)
- Union (all covered reg.), Intersecⁿ (common reg.)
Disjoint event (no overlap in reg.)
- $A \cap B^c$ ⇒ reg. of A outside of B.

→ Events (Part - 3)

Eg. 5 hats are there —

A – no person get their hat

B – every person gets their hat

C – at least 1 person does not get their own hat

D – at least 1 person gets their own hat

$A^c = D$; $B^c = C$

Eg. 6 delivery ⇒ 0, 1, 2, 3, 4, 6

A – no 4s

B – no 6s

C – exactly 20 runs

$A \cup B$ ⟶ no 4s or no 6s

$A \cap B$ ⇒ no 4s and no 6s

$A \cap B \cap C$ → cannot occur

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

] De-Morgan's Laws

→ Introduction to Probability

- Chance of happening sth. ?
- Probab. th. – 'Math' theory to assign chances to events of an expt.

Assume a few things ⟶ axioms

- Probab. is func^n P that assigns to each event a real no. btw. 0 and 1. The entire probab. space (sample space, events, probab. func^n) should satisfy two axioms.

1. $P(S) = 1$
2. If E_1, E_2, E_3 are disjoint events

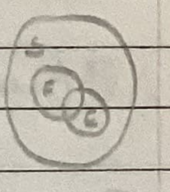
$$P(E_1 \cup E_2 \cup E_3 \ldots) = P(E_1) + P(E_2) + P(E_3) \ldots$$

Eg. Toss a coin ; $S = \{h, t\}$
$P(\phi) = 0$, $P\{h\} = 0.5$, $P\{t\} = 0.5$, $P\{h,t\} = 1$

→ Basic Prop. of Probability
- $P(\phi) = 0 \Rightarrow$ probability of empty is 0.
- $P(E^c) = 1 - P(E)$

- If event E is a subset of event F, i.e. $E \subseteq F$,
  then $P(F) = P(E) + P(F \setminus E)$
  $\Longrightarrow P(E) \leq P(F)$

- If E & F are events,
  $P(E) = P(E \cap F) + P(E \setminus F)$
  $P(F) = P(E \cap F) + P(F \setminus E)$



- If E & F are events,
  $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

⇒ Working with probab. spaces
- Tossing a coin $\Rightarrow S = \{h, t\}$, $E: \phi, \{h\}, \{t\}, \{h, t\}$,
  $P(S) = 1$, $P(\phi) = 0$, if $P\{h\} = p$, then $P(t) = 1 - p$
- A waiter & cashier is needed. D & M from Delhi, R & V from Mumbai. First take waiter → cashier
  A ⇒ cashier is from Delhi ⇒ (D, M) (M, D) (R, D) (R, M) (V, D) (V, M)
  B ⇒ at least posn by delhite
- chance of catching > 400 kg of fish is 35%
  " " " > 500 kg " " " 10%

The chance of getting btw. 400 and 500 kg of fish →
$A \setminus B = 0.25$

- A chance of rain tom — 60%
  B " of max temp > 30° — 70%
  $A \cap B = 40\%$
  $\cancel{(A \cap B)^c} \neq (A^c \cap B^c) = (A \cup B)^c = 0.1$

  $A \cup B = 0.6 + 0.7 - 0.4 = 1.3 - 0.4 = 0.9$

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→ **Distributions**

1. Throw a dice, → $S = \{1, 2, 3, 4, 5, 6\}$

$p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$ } all are equally likely

$$\boxed{P(E) = \frac{\text{no. of outcomes in } E}{\text{no. of outcomes in } S}}$$ } uniform distribuⁿ on a finite space

Eg. 5 red & 8 blue marbles in urn. Pick a marble at random

S = 13 marbles

uniform distribution

$P(\text{red}) = \frac{5}{13}$ $P(\text{blue}) = \frac{8}{13}$

Eg. Throw 2 dice. P is that sum of 2 nos. is 8?

(2,6) (3,5) (4,4) (5,3) (6,2) ⇒ $\frac{5}{36}$

Eg. 50 keys option. How can you describe the outcomes & S? Is the uniform distribuⁿ?

outcome: { ✓ ✗ ✓ ✗ ✗ ✓

Uniform

Eg. 3 hats get mixed. P that none of the person gets the hat.

P_1	P_2	P_3
H_2	H_3	H_1
H_3	H_1	H_2

$P = \frac{2}{6} = \frac{1}{3}$

⇒ **Definition of Conditional Probab.**

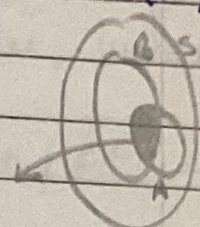
- expts. contain a seq. of steps — one after another
- the initial probab. space : splits into event that

occured and conditional probab. space given that the event occured

- (Conditional probab. space given B), sample space: B

Probab. func^n: $P(A \cap B) / P(B) = P(A/B)$

$$\boxed{P(A \cap B) = P(B)\, P(A/B)}$$



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

⇒ Egs. of Conditional Probab.

Eg. Throw a die → 1, 2, 3, 4, 5, 6 ⇒ P(1/6)

E: even no $P(E) = 1/2$

$$P((2)|E) = \frac{P(2) \cap P(E)}{P(E)} = \frac{[illegible]}{1/2} = \frac{1}{3}$$

Eg. 2 urns with coloured marbles

$P(\text{red} | \text{urn } 1) = \frac{7}{13}$, $P(\text{red} | \text{urn } 2) = \frac{5}{13}$

$P(\text{blue} | \text{urn } 1) = \frac{6}{13}$, $P(\text{blue} | \text{urn } 2) = \frac{8}{13}$

Eg. 15 students – 4 from State 1, 8 from S 2 & 3 from S 2. 3 students are chose at ran, ⇒ S_1, S_3, S_1

A $\frac{4}{15}$ $P(S_3 \cap S.$ $P(A \cap B \cap C)$

B $\frac{3}{14}$ $= P(B)\,P(A|B)$

$4/15$ $= P(A).\, P(B \cap C | A)$

$\frac{4}{15}$ $= P(B|A).\, P(C|A \cap B)$

$$\boxed{\frac{4}{15} \times \frac{3}{14} \times \frac{3}{13}}$$

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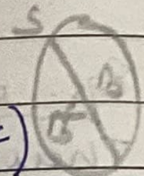
Eg. A family has 2 children. What is the prob. that both are girls, given that at least 1 is a girl.

at least 1 $\Rightarrow$ (G, G) (G, B) (B, G) $\quad \frac{3}{4}$

$$\frac{1}{2} \times \frac{1}{2} = \frac{A \cap A}{4}$$

$$P(B|A) = \frac{P(B \cap A)}{3} \Big/ \frac{3}{4}$$

$\Rightarrow$ Law of Total Probab.



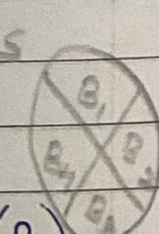
- $P(A) = P(A \cap B) + P(A \cap B^c)$
$\quad\quad = P(A|B)\,P(B) + P(A|B^c)\,P(B^c)$

Eg. Old eg.

$$P(red) = \left(\frac{7}{13} \times \frac{1}{2}\right) + \left(\frac{5}{13} \times \frac{1}{2}\right) = \frac{6}{13}$$

Eg. If u $\uparrow$, 60% chance that unemp. $\uparrow$, u do not inc., 30% " " " $\uparrow$. If the economist there is a 40% chance $\uparrow$ u, by how much unemp will $\uparrow$?

$$P(ui) = P(ui|B)\,P(B) + P(ui|B^c)\,P(B^c)$$
$$= (0.6 \times 0.4) + (0.3)(0.6)$$
$$= 0.24 + 0.18 = 0.42 \Rightarrow 42\%$$



- $B_1, B_2, B_3 \dots$ partition of S

$$P(A) = P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2) \dots$$

Eg. 5 coins — 2 are double-head, 1 is double-tail, 2 are normal. He picks a head (p)

$$P(h) = P(h|hh)\,P(hh) + \cancel{P(h|tt)\,P(tt)} + P(h|n)\,P(n)$$
$$= \left(1 \times \frac{2}{5}\right) + \frac{1}{2} \times \frac{2}{5} = \frac{3}{5}$$

→ Bayes' Theorem

Eg. Old urn eq. ⇒ What abt. P(urn 1 | red) or P(urn 1 | blue)?

Eg. 1% of people have flu. Flu test → 95% with flu is +ve, 2% w/o disease also +. A person is +ve, what is the p that person has swine flu.

$P(F) = 0.01$ $\quad P(+|F) = 0.95$ $\quad P(+|F^c) = 0.02$

$P(F|+) = ?$

Eg. $P(A), P(B)$ ⇒ given $\quad P(A|B)$ & $P(A|B^c)$ given

Find → $P(B|A)$ & $P(B^c|A)$?

$$\boxed{P(B|A) = \frac{P(B)\,P(A|B)}{P(A)}}$$ → Bayes' Theorem

- Bayes' thm. along with law of total probab. can solve many problems?

1 → $P(\text{urn } 1 | \text{red}) = \frac{P(U_1)\,P(\text{red}|U_1)}{P(\text{red})}$ = [illegible] $= \frac{7}{12}$

2 → $P(F|+) = \frac{0.01 \times 0.95}{P(+)}$ = 32%

$P(+) = 0.95 \times 0.01 + 0.02 \times 0.99 = 0.0293$

Eg. MCQ with 4 choices know the correct ans. with $p = 0.75$. She chooses ran, if it is correct → what is the probab that she knew the ans.

$P(C|K) = 1 \quad P(C|DK) = 1/4 \quad P(K) = \frac{3}{4}$

$P(DK) = \frac{1}{4}$

$P(C) = \frac{13}{16} \quad P(K|C) = \frac{3/4}{13/16} = \frac{3}{4} \times \frac{16}{13} = \frac{12}{13}$

Eg. A fair die, toss as the head came. Given 5 heads are obt. → that die shows was 5.

$P(1) \ldots P(6) = \frac{1}{6}$

$P(5 \text{ heads} \mid 1,2,4,3) = 0$

$P(5 \text{ heads} \mid \text{die } 5) = \frac{1}{32}$

$P(5 \text{ heads} \mid \text{die } 6) = \frac{6}{64}$

$P(h) = \frac{1}{6} \times \frac{1}{32} + \frac{1}{6} \times \frac{3}{32}$

$= \frac{1}{48}$

$P(\text{die } 5 \mid 5 \text{ heads}) = \dfrac{\frac{1}{6} \times \frac{1}{32}}{\frac{1}{48}} = \frac{1}{4}$

⇒ **Independence of Events**

- The 2 events A & B are independent, if –

$$\boxed{P(A \cap B) = P(A)\,P(B)}$$

- Probab. of A is unaffected by occurrence of B, ie A & B are independent
- For disjoint / mutually exclusive event A & B,

$$P(A \cup B) = P(A) + P(B)$$

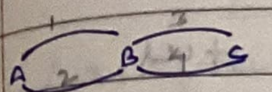
Eg. Toss a coin thrice : A → first toss is heads
B → second toss is heads

$$P(A \cap B) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A) \times P(B)$$

So, A & B are independent

- Disjoint events are never independent. For events to be independent, they should have a non-empty intersecⁿ
- Events A, B, C are mutually independent, if
  - → $P(A \cap B) = P(A)\,P(B)$
  - → $P(B \cap C) = P(B)\,P(C)$
  - → $P(A \cap C) = P(A)\,P(C)$
  - → $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

- A & B are independent $\Rightarrow A^c$ & B^c are independent
  $\Rightarrow A$ & B^c are independent
- n events mutually independent $\Rightarrow$ any subset with or w/o complement are independent

Eg.  . Each roads get blocked with probab. p. What is the probab. that there is an open route from A to B given there is no open route from $A \to C$

$P\left(\text{open route } A \to B\right) = P(1 \text{ is open or } 2 \text{ is open}) = 1 - p^2$

$P\left(\text{no open route}\right) = P(1 \text{ [illegible] or blocked}) = p$

⇒ Repeated independent trial & Bernoulli distribution

- $P(\text{success}) = p$

Bernoulli trial : $S = \{\text{success, failure}\}$

$\therefore P(\text{failure}) = 1 - p$

This distribution is bernoulli (p)

- We can repeat a bernoulli test multiple times independently. $S = \underbrace{2^n \text{ outcomes}}$

Eg. $P(000) = (1-p)^3 \qquad P(101) = p \times p(1-p) = p^2(1-p)$

Eg. Toss a coin 5 times, 32 outcomes, using uniform distribution

- The actual seq. of 0 & 1s not matter. Only the no. of success matters.

$$\boxed{P(b_1, b_2, \ldots b_n) = p^w (1-p)^{n-w}}$$

where, w is the no. of success

⇒ Binomial Distribution

- $B(n, p)$

$p \hookrightarrow$ probab. of success

$n \hookrightarrow$ no. of trials

$$\boxed{P(B(n,p)) = {}^{n}C_{k}\, p^{k} (1-p)^{n-k}}$$

- starts at $(1-p)^n$ → inc. & reach peak → falls to p
- Peak is Near (np)

Eg. $P(d) = 0.1$, $n = 100$, $k = 20$, test true

$$p = {}^{100}C_{20} (0.1)^{20} (0.9)^{80}$$

Eg. $n = 10$ times (a) no. of heads is multiple of 3
(b) no. of heads is even

① $P = \left(\frac{1}{2}\right)^{10} + {}^{10}C_3 \left(\frac{1}{2}\right)^{10} + {}^{10}C_6 \left(\frac{1}{2}\right)^{10} + {}^{10}C_9 \left(\frac{1}{2}\right)^{10}$

② $\dfrac{1 + {}^{10}C_2 + {}^{10}C_4 + {}^{10}C_6 + {}^{10}C_8 + {}^{10}C_{10}}{2^{10}}$

Eg. A bit Alice → Bit has flip of 0.1
(a) 5 bits, what is probab. at most 2 gets flipped
$P(B(5, 0.1)) \in \{0, 1, 2\}$
(b) 10 bits, what is probab. at most 2 gets flipped
$P(B(10, 0.1)) \in \{0, 1, 2\}$

⇒ Geometric Distribution =
- u tossing coin, till you get the head
$P(1) = P(\text{heads in } 1^{st} \text{ toss}) = 1/2$
$P(2) = 1/2 \times 1/2 = 1/4$

$$\boxed{P(k) = \left(\frac{1}{2}\right)^{k}}$$

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- In ludo, no start till you get a 1.

$P(1) = \frac{1}{6}$ $\quad P(2) = \frac{5}{6} \times \frac{1}{6}$ $\quad P(3) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

$$\boxed{P(k) = \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right)}$$

- No. of trials till you get first success, $G(p)$

$P(G=1) = p$ $\quad P(G=2) = (1-p)\,p$

$P(G=3) = (1-p)^2\,p$

$$\boxed{P(G=k) = (1-p)^{k-1}\,p}$$

$$\boxed{P(G \leq k) = 1-(1-p)^k} \Rightarrow \text{Impt. identity}$$

$\Rightarrow P(G>k) = (1-p)^k$

- Keeps on decreasing, but, $p<1$, never goes all the way to 0.

Eg. Ludo (1). What is p she needs < 6, < 11, < 21

$P(G \leq 5) = 1 - \left(1 - \frac{1}{6}\right)^5 = 0.59$

$P(G \leq 10)$ $\quad P(G \leq 20)$

Eg. $P_1 \Rightarrow 0.4$ $\quad P_2 \Rightarrow 0.7$ shoots

1. P that P_1 wins before the 3rd round
2. p that P_1 wins

1. 1_{P_1}, $0_{P_1} 0_{P_2} 1_{P_1}$

0.4 $\quad 0.6 \times 0.3 \times 0.4$

0.072

0.472

2. 1_{P_1}, $0_{P_1} 0_{P_2} 1_{P_1}$, ...

$0.4 + 0.18 \times 0.4 + (0.18)^2 \times 0.4$ ....

$= \frac{0.4}{1-0.18} = \frac{0.4}{0.82}$

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→ Probability in Python

- Monte Carlo simulations → verification
- Available as a Python NB in google colab

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→ Random Var. & Events

- RV - numerical values computed from the outcomes of the expt.
- RV is a func^n with domain as S of expt. & range of the set of real nos.; from sample space to real line
Eg. Throw a die. A RV X can be defined as
$x(1) = x_1$, $X(2) = x_2$, ..., $X(6) = x_6$
- If X is a RV, $\boxed{(X < x) = \{s \in S : X(s) < s\}}$ is an event for all real x
So, $(X > x)$, $(X = x)$, $(X \leq x)$, $(X \geq x)$ are all events.
Eg. In throwing die,
$X = 1$ (die has 1) ; $X < 4$ (E is $\{1, 2, 3\}$) ;
$E\{2, 5\} : (X = 2) \cap (X = 5)$
Eg. Throw a die, [RV]E(2, 4, 6) = 1 [RV]E(1, 3, 5) = 0
$E = 0$; event $\{1, 3, 5\}$ $E < 0$; null event
$E = 1$; event $\{2, 4, 6\}$ $E \leq 1$; event $\{1, 2, 3, 4, 5, 6\}$
event $\{2, 5\}$ can't be expressed in terms of E.
- Instead of assigning probab. to outcomes, we study RV and assign probab. to events defined by them.

→ Discrete RV & its PMF
- The range of RV is the set of values taken by it.
Eg. Throw a die, X = no. shown on die $X = \{1, 2, 3, 4, 5\}$
X = is odd (0) or one (1) $X = \{0, 1\}$
- Discrete RV - A RV is said to be discrete if its range is a discrete set. Any interval is not discrete (a, b)
- PMF of a discrete RV X with range set T is the func^n $T \to [0, 1]$ defined as $f_X(t) = P(X = t)$ for $t \in T$
$(X = t)$ is an event
$P(X = t) = P$(all outcomes that result in X taking value t

- Any P(event) defined using X can be computed using PMF.

Eg. 3 coin tosses. X RV

(a) How many heads will come?

(b) Which will be the first flip that shows head? Y

$X = \{0, 1, 2, 3\}$

$Y = \{none, 1, 2, 3\}$

out	P(out)	X	Y
HHH	1/8	3	1
HHT		2	1
HTH		2	1
HTT		1	1
THH		2	2
THT		1	2
TTH		1	3
TTT		[illegible]	none

(a) $f_X(0) = \frac{1}{8}$, $f_X(1) = \frac{3}{8}$, $f_X(2) = \frac{3}{8}$, $f_X(3) = \frac{1}{8}$

(b) $f_Y(none) = \frac{1}{8}$, $f_Y(1) = \frac{4}{8}$, $f_Y(2) = \frac{2}{8}$, $f_Y(3) = \frac{1}{8}$

Eg. a 3-digit no. (000 - 999). Matches exactly → 2 lakh, if matches exactly 2 of the 3 digits → ₹20000, otherwise nothing. X be the value of ticket. Find distribution of X.

X	$f_X(t)$
0	$(1000-28)/1000$
20000	$27/1000$
200000	$1/1000$

$X = \{0, 20K, 2 lakh\}$

⇒ Prop. of PMF

- RV and their PMF are studied w/o much mention of expt or S

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- Prop. of PMF $\rightarrow$ $\boxed{0 \leq f_x(t) \leq 1}$
$\rightarrow$ $\boxed{\sum_{t \in T} f_x(t) = 1}$

Eg.

t	-1	1	2	4
$f_x(t)$	0.5	0.25	0.125	_

Find $f_x(4)$. $\Rightarrow 0.125$
Find Range of X $\{-1, 1, 2, 4\}$
Find $P(X > 3)$ 0.125 $[f_x(4)]$
Find $P(X < 3/2)$
< 1.5
$= 0.75$

Eg. $f_x(k) = \frac{c}{3^k}$ for $k = 1, 2, 3 \ldots$ Find c
Find $P(X > 10)$
Find $P(X > 10 \mid X > 5)$

$c \rightarrow \frac{c}{3} + \frac{c}{3^2} + \frac{c}{3^3} \ldots = 1 \Rightarrow c = 2$

$\boxed{a + a^2 + a^3 \ldots = \frac{a}{1-a} \text{ if } a < 1}$

$P(X > 10) = \frac{2}{3^{11}} + \frac{2}{3^{12}} \ldots$

$\frac{2/3^{11}}{1 - 1/3}$ $= \frac{1}{3^{10}}$

$P(X > 10 \mid X > 5) = \frac{P(A \cap B)}{P(B)} = \frac{P(X > 10)}{P(X > 5)} = \frac{1/3^{10}}{1/3^{5}} = \frac{1}{3^5}$

$\Rightarrow$ Common Distributions
- Uniform RV $X \sim U(T)$ Range: Finite set T
Eg. Fair coin $X \sim U\{0, 1\}$ PMF: $f_x(t) = \frac{1}{|T|}, t \in T$

- Bernoulli RV $X \sim B(p)$, where $0 \leq p \leq 1$

  Range: $\{0, 1\}$ PMF: $f_X(0) = 1-p$, $f_X(1) = p$

  Bernoulli trial, probab. of success

- Binomial RV $X \sim Bin(n, p)$ Range: $\{0, 1, 2 \ldots n\}$

  PMF: $f_X(k) = {}^nC_k \, p^k (1-p)^{n-k}$

  no. of successes in n independent $B(p)$ trials

- Geometric RV $X \sim G(p)$, where $0 \leq p \leq 1$

  Range: $1, 2, 3 \ldots$ PMF: $f_X(k) = (1-p)^{k-1} p$

  no. of trials for 1st success in repeated independent $B(p)$ trials

- -ve Binomial RV $X \sim -ve\,B(a, p)$

  Range: $\{a, a+1, a+2 \ldots\}$

  PMF: $f_X(k) = {}^{k-1}C_{a-1} (1-p)^{k-a} p^a$

  No. of trials of a successes in repeated $B(p)$ trials

  $\hookrightarrow a = 1 \rightarrow -ve\,B(a, p) \Rightarrow G(p)$

- Poisson RV $X \sim Pois.(\lambda)$, where $\lambda > 0$

  Range $\{0, 1, 2, 3 \ldots\}$ PMF: $f_X(k) = \frac{e^{-\lambda} \lambda^k}{k!}$

$k = 0$	1	2	3	
$f_X(k)$ $e^{-\lambda}$	$e^{-\lambda} \lambda$	$\frac{e^{-\lambda} \lambda^2}{2!}$	$\frac{e^{-\lambda} \lambda^3}{3!}$	$\ldots$

- Hyper geo. RV $X \sim HypG(N, a, m)$

  Consider a popⁿ of 'N' persons with 'a' of type 1 and 'N - a' of type 2. Select 'm' person randomly w/o replacement.

  X = no. of person of type 1 selected

  $X \in \max(0, m-(N-a)), \ldots, \min(a, m)$

  PMF: $f_X(k) = \frac{{}^aC_k \; {}^{(N-a)}C_{(m-k)}}{{}^NC_m}$

→ Poisson RV

- Queues & meteorites
- events occurring over a period of time
  ↳ arrival of a visitor to a website
  ↳ arrival rate → could be constant
  ↳ given arrival, the time for next arrival could be independent
- the no. of arrivals in a fixed period of a time becomes a ~ Pois. (λ)

Eg. Radioactive decay
In 2608 time intervals of 7.5 s each

particles → 0 1 2
times → 57 203 383
fraction → 0.022 ..

emission rate = $\frac{\text{total part.}}{2608}$ = 3.867 (λ)

$$\boxed{\text{Prob. (particles} = k) = \frac{e^{-\lambda}\lambda^k}{k!}}$$

this fracⁿ comes from the model

- Poisson Fit – fracⁿ of times k particles are emitted fits the Poisson model closely

⇒ Functions of 1 RV

- $f(x)$ is $f: R \to R$, $f(X)$ can be a composition of 2 function. So, $f(X)$ is a RV in the same probab. space.

Eg. $X \sim U\{-2,-1,0,1,2\}$ & $f(x) = x^2$. Find R & PMF of $f(X)$.

t	$f_X(t)$	t^2
-2	1/5	4
-1	"	1
0	"	0
1	"	1
2	"	4

R = {0, 1, 4}

$X \in \{-2,-1,0,1,2\}$

$f(X) = X^2$

= {0, 1, 4}

0	1	4
1/5	2/5	2/5

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Eg. $X \sim G(0.5)$ & $f(x) = \begin{cases} x & x < 5 \\ 5 & x \geq 5 \end{cases}$

Find range & distribu^n of $f(X)$

$k = 1, 2, 3, 4, 5, 6, 7$

$f(k) = 1, 2, 3, 4, 5, 5, 5$

1	2	3	4	5
$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^4}$

Range $= \{1, 2, 3, 4, 5\}$

- X is the RV with PMF: $f_X(t)$

$f(X)$ is the RV whose PMF

$$f_{f(X)}(a) = \sum_{t : f(t) = a} f_X(t)$$

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1.1 Joint PMF with 2 discrete RV.

Eg. 1 coin tossed thrice. $X_i = 1$ if i^{th} is head & $X_i = 0$ if i^{th} toss is tails, $i = 1, 2, 3$. Together the 3 RV completely describe the outcome of the experiment

Eg. 2 digit no. 00 - 99. Let X. RV in units place. Let Y RV be the remainder when no. is divisible by 4.

$X \in U(0,1,2,3,4,5,6,7,8,9)$

$Y \in U(0,1,2,3)$

X & Y are not independent

Eg. X = no. of runs in over Y = no. of wickets in over
If $Y = 0$, X is greater than for $Y = 2$

- Suppose X & Y are discrete RV in same probab. space. Let range be T_X & T_Y respectively. The joint PMF of X & Y, f_{XY} is a func^n of $T_X \times T_Y$ to $[0,1]$

$$\boxed{f_{XY}(t_1, t_2) = P(X = t_1 \ \& \ Y = t_2),\ t_1 \in T_X,\ t_2 \in T_Y}$$

- We write it as a table / matrix

Eg. Toss coin twice, $X_i = 1$ (heads), $X_i = 0$ (tails)

$f_{X_1 X_2}(0,0) = P(X_1 = 0, X_2 = 0) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$f_{X_1, X_2}(0,1) = \frac{1}{4}$

- Each entry is btw. 0 & 1
- Sum of all entries = 1

$t_2 \downarrow$ \ $t_1 \rightarrow$	0	1
0	1/4	1/4
1	1/4	1/4

Eg. 2 digit eg. $f_{XY}(0,0) = P(X=0 \ \& \ Y=0) = \begin{pmatrix} 00, 20, 40, \\ 80, 60 \end{pmatrix} = 1/20$

$f_{XY}(4,2) = P(X=4 \ \& \ Y=2) = \begin{pmatrix} 14, 34, 54, 74 \\ 94 \end{pmatrix} = 1/20$

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L1.2 Marginal PMF of discrete RV

- The PMF of the indi. RV X & Y are cfd as marginal PMFs. It can be shown that →

Marginalization
$$\begin{cases} f_X(t) = P(X=t) = \sum_{t' \in T_Y} f_{XY}(t, t') \\ f_Y(t) = P(Y=t) = \sum_{t' \in T_X} f_{XY}(t', t) \end{cases}$$

Eg. Toss a coin → table

$t_2 \downarrow$ \ $t_1 \rightarrow$	0	1	$f_{X_2}(t_2)$
0	1/4	1/4	1/2
1	1/4	1/4	1/2
$f_{X_1}(t_1)$	1/2	1/2	

Marg. PMF of X_1
↳ add over the columns

$f_{X_1}(0) = f_{X_1X_2}(0,0) + f_{X_1X_2}(0,1)$

$f_{X_1}(1) = f_{X_1X_2}(1,0) + f_{X_1X_2}(1,1)$

Marginal PMF of X_2 ⇒ add over the rows

Eg.

t_2 \ t_1	0	1	
0	0.05	0.35	0.4
1	0.25	0.35	0.6
$f_X(t_1)$	0.3	0.7	

⇒ same marginal PMF from different joint PMFs

L1.3 Conditional Distribution of 1 RV given another

- X is discrete RV with range T_X & A is an event. The conditional PMF of X is
$$\boxed{Q(t) = P(X=t \mid A)}$$
conditional

for 1 RV,
$$\boxed{f_{X|A}(t) = \frac{P((X=t) \cap A)}{P(A)}}$$

- X & Y are 2 RVs with joint PMF f_{XY}. The condi. PMF of Y given X = t

$$Q(t') = P(Y=t' \mid X=t) = \frac{P(Y=t',\ X=t)}{P(X=t)} = \frac{f_{xy}(t,t')}{f_x\, t}$$

$$\boxed{f_{xy}(t,t') = f_{Y|X=t}(t')\, f_x(t)}$$

Eg.

$t_2 \backslash t_1$	0	1	2	$f_Y(t_2)$
0	1/4	1/8	1/8	1/2
1	1/8	1/8	1/4	1/2
$f_X(t_1)$	3/8	1/4	3/8	

Eg.

$t_2 \backslash t_1$	0	1	2
0	1/12	0	3/12
1	2/12	1/12	0
2	3/12	1/12	1/12

$$\boxed{f_{xy}(t_1,t_2) = f_{Y|X=t_1}(t_2) \cdot f_x(t_1) = f_{X|Y=t_2}(t_1) \cdot f_Y(t_2)}$$

1.1.4 Eqs on Joint, Marginal & Conditional Probab.

Eg. Throw a die, & toss a coin as many-times no. shown on die. Let X be the no. of die. Let Y be no. of heads. What is joint PMF?

$X \sim U(1,2,3,4,5,6)$ $\quad f_X(t) = 1/6$

$Y|X=t \sim Bin(t, 1/2)$ $\quad f_{Y|X=t}(t') = {}^tC_{t'} \left(\frac{1}{2}\right)^{t'} \left(\frac{1}{2}\right)^{t-t'}$

$= {}^tC_{t'} \left(\frac{1}{2}\right)^t$

$f_{XY}(t, t') = \frac{1}{6} \times {}^tC_{t'} \left(\frac{1}{2}\right)^t$ $\quad t' = 0, 1, 2, \ldots t$

$t' \backslash t$	1	2	3	4	5	6
0	1/12	1/24				
1	1/12	1/12				
2	0					

Eq. $N \sim Pois.(\lambda)$. Given $N = n$, toss a fair coin n times and denote the no. of heads obtained by X. What is the distribuⁿ of X? $f_X(k) = ?$

$$f_N(n) = \frac{e^{-\lambda} \lambda^n}{n!}, \quad n = 0, 1, 2, 3 \ldots$$

$X|N=n \sim Bin(n, 1/2)$ $\quad f_{X|N=n} = {}^nC_k \left(\frac{1}{2}\right)^n$

$$f_{NX}(n,k) = \frac{e^{-\lambda} \lambda^n}{n!} \times {}^nC_k \left(\frac{1}{2}\right)^n \quad \begin{array}{l} n = 0, 1, 2 \ldots \\ k = 0, 1, \ldots n \end{array}$$

$$f_X(k) = \sum_{n=0}^{\infty} f_{NX}(n,k)$$

$$= \sum_{n=k}^{\infty} \frac{e^{-\lambda} \lambda^n}{\cancel{n!}} \times \left(\frac{1}{2}\right)^n \times \frac{\cancel{n!}}{k!(n-k)!}$$

$$= \frac{e^{-\lambda} \lambda^k}{k! \, 2^k} \underbrace{\sum_{n=k}^{\infty} \frac{\lambda^{n-k}}{(n-k)! \, 2^{n-k}}}_{e^{\lambda/2}}$$

$$= \frac{e^{-\lambda/2} (\lambda/2)^k}{k!}$$

$X \sim Pois(\lambda/2)$

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Eg. X = no. of runs Y = no. of wickets $Y \sim \begin{pmatrix} 13/16 & 1/8 & 1/16 \\ 0, & 1, & 2 \end{pmatrix}$

$X | Y=0 \sim U(6, 7, 8, 9, 10, 11, 12)$

$X | Y=1 \sim U(2 - 8)$

$X | Y=2 \sim U(0 - 6)$

4.1.5 Joint PMF of >2 discrete RV

- Suppose $X_1, X_2, \ldots X_n$ are discrete RV. Let range be T_{X_1}

The joint pmf...

$$f_{X_1 \ldots X_n}(t_1, \ldots t_n) = P(X_1 = t_1 \ \& \ldots \& \ X_n = t_n)_{t_i \in T_{X_i}}$$

Eg. Toss coin 3 times. $X_i = 1$ i^{th} toss is heads

t_1	t_2	t_3	$f_{X_1 X_2 X_3}(t_1, t_2, t_3)$
0	0	0	
0	0	1	
0	1	0	$1/8$
0	1	1	
1	0	0	
1	1	0	
1	1	1	

Eg. 3 digit no. 000 - 999. $X \to$ first digit from left
Y = no. mod 2, Z = first digit from right

$X = \{0, 1, 2 \dots 9\}$ $f_{XYZ}(0,0,0) = \frac{10}{1000} = 0.01$

$Y = \{0, 1\}$

$Z = \{0, 1, 2 \dots 9\}$

$f_{XYZ}(1,1,1) = 0.01$ $f_{XYZ}(1,0,1) = 0$

Eg. Over has 6 deliveries. Let X_i denote the no. of runs scored in i^{th} delivery

$X_i \in \{0,1,2,3,4,5,6,7,8\}$

$i = 1, 2 \dots 6$

Too many possibilities ---

L1.6 Marginal PMF of multiple discrete RV

- The PMF of the indi. RV $X_1 \dots X_n$ are c/d marginal PMFs.

$$f_{X_1}(t) = P(X_1 = t) = \sum_{t_2' \in T_{X_2}, \dots, t_n' \in T_{X_n}} f_{X_1 \dots X_n}(t, t_2', \dots t_n')$$

Eg. Toss coin thrice eg. Joint PMF: $f_{X_1 X_2 X_3}(t_1, t_2, t_3) = \frac{1}{8}$

$$f_{X_1}(0) = f_{X_1X_2X_3}(0,0,0) + (0,0,1) + (0,1,0) + (0,1,1)$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$\Rightarrow f_{X_1}(0) + f_{X_2}(1) = 1$$

Eg. 3 digit eg. $X \sim U(0, 1, \dots, 9)$ $Y \sim U(0, 1)$

$Z \sim U(0, 1, \dots, 9)$

Eg. IPL Powerplay eg.

We can assign probab. in the same proporⁿ as data ⟶ it seems reasonable

- Suppose $X_1, X_2, X_3 \sim f_{x_1 x_2 x_3}$. What abt. jt. pmf. of $f_{x_1 x_2}$? $f_{x_2 x_3}$ ...

$$f_{x_1 x_2}(t_1, t_2) = P(X_1 = t_1 \;\&\; X_2 = t_2) = \sum_{t_3' \in T_3} f_{x_1 x_2 x_3}(t_1, t_2, t_3')$$

similarly, for $f_{x_2 x_3}$ and $f_{x_1 x_3}$

- Sum over everything you do not want
  ↳ Marginalisation ⟹ this extends to 4 RVs also

Eg. $X_1 X_2 X_3 \sim f_{x_1 x_2 x_3}$

$$\left.\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \\ & \vdots & \\ 1 & \vdots & 1 \end{matrix}\right\} \frac{1}{9}$$

$f_{x_1 x_2}$

$t_2 \downarrow$ \ $t_1 \rightarrow$	0	1
0	3/9	2/9
1	2/9	2/9

4.1.7 Conditioning

- a wide variety of conditioning is possible when there are many RV

$$X_1 \mid X_2 = t_2 \sim f_{X_1 \mid X_2 = t_2}(t_1) = \frac{f_{x_1 x_2}(t_1, t_2)}{f_{x_2}(t_2)}$$

$$X_1, X_2 \mid X_3 = t_3 \sim f_{X_1 X_2 \mid X_3 = t_3}(t_1, t_2) = \frac{f_{x_1 x_2 x_3}(t_1, t_2, t_3)}{f_{x_3}(t_3)}$$

$$X_1 \mid X_2 = t_2, X_3 = t_3 \sim f_{x_1 \mid x_2 = t_2, x_3 = t_3}(t_1) = \frac{f_{x_1 x_2 x_3}(t_1, t_2, t_3)}{f_{x_2 x_3}(t_2, t_3)}$$

Eg.

t_1	$\cancel{t_2}$ t_2	t_3	t_4
0	0	0	0
0	0	0	1
0	0	1	1
0	0	2	0
0	1	1	0
0	1	1	1
0	1	2	0
1	0	0	1
1	0	2	0
1	0	2	1
1	1	0	1
1	1	1	0

$f_{x_1 \dots x_4}$: $1/12$

$T_{X_4} = T_{X_2} = T_{X_1} = \{0,1\}$

$T_{X_3} = \{0,1,2\}$

$\{X_1 \mid X_3 = 0, X_4 = 1\} \sim \begin{pmatrix} 1/3 & 2/3 \\ 0 & 1 \end{pmatrix}$

~~X_3, X_4~~

$(X_1 \mid X_2 = 0) \sim \begin{Bmatrix} \cancel{4/12} \; 7/12 & \cancel{3/12} \; 7/12 \\ 0 & 1 \end{Bmatrix} = \begin{Bmatrix} 4/7 & 3/7 \\ 0 & 1 \end{Bmatrix}$

$(X_3, X_4 \mid X_1 = 0) \sim$

t_4/t_3	0	1	2
0	1/7	1/7	2/7
1	1/7	2/7	0

- $f_{x_1 \dots x_4}(t_1 \dots t_4) = P(X_1 = t_1, X_2 = t_2, X_3 = t_3, X_4 = t_4)$

$= P(X_1 = t_1 \mid X_2 = t_2, X_3 = t_3, X_4 = t_4)$

$P(X_2 = t_2 \mid X_3 = t_3, X_4 = t_4)$

$P(X_3 = t_3 \mid X_4 = t_4) \; P(X_4 = t_4)$

Eg. $f_{x_1 x_2 x_3}(0,0,0) = f_{x_3}(0) \; f_{x_2 \mid x_3 = 0}(0) \; f_{x_1 \mid x_2 = 0, x_3 = 0}(0)$

$1/9 = 3/9 \cdot 2/3 \cdot 1/2$

Week-2

2.1 Independence of 2 RV

$$f_{xy}(t_1, t_2) = f_x(t_1) \times f_y(t_2)$$

Gen: $f_{xy}(t_1, t_2) = f_x(t_1) \times f_{y|x=t_1}(t_2)$

Independent: $f_{y|x_1=t_1}(t_2) = f_y(t_2)$

- If X & Y are independent,
  [Joint PMF = prod. of marginal PMF.
  Conditional PMFs = Marginal PMFs

Eg.

$t_2 \backslash t_1$	0	1	f_y
0	1/4	1/4	1/2
1	1/4	1/4	1/2
f_x	1/2	1/2	

$\Rightarrow$ Independent

Eg.

$t_2 \backslash t_1$	0	1	2	f_y
0	1/9	1/9	1/9	
1	1/9	1/9	1/9	
2	1/9	1/9	1/9	
f_x				

$\Longrightarrow$ Independent

$\Rightarrow$ Independent don't look over surface, multiply and then check.

- To show X & Y are dependent, verify
$$f_{xy}(t_1, t_2) \neq f_x(t_1) \times f_y(t_2) \quad \left(\text{for any 1 value also}\right)$$

Eg. 2 digit eg. 00-99. X → digit at units place, Y is remainder when no. is divided by 4.

$X \sim U(0, 1, 2 \ldots 9)$ → 1/10

$Y \sim U(0, 1, 2, 3)$ → 1/4

$f_{xy}(1, 0) = 0 \neq f_x(1) \times f_y(0)$

$\Longrightarrow$ Dependant

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L 2.2 Independence of Multiple RV

$$f_{X_1 \dots X_n}(t_1, \dots t_n) = f_{X_1}(t_1) \dots f_{X_n}(t_n)$$

- All subsets of independent RVs are independent

Eg. $X_i = 1$ if i^{th} is heads ... $i = 1, 2, 3$, Joint PMF $= 1/8$

$$f_{X_1 X_2 X_3}(t_1, t_2, t_3) = f_{X_1}(t_1) \times f_{X_2}(t_2)\, f_{X_3}(t_3)$$

$\therefore$, X_1, X_2, X_3 are independent RV.

Eg. 3 digit 000 - 999 . $X \to$ first digit from left, Y = no. mod 2, $Z \to$ first " " right

$\to (X, Z)$ & (X, Y) are independent, but, (Y, Z) are not independent . $\therefore$, X, Y, Z are not independent.

- RV $X_1, \dots X_n$ are said to be i.i.d (independent & identically distributed) if,
  - they are independent
  - the marginal PMFs f_{X_i} are identical
- Repeated trials of an expt. creates i.i.d.

$$\boxed{X_1, X_2 \dots X_n \sim \text{i.i.d } X}$$

Eg. Let $X_1, \dots X_n$ be iid with $G(p)$ distribu^n. What is probab that all of these RV are > than some +ve int j.

$$X \sim G(p) \qquad P(X = k) = (1-p)^{k-1} p$$

$$P(X_1 > j, X_2 > j \dots) = P(X_1 > j) \dots = (P(X > j))^n$$

$$P(X > j) = \sum_{k=j+1}^{\infty} (1-p)^{k-1} p = (1-p)^j$$

$$= (1-p)^{jn}$$

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Eg. $X \sim \begin{pmatrix} 1/2 & 1/4 & 1/8 & 1/16 & 1/16 \\ 0, & 1, & 2, & 3, & 4 \end{pmatrix}$ let $X_1 \dots X_n$ be i.i.d

(i) p that 4 is missing in the sample?

$P(X \neq 4)^n = \left(\frac{15}{16}\right)^n$

(ii) 4 appears exactly once in the samples

$P(X_1 = 4, X_2 \dots X_n \neq 4)$ or $(X_2 = 4 \dots X_n \neq 4) \dots$

$= n P(X=4) (P(X \neq 4))^n = n\left(\frac{1}{16}\right)\left(\frac{15}{16}\right)^{n-1}$

(iii) 3 & 4 appear at least once in the sample

$P(3 \text{ atleast once} \cap 4 \text{ atleast once})$

$P(A^c) = \left(\frac{15}{16}\right)^n \quad P(B^c) = \left(\frac{15}{16}\right)^n \qquad A \cap B = (A^c \cup B^c)^c$

$P(A^c \cap B^c) = \left(\frac{14}{16}\right)^n \qquad P(A^c \cup B^c) = 2\left(\frac{15}{16}\right)^n - \left(\frac{14}{16}\right)^n$

$P(A \cap B) = 1 - \left(2\left(\frac{15}{16}\right)^n - \left(\frac{14}{16}\right)^n\right)$

Eg. $X \sim G(p) \rightarrow P(X > n)$

$P(X > m+n \mid X > m)$

this 2 are equal as per memory prop.

① ~~$P(X > n)$~~

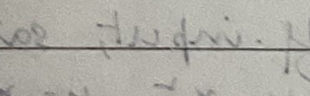
② $P(X > m+n \mid X > m) = \frac{P(X > m+n)}{P(X > m)} \quad \frac{(1-p)^{m+n}}{(1-p)^m}$

$(1-p)^n$

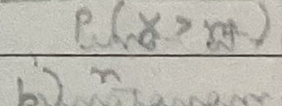
L2.3 Visualizing func^n of 1 RV, one-to-one func^n

- $U\{1 \dots 10\}$ and Binomial $(10, 0.5)$: PMF plots

These are translation distribution are same only axis moves)



Stem plots



$Y = X - 5$ (table method) $\rightarrow$ one-to-one func^n

x	$P(X=x)$	$y = x-5$
0	} 1/11	-5
1		-4
2		-3

$y = \{-5, -4 \dots 4, 5\}$

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- $y = 2^x$ → one-to-one func^n
  $B \sim (10, 0.5)$

0	0.00097..	0
1	"	1
2	0.043...	4
⋮		⋮

- 1-to-1, diff. input → diff. output (monotonic func^{ns}) $x-5$, 2^x
  Table method,
  $$\boxed{P(Y = f(x)) = P(X = x)}$$

L2.4 Visualizing func^{ns} of 1 RV, many-to-1 func^{ns}

- $Y = (X-5)^2$ $X \sim U(0, 1 ... 10)$

x	$P(X=x)$	$y = (x-5)^2$	y	$P(Y=y)$
0	$1/11$	25	0	$1/11$
1		16	1	$2/11$
2		9	4	$2/11$
⋮		⋮	9	$2/11$
10		25	16	$2/11$
			25	$2/11$

⇒ $y = (x-5)^2$ → is not one-to-one
↳ values of p.m.f. change

- many-to-one, for 2> diff. input, same output, non-monotonic func^{ns} $y = x^2$, $y = x(1-x)$, $y = xe^{-x}$

$$\boxed{P(Y = y_0) = P(X = X_1) + \dots + P(X = X_m) \quad y_0 = X_1 + \dots + X_m}$$

L2.5 Eg.s of func^{ns} of 1 RV

Eg. $X \sim U(-5, ... 5)$ $(1/11)$

$$f(x) = \begin{cases} x & x > 0 \\ 0 & x \le 0 \end{cases}$$

Find the distribu^n of $Y = f(x)$

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$\left.\begin{array}{c}-5\\-4\\-3\\-2\\-1\\0\\1\\2\\3\\4\\5\end{array}\right\}$ 0 0 0 0 0 0 1 2 3 4 5

Y	0	1	2	3	4	5
$P(Y=y)$	$6/11$	$1/11$	$1/11$	$1/11$	$1/11$	$1/11$

Eg. $X \sim U(-500, \ldots 500)$. $f(x) = \max(x, 5)$ Find $Y = f(x)$

$= \begin{cases} x & x \geq 5 \\ 5 & x < 5 \end{cases}$

x: $-500, \ldots, 5, 6, \ldots, 500$ — $1/1001$ — $5, 6, \ldots, 500$

$Y = \{5, \underbrace{6, \ldots, 500}_{1/1001\ (495\text{ entries})}\}$

$\frac{506}{1001}$

L2.6 Introducⁿ to funcⁿ of 2RV

- $X, Y \sim$ iid $U(0,1)$, $Z = X + Y$ (many-to-one funcⁿ)

x	y	$f_{XY}(x,y)$	z
0	0	1/4	0
0	1	1/4	1
1	0	1/4	1
1	1	1/4	2

z	0	1	2
	1/4	1/2	1/4

- $Z = \max(x, y)$

x	y	$f_{XY}(x,y)$	z
0	0	1/2	0
0	1	1/4	1
0	2	1/8	2
1	0	1/16	1
1	1	1/32	1
1	2	1/32	2

$z \in \{0, 1, 2\}$: $\left(\frac{1}{2}, \frac{11}{32}, \frac{5}{32}\right)$

Eg. Pair of fair dice thrown. Distribuⁿ of sum, max, min
→ table method becomes cumbersome and not possible

desmos

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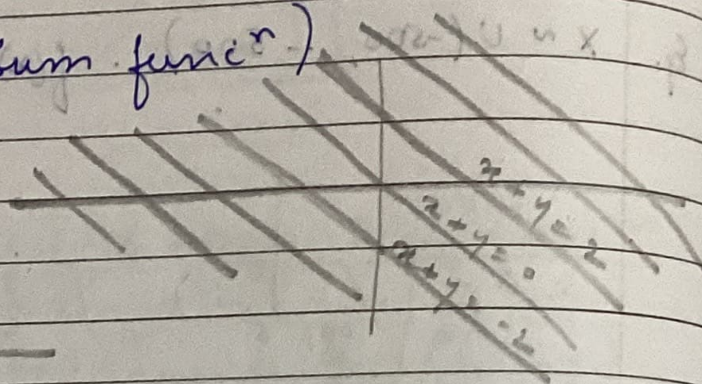
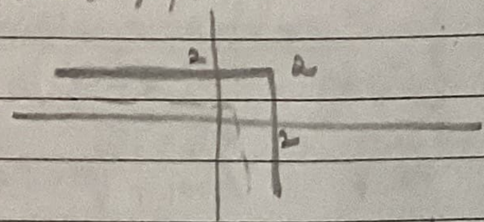
L2.7 Visualizing func^n of 2RV

- $g(x,y)$: func^n → 3D plot, but not useful
- Contours I : values of (x,y) that result in $g(x,y) = c$
  ↳ make a plot of those (x,y) for diff. c
- Regions II : values of (x,y) that result in $g(x,y) \le c$

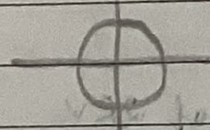
↳ $g(x,y) = x + y$ (Sum func^n)



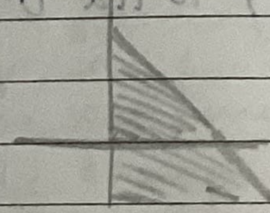
$\max(x,y) = c$



$x^2 + y^2 = c$



- $x + y \le c$ (regions)



$\max(x,y) \le c$

$x^2 + y^2 \le c$

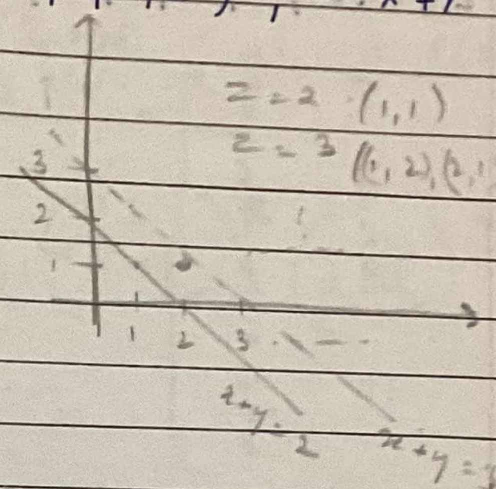
L2.8 Sum of 2RV

- $X, Y \sim f_{XY}$. Let $Z = g(X,Y)$
1. → find range of Z
2. → Add over the contours

$$P(Z = z) = \sum_{(x,y):\, g(x,y) = z} f_{XY}(x,y)$$

Eg. $X, Y \sim$ iid $U \sim (1,2,3,4,5,6)$, $Z = X + Y$

$Z = (2, 3, \dots, 11, 12)$



z	2	3	4	5 ...	12
p	1/36	2/36	3/36	4/36	1/36

L2.9 Max. of 2 RV

Eg. $X, Y \sim$ iid $U(1,2,3,4,5,6)$, $\max(X, Y)$

$Z \in (1, 2, 3, 4, 5, 6)$

	(1,1)	(1,2) (2,1) (2,2)	(1,3) (2,3) (3,3) (3,2) (3,1)			
$P(Z=z)$	1/36	3/36	5/36	7/36	9/36	11/36

- iid $U\{1, \dots, n\}$: Sum

$X, Y \sim$ i.i.d $U(1, 2 \dots n)$, $W = X + Y$

1. Range $\rightarrow \{2, 3, \dots, 2n\}$
2. PMF

$$P(W = w) = \begin{cases} \frac{w-1}{n^2}, & 2 \leq w \leq n+1 \\ \frac{2n - w + 1}{n^2}, & n+2 \leq w \leq 2n \end{cases}$$

- iid $U\{1, \dots, n\}$: $\text{Max}(x, y)$ — $x, y \sim$ iid $U\{1, \dots, n\}$, $Z = \max(X, Y)$

$\rightarrow Z \in (1 \dots n)$

$\rightarrow P(Z = z) = \frac{2z - 1}{n^2}$

L2.10 Func^ns of RVs

Eg. Twice thrown dice. P that sum of the 2 nos. is 6. ?

what is the PMF of sum?

Eg. $L \sim U(5, 7, 9, 11)$ with $\frac{1}{4}$ written above, $l = l$, $B \sim U(l-1, l-2, l-3)$

t_1	t_2	(t_1, t_2)
5	4, 3, 2	
7	6, 5, 4	$\frac{1}{12}$
9	8, 7, 6	
11	10, 9, 8	

$A = 20, 15, 10, 42, 35, 28, 72, 63, 54, 110, 99, 88$

$\underbrace{}_{P = \frac{1}{12}}$

Eg. $X_1 \dots X_n$ be the result of n i.i.d $B(p)$ trials. The sum of n RV $(X_1 + \dots + X_n)$ is $Bin(n, p)$.

Eg. $X \sim U(0,1,2,3)$ & $Y \sim U(0,1,2,3)$. PMF of $Z = X + Y$

$(0, \dots 6)$

	0	1	2	3	4	5	6
P	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$				$\frac{1}{16}$

- suppose X & Y take int. values & let the jt. pmf be f_{xy}. $Z = X + Y$

$$P(Z = z) = P(X + Y = z) = f_{xy}(x, z - x) = f_{xy}(z - y, y)$$

- If X & Y are indep.

$$f_{X+Y}(z) = \sum_{-\infty}^{\infty} f_X(x) \, f_Y(z - x) \quad \Big\} \text{ Convolution}$$

Eg. $X \sim Poi(\lambda_1)$ & $Y \sim P(\lambda_2)$

1. PMF of $Z = X + Y$
2. find conditional dist. of $X | Z$

(1)

$$f_Z(z) = \sum_{0}^{\infty} f_X(x) \cdot f_Y(z - x) = \sum_{0}^{z} \frac{e^{-\lambda_1} \lambda_1^{x}}{x!} \cdot \frac{e^{-\lambda_2} \lambda_2^{z-x}}{(z-x)!}$$

$$= \frac{e^{-\lambda_1 - \lambda_2}}{z!} \sum_{0}^{z} \frac{z!}{x!(z-x)!} \lambda_1^{x} \lambda_2^{z-x}$$

(2) $P(X=k \mid Z=n) = \dfrac{P(X=k, Z=n)}{P(Z=n)} = \dfrac{P(X=k)\,P(Y=n-k)}{P(Z=n)}$

$= e^{-\lambda_1} \lambda_1^{k}/k! \; e^{-\lambda_2}$

$X|Z \sim Bin\left(n, \dfrac{\lambda_1}{\lambda_1+\lambda_2}\right)$

$Y|Z \sim Bin\left(n, \dfrac{\lambda_2}{\lambda_1+\lambda_2}\right)$

$= \dfrac{n!}{k!\,(n-k)!} \left(\dfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k} \left(\dfrac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-k}$

- If X & Y are indept, the funcns of them are also independent.
- Funcns of non-overlapping sets of indep. RV are also independent

L2.11 Max. & Min. of 2 RVs

- X, Y — f_{XY} $\quad Z = \min(X, Y)$

Eg. Throw a dice, Z = min. of 2 nos. seen

z =	1	2	3	4	5	6
$P(Z=z)$ →	11/36	9/36	7/36	5/36	3/36	1/36

- $f_Z(z) = P(\min(X,Y) = z)$
  $= f_{XY}(z,z) + \sum_{t_2 > z} f_{XY}(z, t_2) + \sum_{t_1 > z} f_{XY}(t_1, z)$

- CDF of a RV X is a funcn $F_X : \mathbb{R} \to [0,1]$
  $F_X(x) = P(X \leq x)$

If X & Y are independent,
$F_Z(z) = F_X(z) \times F_Y(z)$
$= P(X \leq z) \times P(Y \leq z)$

⇒ CDF of max. is prod. of CDFs.
⇒ CDF of min. is $F_X + F_Y - F_X F_Y$

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Eg. $X_1 \dots X_n \sim$ iid X. Find min $(X_1 \dots X_n)$
max $(X_1 \dots X_n)$

② $P(\max(X_1, \dots X_n) \leq z) = (P(X \leq z))^n$
$= (F_X(z))^n$

① $P(\min) = (P(X \geq z))^n$

Eg. $X \sim G(p)$, $Y \sim G(p)$ be independent. Find dist. of $\min(X, Y)$

$P(\min(X,Y) \geq k) = P(X \geq k)\, P(Y \geq k)$
$= (1-p)^{k-1} (1-p)^{k-1}$
$= ((1-p)^2)^{k-1}$

$P(\min(X,Y) \geq k+1) = ((1-p)^2)^k$
$= ((1-p)^2)^{k-1} (1-(1-q))$

$\min(X,Y) \sim G(1-q)$ $\quad q = (1-p)^2$

Week - 3

3.1 Expected Value of Random Variable

- summarizing the data, avg. value of dataset, avg. represent how large nos. are doing,
- EV of a RV represents the average value

Eg. Place bet for 2 dies — under 7 get money
— over 7 get money
— = 7 get 4 times money

- Suppose, X a RV, PMF f_x. The EV of X is

mean of x / avg. value of x $\boxed{E[X] = \sum_{t \in T_X} t\, f_X(t)} = \sum_{t \in T_X} t\, P(X=t)$

→ E[X] may or may not belong to X
→ E[X] has the same units as X

Eg.- $X \sim B(p)$ $1 \to p$, $0 \to 1-p$

$\boxed{E[X] = p} = P(X=1)$

- $X \sim U(1,2,3,4,5,6)$

$E[X] = 1\times\frac{1}{6} + 2\times\frac{1}{6} + \cdots + 6\times\frac{1}{6} = 3.5$

- $X \sim \{200, 20, 0\}$ with probabilities $1/1000$, $27/1000$, $972/1000$

$E[X] = \frac{1}{5} + \frac{20\times 27}{1000} + 0 = 0.74$

- $X \sim \{-2, -1, 0, 1\}$ with probabilities $1/5$, $1/5$, $1/2$, $1/10$

$E[X] = -\frac{2}{5} - \frac{1}{5} + 0 + \frac{1}{10} = -\frac{5}{10} = -\frac{1}{2}$

Eg. $U\{a, a+1, \ldots, b\}$

$E[X] = a \times \frac{1}{b-(a+1)} + (a+1)\frac{1}{b-a+1}$

$\boxed{E[X] = \frac{a+b}{2}}$

↳ for uniform distribution

- $X \sim G(p)$
$$E[X] = \sum_{t=1}^{\infty} t (1-p)^{t-1} p$$

- $X \sim Pois(\lambda)$
$$E[X] = \sum_{t=0}^{\infty} t \frac{e^{-\lambda} \lambda^t}{t!}$$

- $X \sim Bin(n,p)$
$$E[X] = \sum_{t=0}^{n} t \, {}^nC_t \, p^t (1-p)^{n-t}$$

- we can simply the sum by.
  → Difference eqn (DE) →
  → Geometric prog. (GP) → $\frac{a_1}{1-r}$
  → Exponential funcn → $\sum_{t=0}^{\infty} \frac{e^{-\lambda} \lambda^t}{t!} = 1$
  → Binomial formula → $(a+b)^n$

- $X \sim G(p) \Rightarrow \boxed{E[X] = 1/p}$
- $X \sim Pois(\lambda) \Rightarrow \boxed{E[X] = \lambda}$
- $X \sim Bin(n,p) \Rightarrow \boxed{E[X] = np}$

L3.2 Prop. of EV

$$E[X] = \sum_{t \in T_X} t \, P(X=t)$$

- c as RV X with $P(X=c)=1 \Rightarrow E(c) = c$
- X takes only non-negative values → $E(X) \geq 0$
- $X_1 \dots X_n$ have joint PMF $f_{X_1 \dots X_n}$. Let $g: T_{X_1} \times \dots \times T_n \to R$, let $Y = g(X_1 \dots X_n)$

$$E[g(X_1, \dots X_n)] = E[Y] = \sum_{t_i \in T_X} g(t_1 \dots t_n) f_{X_1 \dots X_n}(t_1 \dots t_n)$$

↳ EV of a funcn of RV

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Eg. $X \sim U(-2,-1,0,1,2)$; $g(X) = X^2 \sim \begin{pmatrix} 1/5 & 2/5 & 2/5 \\ 0, & 1, & 4 \end{pmatrix}$

$E[g(x)] = 0 \times \frac{1}{5} + 1 \times \frac{2}{5} + 4 \times \frac{2}{5} = 2$

or

$E[g(x)] = (-2)^2 \times \frac{1}{5} + (-1)^2 \times \frac{1}{5} + (0)^2 \times \frac{1}{5} + (1)^2 \times \frac{1}{5} + (2)^2 \times \frac{1}{5} = 2$

Eg. $(X,Y) \sim U\{(0,0),(1,0),(0,1),(1,1),(-1,1),(1,-1)\}$

$g(X,Y) = X^2 + XY + Y^2 \sim \begin{Bmatrix} 0, & 1, & 3 \end{Bmatrix}$
$\quad 1/6 \quad 4/6 \quad 1/6$

$E[g(x,y)] = 0 + \frac{4}{6} + \frac{3}{6} = \frac{7}{6}$

or $E[g(x,y)] = \left(\frac{1}{6} \times 0\right) + \left(\frac{1 \times 1}{6}\right) + \cdots + \left(\frac{1 \times 1}{6}\right) = \frac{7}{6}$

- $\boxed{E[cX] = c\,E[X]}$ ⟹ linearity of Expected Value
- $\boxed{E[X+Y] = E[X] + E[Y]}$
- $\boxed{E[aX + bY] = aE[X] + bE[Y]}$

Eg. the upper (x,y) eg.

$E[g(x,y)] = E[x^2 + xy + y^2]$
$= E[x^2] + E[xy] + E[y^2] = 7/6$

Eg. EV of 2 nos. ~~sum~~ seen ofr 2 nos. seen

X = first no. Y = second no.

$E[X+Y] = E[X] + E[Y] = 3.5 \times 2 = 7$

- $Y \sim Bin(n,p) \Rightarrow E[Y] = np$
- A RV X with $E[X] = 0$ is said to be a zero-mean RV. ⟹ $X + c$

$P(X + c = t + c) = P(X = t)$ ⟹ translated PMF

- $Y = X - E[X]$ & $E[Y] = 0$
  So, $X - E[X]$ is a 0-mean RV

Eg. 10 balls in 3 bins. What is the Expected no. of empty bins?

$X_i = \begin{cases} 1 & \text{if bin } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$ $\quad i = 1, 2, 3$

$P(X_i = 1) = \left(\frac{2}{3}\right)^{10}$, $P(X_i = 0) = 1 - \left(\frac{2}{3}\right)^{10}$

empty bins $= X_1 + X_2 + X_3$

$E[Y] = 3E[X_1] = 3 \times \left(\frac{2}{3}\right)^{10}$

L 3.3 Simulaⁿ of EV in Python
- casino die game

- import scipy.stats as st
  ↳ it has all common distributions

- balls and bin problem

L 3.4 Var & Std. deviaⁿ
- sometimes, EV are same for altogether diff. RVs. How good is EV for describing RV?
- The var. of RV X, first moment, second moment better

second central moment $\boxed{Var(X) = E[(X - E[X])^2]} = \boxed{E(X^2) - (E(X))^2}$

& standard deviaⁿ of X,

$\boxed{SD(X) = \sqrt{Var(X)}}$

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Eg. $Y \sim \{\overset{1/2}{9}, \overset{1/2}{11}\}$ $(E[Y])^2 = (10)^2 = 100$

$\frac{81 \times 1}{2} + \frac{121 \times 1}{2} = \frac{81 + 121}{2} = \frac{202}{2} = 101$

$Var(Y) = 1$
$SD(Y) = 1$

Eg. Throw a die $U \sim (1, 2, 3, 4, 5, 6)$
$(E(X))^2 = (3.5)^2$
$= \text{sth}$

$Var(X) = (1 - 3.5)^2 \frac{1}{6} + (2 - 3.5)^2 \frac{1}{6} \ldots$

$= \frac{35}{12}$

- $Var(aX) = a^2 Var(X)$ — $Var(X + a) = Var(X)$
- $SD(aX) = |a| SD(X)$ — $SD(X + a) = SD(X)$

Scaling and Translatⁿ

- for dependant / independent, for RV X & Y
  $\boxed{E[X + Y] = E[X] + E[Y]}$
- for independent RVs —
  (i) $E[XY] = E[X] \cdot E[Y]$
  (ii) $Var[X + Y] = Var[X] + Var[Y]$
  Impt.

Eg. Var. of sum of 2 die
V.V.V. Imp' X → first Y → second X & Y are indep.
$\frac{35}{12} + \frac{35}{12} = \frac{35}{6}$

Distri.	EX	Var.
$B(p)$	p	$p(1-p)$
$Bin(n, p)$	np	$np(1-p)$
$G(p)$	$1/p$	$(1-p)/p^2$
$Pois(\lambda)$	λ	λ
$U \sim (1, \ldots n)$	$(n+1)/2$	$(n^2-1)/12$

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- A RV X is said to standardized if $E[X]=0$, $V(X)=1$

X,
$$\boxed{Y = \frac{X - E[X]}{SD(X)}}$$, is a standardized RV

→ PMF of Y is 'very similar' to PMF of X.

L3.5 Covariance & its prop.

- X & Y are RVs,

$$\boxed{Cov(X,Y) = E[(X-E[X])(Y-E[Y])]}$$

→ it is positive, if X is higher, Y is also higher
→ it is -ve, if X is higher, Y is lower
→ it is 0, then X & Y are 'uncorrelated'

Eg. X is ht., Y is wt. of person
→ ~~we expect cov(X,Y) to be +ve~~

Eg.

	X	-1	0	1	
Y	-1	1/15	2/15	2/15	5/15
	0	2/15	1/15	2/15	5/15
	1	2/15	2/15	1/15	5/15
		5/15	1/3	1/3	

$E[X] = E[Y] = 0$

$Cov(X,Y) = (-1)(-1)\frac{1}{15} + (-1)(1)\frac{2}{15} + (-1)(1)\frac{2}{15}$

$+ (1)(1)\frac{1}{15} = \frac{-2}{15}$

- $\boxed{Cov(X,X) = Var(X)}$
- $\boxed{Cov(X,Y) = E[XY] - E[X]E[Y]}$
- it is sym. & is a linear quantity

- if X & Y are independent, X & Y are uncorrelated, i.e. $Cov(X,Y) = 0$
- if X & Y are uncorrelated, they may be dependant

Correlation coefficient

- The correlation coeff. of 2 RVs X & Y, by $\rho(x,y)$

$$\boxed{\rho(x,y) = \frac{Cov(X,Y)}{SD(X)\,SD(Y)}}$$

- $-SD(X)\,SD(Y) \leq Cov(X,Y) \leq SD(X)\,SD(Y)$

$\Rightarrow \boxed{-1 \leq \rho(X,Y) \leq 1}$

- $\rho(x,y) \Rightarrow$ summarizes the trend
  $\Rightarrow$ dimensionless quantity

$\Rightarrow \rho = 1$ or $-1 \rightarrow Y = aX + b$
  ↳ Y is a linear func^n of X

$\Rightarrow |\rho(X,Y)| \sim 1 \Rightarrow$ strongly correlated

Eg.

Y \ X	0	1
0	$1/4 - x$	$1/4 + x$
1	$1/4 + x$	$1/4 - x$

$E[X] = E[Y] = 1/2$

$E[XY] = \frac{1}{4} - x$

$V(X) = V(Y) = \frac{1}{4}$ $\quad SD(X) = SD(Y) = \frac{1}{2}$ $\quad Cov(x,y) = -x$

$\rho(x,y) = \frac{-x}{1/4}$

Eg.

Y \ X	c	c+1
d	[same	]
d+1		

$E[Y] = d + 1/2$ $\quad E[X] = c + 1/2$

$E[XY] = cd + \frac{c}{2} + \frac{d}{2} + \frac{1}{4} - x$

$Cov(x,y) = -x$ $\quad \rho(x,y) = -4x$

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L3.7 Bounds in probab. using mean and var.

- RV X ⇒ $\mu_x \Rightarrow E[X]$
  $\sigma_x^2 \Rightarrow Var(X)$
  $\sigma_x \Rightarrow SD(X)$

- avg. says sth. abt. distribuⁿ of marks
- $(X-\mu)$: measures the dist. of X from the mean μ. It can be +ve / −ve
- Std units : The no. of std. deviaⁿ that a realizaⁿ of a RV is away from the mean. $\boxed{\frac{X-\mu}{\sigma}}$

X ⇒ should be within $\mu - c\sigma$ and $\mu + c\sigma$

Eg. Throw a pair of die $\mu = 7$, $\sigma = 2.42$
X = sum of 2 nos.

$P(|X-\mu| \le \sigma) = P(4.58 \le X \le 9.42)$
$= P(X \in 5,6,7,8,9) = \frac{2}{3}$

$P(|X-\mu| \ge 2\sigma) = \frac{2}{36} = 0.056$

- Markov's inequality → X RV, taking non-negative values with a finite μ. Then,

$$\boxed{P(X \ge c) \le \frac{\mu}{c}}$$

- Chebyshev's inequality → X RV, mean μ & a finite var σ^2, then, X could be +ve/−ve

$$\boxed{P(|X-\mu| \ge k\sigma) \le \frac{1}{k^2}}$$

- $P(|X-\mu| \ge 2\sigma) \le \frac{1}{4}$

But, $X \sim Bin(10, 0.5)$, $\mu = 5$, $\sigma = 1.58$

$$P(|X-5| \geq 2\sigma) = P(X \in 0, 1, 9, 10) = 0.021$$

- Mean μ, through Markov, bounds the probab. that a non-negative RV takes values much larger than the mean.
- μ & σ, through Chebyshev, bound the probab. that X is away from μ by $k\mu$.

Week-4

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L4.1 Introduction (Continous RV)

- Meteorite wt. (0.01gm - 60 tonnes) → 45000+ data au
- Bin (n, p), $n \uparrow$, and p is constant Eg. $^{1000}C_{500}$
- Meteorite data → (0.01 gm to 60 ton) $\xrightarrow{\log_2}$ [-6.6, 25.8]
- histogram of log of wts → precision is reduced
- we will work with intervals and histograms

L4.2 Cumulative distribution func^n

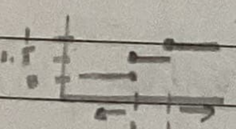
- CDF of a RV X, is a func^n from $\mathbb{R} \to [0,1]$

$$\boxed{F_X(x) = P(X \leq x)}$$

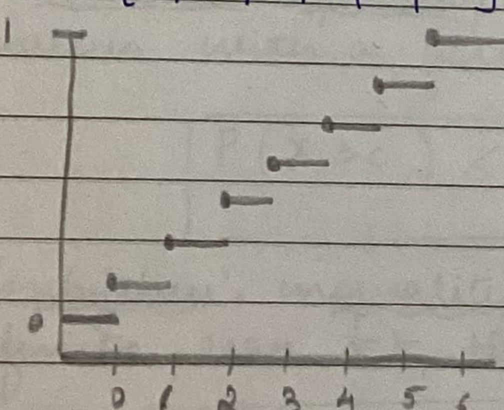
- $F_x(b) - F_x(a) = P(a < X \leq b)$
- F_x : non decreasing func^n taking non-ve values
- $x \to -\infty$, F_x goes to 0 ; $x \to \infty$, F_x goes to 1

Eg. $X \sim \left\{ \overset{1-p}{0}, \overset{p}{1} \right\}$

$$F_x(x) \begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$



Eg. $X \sim \{1, 2, 3, 4, 5, 6\}$



$$F_x(x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \leq x < 2 \\ 2/6 & 2 \leq x < 3 \\ 3/6 & 3 \leq x < 4 \\ 4/6 & 4 \leq x < 5 \\ 5/6 & 5 \leq x < 6 \\ 1 & 6 \leq x \end{cases}$$

- $X \sim \left\{ \overset{p_1}{x_1}, \overset{p_2}{x_2}, \overset{p_3}{x_3}, \overset{p_3}{x_4}, \overset{p_5}{x_5} \right\}$

$$F_x(x) = \begin{cases} 0 & x < x_1 \\ p_1 & x_1 \leq x < x_2 \\ p_1 + p_2 & x_2 \leq x < x_3 \\ p_1 + p_2 + p_3 & x_3 \leq x < x_4 \\ " + p_4 & x_4 \leq x < x_5 \\ " + p_5 & x_5 \leq x \end{cases}$$

Eg. $X \sim U\{1, 2, \dots 10\}$ $\quad F_{x}(x) = \begin{cases} 0 & x < 1 \\ k/100 & k \le x < k+1 \\ 1 & x \ge 100 \end{cases}$

$P(3 < X \le 10) = F_x(10) - F_x(3) = 7/100$
$P(3.2 < X \le 10.6) = F_x(10.6) - F_x(3.2) = 7/100$
$P(X \le 17) = F_x(17) = 17/100$
$P(X \le 17.3) = F_x(17) = 17/100$
$P(X > 87) = 1 - F_x(87) = 13/100$
$P(X > 87.4) = 1 - F_x(87.4) = 13/100$

L4.3 Approx. of CDF from discrete to continuous

- A func^n $F: \mathbb{R} \to [0, 1]$ is said to be CDF, if –
  and F is continuous from the right.
  no need for a step-like structure, it can be smooth and continuous
- It goes from $0 \to 1$, with non-decreasing curve.

Eg. $X \sim U\{1, 2, \dots, 100\}$ $\quad F(x) = \begin{cases} 0 & x \le 0 \\ x/100 & 0 \le x \le 100 \\ 1 & x \ge 100 \end{cases}$

$\left.\begin{array}{l} F(10.6) - F(3.2) = 7.4/100 \\ F_x(17.3) = 17.3/100 \end{array}\right\}$ prev. eg.

Eg. Bin $(100, 0.6)$

$F_x(k) = \sum_{j=0}^{k} {}^{100}C_j \, (0.6)^j (0.4)^{n-j}$ $\qquad F_{(x)} = \dfrac{1}{1 + \exp\left(\dfrac{-1.65(x-60)}{\sqrt{24}}\right)}$

$P(40 < X \le 50) = 0.0271, \quad F(50) - F(40) = 0.0310$

- Better approx. are possible, as n grows.

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L4.4 Gen. RV and Conti. RV

- $P(X \le x) = F(x)$ $\Rightarrow$ RV with CDF $F(x)$
- $P(a < X \le b) = F(b) - F(a)$
- If $F(x)$ rises from F_1 to F_2 at x_1, $P(X = x_1) = F_2 - F_1$
- If $F(x)$ is continuous at x_0, $P(X = x_0) = 0$

pmf

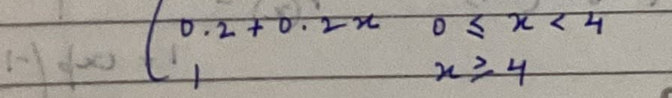
Eg. $F(x) = \begin{cases} 0 & x < 0 \\ 0.5x + 0.1x & 0 \le x \le 5 \\ 1 & x > 5 \end{cases}$

$\rightarrow P(1.99 < X \le 2.01) = F(2.01) - F(1.99) = 0.002$

$\rightarrow P(X = 2.000\ldots) = 0$ $\Rightarrow$ F_X is continuous at such a pt.

as precision $\uparrow$, probab. $\downarrow$

- A RV X with CDF $F_X(x)$ is said to be a conti RV if $F_X(x)$ is continuous at every x.
  - $\rightarrow$ no jumps
  - $\rightarrow$ $P(X = x) = 0$ for all x
  - $\rightarrow$ $P(a < X \le b) = F(b) - F(a) = P(a < X < b)$
  $= P(a \le X \le b) = P(a \le X < b)$



discrete mix continuous

F $\rightarrow$ always inc.

Eg. $F_X(x) = \begin{cases} 0 & x < -5 \\ 0.2 & -5 \le x < 0 \\ 0.2 + 0.2x & 0 \le x < 4 \\ 1 & x \ge 4 \end{cases}$

$P(X > 2) = 1 - P(X \le -2)$
$= 1 - F(-2)$
$= 1 - 0.2 = 0.8$

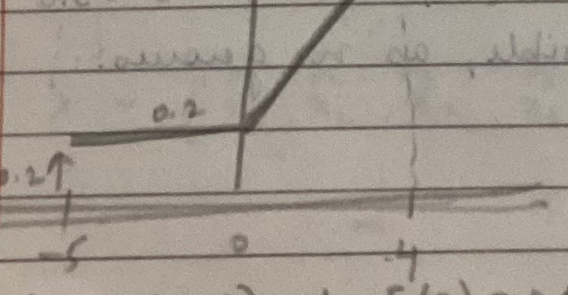
$P(X \le -3) = F(-3)$

$P(X = -3) = 0$

$P(X < -3) = 0.2$

$P(-3 < X < -1) = F(-1) - F(-3) = 0$

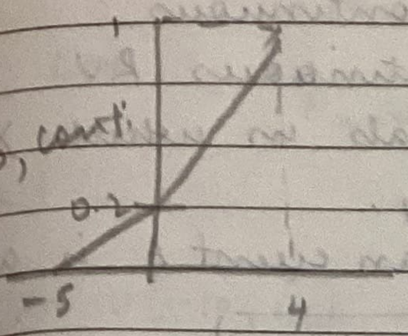
$P(-1 < X < 1) = F(1) - F(-1) = 0.2$



$P(X \ge 3) = P(X > 3) = 1 - F(3) = 0.2$

$P(X = -5) = 0$ $\Rightarrow$ Not a conti. RV

Eg. $F_x(x) = \begin{cases} 0 & x < -5 \\ 0.04x + 0.2 & -5 \le x < 0 \\ 0.2 + 0.2x & 0 \le x < 4 \\ 1 & x \ge 4 \end{cases}$



C4.5 Probab. density funcⁿ

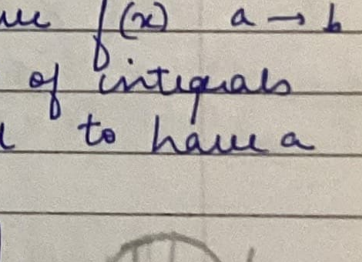
- $\dfrac{dF(x)}{dx} = f(x)$ $\quad F(x) = \int f(x)\, dx$ Indefinite integral
- $\int_a^b f(x)\, dx = F(b) - F(a) \rightarrow$ Definite integral

→ It equals the area under the curve $f(x)$ $a \rightarrow b$

→ Table of integrals → wiki → list of integrals

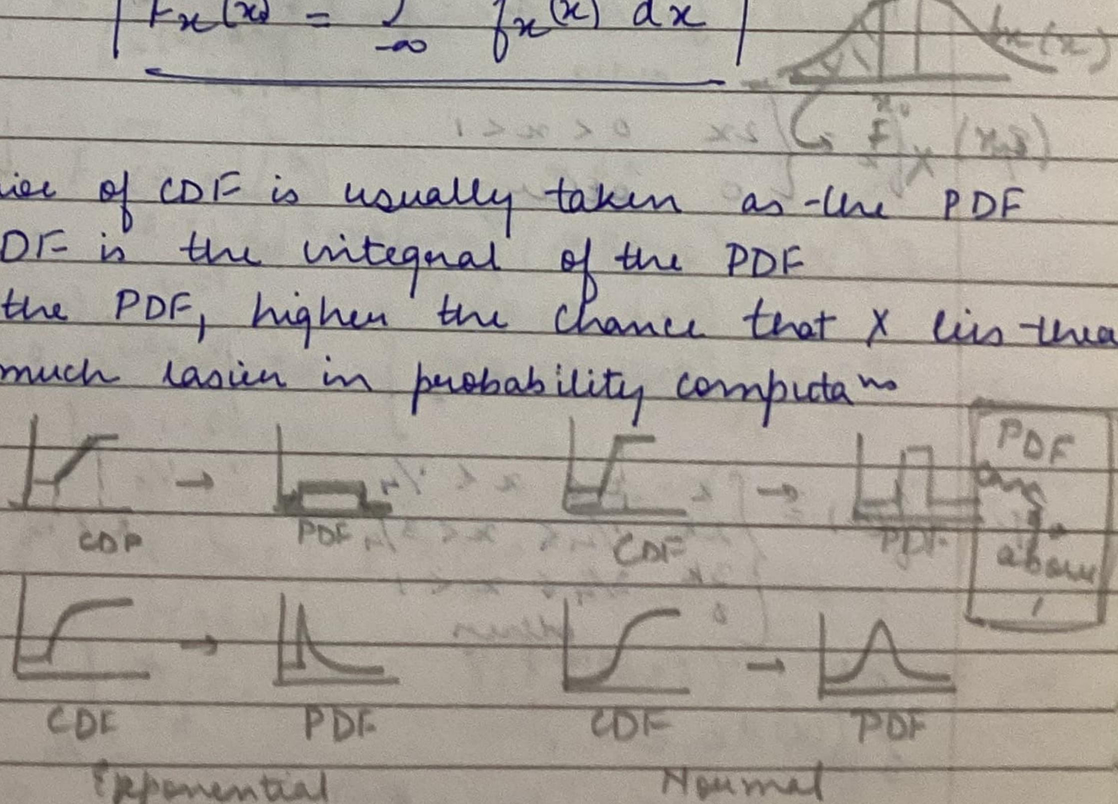
- A conti. RV X with CDF $F_x(x)$ is said to have a PDF $f_x(x)$ if for all x_0

$$\boxed{F_x(x_0) = \int_{-\infty}^{x_0} f_x(x)\, dx}$$



- Derivative of CDF is usually taken as the PDF
  ↳ CDF is the integral of the PDF
- Higher the PDF, higher the chance that X lies there.

PDF is much easier in probability computaⁿ



- Prop. of PDF →
  (i) $f(x) \geq 0$ (always non-negative)
  (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$
  (iii) $f(x)$ is piecewise continuous
- $P(X = x) = 0 \Rightarrow$ for continuous RV
- supp(X) contains intervals in which X can fall with +ve probability.
- For a RV X with PDF f_X, an event A is a subset of the R

$$\boxed{P(A) = \int_A f(x)\,dx}$$

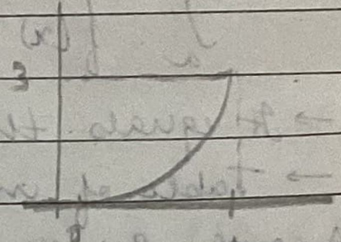
Eg. $f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{other} \end{cases}$ Show f is DF.

$\int_0^1 3x^2\,dx = x^3 \Big|_0^1 = 1$

$P(X = 1/5) = (2/5) = 0$

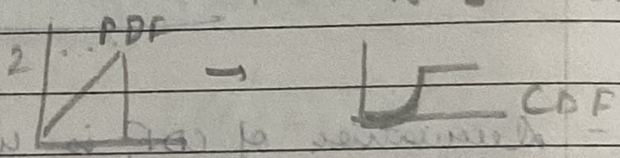
$P(X \in [1/5 - \epsilon, 1/5 + \epsilon])$

$= x^3 \Big|_{1/5-\epsilon}^{1/5+\epsilon} = \boxed{\frac{6}{25}\epsilon + 2\epsilon^3}$



$P(X \in [2/5 - \epsilon, 2/5 + \epsilon]) = x^3 \Big|_{2/5-\epsilon}^{2/5+\epsilon} = \frac{24\epsilon}{25} + 2\epsilon^3$

Eg. $f_X = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{other} \end{cases}$



$P(X \in (0.1, 0.3)) = x^2 \Big|_{0.1}^{0.3} = 0.08$

$\left.\begin{array}{l} [0.1, 0.3] \\ [0.1, 0.3) \end{array}\right\} x^2 \Big|_{0.1}^{0.003} = 0.0009 - 0.01$

Eg. $f(x) = \begin{cases} k & 0 \leq x < 1/4 \\ 2k & 1/4 \leq x < 3/4 \\ 3k & 3/4 \leq x < 1 \\ 0 & \text{other} \end{cases}$ $= \int_0^{1/4} k\,dx + \int_{1/4}^{3/4} 2k\,dx + \int_{3/4}^{1} 3k\,dx = 1$

$\int_0^1 f(x)\,dx = 1$

$k = 1/2$

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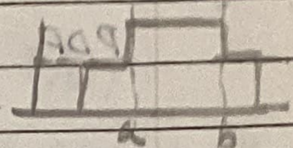
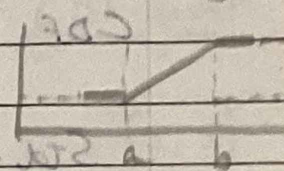
4.6 Common Distributions

- Uniform $\quad X \sim U[a, b]$

PDF $\quad f_x(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{other} \end{cases}$



CDF $\quad F_x(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$



Eg. $X \sim U[-10, 10]$

$f_x(x) = \begin{cases} \frac{1}{20} & -10 \le x \le 10 \\ 0 & \text{other} \end{cases}$

$P(-3 < X < 2) = \frac{5}{20}$

$P(5 < |X| < 7) = \frac{2}{20} + \frac{2}{20} = \frac{4}{20}$

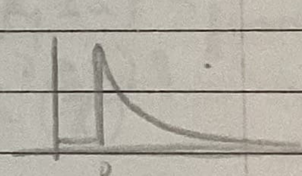
$P(x_0 - \varepsilon < X < x_0 + \varepsilon) = \frac{2\varepsilon}{20}$

x_0 inside $[-9, 9]$

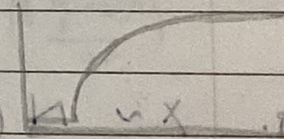
$P(X > 7 \mid X > 3) = \frac{(X > 7,\ X > 3)}{X > 3} = \frac{P(X > 7)}{P(X > 3)} = \frac{3/20}{7/20} = \frac{3}{7}$

- Exponential $\quad X \sim Exp(\lambda)$

PDF $\quad f_x(x) = \begin{cases} \lambda \exp(-\lambda x) & x > 0 \\ 0 & \text{other} \end{cases}$



CDF $\quad F_x(x) = \begin{cases} 0 & x \le 0 \\ 1 - \exp(-\lambda x) & x > 0 \end{cases}$



Eg. $X \sim Exp(2)$

$f_x(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \le 0 \end{cases}$ $\qquad P(5 < X < 7) = \int_5^7 2e^{-2x}\, dx$

$= -e^{-2x}\Big|_5^7 = e^{-10} - e^{-14}$

$P(1 - \varepsilon < X < 1 + \varepsilon) = e^{-2}(e^{2\varepsilon} - e^{-2\varepsilon})$

$P(X > 4) = 1 - P(X \le 4)$

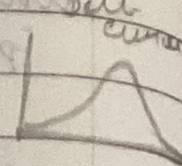
$= 1 - (1 - e^{-8})$

$= e^{-8}$

$P(X > 7 \mid X > 3) = \frac{e^{-14}}{e^{-6}} = e^{-8}$

Memoryless Prop $\Longrightarrow P(X > s + t \mid X > s) = e^{-t}$

- Normal / Gaussian distrib^n $\quad X \sim \text{Normal}(\mu, \sigma^2)$



PDF $\quad f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

CDF $\quad F_x(x) = \int_{-\infty}^{x} f_x(\mu)\, d\mu$

Std. Normal : Normal $(0,1)$

- If $X \sim N(\mu, \sigma^2)$, then $\left(\frac{X-\mu}{\sigma}\right) \sim N(0,1)$

$Z \sim N(0,1)$ PDF $\rightarrow \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$

CDF $\rightarrow \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$

Eg. $X \sim N(\overset{\mu}{2}, \overset{\sigma^2}{5})$

$Z = \frac{X-2}{\sqrt{5}} \sim N(0,1)$

$X = \sqrt{5}\, Z + 2 \qquad X < 5 \Rightarrow \sqrt{5}\, Z + 2 < 5 \Rightarrow Z < 3/\sqrt{5}$

$\cdot\ P(X<5) = P\left(Z < \frac{3}{\sqrt{5}}\right) = F_Z\left(3/\sqrt{5}\right)$

$X > 5 \Rightarrow Z > 3/\sqrt{5} \Rightarrow P(\ \ ) = 1 - F\left(\frac{3}{\sqrt{5}}\right)$

Eg. $X \sim N(3,1)$

$Z = X - 3 \sim N(0,1) \Rightarrow X = Z + 3$

$5 < X < 7 \Rightarrow 5 < Z + 3 < 7 \Rightarrow 2 < Z < 4$

$P(5 < X < 7) = P(2 < Z < 4) = F_Z(4) - F_Z(2)$

$\boxed{P(Z>0) = 1/2}$

14.7 Func^n of conti. RV

Eg. $X \sim U[0,1]$ $\qquad Y = 2X \in [0,2]$ is a RV, CDF of Y?

$F_Y(y) = P(Y \le y) = P(2X \le y) = P(X \le y/2)$

$= \int_0^{y/2} f_X(x)\,dx = y/2$ $\qquad \Rightarrow$ Find CDF first

PDF of Y, $f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2} \Rightarrow Y \sim U[0,2]$

$$\boxed{Y = aX + b \sim U[b, b+a]}$$

- X is conti. RV with CDF F_X and PDF f_X
  $Y = g(X)$ is RV with CDF F_Y
  $\rightarrow F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \in \{x : g(x) \le y\})$
- Suppose X is conti. with PDF f_X. Let $g(x)$ be monotonic for $x \in supp(X)$ with derivative $g'(x) = \frac{dg(x)}{dx}$. Then, PDF of $Y = g(X)$ is

$$\boxed{f_Y(y) = \frac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y))}$$ $\Rightarrow$ for monotonic diff. func^n

$\Rightarrow$ Translat^n $\quad Y = X + a \qquad f_Y(y) = f_X(y - a)$

$\Rightarrow$ Scaling $\quad Y = aX \qquad f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right)$

$\Rightarrow$ Affine $\quad Y = aX + b$
$\qquad f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

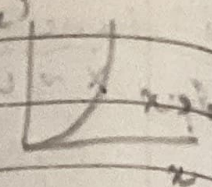
- $X \sim$ Normal $(0,1)$ $\qquad f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$

$Y = \sigma X + \mu \qquad f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$\Rightarrow Y \sim$ Normal (μ, σ^2)

- $X \sim$ Normal (μ, σ^2) $\qquad \Rightarrow$ Affine transform^n of normal RV is also normal.
  $Y \sim$ Normal $(0,1)$
  $Y = \frac{(X-\mu)}{\sigma}$

Eg. $X \sim Exp(\lambda)$. Find the PDF of X^2



$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{else} \end{cases}$$

$g(x) = x^2$

$g'(x) = 2x$

$y = x^2$

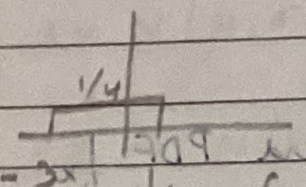
$x = \sqrt{y} \Rightarrow g^{-1}(y) = \sqrt{y}$

$Y = X^2$

$$f_Y(y) = \frac{1}{2\sqrt{y}} \lambda e^{-\lambda \sqrt{y}}$$

Eg. $X \sim U[-3, 1]$. Find PDF of X^2

$g(x) = x^2$ is not monotonic in supp(X)



supp$(Y) = [0, 9]$

$F_Y(y) \Rightarrow P_Y(Y \leq y) = P_X(X^2 \leq y)$

$y \in [0, 1]$: $(X^2 \leq y) \longleftrightarrow -\sqrt{y} \leq X \leq \sqrt{y} \Rightarrow F_Y(y) = \frac{2\sqrt{y}}{4}$

$y \in [1, 9]$: $(X^2 \leq y) \longleftrightarrow -\sqrt{y} \leq X \leq 1 \Rightarrow F_Y(y) = \frac{1 + \sqrt{y}}{4}$

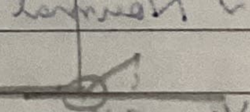
$$f_Y(y) = \begin{cases} \frac{1}{2} \cdot \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ \frac{1}{4} \cdot \frac{1}{2\sqrt{y}} & 1 < y < 9 \end{cases}$$

Eg. $X \sim U[-3, 1]$. Find PDF of $\max(X, 0)$

$$g(x) = \max \begin{cases} 0 & -3 < x < 0 \\ x & 0 < x < 1 \end{cases}$$

$Y = g(x) \in [0, 1]$

$y < 0$: $F_Y(y) = P(Y \leq \text{neg.}) = 0$



$y = 0$: $F_Y(y) = P(Y \leq 0)$

$= P(g(x) \leq 0)$

$= P(-3 < x < 0) = 3/4$

g is not diff.

$\Rightarrow Y$: not continuous

$0 < y < 1$

$F_Y(y) = \frac{3}{4} + \frac{y}{4} = \frac{3+y}{4}$

$F_Y(y) = 1$

L4.6 Expectatn of Conti. RV

- X conti RV, density $f_X(x)$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

for discrete,

$$E[g(X)] = \sum_{x \in T_X} g(x)\, p_X(x)$$

↑ integratn ↑

- Mean / $E[X]$ or μ_X / μ

$$g(X) = X \Rightarrow E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, d(x)$$

- Variance / $Var(X)$ / σ_X^2 / σ^2

$$Var(X) = E[(X-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\, dx$$

$$Var(X) = E[X^2] - E[X]^2$$

- $X \sim U[a,b]$ $\quad f_X(x) = \frac{1}{b-a}$

$$E[X] = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12}$$

- $X \sim Exp(\lambda)$ $\quad f_X(x) = \lambda \exp(-\lambda x) \quad x > 0$

$$E[X] = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}$$

- $X \sim Normal(\mu, \sigma^2)$ $\quad f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$$E[X] = \mu \qquad Var(X) = \sigma^2$$

- Markov inequality $\Rightarrow$ $X \to$ mean μ

$$P(X<0) = 0 \qquad P(X>c) \le \frac{\mu}{c}$$

- Chebyshev inequality $\Rightarrow$ $P(|X-\mu| \ge k\sigma) \le \frac{1}{k^2}$

Eg. $f_X(x) = \begin{cases} 1-|x| & -1 \le x < 1 \\ 0 & \text{other} \end{cases}$ Find CDF of X, $E[X]$, $Var(X)$

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} + x + \frac{x^2}{2} & -1 \le x < 0 \\ \frac{1}{2} + x - \frac{x^2}{2} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$E[X] = 0$$

$$Var(X) = \frac{1}{6}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

Q. X has pdf $f_x(x) = \begin{cases} \frac{1}{2}\cos x & -\pi/2 \le x \le \pi/2 \\ 0 & \text{other} \end{cases}$

CDF of X, $E[X]$, $Var(X)$

$E[X] = 0$

$F_x(x) = \begin{cases} 0 & x < -\pi/2 \\ \frac{1+\sin x}{2} & -\pi/2 < x < \pi/2 \\ 1 & x > \pi/2 \end{cases}$ $\quad Var(X) = \frac{\pi^2}{4} - 2$

Formula Chart

Discrete RV

	PDF	CDF	
- $U \sim \{a, \dots, b\}$	$\frac{1}{n}$, $x=k$; $n = b-a+1$; $k = a, a+1, \dots b$	$\begin{cases} 0 & x<0 \\ \frac{k-a+1}{n} & k \le x < k+1 \\ 1 & x \ge n \end{cases}$	$E[X] = \frac{a+b}{2}$; $Var(X) = \frac{n^2-1}{12}$
- $B(p)$	$\begin{cases} p & x=1 \\ 1-p & x=0 \end{cases}$	$\begin{cases} 0 & x<0 \\ 1-p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$	$E[X] = p$; $Var(X) = p(1-p)$
- $Bin(n,p)$	${}^nC_k p^k (1-p)^{n-k}$	$\begin{cases} 0 & x<0 \\ \sum_{i=0}^{k} {}^nC_i \dots & \\ 1 & x \ge n \end{cases}$	$E[X] = np$; $Var(X) = np(1-p)$
- $Geo(p)$	$(1-p)^{k-1}(p)$	$\begin{cases} 0 & x<0 \\ 1-(1-p)^k & k \le x < k+1 \end{cases}$	$E[X] = \frac{1}{p}$; $Var(X) = \frac{1-p}{p^2}$
- $Poisson(\lambda)$	$\frac{e^{-\lambda}\lambda^k}{k!}$	$\begin{cases} 0 & x<0 \\ e^{-\lambda}\sum_{i=0}^{k} \frac{\lambda^i}{i!} \end{cases}$	$E[X] = \lambda$; $Var(X) = \lambda$

Continuous RV

	$f_x(x)$	$F_x(x)$	
$U \sim [a,b]$	$\frac{1}{b-a}$	$\begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a<x<b \\ 1 & x \ge b \end{cases}$	$E[X] = \frac{a+b}{2}$; $Var(X) = \frac{(b-a)^2}{12}$

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- Exp(λ) $\lambda e^{-\lambda x}$ $x > 0$ $\begin{cases} 0 & x \le 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases}$ $E[X] = \frac{1}{\lambda}$ $Var(X) = \frac{1}{\lambda^2}$

- Normal(μ, σ^2) $\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ No closed form $E[X] = \mu$ $Var(X) = \sigma^2$

- Markov's inequality - RV, non-ve values, finite mean
$$P(X \ge c) \le \frac{\mu}{c}$$

- Chebyshev's inequality - discrete RV, μ & σ^2
$$P(|X - \mu| \ge k\sigma) \le \frac{1}{k^2}$$

Week 1 & Week 2 Rev. Notes

1. Joint PMF $f_{XY}(t_1, t_2) = P(X = t_1, Y = t_2) \rightarrow$ table
2. Marginal PMF $f_X(t_1) = P(X = t_1) = \sum_{t_2 \in T_Y} f_{XY}(t_1, t_2)$ $\rightarrow f_Y$
   $\hookrightarrow$ given joint PMF, the marginal is unique
3. Conditional PMF $f_{X|A}(t) = P(X = t \mid A)$
   $= \frac{P((X = t) \cap A)}{P(A)}$
   This range can be diff from A
   Conditional of 1 RV given another
   $$f_{Y|X=x}(y) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{f_{XY}(x, y)}{f_X(x)}$$
4. We can extend this notion to multiple RV
5. Factors of joint PMF
   $$f_{X_1 X_2 X_3 X_4}(t_1, t_2, t_3, t_4) = f_{X_1 | X_2 = t_2, X_3 = t_3, X_4 = t_4}(t_1) \times f_{X_2 | X_3 = t_3, X_4 = t_4}(t_2) \times f_{X_3 | X_4 = t_4}(t_3) \times f_{X_4}(t_4)$$
6. For X & Y to indep.; $f_{XY}(x, y) = f_X(x) \times f_Y(y)$ for all x & y

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And if even 1 cases fails, the X & Y are dependent. All subsets of indep. RVs are indep

7. i.i.d RV $\longrightarrow$ independent
$\longrightarrow$ marginal PMFs are identical

$\rightarrow$ Let $X_1, X_2 \dots X_n \sim$ i.i.d X (Geo(p))

$$P(X = k) = p^{k-1} \times p$$
$$P(X > j) = (1-p)^j$$
$$P(X_1 > j, X_2 > j \cdots X_n > j) = (1-p)^{n \times j}$$

8. $f(X)$ is a func^n of RV X with PMF $f_X(t)$

$$f_{f(X)}(a) = P(f(X) = a) = P(X \in \{t : f(t) = a\})$$
$$= \sum_{f(t)=a} f_X(t)$$

$\rightarrow$ $X, Y \sim f_{XY}$ $Z = X + Y$

$$P(Z = z) = \sum P(X = x, Y = z - x)$$
$$= \sum f_{XY}(x, z - x)$$

Convolution - If X & Y are indep.

$$f_{X+Y}(z) = \sum f_X(x) \times f_Y(z-x)$$

$\rightarrow$ Let $X \sim$ Pois.(λ_1) & $Y \sim$ Pois.(λ_2)

$$\Rightarrow f_Z(z) \sim \text{Pois.}(\lambda_1 + \lambda_2)$$

$$(X = k \mid Z = n) \sim \text{Bin}\left(n, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right)$$
$$(Y = k \mid Z = n) \sim \text{Bin}\left(n, \frac{\lambda_2}{\lambda_1 + \lambda_2}\right)$$

9. CDF of RV X

$$F_X(x) = P(X \le x)$$

10. $X, Y \sim f_{XY}$ & $Z = \min(X, Y)$

$$f_Z(z) = P(Z = z) = P(\min(X,Y) = z)$$
$$= f_{XY}(z, z) + \sum_{t_2 > z} f_{XY}(z, t_2) + \sum_{t_1 > z} f_{XY}(t_1, z)$$

$$F_Z(z) = P(Z \le z) = 1 - [P(X > z, Y > z)]$$

11. $z = \max(X, Y)$

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Week-5

L5.1 Motivaⁿ

- Iris data set → R.A. Fisher (Statician)
- Iris flower –
  - 3 classes → 0, 1 & 2 (50 instances in each)
  - Data → sepal len (SL), sepal width (SW), petal len (PL), PW
  - Classificaⁿ → given data, find class
- see data → summary (min-max, avg, stdev)
  → we can prepare histograms
- Joint distribuⁿ → Class (discrete) and sepal length (continuous)

L5.2 Joint distribuⁿ: Discrete and conti

- (X, Y) : jointly distributed
- X: discrete with range T_X and PMF $p_X(x)$
- For each $x \in T_X$, a conti RV Y_x with density $f_{Y_x}(y)$
- $Y_x : (Y \mid X = x)$

  $f_{Y_x}(y) = f_{Y|X=x}(y)$
- Marginal density of y

$$\boxed{f_Y(y) = \sum_{x \in T_X} p_X(x)\, f_{Y|X=x}(y)}$$

Eg. $X \sim U\{0,1,2\}$ & $Y|X=0 \sim \text{Normal}(5, 0.4)$, $Y|X=1 \sim \text{Normal}(6, 0.5)$, $Y|X=2 \sim \text{Normal}(7, 0.6)$

1. What is marginal of Y
2. We observe Y to be around y_0. What can you say abt. X.

1. $f_{Y|X=0}(y) = \dfrac{1}{\sqrt{2\pi}\,0.4} e^{-\frac{(y-5)^2}{2(0.4)^2}}$

   $f_{Y|X=1}(y) =$

   $f_{Y|X=2}(y) =$

for 0

$$f_Y(y) \propto \underbrace{\frac{1}{3} \times \frac{1}{\sqrt{2\pi} \times 0.4} e^{-\frac{(y-5)^2}{2(0.4)^2}}}_{\text{for } 0} + \underbrace{\ldots}_{\text{for } 1} + \underbrace{\ldots}_{\text{for } 2}$$

↳ "Mixture of Gaussians" distribution

- X & Y are jointly distributed with $X \in T_X$ discrete and conditional densities $f_{Y|X=x}(y)$.

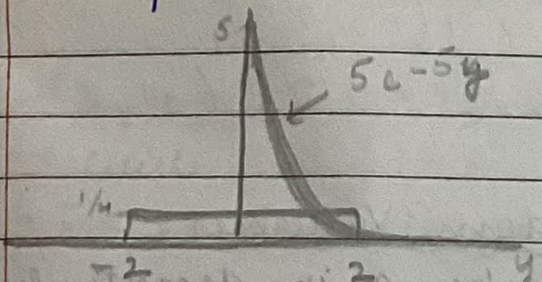
$$P(X = x \mid Y = y_0) = \frac{p_X(x)\, f_{Y|X=x}(y_0)}{f_Y(y_0)}$$

↳ marginal density of Y

If X & Y are independent,

$$P(X = x \mid Y = y_0) = p_X(x)$$

Eg. $X \sim U\{-1, 1\}$. Let $Y|X=-1 \sim U[-2, 2]$, $Y|X=1 \sim Exp(5)$. Find dist. of X given $Y = -1, 1, 3$



$X|Y=-1$: $P(X=-1 \mid Y=-1)$

$= p_X(-1) \cdot f_{Y|X=-1}(-1)$

$$f_Y(y) = \underbrace{\frac{1}{2} \times \frac{1}{4}}_{[-2,2]} + \underbrace{\frac{1}{2} \times 5e^{-5y}}_{Exp(5)}$$

$$\begin{cases} 0 & y < -2 \\ \frac{1}{8} & -2 < y < 0 \\ \frac{1}{8} + \frac{1}{2} 5e^{-5y} & 0 < y < 2 \\ \frac{1}{2} 5e^{-5y} & y > 2 \end{cases}$$

$P(X = 1 \mid Y = -1)$

$X|Y=1$: $P(X=-1 \mid Y=1)$

$$= \frac{\frac{1}{2} \times \frac{1}{4}}{\frac{1}{8} + \frac{1}{2} 5e^{-5}}$$

$P(X = -1 \mid Y = 1)$

$= 1 -$

Eg. 60% of adults in age grp (45 - 50) are male. height of male $\sim N(160, 10)$ and of female $\sim N(150, 5)$. A random person is 155 cm. Is that male / female.

$X \sim \{M, F\}$ $\quad Y \mid X = M \sim N(160, 10)$

$Y \mid X = F \sim N(150, 5)$

$X \mid Y = 155$, $\quad P(X = M \mid Y = 155) = \dfrac{0.6 \times \frac{1}{\sqrt{2\pi} \times 10} e^{-\frac{5^2}{200}}}{0.6 \times \quad + 0.4 \times}$

$P(X = F \mid Y = 155) = 1 - P(X = M \mid Y = 155)$

Eg. $Y = X + Z$ $\quad X \sim U\{-3, -1, 1, 3\}$ & $Z \sim N(0, \sigma^2)$ are indep. What is dist. of Y? Find dist. of $(X \mid Y = 0.5)$

$f_{Y \mid X = -3}(y) = (-3 + z) \sim N(-3, \sigma^2)$

$Y \mid X = -1 \Leftrightarrow (-1 + z) \sim N(-1, \sigma^2)$

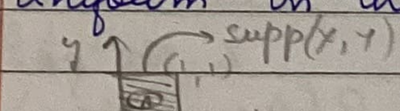
L5.3 Jointly Continuous Random Variables

- A func^n $f(x,y)$ is said to be a jt. density func^n if
  - → $f(x,y) \geq 0$ → f is non-negative
  - → $\iint_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$
  - → $f(x,y)$ is piecewise continuous in each var.

$$P((X,Y) \in A) = \iint_A f(x,y)\, dx\, dy \qquad f(x,y) = f_{XY}(x,y)$$

$$\text{supp}(X,Y) = \{(x,y) : f_{XY}(x,y) > 0\}$$

- Uniform in the unit sq. $\quad f_{XY}(x,y) = \begin{cases} 1 & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{other} \end{cases}$



$$\int_0^1\!\!\int_0^1 1\, dx\, dy = \int_0^1 x\Big|_0^1 dy = \int_0^1 dy = y\Big|_0^1 = 1$$

$$P((X,Y) \in A) = \iint_A 1\, dx\, dy = \text{Area of } A$$

Eg. $P(0 < x < 1/2,\ 0 < y < 1/2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

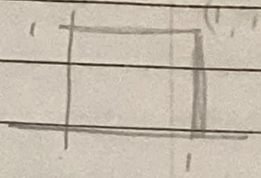
Eg. $P(0 < X < 0.1,\ 0 < Y < 0.1) = 0.01$

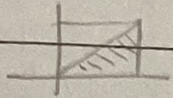
$P(0.5 < X < 0.6,\ 0 < Y < 0.1) = 0.01$

$P(0.9 < X < 1,\ 0.9 < Y < 1) = 0.01$

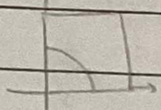
$P(0 < X < 0.1) = 0.1$

$P(0.5 < Y < 0.6) = 0.1$



$P(X > Y)$  $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

$P(X > 2Y)$  $= \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{4}$

$P(X^2 + Y^2 < 0.25)$  $\frac{1}{4} \times \pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{16}$

2D Uniform distribution – Fix some reg. D in R^2 with total area $|D|$. $(X,Y) \sim U(D)$

$$f_{XY}(x,y) = \begin{cases} \frac{1}{|D|} & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$

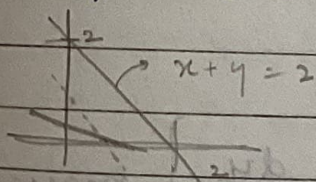
→ Rectangle: $D = [a,b] \times [c,d] = \{(x,y) : a < x < b,\ c < y < d\}$

→ Circle: $D\ \{(x,y) : (x - x_0)^2 + (y - y_0)^2 \leq a^2\}$

→ For any sub reg. A of D, $P((X,Y) \in A) = |A| / |D| = \frac{\text{Area}(A)}{\text{Area}(D)}$

Eg. $(X,Y) \sim U(D) \rightarrow D = \{(x,y) : x + y < 2,\ x > 0,\ y > 0\}$

Sketch, $P(X + Y < 1)$, $P(X + 2Y > 1)$



$P(X + Y < 1) = \frac{1/2}{2} = \frac{1}{4}$ $\quad P(X + 2Y > 1) = \frac{2 - \frac{1}{4}}{2} = \frac{\frac{7}{4}}{2} = \frac{7}{8}$

Eg. $f_{XY}(x,y) = \begin{cases} x + y & 0 < x,\ y < 1 \\ 0 & \text{other} \end{cases}$. Show valid density

$P(X < 1/2,\ Y < 1/2)$

& $P(X + Y < 1)$

$$\int_{y=0}^{1}\int_{x=0}^{1}(x+y)\,dx\,dy \Rightarrow \int_0^1 (x+y)\,dx = \frac{x^2}{2}+yx\Big|_0^1 = \frac{1}{2}+y$$

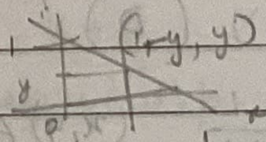
$$\int_0^1 \frac{1}{2}+y\,dy = \frac{y}{2}+\frac{y^2}{2}\Big|_0^1 = 1$$

$$P(X<1/2,\ Y<1/2) = \int_0^{1/2}\int_0^{1/2}(x+y)\,dx\,dy$$

$$= \int_0^{1/2}\frac{x^2}{2}+yx\Big|_0^{1/2} = \frac{1}{8}+\frac{y}{2}$$

$$= \frac{y}{8}+\frac{y^2}{4}\Big|_0^{1/2} = \frac{1/2}{8}+\frac{1/4}{4} = \frac{1}{8}$$

$$P(X+Y<1) = \int_0^1\int_{x=0}^{1-y}(x+y)\,dx\,dy$$



$$= \frac{x^2}{2}+yx\Big|_0^{1-y} = \frac{(1-y)^2}{2}+\frac{(1-y)y}{1}$$

$$= \frac{(1-y)^2+2y(1-y)}{2}$$

$$= \frac{1+y^2\cancel{-2y}\cancel{+2y}-2y^2}{2}$$

$$= \frac{1-y^2}{2}$$

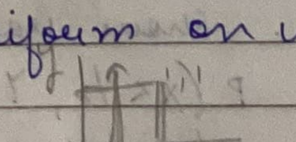
$$\int_0^1 \frac{1-y^2}{2}\,dy = \frac{1}{3}$$

LS.4 Marginal Densities

- For jt. density, $f_{XY}(x,y)$

$$\left.\begin{aligned} f_X(x) &= \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy \\ f_Y(y) &= \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx \end{aligned}\right\}$$ marginal densities ↓ PDFs of X & Y individually

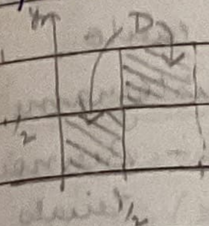
- Uniform on unit sq.



$$f_X(x_0) = \int_{y=0}^{1} f_{XY}(x_0,y)\,dy = 1 \quad 0<x_0<1$$

$X \sim U[0,1]$
$Y \sim U[0,1]$

- $(X,Y) \sim U(D) \rightarrow D = [0, \tfrac{1}{2}] \times [0, \tfrac{1}{2}] \cup [\tfrac{1}{2}, 1] \times [\tfrac{1}{2}, 1]$



Area $(D) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ $\qquad f_{xy}(x,y) = \begin{cases} 2 & (x,y) \in D \\ 0 & \text{other} \end{cases}$

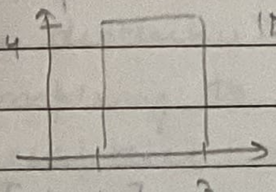
$x_0 \in [0, 1/2] \quad f_x(x_0) = 1$

$x_0 \in [1/2, 1] \quad f_x(x_0) = 1$

$X \sim U[0,1]$

$Y \sim U[0,1]$

- Marginals do not determine joint uniquely
- $(X,Y) \sim U(D) \quad D = [1,3] \times [0,4]$



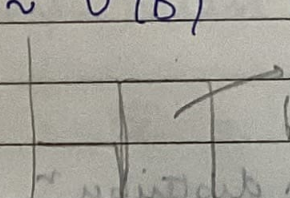
$|D| = 8$

$f_{xy} = \frac{1}{8}$ $\qquad f_x(x) = 4 \times \frac{1}{8} = \frac{1}{2}$

$X \sim U[1,3]$

$Y \sim U[0,4]$

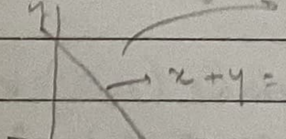
- $(X,Y) \sim U(D) \quad D = [0,1] \times [0,1] \cup [1,2] \times [0,2]$



$f_{xy} = \frac{1}{3}$ $\qquad f_x(x) = \begin{cases} \frac{1}{3} & 0 < x < 1 \\ \frac{2}{3} & 1 < x < 2 \end{cases}$

$f_y(y) = \begin{cases} \frac{2}{3} & 0 < y < 1 \\ \frac{1}{3} & 1 < y < 2 \end{cases}$

- $(X,Y) \sim U(D) \quad D = \{(x,y) : x + y < 2,\ x > 0,\ y > 0\}$



$f_{xy} = \frac{1}{2}$

$f_x(x) = \int_0^{2-x} \frac{1}{2}\, dy = 1 - \frac{x}{2} \quad 0 < x < 2$

$f_y(y) = \int_0^{2-y} \frac{1}{2}\, dx = 1 - \frac{y}{2} \quad 0 < y < 2$

Q. $f_{xy}(x,y) = \begin{cases} x + y & 0 < x, y < 1 \\ 0 & \text{others} \end{cases}$

$f_x(x) = \int_0^1 (x+y)\, dy = xy + \frac{y^2}{2} \Big|_0^1 = x + \frac{1}{2} \quad 0 < x < 1$

$f_y(y) = y + \frac{1}{2} \quad 0 < y < 1$

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Week-7

L7.1 Stats from i.i.d samples
- iid samples — Bernoulli trials, monte carlo simulations, computing histograms
- Occurrence of an event A is success — Bernoulli trials

  $p = P(A)$

  Indep. repetitionⁿ / trials of expt., n times

  $X_i = 1$ if A occurs in i^{th} trial, $P(A) = P(X_i = 1)$

  iid Bernoulli samples : $X_1, X_2 \ldots X_n$
- Repeat n times simulaⁿ independently

  $$P(A) \approx \frac{n_A}{n}$$
- Bin : $[a, b]$

  n_b : no. of x_i that fall inside $[a, b]$

  Event A = $(a < X < b)$

  $$P(a < X < b) \approx \frac{n_b}{n}$$
- i.i.d samples hold info. on distribuⁿ

  Data → modelled as obs. from i.i.d repetiⁿ of an expt.
- Analysis → metric → 'good'

  → no: of samples v/s good metrics

Eg. 20 i.i.d $B(p)$ with p?

Sampling 1 → 1, 1, 0, 1, 0, 0 . . . .

Sampling 2 → 0, 0, 1, 1, 1, 0 . . . .

⇒ p same for all sampling

⇒ but samples aren't same

⇒ even if var, the p must guarantee

- Statistical problem —

  Model: $X_1, X_2 \ldots X_n \sim$ iid X

  Given: $x_1, x_2, x_3 \ldots$ from 1 sampling

  ⇒ distribuⁿ of X is partially known

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Goal: Procedure to find info. about distribuⁿ of X

7.2 Empirical distribution and descriptive stats

- Empirical distribuⁿ → Let $X_1, X_2 \ldots$ iid samples.
  Let $\#(X_i = t)$ denote no. of times t occurs.

$$p(t) = \frac{\#(X_i = t)}{n}$$

Eg. $n = 20$ ⇒ 1 1 0 1 0 0 0 1 0 1 1 0 1 1 0 1 0 1 1 1

Empirical $\sim \left\{ \overset{8/20}{0}, \overset{12/20}{1} \right\}$

- this distribuⁿ is random. t and $p(t)$ may Δ from 1 sampling to another
- Descriptive stats : Prop. of empirical distribuⁿ
  ↳ mean, var., probab.
- $\boxed{\text{Sample Mean } (\overline{X}) = \frac{X_1 + X_2 + \ldots + X_n}{n}}$

$$\overline{x} = \frac{x_1 + x_2 + \ldots x_n}{n}$$ → sample mean

(Above) eg. $\overline{X} = \frac{12}{20}$

Eg. $B(0.5)$ samples $X_1 \ldots \sim$ iid $\left\{ \overset{0.5}{0}, \overset{0.5}{1} \right\}$ ↳ distribuⁿ mean $= \frac{1}{2}$

$n = 5$ sample 1 : $\overline{X} = 3/5$

— $4/5$

Eg. $N(0,1)$ samples $X_1 \ldots \sim$ i.id $N(0,1)$ ← distribⁿ mean

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ $n = 5$

sample means $= -0.25 / 0.17 / 0.11$

- Let $X_1, \ldots X_n$ be iid samples of distribuⁿ with μ and σ^2.

Sample $\boxed{E[\overline{X}] = \mu}$ $\boxed{\text{Var}(\overline{X}) = \frac{\sigma^2}{n}}$

- var. of sample mean decreases with n
- as n increases :-
  (i) var. of sample mean → 0

(ii) the spread of sample mean will dec.
(iii) sample mean will take values close to the distribuⁿ mean.

- Sample Var $\boxed{(s^2) = \frac{(X_1-\bar{X})^2 + (X_2-\bar{X})^2 \dots (X_n-\bar{X})^2}{n-1}}$

- $\boxed{E[S^2] = \sigma^2}$

- Var. of distribuⁿ → constant R and not random
  Sample Var → random var. with mean equal to distribuⁿ var.

Eg: $B(1/2)$ $\mu = 0.5$, $\sigma^2 = 0.25$
$N(0,1)$ $\mu = 0$, $\sigma^2 = 1$

- $X_1 \dots X_n \sim X$ iid samples from distri of X.

Sample proportion $\boxed{S(A) = \frac{\#(X_i \text{ for which } A \text{ is true})}{n}}$

$\boxed{E[S(A)] = P(A)}$ $\boxed{\text{Var}(S(A)) = \frac{P(A)(1-P(A))}{n}}$

as $n \uparrow$, $S(A) \approx P(A)$, $E[S(A)] = P(A)$
$\text{Var}(S(A)) = 0$

L7.3 Illustrations with Data

- 3 classes of iris — 0, 1, 2 ⟹ 50 each
  ↳ SL, SW, PL, PW
- Model: iid samples acc. to some unknown distribution
  ↳ how good is the iid samples model
- Taj Mahal Air Quality → data.

- do some descriptive stats and get sample mean, variance and few proportions
- IPL - runs scored in deliveries 0.1, 0.2, 0.3 ↳ 1598 innings
- Clear trend from samples → runs scored later are more higher

L7.4 Sum of indep. RV

- $X_1 \dots X_n$ be a RV, $S = X_1 + \dots + X_n$ ⇒ sum

$$\boxed{E[S] = E[X_1] + \dots + E[X_2]}$$

If $X_1, \dots X_n$ are pairwise correlated,

$$E[X_i X_j] = E[X_i]\, E[X_j]$$

$$\boxed{Var(S) = Var(X_1) + \dots + Var(X_2)}$$

→ Mean of sum is sum of means
→ If uncorrelated, variance of sum is sum of variances.

Independence ⇒ Uncorrelated

- $S = a_1 X_1 + \dots + a_n X_n$ a_i are constants

$E[S] = a_1 E[X_1] + \dots + a_n E[X_n]$

$Var(S) = a_1^2 Var(X_1) + \dots + a_n^2 (Var(X_2))$

⇒ iid samples - $X_1 \dots X_n \sim$ iid

$E[S] = (a_1 + \dots + a_n)\, E[X]$

$Var(S) = (a_1^2 + \dots + a_n^2)\, Var(X)$

- Weak law of large nos.

$X_1 \dots X_n \sim$ iid (X)

$\mu = E[X]$, $\sigma^2 = Var(X)$, $\bar{X} = (X_1 \dots + X_n)/n$

$E[\bar{X}] = \mu$ $Var(\bar{X}) = \frac{\sigma^2}{n}$

$$\boxed{P(|\bar{X} - \mu| > \delta) \leq \frac{\sigma^2}{n\delta^2} \to 0}$$

↳ $P(\bar{X} > \mu + \delta)$
$P(\bar{X} < \mu - \delta)$

- with probab. $\geq \left(1 - \frac{\sigma^2}{n\delta^2}\right)$, sample mean lies in $[\mu-\delta, \mu+\delta]$

- n (iid) samples
  - → Bern. (p)
    $P \geq \left(1 - \frac{p(1-p)}{n\delta^2}\right)$, $\mu \sim [p-\delta, p+\delta]$
  - → $U(-M \ldots M)$
    $P \geq \left(1 - \frac{M(M+1)}{3n\delta^2}\right)$, $\mu \sim [-\delta, \delta]$
  - → $N(0, \sigma^2)$
    $P \geq \left(1 - \frac{\sigma^2}{n\delta^2}\right)$, $\mu \sim [-\delta, \delta]$
  - → $U[-A, A]$
    $P \geq \left(1 - \frac{A^2}{3n\delta^2}\right)$, $\mu \sim [-\delta, \delta]$

L7.5 Concentration phenomenon

- Markov inequality : X → +ve values

$$\boxed{P(X \geq t) \leq \frac{E[X]}{t}}$$

- Chebyshev:

$$\boxed{P((X - E[X])^2 \geq t^2) \leq \frac{Var(X)}{t^2}}$$

- Chernoff inequality : $E[X] = 0$

$$P(X \geq t) = \boxed{P(e^{\lambda X} \geq e^{\lambda t}) \leq \frac{E[e^{\lambda X}]}{e^{\lambda t}}}, \quad \lambda > 0$$

- MGF (moment generating func^n) of X : $E[e^{\lambda X}]$
  - → Pick λ that provides best bound
  - → use upper bound on MGF

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What is centralising?

→ X : RV with mean $E[X]$

→ centralised version of X: $X - E[X]$

Centralised $B(1/2)$

→ $B(1/2)$: $\{0, 1\}$ with prob. $1/2$, $1/2$, mean $= 1/2$

→ Centralised : $X \sim \{-1/2, 1/2\}$, mean $= 0$

$$E[e^{\lambda X}] = \frac{e^{\lambda/2} + e^{-\lambda/2}}{2}$$

Bound on MGF:

$$E[e^{\lambda X}] = \frac{e^{\lambda/2} + e^{-\lambda/2}}{2} \leq e^{\lambda^2/4}$$

$X_1, \dots X_n \sim$ iid X

$S = X_1 + \dots + X_n$, MGF of S? $\quad \bar{X} = \frac{S}{n}$

$$\boxed{E[e^{\lambda S}] = E[e^{\lambda X}]^n}$$

↳ MGF of sum of indep. RVs is product of the indi. MGFs.

$$E[e^{\lambda S}] = \left(\frac{e^{\lambda/2} + e^{-\lambda/2}}{2}\right)^n \leq e^{n\lambda^2/4}$$

↳ Upper bound is much easier than MGF.

Chernoff bound for $B(n, 1/2)$

$$P(S > t) \leq \frac{E[e^{\lambda S}]}{e^{\lambda t}} \leq e^{n\lambda^2/4 - \lambda t}$$

$$\lambda = 2t/n \Rightarrow P(S > t) \leq e^{-t^2/n}$$

Now,

$$P(Y > n/2 + n\delta/2) = P(S > n\delta/2) \leq e^{-n\delta^2/4}$$

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- $X_1, \ldots X_n \sim$ iid , $Y = X_1 + \ldots + X_n$

Concentration Phenomenon

→ exponential bounds for $P(Y > E[Y] + t)$ by bounding $E[e^{\lambda Y}]$ and using chernoff

L7.6 Central Limit Theorem

- MGF ⇒ Let X be a zero-mean RV.

$$\boxed{M_X(\lambda) = E[e^{\lambda X}]} \quad \{f : \mathbb{R} \to \mathbb{R}\}$$

→ X : discrete with PMF f_X

$$M_X(\lambda) = f_X(x_1) e^{\lambda x_1} + f_X(x_2) e^{\lambda x_2}$$

→ X : continuous with PDF f_X and support T_X

$$M_X(\lambda) = \int_{x \in T_X} f_X(x) e^{\lambda x} dx$$

Eg. 1. $X \sim \{0\}$ (probability 1)

→ $M_X(\lambda) = 1 \times e^0 = 1$

2. $X \sim \{-p, 1-p\}$ (probabilities $1-p$, p) ← Centralised B(p)

→ $M_X(\lambda) = (1-p) e^{-p\lambda} + p e^{(1-p)\lambda}$

3. $X \sim \{-1, 0, 2\}$ (probabilities $1/2$, $1/4$, $1/4$)

→ $M_X(\lambda) = 0.5 e^{-\lambda} + 0.25 e^0 + 0.25 e^{2\lambda}$

4. $M_X(\lambda) = \left(\frac{1}{3}\right) e^{3\lambda/2} + \left(\frac{1}{6}\right) e^{-3\lambda} + \left(\frac{1}{8}\right) e^{-\lambda} + \left(\frac{1}{8}\right) e^{\lambda} + \frac{1}{8}$

→ $X \sim \{-3, -1, 0, 1, 3/2\}$ (probabilities $1/6$, $1/8$, $1/8$, $1/8$, $1/3$)

5. $\boxed{X \sim N(0, \sigma^2) \\ M_X(\lambda) = e^{\lambda^2 \sigma^2 / 2}}$ V.V.V. Impt. MGF

$E[X] = 0$

$E[X^2] = \sigma^2$

$E[X^3] = 0$

$E[X^4] = 3\sigma^4$

- $E[e^{\lambda X}] = 1 + \lambda \underbrace{E[X]}_{1^{st}\text{ moment}} + \frac{\lambda^2}{2!} \underbrace{E[X^2]}_{2^{nd}\text{ moment}} + \frac{\lambda^3}{3!} \underbrace{E[X^3]}_{3^{rd}\text{ moment}} \ldots$

$$= E\left[1 + \lambda X + \frac{\lambda^2 X^2}{2!} + \frac{\lambda^3 X^3}{3!} + \ldots\right]$$

- $X_1, X_2 \sim$ iid X, $\quad Y = X_1 + X_2$

$X \sim$ iid cen. $B(p)$

$M_X(\lambda) = (1-p)e^{-p\lambda} + pe^{(1-p)\lambda}$

$M_Y(\lambda) = M_X(\lambda)^2 = (1-p)^2 e^{-2p\lambda} + 2p(1-p)\, e^{(1-2p)\lambda} + p^2 e^{2(1-p)\lambda}$

$Y \sim \left\{ \overset{(1-p)^2}{-2p},\ \overset{2p(1-p)}{1-2p},\ \overset{p^2}{2(1-p)} \right\}$

- MGF of sample mean $\rightarrow$ $\bar{X} = (X_1 + \dots + X_n)/n$

$$\boxed{M_{\bar{X}}(\lambda) = \left(\frac{e^{\frac{\lambda}{2n}} + e^{-\lambda/2n}}{2}\right)^n}$$

$M_{\bar{X}}(\lambda) \rightarrow 1$ as $n \uparrow$

- MGF convergence at $1/\sqrt{n}$ scaling

$Y = (X_1 + X_2 \dots X_n)/\sqrt{n}$

$$M_Y(\lambda) = \left(\frac{e^{\frac{\lambda}{2\sqrt{n}}} + e^{\frac{-\lambda}{2\sqrt{n}}}}{2}\right)^n$$

$M_Y(\lambda) \rightarrow e^{\lambda^2\sigma^2/2}$ as $n \uparrow$

- Central Limit Th.

Let $X_1, \dots X_n \sim$ iid X, $E[X] = 0$, $Var(X) = \sigma^2$

$Y = (X_1 + \dots X_n)/\sqrt{n}$

$$\boxed{M_Y(\lambda) \rightarrow e^{\lambda^2\sigma^2/2}}$$

↳ MGF of $N(0, \sigma^2)$

- $P(Y - n\mu \geq \delta n\mu)$ $\quad$: $1 - F\left(\frac{\delta\sqrt{n}\,\mu}{\sigma}\right)$

$(Y - n\mu)/\sqrt{n}$: approx. $N(0, \sigma^2)$

$$\boxed{\frac{Y - n\mu}{\sqrt{n}\,\sigma} \approx N(0,1)}$$

Week-10

L10.1 Intro. to Hypothesis testing

- Is a coin authentic or counterfeit?
  $P(H) = 0.5$ (H_0) $P(H) = 0.6$ (H_A)
  - Null hypothesis (H_0)
  - Alternative hypothesis (H_A)
- 3 times toss coin → HHH, HHT ... TTH
  - some accept H_0 and others reject H_0
  - A be the set which accept H_0
- Every acceptance subset A corresponds to a test
- $X_1 \ldots X_n \in \mathcal{X}^n$ H_0 & H_A
  $A \subseteq \mathcal{X}^n \longrightarrow$ Hypothesis test
  If $X_1 \ldots X_n \in A$, accept H_0
- Metric 1 → significance level (size) of a test → α
  - I → Reject H_0 when H_0 is true
    $\alpha = P(\text{Type I error}) = P(\text{Reject } H_0 \mid H_0 \text{ is true})$
- Metric 2 → Power of a test, $1-\beta$
  - II → Accept H_0 when H_A is true
    $\beta = P(\text{type II error}) = P(\text{Accept } H_0 \mid H_A \text{ is true})$

Eg. $\mathcal{X}^3 = \{HHH \ldots TTT\}$ $H_0: P(H) = 0.5$
$H_A: P(H) = 0.6$

$A = \phi$
→ always reject H_0
→ $\alpha = 1$, $\beta = 0$

$A = \mathcal{X}^3$
→ always accept H_0
→ $\alpha = 0$, $\beta = 1$

- there are some least β for a fixed α.

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- Neyman - Pearson paradigm of hypothesis testing
H_0, H_A, 2 errors, 2 metrics → $\alpha = P(1 \text{ error})$
$1-\beta = 1 - P(2 \text{ error})$

10.2 Size and Power of a test

Eg. 100 coins, T is the no. of heads. Rejects H_0 if $T > C$. $\alpha = ?$ $\beta = ?$
$H_0 \Rightarrow 0.5$
$H_A \Rightarrow 0.6$

→ $A = \{\text{outcome} : T \leq C\}$

$\alpha = P_{100}(\text{Reject } H_0 \mid H_0) = P(A^c \mid P(H) = 1/2)$

$= \sum_{k=C+1}^{100} \binom{100}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{100-k}$

$1-\beta = P(\text{Reject } H_0 \mid \underset{0.6}{H_A}) = P(A^c \mid P(H = 0.6))$

$= \sum_{k=C+1}^{100} \binom{100}{k} 0.6^k (0.4)^{100-k}$

→ ~~higher~~ lower c, higher power

Eg. $X \sim N(\mu, 1)$ $H_0 = \mu = 1$ $H_A = 1$. Rejects H_0 if $X > C$. $\alpha = ?, \beta = ?$

$A = \{X \leq C\}$ $\alpha = P(A^c \mid \mu = -1) = P(N(-1,1) > C)$

$= P\left(\underbrace{\frac{N(-1,1)+1}{1}}_{Z \sim N(0,1)} > \frac{C+1}{1}\right) = 1 - F_Z(C+1)$

F_Z: CDF of $N(0,1)$

$1-\beta = P(A^c \mid \mu = 1) = P\left(\frac{N(+1,1) - 1}{1} > \frac{C-1}{1}\right)$

$= 1 - F_Z(C-1)$

Eg. $X \sim Bin(100, p)$, $H_0 = 0.5$, $H_A \neq 0.5$ Rejects H_0 if $|X - 50| > 10$, $\alpha = ?$ $\beta = ?$