Week-5 Math-1

L1 One-to 1 funcⁿ : Defⁿ & Tests

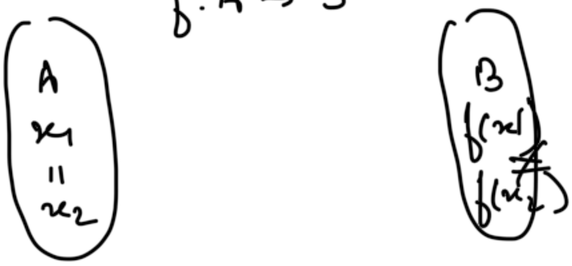
$y = f(x)$ $\qquad\qquad$ $f : A \to B$

$A, B \in R$

Domain $\qquad\qquad$ Codomain

1. one x $\qquad\qquad$ ≥ 1 $f(x)$

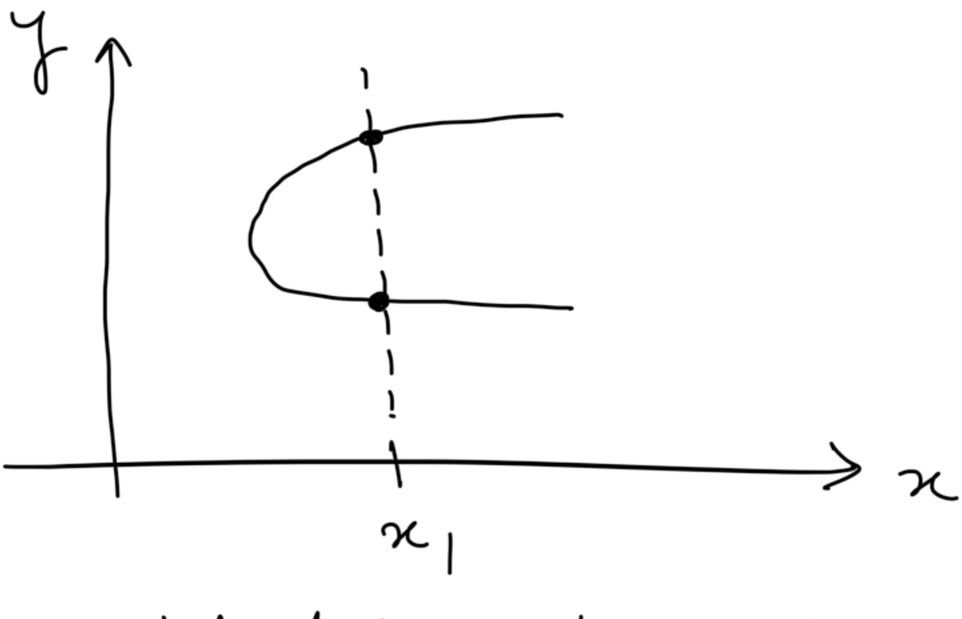
$f : A \to B$



if $(x_1) = (x_2)$
then $f(x_1) \neq f(x_2)$

that means this is not at all a function.

Vertical line test



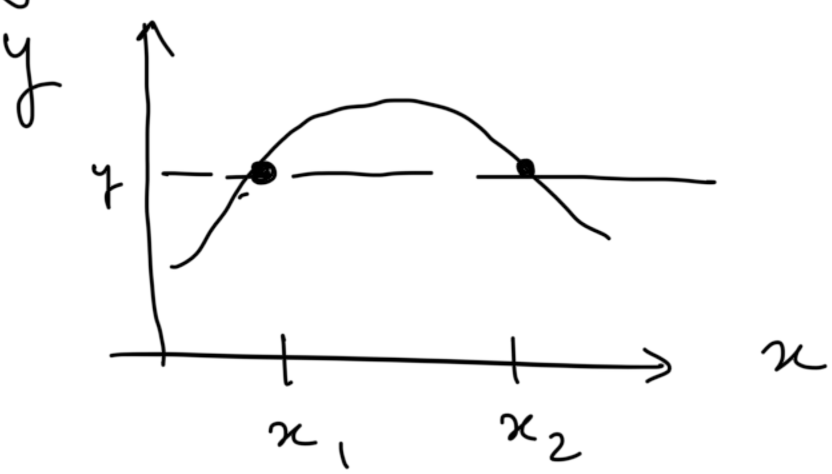
if the vertical lines tests fail, hence, the relation cannot be a funcⁿ

2. more than 1 x one $f(x)$

$$x_1 \neq x_2 \longrightarrow f(x_1) = f(x_2)$$

yes, this is a func^n

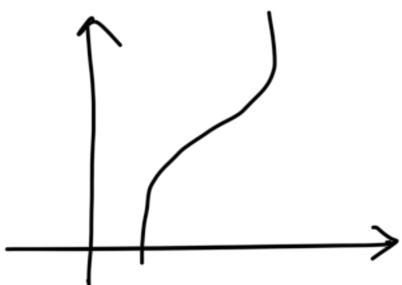
Horizontal line test



if the horizontal line test fails, then, even though the func^n is present, but it won't be a reversible func^n.

3. 1 x 1 $f(x)$

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$



$\Rightarrow$ one-to-one func^n

also, such func^{ns} are reversible functions.

- A func^{n} $f: A \to B$ is c/d one-to-one if for any $x_1 \neq x_2 \in A$,

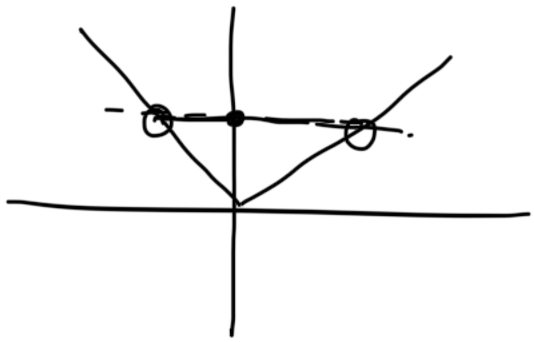
  then $f(x_1) \neq f(x_2)$.

  or if $f(x_1) = f(x_2)$

  then, $x_1 = x_2$.

L2 Eqs. & Theorems

$$f(x) = |x| \Rightarrow \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

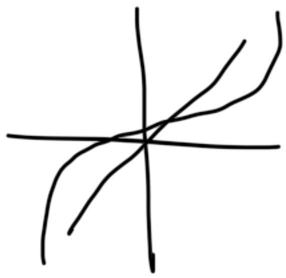


$\Rightarrow$ this is a func^{n} but not a reversible func^{n}

$f(x) = x$

or

x^3



$\Rightarrow$ yes, this is a reversible, 1-to-1 func^{n}.

- If the horizontal line intersects the graph of a func^{n} f in at most 1 pt., then f is one-to-one.

  $\hookrightarrow$ Horizontal Line Test.

- for every $x_1, x_2 \in A$

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$\Rightarrow$ increasing 1-to-1 func^n

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

$\Rightarrow$ decreasing 1-to-1 func^n

- If f is increasing or decreasing func^n then f is one-to-one.

L3 Exponential Func^{ns} : Def^n

a^x $\quad a > 0, \; x \in \mathbb{Q}$

base, exponent

1. what if $x \in \mathbb{R} \setminus \mathbb{Q}$?
2. Why $a > 0$?

Ques. Can we define a^x ($a > 0$) for $x \in \mathbb{R} \setminus \mathbb{Q}$?

Eg. $2^{\sqrt{2}}$, 5^{π}

$\pi = 3.14159 \ldots$ (non-repeating)

$5^3 \Rightarrow 5^{3.1} \Rightarrow 5^{3.14} \Rightarrow 5^{3.15 \ldots} \Rightarrow 5^{\pi}$

$2^{1} \Rightarrow 2^{1.4} \Rightarrow 2^{1.41\ldots} \Rightarrow 2^{\sqrt{2}}$

$\therefore$, a^x is defined for $x \in \mathbb{R}$.

$\hookrightarrow$ this can be proved by seq. theory.

- Laws of exponents -

$s, t \in \mathbb{R}$ and $a, b > 0$

1. $a^s \cdot a^t = a^{s+t}$
2. $(a^s)^t = a^{st}$
3. $ab^s = a^s b^s$

$1^s = 1$, $a^{-s} = \dfrac{1}{a^s}$, $a^0 = 1$ $a > 0$

0^0 is undefined.

- An exponential funcⁿ in std. form is given by $f(x) = a^x$, where $a > 0$, $a \neq 1$.

$\boxed{\begin{matrix} 0 < a < 1 \\ a > 1 \end{matrix}}$

(i) domain of f is $\mathbb{R}$

(ii) $f(x) = 1^x = 1$ (constant)

$\therefore$, $a \neq 1$.

L4 Graphing

$f(x) = 2^x$

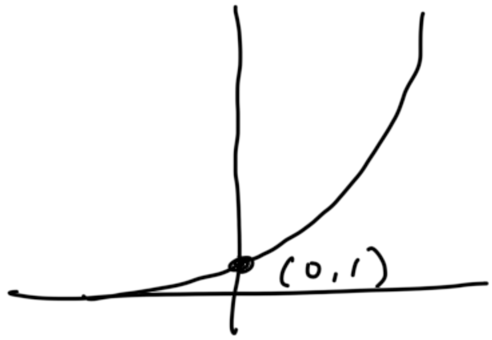
domain = R

range = $(0, \infty)$

y - intercept = 0, 1

x - intercept = Nil

$y = 0 \Rightarrow$ horizontal asymptote



end beha. $\Rightarrow$ $x \to \infty$ $2^x \to \infty$

$x \to -\infty$ $2^x \to 0$

No roots

it is a increasing func^n

so, for all a^x func^{ns} (0, 1) y-int.

when $a > 1$. (1, a)

now, for $0 < a < 1$ $g(x) = a^x$

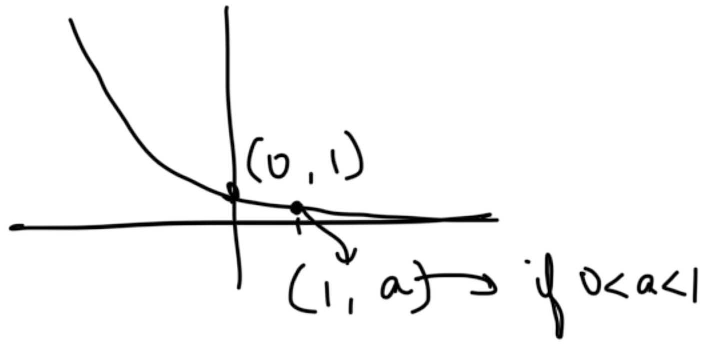
$$g(x) = \left(\frac{1}{5}\right)^x = 5^{-x}$$

domain = R

range = $(0, \infty)$

y intercept = Nil

No root



end beha $\Rightarrow$ $x \to \infty$ $5^{-x} \to 0$

$x \to -\infty$ $5^{-x} \to \infty$

∴, decreasing func^n.

again horizontal asymptote.

1.5 Natural Exponential Func^n

- from the theory of limits

$$\left(1+\frac{1}{n}\right)^n \rightarrow e \quad \text{as} \quad n \rightarrow \infty$$

e is irrational no., $e \approx 2.718\ldots$

- It is generally used in interest rate calcula^n. $\hookrightarrow$ spl. in continuous compounding.

$$₹1 \rightarrow 1\% \rightarrow \left(1+\frac{1}{100}\right)^n$$

but quarterly interest $\left(1+\frac{0.01}{4}\right)^4$

or $\left(1+\frac{0.01}{n}\right)^n \quad n \rightarrow \infty$

$\therefore\; e^{0.01t} \rightsquigarrow t$ no. of years.

- $$\boxed{\left(1+\frac{x}{n}\right)^n} \longrightarrow \boxed{e^x}$$

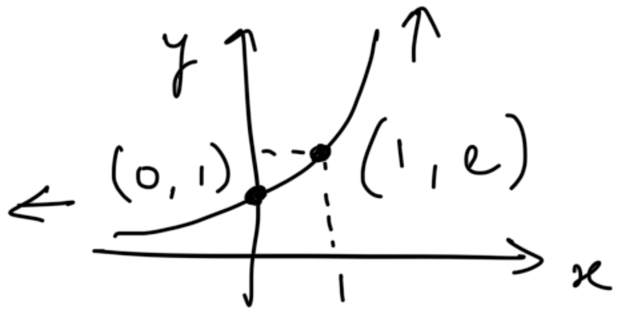
$\hookrightarrow$ euler's no.

- The natural exponential func^n is defined as $f(x) = e^x$.

Domain $= R$

Range $= 0, \infty$

$e > 1$



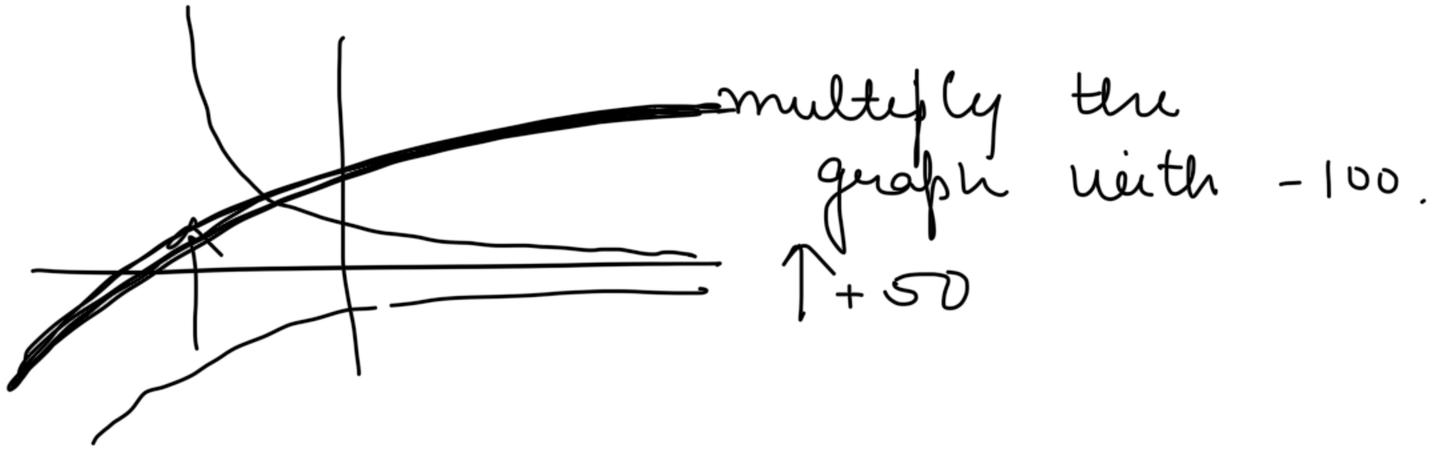
- e is the slope of tangent line to $f(x) = e^x$ at $(1, e)$
- The area under the curve $f(x) = e^x$ from $(-\infty, 1)$ is e.
- for $f(x) = \frac{1}{x}$, $x \in (1, e)$, the area under the curve is 1.

Ques. $R(t) = 50 - 100e^{-0.2t}$

(a) what % of people watch in 10 mins.

$$R(10) = 50 - 100e^{-2} = 36.46.$$

(b) $R(t) = 50 - 100 e^{-0.2t}$



the horizontal symptote looks like 50 units. The highest % you can reach is 50.

(c) $50 - 100e^{-0.2t} = 30$

$$100e^{-0.2t} = 20$$

$$\boxed{\frac{1}{5} e^{-0.2t}} \approx t = 8 \text{ minutes}$$

L6 Composite Func^ns

$f(x) = 0.85x$

$h(x) = x - 3000$

if x is the price of product ~> the price should be ≥ 3000.

$g(x) = 0.85x - 3000$

$\therefore g(x) = f(x) - 3000$

$= \underbrace{h(f(x)) = (h \circ f)(x)}$

such func^ns are called composite func^n

$x = 14000$

$(h \circ f)(x) = h(f(x))$

$= f(x) - 3000$

$= 0.85x - 3000$

$= 8900.$

this is how you do the evalua^n of $(g \circ f)(x)$.

- The composit^n of func^n f & g is denoted by $f \circ g$ & is defined by $(f \circ g)(x) = f(g(x))$

- The domain of the composite func^n $f \circ g$ is the set of all x s.t.

(i) x is in the domain of g
(ii) $g(x)$ is in the domain of f.

$$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$

$\in$ Dom.(g) $\quad$ $\in$ Dom.(f) $\quad$ $\in$ Range(f)

L 7. Eg.

given $f(x) = 3x-4$ $\quad g(x) = x^2$

(a) $(g \circ f)(x)$

$= g(f(x)) = g(3x-4)$
$= (3x-4)^2$

(b) $(f \circ g)(x) = f(g(x))$
$3(x^2) - 4 = 3x^2 - 4$

Eg. $f(x) = x+1$ $\quad g(x) = x^2 - 1$

$f(g(x)) = f(x^2-1) = x^2 - 1 + 1 = x^2$

$g(f(x)) = g(x+1) = (x+1)^2 - 1$
$= x^2 + \cancel{1} + 2x - \cancel{1}$
$= x^2 + 2x$

L 8 Domain of Composite Func^n

$(f \circ g)(x) = f(g(x))$

- The following values must be excluded from input x.
  - $\to x \notin Dom(g) \Rightarrow x \notin Dom(f \circ g)$
  - $\to \{x \mid g(x) \notin Dom(f)\}$ must not be included in $Dom(f \circ g)$

L9 Inverse Functions

not all func^{ns} are reversible. eg. x^2

let's see one-to-one func^n

$g(x) = 4x \qquad h(x) = \frac{x}{4}$

$(g \circ h)(x) = g(h(x)) = 4\left(\frac{x}{4}\right) = x$

$(h \circ g)(x) = h(g(x)) = \frac{(\cancel{4}x)}{\cancel{4}} = x$

- The inverse of a func^n f, f^{-1} is a func^n s.t.

$f^{-1} \circ f(x) = x \quad \forall x \in Dom(f)$
$= Range(f^{-1})$

$f \circ f^{-1}(x) = x \quad \forall x \in Dom(f^{-1})$
Range(f)

f is a one-to-one func^n
$\therefore \Rightarrow f^{-1}$ exists for f.
$\hookrightarrow$ inverse func^n $f^{-1} \neq \frac{1}{f}$

Eg. $g(x) = x^3$ & $g^{-1}(x) = \sqrt[3]{x}$

$$g^{-1}(g(x)) = \sqrt[3]{x^3} = x$$

$$g(g^{-1}(x)) = \left(\sqrt[3]{x}\right)^3 = x$$

Eg. similarly verify,

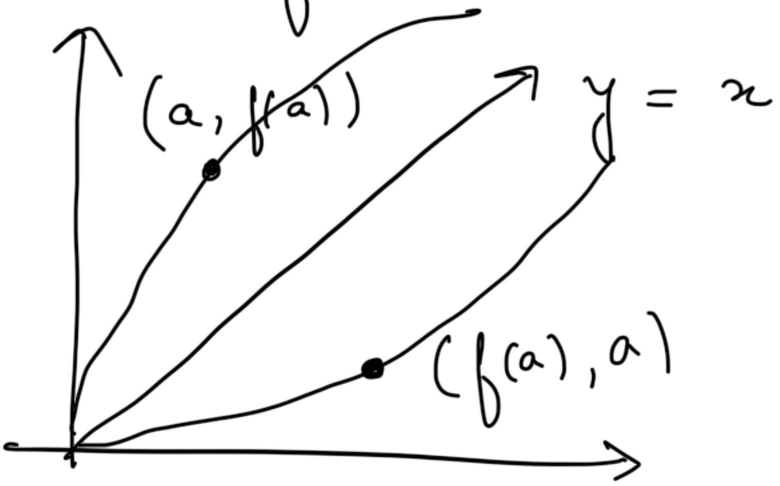
$f(x) = \dfrac{x-5}{2x+3}$ & $g(x) = \dfrac{3x+5}{1-2x}$

- graph of f & f^{-1}

$$a \xrightarrow{f} f(a)$$

$$a = f^{-1}(f(a)) \xleftarrow{f^{-1}} f(a)$$

if $(a, f(a))$ is on the graph of f
then $(f(a), a)$ is on the graph of f^{-1}



Theorem –
The graphs of f & f^{-1} are symmetric across the $y = x$ line.