Video-1 30/6/21

1. Introduction

- data sc. includes maths, stats and computing

→ nos., sets, relations and functions

→ coordinate geometry → lines, slopes, angles

→ quadratic eq^{ns} → polynomials → Exponentials & Logs

→ Graphs – nodes and edges (map of airline timetable)

Video-2 L1.1

2. Natural Nos. and their Operations

→ Natural Nos.

- nos. keep a count of obj. (1, 2, 3, 4 . . .)
- 0 (Indian Origin) – no count at all
- $\mathbb{N} = \{0, 1, 2 \ldots\}$ [No to emphasize 0 is inc.]
- Add, Subtract, Multiply, Divide

→ Integers

- 5 - 6 is not a natural no. ⇒ -1, -2, -3 . . .
- $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2 \ldots\}$ ⇒ No. line $-\infty \longleftrightarrow +\infty$ -1 0 1
- multiplication → $m \times n = \overbrace{m + m + \ldots + m}^{n \text{ times}}$ mn, $m \cdot n$
  ↳ it is repeated addition $(- \times -) = +$ [sign rule]
- repeated multiplication ⇒ $m \times m$ ⇒ m squared ⇒ m^2
  ↳ (exponentiation)

- division is repeated substraction ; quotient of $19 \div 5 = 3$
  $\boxed{19 \bmod 5 = 4}$ remainder $19 \div 5 = 4$

- Factors ⇒ a divides b if b mod a = 0 ⇒ $\boxed{a \mid b}$
  [$4 \mid 20$, $7 \mid 63$, $4 \nmid 19$, $9 \nmid 100$, etc. ⇒ b is a multiple
  ⇒ $a \mid b$ ⇒ a is a factor of b ⇒ occur in pairs
  Eg. 12 = {1, 12}, {2, 6}, {3, 4} Exc ⇒ perfect squares

- Prime nos. → no factors other than 1 and itself
  ⇒ 1 is not a prime - only 1 factor ⇒ 2
⇒ Sieve of Eratosthenes → removes multiples of
  (not a efficient way, but still a good way)

- every no. can be decomposed into prime factors ⇒ prime factorization
⇒ this discomposition is unique

$12 = 2 \cdot 2 \cdot 3$
$= 2^2 \cdot 3$

Video-3 L 1.2

3. Rational Nos.

- cannot represent 19/4 as an integer Frac^n ⇒ $3\frac{4}{5}$
- Rational Nos. → p/q p & q are integers ⇒ Q
- same no can be written in many ways. $3/5 = 6/10$
  ↳ useful to add, subtract, compare rationals
- representa^n is no unique
- reduced form: p/q, where p, q have no common factors
  ↳ come from ratio
- gcd (18, 60) = 6 ⇒ greatest common divisor
- Density → for each integer, we have a next integer and a prev. integer → $m-1 \leftarrow m \rightarrow m+1$
⇒ no integer btw. m & $m+1$ and m & $m-1$
⇒ not possible for rationals ⇒ btw. any 2 rationals we can find another one
⇒ rationals are dense, integers are discrete

Video-4 L 1.3

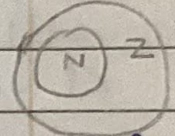
4. Real no. & complex no.

- is every pt. on the no. line a Q? ⇒ No
- $\sqrt{m}$ is x, such that $x \times x = m$ ⇒ Perfect sqs. 1, 49
- $\sqrt{2}$ cannot be written as p/q. → $\sqrt{2}$
- It is irrational + rational ⇒ Real nos. R
- real nos. are dense

- some well-known Irr. nos. $\pi = 3.14\ldots$, $e = 2.718\ldots$
- $\sqrt{-1}$ ⇒ laws of signs of multiplicaⁿ
  ↳ It is a complex number.

Video 5 L 1.4

5. Set theory
- set is a collectⁿ of item. Eg. Factors of 24 $\{1, 2, 3 \ldots, 24\}$
- It may be ∞. Eg. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$
- no requirement that members of a set have uniform type.
- unordered, duplicates doesn't matter.
- Cardinality - no. of items in a set
- in finite sets can be listed out explicitly $\{1, 0\}$
- ∞ sets can't be listed (the platonic solids)
- not every collecⁿ of items is a set
  ↳ Bertrand Russell's Paradox
- Elements - Items in a set
  ↦ Membership : $\boxed{x \in X}$ x is an element of X.
- X is a subset of Y. If every element of X is also an element of Y. $\boxed{X \subseteq Y}$. Eg. $\mathbb{N} \subseteq \mathbb{Z}$



Venn Diag.

- Every set is a subset of itself. $X \subseteq X$
  ↦ $X = Y$, iff. $X \subseteq Y$ & $Y \subseteq X$
- Proper subset : $X \subseteq Y$ but $X \neq Y \Rightarrow X \subsetneq Y$
- The empty set has no elements - ϕ
  ↦ $\phi \subseteq X$, Every element of ϕ is also in X.
- Powerset - set of subsets of set ⇒ $n = 2^n$ subset
  $X = \{a, b\} \Rightarrow$ Powerset $= \{\phi, \{a\}, \{b\}, \{a, b\}\}$
- Proof → $X = \{x_1, x_2 \ldots x_n\}$
  In a subset, either include or exclude each x_i
  2 choices per element, $\underbrace{2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n$ subsets
- bit ~ digit

Date / /

L1.5 Construcⁿ of Subsets and Set Operaⁿˢ

- set comprehensions
- the subset of even integers $\{x \mid x \in \mathbb{Z}, x \bmod 2 = 0\}$
  - → have 1 set ($\mathbb{Z}$), condⁿ to keep the no. or not and collect all x that satisfies the condition
- set of perfect sq. → $\{m \mid m \in \mathbb{N}, \sqrt{m} \in \mathbb{N}\}$ $\{1,4,9..\}$
- set of rationals in red. ⇒ $\{p/q \mid p,q \in \mathbb{Z}, \gcd(p,q)=1\}$
- Intervals
  - → inte. from -6 to 6 $\{z \mid z \in \mathbb{Z}, -6 \leq z \leq 6\}$
  - → closed interval (includes end points) $[0,1]$ $\{x \mid x \in \mathbb{Z}, 0 \leq x \leq 1\}$
  - → open (exclude ends) $(0,1)$ $\{x \mid x \in \mathbb{Z}, 0 < x < 1\}$
  - → left open $(0,1]$ $\{x \mid x \in \mathbb{Z}, 0 < x \leq 1\}$
- Union → (combines) $\cup$ $\{a,b,c\} \cup \{c,d,e\} = \{a,b,c,d,e\}$
- Intersecⁿ → (common) $\cap$ $\{a,b,c,d\} \cap \{a,d,e,f\} = \{a,d\}$
- Set Diff. → element X that are not in Y, $X \setminus Y$ $(X-Y)$
  - $X \setminus Y \neq Y \setminus X$ $\{a,b,c,d\} \setminus \{a,d,e,f\} = \{b,c\}$
- Complement - element not in X, $\bar{X}$ or X^C
  - → define complement relative to larger set universe
  - → complement of prime no. in $\mathbb{N}$ are composite nos.

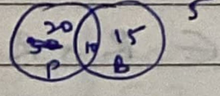
L1.6 Sets : Egs :

- $5 \in \mathbb{Z}$, $\sqrt{2} \notin \mathbb{Q}$; $P \subseteq \mathbb{N}$, $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Q} \subseteq \mathbb{R}$
- ϕ, & set itself is a subset of every set
- sq. of even integers $(0,4,16..)$ $\{x^2 \mid x \in \mathbb{Z}, x \bmod 2 = 0\}$
  - → Generate (from existing set), filter
  - → Transform / Modify
- identity transformaⁿ → take the input and gives output as it is

- real in interval $[-1, 2)$ $\{x \mid x \in \mathbb{R}, -1 \leq x < 2\}$
- cube of 1st 5 N $Y = \{n^3 \mid n \in \{0,1,2,3,4\}\}$
  ↳ for 500 N? $X = \{n \mid n \in \mathbb{N}, n < 500\}$ } makes diffⁿ readable
  (less tedious) $Y = \{n^3 \mid n \in X\}$
- Z whose sq. root is also Z $\{z \mid z \in \mathbb{Z}, \sqrt{z} \in \mathbb{Z}\}$
  ↳ since all are +ve $\Rightarrow \{n \mid n \in \mathbb{N}, \sqrt{n} \in \mathbb{N}\}$
  ↳ also, generate all sq. $\Rightarrow \{n^2 \mid n \in \mathbb{N}\}$
- choose the generator as req.

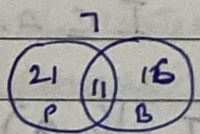
L1.7 Egs. of set problems

- In class, P → 30 student B → 25, 10 both, 5 null
  Total?

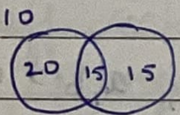


50
3. 15 in $B \setminus P$
1. 10 in $P \cap B$
4. 5 in $\overline{P \cup B}$
2. 20 in $P \setminus B$

- 55 students, 32 → P, 11 took both, 7 neither
  B but not P?
  $55 - 7 = 48 - 32 = 16$



- 60 students, 35 → P, 30 → B, 10 took neither
  both?
  $60 - 10 = \underline{50}$



$35 + 30 - x = 50$
$x = 15$

- set notaⁿ → concisely describe collecⁿ of obj.

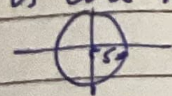
L1.8 Relations

- $X \cup Y$, $X \cap Y$, $X \setminus Y$, X^c (w.r.t Y)
- Cartesian prod. ⇒ $A \times B = \{(a,b) \mid a \in A, b \in B\}$
  Eg. $A = (0,1)$ $B = (2,3)$ $\Rightarrow \{(0,2),(0,3),(1,2),(1,3)\}$
- in a pair, order is impt.
- for set of nos., product is as 2-d space

- cartesian product + set comprehension
  $\{(m,n) \mid (m,n) \in N \times N,\ n = m+1\}$ $\{(0,1),(2,3)\ldots\}$
- Binary relaⁿ → subset of prod. $R \subseteq A \times B$
  Notⁿ ⇒ $(a,b) \in R$, aRb

Eq. T → Teachers, C → courses $A \subseteq T \times C$
$A = \{(t,c) \mid (t,c) \in T \times C,\ t \text{ teaches } c\}$
Sheila → Bio
Ajay → Eng

Eq. P → people, $M \subseteq P \times P$
$M = \{(m,c) \mid (m,c) \in P \times P,\ m$ is the mother



Eq. Points at dist. 5 from (0,0)
- cartesian prod. of more than 2 sets

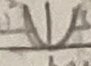
Eq. Pythagorean triples $\{(a,b,c) \mid (a,b,c) \in N \times N \times N,\ a,b,c > 0,\ a^2+b^2=c^2$

Eq. sq. corner (x,y) corner $\in R \times R$
↳ 4 such, we get a sq. $Sq \subseteq R^2 \times R^2 \times R^2 \times R^2$

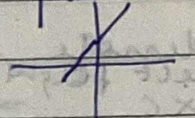
- Identity Relaⁿ $I \subseteq A \times A$ $I = \{(a,b) \mid (a,b) \in A \times A, a=b$
  $I = \{(a,a) \mid (a,a) \in A \times A\}$ / $\{(a,a) \mid a \in A$
- Reflexivity - element is related to itself
  ↳ $R \subseteq A \times A$, $I \subseteq R$
  ↳ $\{(a,b) \mid (a,b) \in N \times N,\ a,b > 0,\ a \mid b\}$
- Symmetric - $\{(a,b) \in R$, iff. $(b,a) \in R\}$
  Eq. Reduced fractions, $|a-b| = 2$
- Transitive - if $(a,b) \in R$, $(b,c) \in R$, then $(a,c) \in R$
  → $2 \mid 6$, $6 \mid 36$ ∴ $2 \mid 36$
- Antisym. → $(a,b) \in R$ & $a \neq b$ → $(b,a) \notin R$
- Equivalence relaⁿ → reflexive + sym + transitive

Eq. same mod 5 7 mod 5 = 22, 22 mod 5 = 2
- measures time → 24 hrs into 2, 20:am = 2 pm
- Eq. relaⁿ partitions a set. grps of eq. elements are c/d equivalence classes.

L1.9 Functions

- a rule to map input → output   Eg convert x to x^2
- need to specify input & output sets  $x \mapsto x^2$, $sq(x) = x^2$
  parabola

- Domain → input set   domain (sq) $= \mathbb{R}$
- Codomain → output set of possible values   cod.(sq) $= \mathbb{R}$
- Range → actual output   range(sq) $= \mathbb{R} \geq 0$
- $f: X \mapsto Y$; domain of f is X, codomain is Y
- associate a relaⁿ R_f of funcⁿ f $R_{sq} = \{(x,y) \mid x, y \in \mathbb{R}, y = x^2\}$
- $R_f \subset$ domain$(f) \times$ range(f)
- Prop. of R_f
  - → define on entire domain, $x \in$ dom.(f), $(x,y) \in R_f$
  - → single-valued, $x \in$ domain(f) → exactly 1 output
  - ↳ drawing f as a graph is plotting R_f

Eg. $f(x) = 3.5x + 5.7$ 
- ↳ (line) slope(2) intercept)
- → ∆ing the slope and intercept we get diff. lines
- ↳ domain $= \mathbb{R}$, codomain = Range $= \mathbb{R}$

Eg. $x \mapsto \sqrt{x}$; Is this a funcⁿ? $\sqrt{25}$ has 2 opⁿˢ → +sq. root
- ↳ Domain ($\mathbb{R} \geq 0$) depends on codomain → $\mathbb{R}$

- Injective (one-to-one) - diff. inputs → diff. output   Eg. $f(x) = 3x + 5$
- Surjective - range is codomain → onto $y \in$ codomain
  $x \in$ domain(f) $f(x) = -7x + 10$; $f(x) = y$
- Bijective - 1-1 corr. btw. domain & codomain
  - → every $x \in$ domain(f) maps to distinct $y \in$ codomain(f)
  - → $y \in$ codomain(f) has a unique pre-image $x \in$ domain(f) such that $y = f(x)$
  - → for ∅ sets we can count the items
  - ↳ we can say bijecⁿ results in same cardinality for codomain and domain

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L1.10 Relations Eg.

- $A \times B$ cartesian prod., order is impt.
- $A = \{1, 4, 7\}$, $B = \{1, 16, 49\}$ (1,1), (1,16), (1,49) . . . $(m \times n)$ pairs
  $A \times B \neq B \times A$, we can have $B \times B$ (identical + ...)
- we can take cartesian prod. of > 2 sets, $A \times B \times A$
- a relaⁿ picks out certain tuples from cart. prod.

Eg. $S \subseteq A \times B = \{(1,1), (4,16), (7,49)\}$ $\{(a,b) \mid (a,b) \in A \times B, b = a^2\}$

- divisibility → pair ℕ (d, n), such that $d \mid n$. (7,63)
  go to integers ⟸ $D = \{(d,n) \mid (d,n) \in \mathbb{N} \times \mathbb{N}, d \mid n\}$
- prime power - pair of natural nos (prime) 5^5
  and $n = p^m$ for some natural no. m
  - Egs. (3,1) (5,625), (7,343) $P = \{p \mid p \in \mathbb{N}, f(p) = \{1, p\}\}$
  - $PP = \{(p,n) \mid (p,n) \in P \times \mathbb{N}, n = p^m$ for $m \in \mathbb{N}\}$
- Airline routes - Eg. Bang., Kolkata . . .
- Let C denote the set of cities served by airli...
  - $D \subseteq C \times C$ (direct flight) → so, is D reflexive / irreflexive
  - is D sym. ⇒ yes, like chennai ⇌ delhi
- Tables as relaⁿ → Dist $\subseteq C \times C \times \mathbb{N}$

Source	destinaⁿ	Dist. (km)
Bangl.	chennai	290
Delhi	chennai	1752

} some entries are useless
} D→C, C→D, dist. same

- distances are sym., even if D is not

Eg. Students data
Roll no. | name | DOB . . .
] some columns are spl like Roll no. → (key)

- Name is not a key, becoz 2 people have same name
- funcⁿ of roll nos. ⟶ (name, DOB)
- such tables are cld. (key, value) pairs
- we can join more than 2 tables and join them

$\{(r, n, s, g) \mid (r, n, d) \in$ Students, $(r', s, g) \in$ grades, $r = r'\}$

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L1.11 Func^n - Eqs.

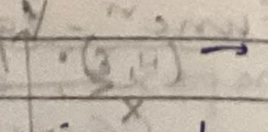
- $x \mapsto x^2$, $g(x) = x^2$; domain, codomain, range
- $R_g = \{(x, y) \mid (x,y) \in R, y = x^2\}$
- what range of values does the output span
- $f(x)^2 = x^2$, is always +ve, $0 \to \infty$
  $f(x) = x^3 - 3x^2 + 5$, $-\infty \to \infty$
  $f(x) = 5 \sin x$, $-5 \to 5$
- $f(x) = x^2$, min at $x = 0$, max = no
- $f(x) = x^3 - 3x^2 + 5$, has no global max. and min.
  but local max, $x = 0$; local min, $x = 2$
- $f(x) = x^3 - 3x^2 + 5$ grows faster than $g(x) = x^2$

Eg. $G(y)$ be the no. of data sc. gruds. in year y
$J(y)$ jobs of data sc. $\Rightarrow$ we hope for $G(y)$ and
$J(y)$ should grow at similar rates

L1.12 Prime nos. $\rightarrow$ used in modern ATMs cryptography!

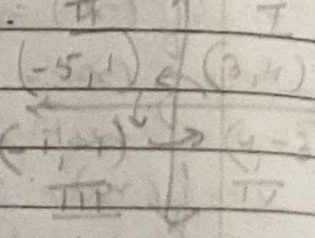
- 2, 3, ... 5 $\rightarrow$ has only 2 factors 1 & p, so 1 is not
- is the set P finite or ∞? $\rightarrow$ so there has to be largest
- is there a largest prime?
- Euclid of Alexandria $\Rightarrow$ ∞ set of prime nos.
- if $n \mid (a+b)$ and $n \mid a$, then $n \mid b$ $\quad$ $1 \mid (4+7)$
  $n \mid (a+b)$, $a + b = u \cdot n$ $\quad$ $n \mid a$, $a = v \cdot n$
  $\therefore$, $a + b = v \cdot n + b = u \cdot n \Rightarrow b = (u - v)n$
- Let's say set is finite $\{p_1, p_2 \cdots p_k\}$, $n = p_1 \times p_2 \cdots + 1$
  $\hookrightarrow$ n is a composite no. , at least 1 prime p_j, $p_j \mid n$
- Since, $p_j \mid p_1 \cdot p_2 \cdots p_k \Rightarrow$ from above result $p_j \mid 1$
  $\therefore$, this isn't possible. So, n is a prime, which is bigger than p_k.
- extensively studied (no. theory)
- Let $\pi(x)$ denote the no. of primes $\leq x$ $\quad$ $4 \rightarrow (2, 3)$
- The prime no. thm. says that $\pi(x)$ is approx. $\dfrac{x}{\log(x)}$
  for large values of x

L2.1 Rectangular Coordinate sys.



→ Coordinate plane

- axis, pts., line
- a reference sys helps in locating points
- coordinates in x & y points → X & Y axis
- X & Y axis meet at 90°, ∴ ⊥ perpendicular
- ∴, rectangular coordinate sys.
  (higher)
- point of intersecⁿ of 2 axes is origin; (0, 0) (x, y)
- Quadrants in the coordinate sys.
- (5, 0) ? → not on any quadrant
  ↳ it is on X - axis
- coordinates axes has 6 parts (4 + 2) (quad. axis)
- graph the points easily



L2.2 Distance Formula

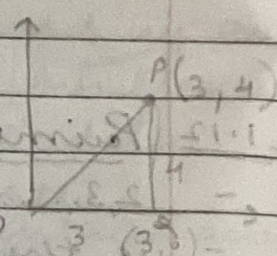
- distance of a pt. from origin

eg. (3, 4) from origin

$OP^2 = OQ^2 + QP^2$

$OP = 5$ units

(By Pythagorean Th.)



- dist. btw. any 2 pts. $P(x_1, y_1)$ & $R(x_2, y_2)$

eg. P (5, 6) & R (−1, 2)

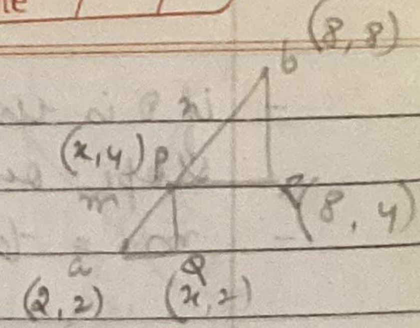
$$\boxed{\text{Dist.} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

$\Rightarrow \tan^2\theta + 1 = \sec^2\theta$

L2.3 Section Formula

- the pt. P cuts the line segment AB in m : n ratio, we have to find coordinates of P
- $A \rightarrow (x_1, y_1)$ $B \rightarrow (x_2, y_2)$, $P = (x, y)$
- $\Delta AQP \sim \Delta PRB$ (by AAA similarity) ⇒ so their sides are in ratio

$$\frac{m}{n} = \frac{AP}{PB} = \frac{AQ}{PR} = \frac{QP}{RB}$$



$$\frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

$$\boxed{x = \frac{mx_2 + nx_1}{m+n}}, \quad \boxed{y = \frac{my_2 + ny_1}{m+n}}$$

L2.4 Area of Triangle

- area of a using coordinate sys.
- the vertices are $(x_1 y_1)$ A ; $(x_2 y_2)$ B & $(x_3 y_3)$ C
- ar (trap.) = $\frac{1}{2} \times$ (sum of parallel sides) $\times$ height

$$= \frac{1}{2}(y_1 + y_3)(x_3 - x_1)$$

- ar(s_1 trap) = $\frac{1}{2}(y_1 + y_2)(x_2 - x_1)$ ar(s_2 trap.) = $\frac{1}{2}(y_1 + y_2)(x_3 - x_2)$
- ar (Δ) = ar(trap.) − ar(s_1 trap.) (ar s_2 trap.)

$$\boxed{ar(\Delta) = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]}$$

⇒ all vertices in anti-clockwise direcⁿ

L2.5 Slope of a line

- rise ^{y} and run $^{\to x}$ ratio
- A (x_1, y_1) & B (x_2, y_2) → on the line
- slope $m = \frac{y_1 - y_2}{x_1 - x_2} = \tan\theta$
  ↳ θ is the inclinaⁿ of line with +ve x-axis, measured in anticlockwise direcⁿ
- $0° \leq \theta \leq 180°$ $\tan 90° \Rightarrow$ not defined
- line parallel to X-axis, 0°, $m = \tan 0 = \tan 0 = 0$
- inclinaⁿ of vertical line is 90°, m is undefined

- if θ is the inclina^n of line l, then $\tan\theta$ is l^d slope or gradient of line l ; $\theta \neq 90°$, $m = \tan\theta$
- $m = \tan(180-\theta) = -\tan\theta = \frac{y_1 - y_2}{x_1 - x_2}$

$\Rightarrow$ consistent in all aspects

L2.6 ∥ & ⊥ lines

- can there be many lines with same slope? ∞
  many ∥ lines with same slope
- no, u can't uniquely determine the line by slope
- slope is useful $\Rightarrow$ ∥ & ⊥ lines cond^n

let - l_1 & l_2 are non-vertical lines with slopes m_1 & m_2, $\angle$ of inclina^n α & β

$\rightarrow l_1 \parallel l_2$, $\alpha = \beta \Rightarrow \tan\alpha = \tan\beta \Rightarrow m_1 = m_2$

$\hookrightarrow m_1 = m_2 \Rightarrow \tan\alpha = \tan\beta \Rightarrow 0° \leq \alpha, \beta \leq 180°$

$\therefore \alpha = \beta \Rightarrow l_1 \parallel l_2$

- 2 non-verticals lines l_1 and l_2 are ∥ iff. their slopes are equal

let - l_1, l_2; 2 non-vertical lines, slopes m_1 m_2 and slopes of inclina^n α, β

$\rightarrow l_1 \perp l_2$, $90 + \alpha = \beta \Rightarrow \tan\beta = \tan(90+\alpha)$

$= \cot\alpha = -\frac{1}{\tan\alpha}$

$\therefore m_2 = \frac{-1}{m_1} \Rightarrow m_1 \times m_2 = -1$

$\hookrightarrow m_1 \times m_2 = -1$, $\tan\alpha \cdot \tan\beta = -1 \Rightarrow \tan\beta = \frac{-1}{\tan\alpha}$

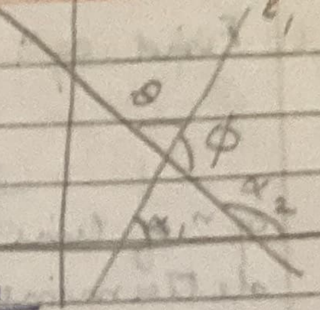
$\tan\alpha = -\cot\beta = \tan(90+\beta) / \tan(90-\beta)$

$\Rightarrow \alpha$ & β differ by $90° \Rightarrow l_1 \perp l_2$

- 2 non-vertical lines l_1 and l_2 are ⊥ iff. $m_1 \times m_2 = -1$

let - l_1, l_2, m_1, m_2, α_1, α_2

l_1 & l_2 intersect and let ϕ & θ be adjacent



$\angle$s formed by l_1 and l_2

$\theta = \alpha_2 - \alpha_1$

for $\alpha_1, \alpha_2 \neq 90^\circ$

$\tan\theta = \tan(\alpha_2 - \alpha_1)$

$$= \frac{\tan\alpha_2 - \tan\alpha}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{m_2 - m_1}{1 + m_1 m_2}$$

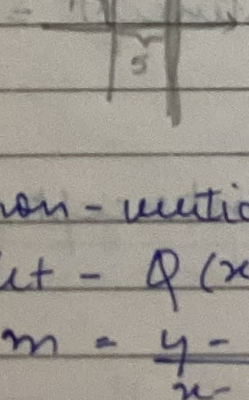
$$\tan\phi = \tan(180 - \theta) = -\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

- 3 pts. are co-linear, their slopes are equal,
  ↳ Rela^n of colinearity with slopes

L2.7 Representa^n of a line (Part - 2)

- how to represent a line uniquely?
- line l, definite exp. that describes the line in terms of coordinate plane
- given pt. P, satisfy the exp. for line l, the pt P is on line l
- horizontal line - iff. it ∥ to X-axis, locate by specifying the Y-axis value; $y = a \Rightarrow (x, a)$
- vertical line - iff. it ∥ to Y-axis, X-axis coordinate; $x = b \Rightarrow (b, y)$

Eg. Eq^n of line ∥ to axis and passing thru (5, 7)



Horizontal line is $y = 7$

vertical line is $x = 5$

- non-vertical l, m & pt. P (x_0, y_0), find the eq^n
  Let - $Q(x, y)$ another pt. on line l

$$m = \frac{y - y_0}{x - x_0} \qquad (y - y_0) = m(x - x_0)$$

(Point - slope form)

Eg. Find eq^n, $m = -2$, $P = (5, 6)$
$(y-6) = -2(x-5) \Rightarrow y = 16 - 2x$ is an eq this line
(10) (3) $\rightarrow (3, 10)$

- eq^n of line : 2-point form $\Rightarrow$ 2 pts uniquely determine the line
line l, $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x, y)$] colinear
acc. to concept of coliniarity $m = \frac{y_1 - y}{x_1 - x} = \frac{y_1 - y_2}{x_1 - x_2}$

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$ 2-Pt. form

Eg. Find eq^n, $(5, 10)$ & $(-4, -2)$
$Q = (x, y)$ $m = \frac{y-10}{x-5} = \frac{-12}{-9} = \frac{4}{3}$ $y - 10 = \frac{4}{3}(x-5)$
$y - 10 = \frac{4x}{3} - \frac{20}{3} \Rightarrow 3y = 4x + 10$

L2.8 Representa^n of a line (Part 2)

- eq^n of line : slope - intercept form
- line l with slope m cut Y-axis at c → y-intercept
$y - c = m(x - 0) \Rightarrow y = mx + c$ slope-intercept form

- line l with slope m cut X at d → x-intercept
$y = m(x - d) = mx - md$

Eg. Find the eq^n, $m = 1/2$, y-intercept $= -3/2$
$y = \frac{1}{2}x - \frac{3}{2} \Rightarrow 2y = x - 3$

Eg. Find the eq^n, $m = 1/2$, x-intercept $= 4$
$y = \frac{x}{2} - 2 \Rightarrow 2y = x - 4$

- eq^n of line : intercept form

- a line x-intercept at a & y-intercept at b
  2 pts. $(a,0)$ & $(0,b)$
  $$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

eg. eq^n of line x-intercept $= -3$, y-intercept at 3
$\frac{x}{3} + \frac{y}{3} = 1 \Rightarrow \frac{y}{3} = 1 + \frac{x}{3} \Rightarrow y = x + 3$

L2.9 General Eq^n of line
- slope pt. form $(y - y_0) = m(x - x_0)$
- slope intercept form y intercept $y = mx + c$
  x intercept $y = mx - md$
- 2 pt. form $(y - y_1) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$
- intercept form $x = a$, $y = b$ $\frac{x}{a} + \frac{y}{b} = 1$
- gen. form $\Rightarrow Ax + by + c = 0$ [Linear Polynomial in 2 Variable]
  ↳ (A, b can't be simultaneously 0)
  gen. linear eq^n of line

Eg. eq^n of line $3x - 4y = -12$ find slope, x-intercept, y-intercept
$m = \frac{-A}{B} = \frac{-3}{(-4)} = \frac{3}{4}$
$3x - 4y + 12 = 0$ $x = -4$ $y = 3$

L2.10 Eq^n of || & ⊥ line in gen. form
eg. Show that $a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$ $b_1, b_2 \neq 0$
1 → || if $a_1b_2 = a_2b_1$
2 → ⊥ if $a_1a_2 + b_1b_2 = 0$

1. for ∥ line $m_1 = m_2 \Rightarrow m_1 = \frac{-a_1}{b_1}, m_2 = -\frac{a_2}{b_2}$ $\cancel{-}\frac{a_1}{b_1} = \cancel{-}\frac{a_2}{b_2}$

$\Rightarrow a_1 b_2 = a_2 b_1$ ∴ ∥ line

2. $m_1 \times m_2 = -1 \Rightarrow m_1 = -\frac{a_1}{b_1}, m_2 = \frac{-a_2}{b_2}$

$\cancel{-}\frac{a_1}{b_1} \times \cancel{-}\frac{a_2}{b_2} = -1 \Rightarrow a_1 a_2 = -b_1 b_2$

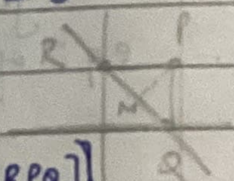
$\Rightarrow a_1 a_2 + b_1 b_2 = 0$

L2.11 Eqn of a ⊥ line passing threw pt.

Eq. Eqn of line ⊥ to $x - 2y + 3 = 0$, passing threw $P(1, -2)$

L2.12 dist. of a line from a given pt.

- dist. of a point from a line
- $P(x_1, y_1)$, line l $Ax + By + C = 0$



$m = -\frac{A}{B}$, $x = -\frac{C}{A}$, $y = -\frac{C}{B}$

$ar(\Delta RPQ) = \frac{1}{2} \times QR \times PM \Rightarrow PM = \frac{2(ar[\Delta RPQ])}{QR}$

→ find the length of QR (by dist. formula)

ar $\Delta RPQ = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

$= \frac{1}{2}\left(\frac{|C|}{|AB|}\right)(|Ax_1 + By_1 + C|)$

$PM = \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}$

$QR = \frac{|C|}{|AB|}\sqrt{A^2 + B^2}$

- dist. btw. 2 ∥ lines, l_1 & l_2 two ∥ line, m

$l_1 \Rightarrow y = mx + c_1$ ✓ dist. $= \frac{|c_1 - c_2|}{\sqrt{1+m^2}}$

$l_2 \Rightarrow y = mx + c_2$

gen. formula ✓ $d = \frac{|c_1 - c_2|}{\sqrt{A^2+B^2}}$

Eg. Find the dist. P(3, −8) line $3x - 4y - 26 = 0$

$\frac{9 + 20 - 26}{5} = \frac{29 - 26}{5} = \frac{3}{5}$

Eg. dist. btw. ∥ lines $3x - 4y + 20$, $3x - 4y + 5 = 0$

$\frac{2}{5}$

L2.13 Straight Line Fit

Eg. $V = IR$ → this represents a line passing thru origin

$y = mx$ → which line is fit / better

1 line will not fetch result,

x_i	y_i
1	2
5	4
7	8
8	9
9	8.7
10	9

2 lines, $y = x$, $y = 2x \Rightarrow$ both are correct, but which is better

$\sum_{i=1}^{6}(y_i - x_i)^2 = 5.09$ $\quad \sum_{i=1}^{6}(y_i - 2x_i)^2 = 328.49$] from the set of obs. (x_i, y_i) $i = 1, 2, 3, 4, 5, 6$

the one will which has least diff.

⇒ the diff. least (5.09) should take because it is more precise, ∴, $y = x$ line should be used.

∴, $m = 1\Omega$

- dist. of set of pts. from a line (slope-intercept)

(x_i, y_i) $i = 1, 2, \ldots n$, and a line $y = mx + c$

The sq. sum of the dist. of set of pts. from the

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$$SSE = \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

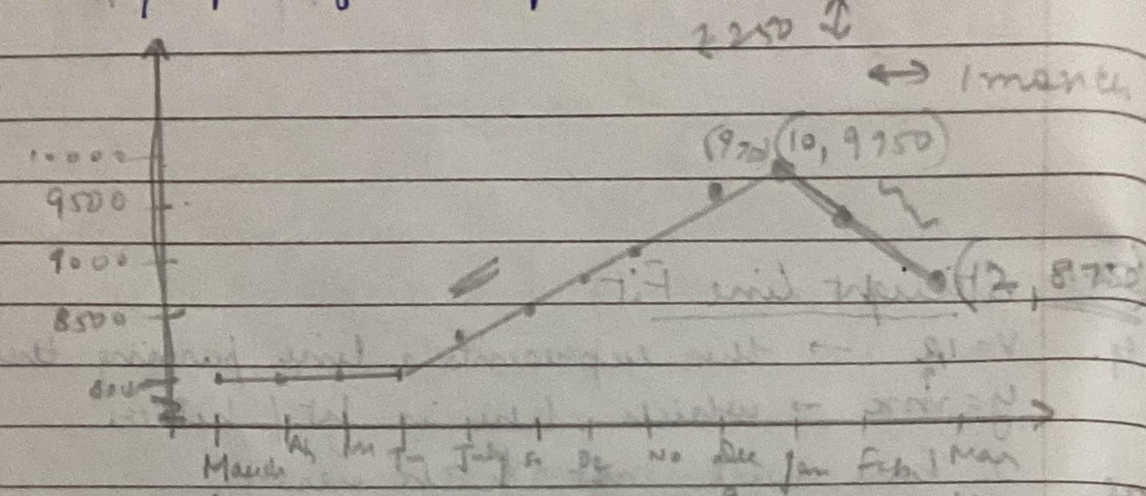
(sum sq. error)

- now, have to find that fits the given set of pts. ⇒ then use SSE (it should be minimum)

⇒ Tutorial

Eg. mobile A, ₹8000 March 2019 const June ₹250 each month ↑
mobile B, ₹6000, Jan 2020 ∴, A ↓ ₹500 per month from Jan constant in March 2020

(a) draw graph of A's price



August = (5, 8500)
Oct = (7, 9000)

$$\frac{y - 8500}{x - 5} = \frac{9000 - 8500}{7 - 5}$$

$y - 8500 = 250x - 1250$

$y = 250x + 7250$

For next line

$$\frac{y - 8750}{x - 12} = \frac{9000}{} = -500$$

$y - 8750 = -500x + 6000$

$y = -500x + 14750$

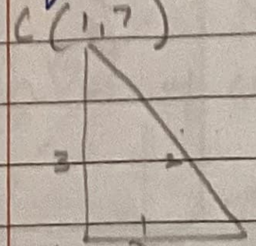
(b) Price of mobile A in dec. ₹ 9500

(c) cal. slope from Jan to March -500

(d) cal. price of March 2020 8750

→ Tutorial

Eg. ΔABC, ₹10 / sq. how to pay for whole field? fencing using ₹5/unit, pay for 3 rounds of fencing



$\frac{1}{2}(x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2))$

$\frac{1}{2}(1(-6)+9(6)+\cancel{7(0)})$

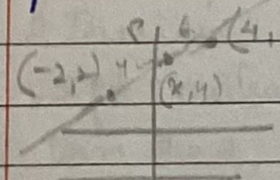
$\frac{1}{2}(-6+54) = \frac{48}{2} = 24 \Rightarrow$ ₹240

$1 \to \sqrt{(8)^2+0} = 8$ $2 \to \sqrt{(8)^2+(6)^2} = 10$ $3 \to 6$

$18 + 6 = 24 \to$ ₹$120 \times 3 =$ ₹360

→ Tutorial

Eg. Abdul $(-2,2)$ & Rani $(4,10)$ started toward each other, pos^n P speeds 60km/hr & 90 km/hr & meet in 4 min, pos^n P? 1 unit = 1 km



$\sqrt{(6)^2+(8)^2}$

10 unit

(Use Sec^n Formula)

$\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}$

$(0.4, 5.2)$

→ Tutorial

Eg. a line $7y-8x-56=0$, mirror image, Y- axis, a new-line is formed, what'll the eq^n of new line? If A is the set of all inside area of 2 lines and X axis ⟹

Date / /

- set of y coordinates & x coordinates in set A

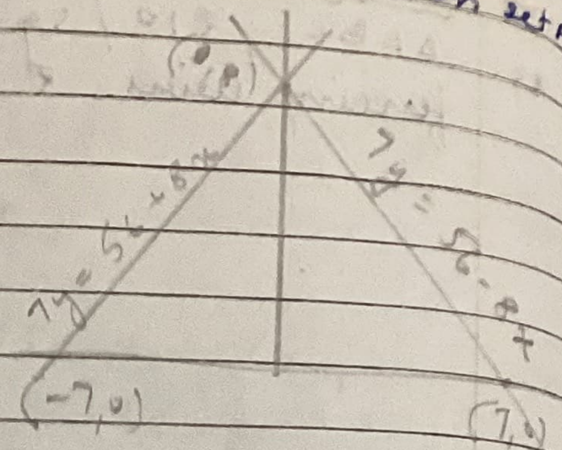
$-8x + 7y = 56 \Rightarrow (+7, 0)$

$(0, 8)$

$x = -7$

$y = 6$

$Ax + By + C = 0$



$$\frac{y - y_2}{x - x_2} = \frac{y_1 - y_2}{x_1 - x_2}$$

$7y = 8x + 56$

$m = -\frac{A}{B}$ $\frac{0-8}{-7-0}$ $\frac{-8}{7}$

$\frac{1}{2}(56 + 56)$

(a) y coordinat $[0, 8]$

(b) x coordinats $[-7, 7]$

⇒ Tutorial

Q. Cell phone plan with 100 free min., ₹ 50 monthly fee, 20 paise for addnal min. Bill amt. for 700 min./month

$50 + 60 = 110$

$300 \times 0.2 = 60$ $700 - 400 = 300$ m

⇒ Tutorial

Eg.

K (4, 4)

L (6.5, 6.5)

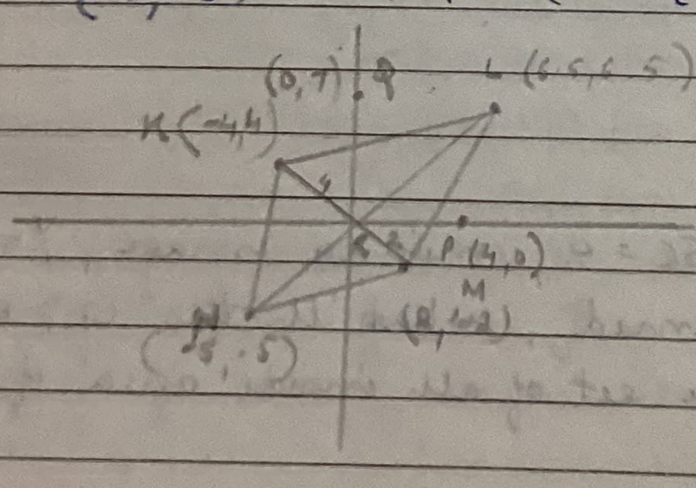
M (2, -2)

N (-5, -5)

R is the pt. of intersecⁿ of KM & LN

Cand cut KM ⇒ $\frac{KM}{RM} = \frac{4}{2}$

P, Q = (4, 0) (0, 7)



RP & RQ are ⊥

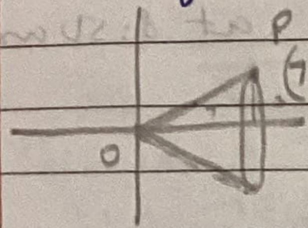
2, 3, 4 Lx

∠LRP + ∠PRM = 90°

⇒ Tutorial

Eg. O (0,0) & P (7,3), line segment OP rotate 360° around X-axis. $V = \frac{1}{3}\pi r^2 h$

(a) Vol. of cone generated by rotaⁿ



$V = \frac{1}{\cancel{3}} \times \frac{22}{\cancel{7}} \times \cancel{3} \times 3 \times \cancel{7} = 66$ unit³

(b) if rotaⁿ is done around Y-axis, what will be the vol

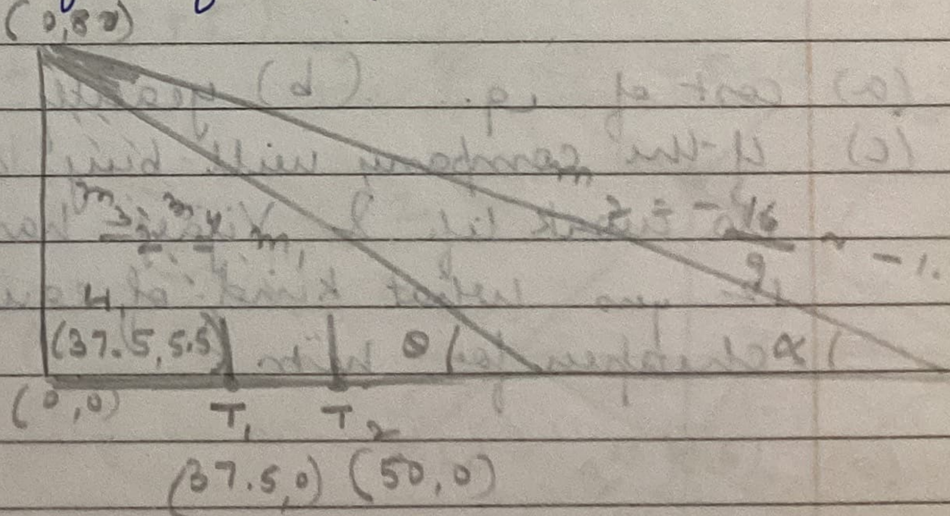
$V = \frac{1}{\cancel{3}} \times \frac{22}{\cancel{7}} \times \cancel{7} \times 7 \times \cancel{3} = 154$ unit³

⇒ Tutorial

Eg. Sanya hears a sound, balcony ht. from ground 80 ft. Torch light that make ∠s btw θ & α with ground. 2 Thieves ht. 5.3 ft & 5 ft at dist. of 37.5 ft & 50 ft away from bulcy

(a) $\tan\theta = 2$, $\tan\alpha = \frac{16}{9}$, can Sanya see thieves

(b) She moves her Torch, she can see the ground from a dist. of 48 ft. Can she see?



$m_1 = \frac{-74.7}{37.5} \approx -1.992$

$m_2 = \frac{-80}{50} \approx -1.6$

$m_1 < m_3 < m_2$

$m_4 > m_2 > m_1$

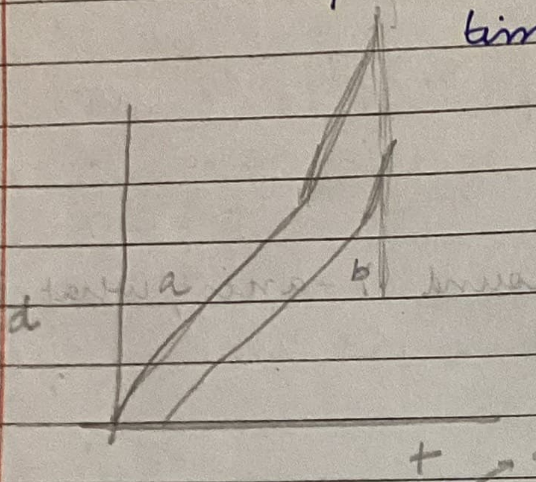
so thief 1 visible but thief 2 isn't visible

⇒ Tutorial

- Suresh & Ramesh, office starts 9:30 am
8:50 am 9:00 am 60 km/h
at 9:20 Suresh +30 km/h each

⇒ they reach office on time
time begins at 8:50 am

(a)



a → suresh
b → ramesh

(b) final pos^n of (t, d) of Ramesh & suresh
(40, 35), (40, 45)
should have same x - coordinate

⇒ Tutorial

Eq. eq A B
work life 3ys 4y'
values dec.
by eq^n

A → $5x + 12.5v_A - 62.5 = 0$
B → $6x + 12v_B - 72 = 0$
v_A & v_B are value in 1000s
x → no. of yrs from date of purchase

(a) cost of eq. (b) yearly deprecia^n of eq^n
(c) if the company will buy an equipment after its work life & Vijay has eq. of such eq. for 12 yrs. what kind of equipment 'll be cheaper for him?

(a) $12.5\,v_A = 62.5$ $12\,v_B = 72$

$v_A = \frac{62.5}{12.5} = 5000$ $v_B = 6000$

(b) A: $12.5\,u_A = -5x + 62.5$ B: $12\,u_B = -6x + 72$

$y = mx + c$

$y = \frac{-5}{12.5}x + 62.5$ $u_B = \frac{-6}{12}x + 72$

$= -0.5x + 72$

$u_A = -0.4x + 5$

$= -400 \,|\, yr$ $u_B = -500/yr$

(c) A: $5000 \rightarrow -400/yr$ B: $6000 \rightarrow -500/yr$

3 yrs → ₹1200 4 yrs — ₹2000

$5000 - 1200 = ₹3800$ $6000 - 2000 = ₹4000$

12 yrs 12 yrs

$3000 + 3600$ $6000 + 4000$

$= ₹8600$ $= ₹10000$

⇒ Tutorial

eg. $l_1 \Rightarrow 6x + 12y = 72$ if a line$_3$ $\parallel l_1$, passes $(-5, 0)$

$l_2 \Rightarrow -6x + 5y = 30$ $l_4 \perp l_3$, passes $(0, -\frac{5}{2})$

(a) cardinality A → set of all pts. common to all 2 of mentioned lines

$m_1 = \frac{-A}{B} = \frac{-6}{12} = \frac{-1}{2}$ for ∥ line $m_1 = m_3 \Rightarrow -\frac{1}{2}$

$(y - 0) = \frac{-1}{2}(x + 5) \Rightarrow x + 2y + 5 = 0$ l_3

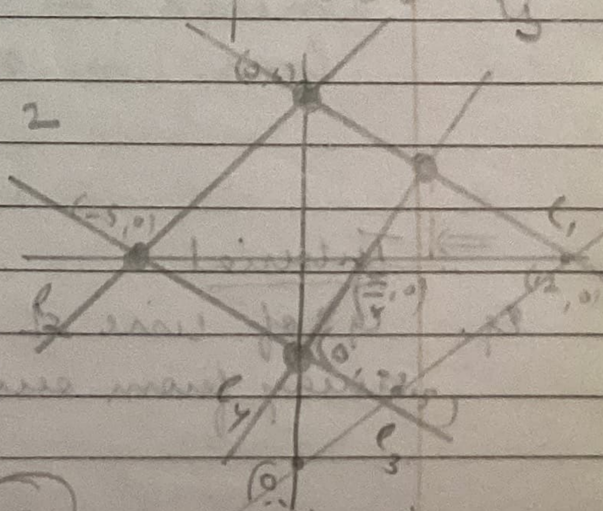
$m_3 \times m_4 = -1$

$m_4 = \frac{-1}{m_3} = \frac{-1}{-1/2} = 2$

$\left(y + \frac{5}{2}\right) = 2(x - 0)$

$y + \frac{5}{2} = 2x$

$-4x + 2y + 5 = 0$ l_4



5 common pts

(b) R is a set of all pts. inside the eq., find Range & domain

(x, y) → codomain
↓
Domain

$y \Rightarrow [-2.5, 6]$ Range
$x \Rightarrow [-5, 3.4]$ Domain

(c) $l_5 \Rightarrow x + 2y = 12$, find the card. set B which pts. common to l_1 and l_5

∞

$x = 0$ 12
$y = +6$ 0
$(0, +6)$

⇒ Tutorial

eg. Lincoln & Lina →

6 shar → M
1 shar → N
₹400

$\cancel{x}$ → 4 shar → M
3 shar → N
360

M → x → 60
N → y → 40

$(6x + y = 400) \times 3 \Rightarrow$

$4x + 3y = 360$

$18x + \cancel{3y} = 1200$
$4x + \cancel{3y} = 360$
$-\quad\quad -$
$14x = 840$

$360 + y = 400$
$y = 40$

Lincoln → M → 360, N = 40
Lina → M → 240, N = 120

⇒ Tutorial

eg. eqⁿ of line ⊥ to $y - 5x = 0$ and $\frac{4}{\sqrt{26}}$ unit from away from origin

$y = 5x$ $\quad -5x + y$ $\quad m_1 \times m_2 = -1$

$m_1 = 5$ $\quad m_2 = -\frac{1}{5}$ $\quad \frac{4}{\sqrt{26}} = \frac{x_1 + 5y + c}{\sqrt{26}}$

$y = mx + c$

$y = -\frac{1}{5}x + c \Rightarrow 5y + x = c$ $\quad c = \pm 4$

$5y + x = \pm 4$

$5y + x = 4$ ✓ $\quad 5y + x = -4$ ✓

⇒ Tutorial

eg. ar($\triangle ABC$)

C (6, 7)

B (0, 5)

A (4, 2)

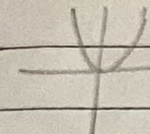
$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

$\frac{1}{2}(4(\cancel{-2}) + \cancel{6(-3)} + 0)$

$\frac{17}{2}$ sq. units

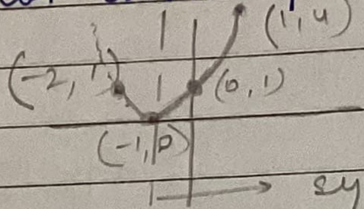
L 3.1 Quadratic Func^n

- it is described as $f(x) = \overset{\text{Quadratic term}}{ax^2} + \underset{\text{linear term}}{bx} + \overset{\text{const}}{c}$, $a \neq 0$
- quadratum → sq.
- graph of any quad. func^n → parabola
- to graph, we need to plot the ordered pairs on the coordinate plane $y = x^2$



- $f(x) = x^2 + 2x + 1$ 1 → generate a table
  2 → plot these pts 3 → connect them

0, 1
1, 4
−1, 0
−2, 1



- all parabolas have axis of symmetry, pts. of each other side, exactly match each other
- pts. at with axis of sym. meets parabola → vertex
- y-intercept (when $x = 0$) → c
- eq^n of axis of sym. ⇒ $\boxed{x = \frac{-b}{(2a)}}$ ✓ x-intercept

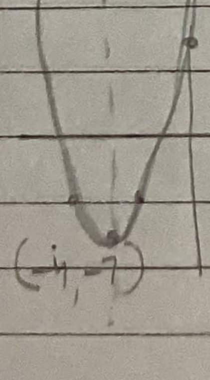
Eg. $f(x) = x^2 + 8x + 9$

y-intercept = 9   eq^n = $\frac{-b}{2a} = -\frac{8}{2}$ −4

x-intercept = −4

vertex = (−4, −7)

x, y
(−4, −7)
(−3, −6)
(−5, −6)
(0, 9)

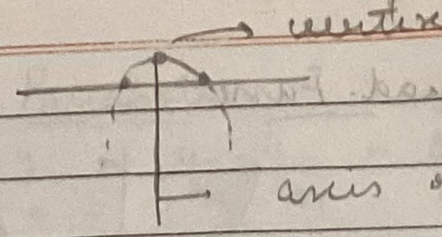


Eg. $f(x) = -x^2 + 1$

(0, 1)   eq^n = $\frac{-b}{2a} = \frac{-0}{2(1)} = 0 = x$

y-axis is the axis of sym

(0, 1)
(1, 0)
(-1, 0)



- y-coordinate of the vertex of a func^n is min. or max value attained the func^n
- $a > 0$ ⇒ opens up, with min. value] only 1 at a time
- $a < 0$ ⇒ opens down, with max. value]
- $R \cap \{ f(x) \mid f(x) \geq f_{min} \}$ ($a > 0$) $R \cap \{ f(x) \mid f(x) \leq f_{max} \}$ ($a < 0$)

L3.2 Eg^s of Quad. Func^{ns}

Eg. $f(x) = x^2 - 6x + 9$ $\quad a = 1,\ b = -6,\ c = 9$ $\quad$ U opens up, min. value

domain $= R$ $\quad \frac{6}{2} = 3$ $\quad 9 - 18 + 9 = 0$ $\quad (3, 0)$

Range $= R \cap \{ f(x) \mid f(x) \geq 0 \}$

Eg. tour bus $= 500$ cus/day ₹40 / person, would loose 10 passanger / day for each ₹4 fare hike. How much should be fare to get max. income

1 unit of hike = ₹4 ⇒ x $\quad$ no. of passengers $= (500 - 10x)$

price / pass. $= (40 + 4x)$

income $= (40 + 4x)(500 - 10x)$

$= 20000 - 400x + 2000x - 40x^2$

$= -40x^2 + 1600x + 20000$

$a < 0$ $\quad \frac{-b}{2a} = \frac{1600}{80} = 20 = x$

max. possible

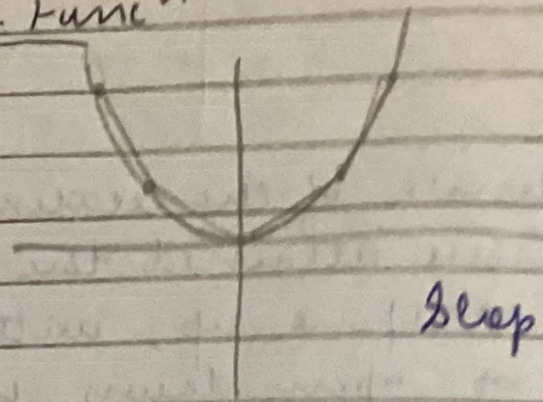
$y = 36000$ $\quad | \quad 40 + 80 = ₹ 120/$ person

- $f(x) = ax^2 + bx + c,\ a \neq 0$, determine slope of f

L3.3 **Slope of Quad. Func^n**

- $y = x^2$

		$y_i - y_{i-1}$
0, 0		
1, 1		1
2, 4		3
-1, 1		-3
-2, 4		



Slope $= 2x$

Slope of f $= 2ax + b$ $\Rightarrow f(x) = ax^2 + bx + c$

- for slope $= 0$, func^n becomes min / max when $2ax + b = 0 \Rightarrow x = \frac{-b}{2a}$

L3.4 **Solu^n of Quad. Eq^n using graph**

- also c/d roots of quad. eq^n (solu^n)
- if a quad. func^n is set = to a value, then the result is a quad. eq^n
- if $ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c = \mathbb{Z}$, then the quad. eq^n is in its std. form
- 0's of a func^n are x-intercepts of its graph and it is sol^n of quad. eq^n

Eg. 1. $x^2 + 6x + 8 = 0$

2. $x^2 + 2x + 1 = 0$

3. $x^2 + 1 = 0$

(1) (x, y) $-3, -1$

$-2, 0$

$-4, 0$

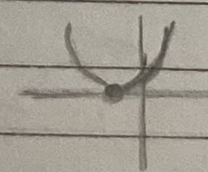
2 real root

(2) $-\frac{2}{2} = -1, 0$

$0, 1$

$1, 4$

-2

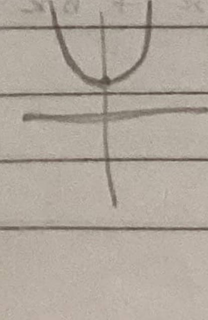


1 real root

(3) $x^2 + 1 = 0$ $x = 0$

$y = 1$

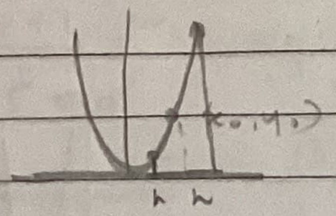
$(0, 1)$



no real root

L3.5 Slope: Line & Parabola

- slope = $\frac{\text{rise} \leftarrow y}{\text{run} \leftarrow x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$
- rate of Δ is constant in straight line $y = mx + b$
  linear func^n $\Leftarrow$ $f(x) = mx + b$
- $y = x^2$



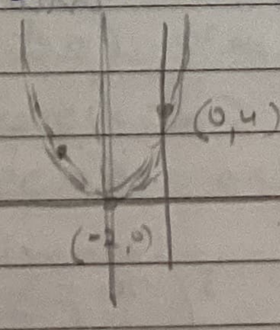
$g(x) = 2x_0$

- $f(x) = ax^2 + bx + c, \quad a \neq 0$
  For any x_0, $\boxed{\text{slope}\Big|_{x = x_0} = 2ax_0 + b}$
- rate of Δ = slope(x) = 0 $\rightarrow$ $x = \frac{-b}{2a}$

since, rate of Δ is 0, so prior to rate was +ve
$\Rightarrow$ now coming to 0.

L3.6 Summary Lect (Quad. Func^{ns})

- $f(x) = mx + b \quad m \neq 0$; $f(x) = ax^2 + bx + c$; $a \neq 0$ (non-vertical line)
  straight line quad. func^n
- Graph; 1. $x = \frac{-b}{2a}$ (axis of sym.) 2. y-intercept
  3. $(x_1, y_1), (x_2, y_2)$
- Eg. $f(x) = x^2 + 4x + 4$ $\quad \frac{-b}{2a} = \frac{-4}{2}$ $\quad (-2, 0)$ vertex



(0, 4)
(1, 9)
(-3, 1)

$m = 2ax + b$
$= 2x + 4$ $\nearrow$ at $x = 0$, eq^n & its behaviour

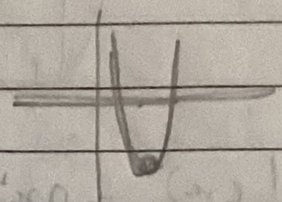
- $a > 0$; min $\Rightarrow$ $a < 0$ Max value

4.3.2 **Solⁿ of quad. eqⁿ using factorizaⁿ**

- $y = f(x) = a(x-p)(x-q)$, where p & q are x intercepts for the funcⁿ
  $y = a(x-p)(x-q) \Rightarrow$ intercept form

Eg. $y = 3(x-1)(x-5)$
1, 5 are x-intercepts $\rightarrow \frac{1+5}{2} = \frac{6}{2} = 3$

$y = 3(2)(-2) = -12$
$(1,0), (5,0), (3,-12)$



- intercept to std. form
  $y = 3(x-1)(x-5)$

FOIL Method
first terms the outer, the inner and last terms

$(ax+b)(cx+d) = acx^2 + axd + cxb + bd$
$= acx^2 + (ad+cb)x + bd$

- prod. of coeff. of x^2 & constant is $abcd$.
- prod. of 2 terms in coeff. x is also $abcd$.

Eg. quad. eqⁿ $2/3$ & -4 roots in std. form
$\left(x - \frac{2}{3}\right)(x+4) = 0 \Rightarrow x^2 + 4x - \frac{2x}{3} - \frac{8}{3} = 0$

$3x^2 + 12x - 2x - 8 = 0 \rightarrow 3x^2 + 10x - 8 = 0$

- std. to intercept form

Eg. $f(x) = 5x^2 - 13x + 6 = 0$ into intercept
$5x^2 - 13x + 6 = acx^2 + (ad+cb)x + bd$
$abcd = 30 \Rightarrow 5x^2 - 10x - 3x + 6 = 0$
$5x(x-2) - 3(x-2) = 0 \quad (5x-3)(x-2) = 0$
$x = \frac{3}{5}, 2$
$5\left(x - \frac{3}{5}\right)(x-2) = 0$

eg. $x^2 = 8x$ $x = 8$ $x(x-8)$ 0, 8 are the roots

eg. $x^2 - 4x + 4 = 0$ $x^2 - 2x - 2x + 4 = x(x-2) - 2(x-2)$
$(x-2)(x-2) = 0$ 2, 2 are the roots
→ 2 is repeated real root

eg. $x^2 - 25 = 0$ $x^2 = 25$ $x = 5$
$(5, -5)$ $(x-5)(x+5) = 0$

L3.8 Solⁿ using sq. method
- also related to the quad. formula

eg. $x^2 + 10x - 24 = 0$ ⟹ $x^2 + 10x = 24$
$(x+12)(x-2) = 0$ $(x+a)^2 = x^2 + 2ax + x^2$
$(-12, +2)$ $2a = 10$
$a = 5$
$x^2 + 10x + 25 = 24 + 25 = 49$
$(x+5)^2 = (7)^2$

Caution: → $= -x$ u cannot use it - then, after adding a if it still remain negative, do not use it.

$x + 5 = \pm 7$
$x = -5 - 7$ $x = -5 + 7$
$= -12, +2$

L3.9 Quad. Eqⁿ with Irrational roots

Eg. $x^2 - 4x + 4 = 32$
$(x-2)^2 = 32$ → 2 root,
$x - 2 = \pm 4\sqrt{2}$ $(2 + 4\sqrt{2})$ $(2 - 4\sqrt{2})$

- $ax^2 + bx + c = 0$
- Quad. Formula

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$ → discriminant

$b^2 - 4ac$	
>0	2
<0	0
$=0$	1

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2-4ac>0$ perfect sq. — 2 real, rational roots

no " " — 2 real, irrational roots

$b^2-4ac=0$ — 1 real, rational root

$b^2-4ac<0$ — no real root

eg. find discriminant and type of roots

① $9x^2-12x+4=0$ $\quad b^2-4ac = 144 - 144 = 0$

1 real root

② $2x^2+16x+33=0$ $\quad 256-264=-8$

no real root

method	can be used	preference
- graphing	occasionally	best used to verify the ans. found is algebraically
- factoring	"	if c terms are 0, or factors are easy to find
- completing the sq.	always	when b is even
- qud. formula	"	when other method fail

- why $x=\frac{-b}{2a}$ is the axis of sym.

$f(x) = ax^2+bx+c = a\left(x^2+(b/a)x + c/a\right)$

$= a\underline{\left(x+\frac{b}{2a}\right)^2} + \left(c-\frac{b^2}{4a}\right) \rightarrow$ constant

↳ everything is sym. around this line

⇒ Tutorial 1

Q. (a) min. value of y where $y = x^2 + x + 2$

(b) x intercept of $y = x^2 + x + 2$

(c) length of line segment there the y-intercept and pt. $(-2, 4)$

(a) $\frac{-b}{2a} = \frac{-1}{2}$

$\frac{1}{4} - \frac{1}{2} + \frac{2}{1} = \frac{1-2+8}{4} = \frac{9-2}{4}$

$\frac{7}{4}$ ✓

(b) $\frac{-b}{2a} = \frac{-1}{2}$ ✓

$b^2 - 4ac$ $\quad 1 - 8$ ×

roots ×

(c) $(0, 2) \rightarrow (-2, 4)$ $\quad \sqrt{(-2-0)^2 + (4-2)^2}$ units $\sqrt{4+4} = \sqrt{8}$

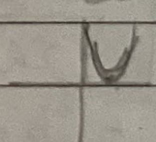
$2\sqrt{2}$ units

→ Tutorial 2

Q. Find k → $y = x^2 - 6x + k$, touch x-axis at 1 pt.

$\frac{-b}{2a} = \frac{6}{2} = 3$

x $\quad y = k$



3, 0 $\quad (0, k)$

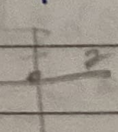
$9 - 18 + k = 0$

$k = 9$

⇒ Tutorial 3

Q. $x = y^2 - 6y + 8$, there are 2 stops at $x = 0$, aarav's home is at origin, min. dist. to travel to catch train

2 units



$y = 2, 4$

⇒ Tutorial 4

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Eg. $40 f(x) = 88x - x^2 + 1200$, which speed $\to x$, fuel economy $f(x)$. Acc. to this, max. economy of vehicle, what is speed for the same

$f(x) = \left(\frac{88}{40}\right)x - \left(\frac{1}{40}\right)x^2 + 30$

$a = \frac{-1}{40}$

$b = \frac{88}{40}$

$\frac{-b}{2a} = \frac{-\frac{88}{40}}{2\left(\frac{-1}{40}\right)}$ $x = 44$ kmph

$y = 126.8 - 48.4 = 78.4$ kmpl

→ Tutorial - 5

Eg. prod. rate of fac. f_1 & f_2 $R = f_1 f_2$

$R = (ax+b)(-cx+d)$

$= -acx^2 + axd - bcx + bd$

$= (-ac)x^2 + (ad - bc)x + bd$

$\frac{-b}{2a} = \frac{(ad-bc)}{2ac}$

$R = (-ac)\left[\frac{(ad-bc)}{2ac}\right]^2 + \frac{(ad-bc)^2}{2ac} + bd$

$= -\frac{(ad-bc)^2}{4ac} + \frac{(ad-bc)^2}{2ac} + \frac{bd}{1}$

$= \frac{-(ad-bc)^2 + 2(ad-bc)^2 + 4abcd}{4ac}$

$= \frac{-(a^2d^2 + b^2c^2 - 2abcd) + 2(a^2d^2 + b^2c^2 - 2abcd) + 4abcd}{4ac}$

$= -a^2d^2 - b^2c^2 + 2abcd + 2a^2d^2 + 2b^2c^2 - 4abcd + 4abcd$

$= \frac{a^2d^2 + b^2c^2 + 2abcd}{4ac} = \frac{(ad+bc)^2}{4ac}$

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→ Tutorial - 6

Q. $f(x) = -x^2 + 8x + 6$, P & Q that they 2 units away from axis of sym., V → vertex

$\frac{-b}{2a} = \frac{-8}{-2} = 4$ V = (4, 22) $b^2 - 4ac$ $\sqrt{64 + 16}$ ... $\sqrt{48}$ $4\sqrt{3}$

(4) $-16 + 32 + 6$
$38 - 16 = 22$

(5) $-25 + 40 + 6 \Rightarrow 46 - 25$
$= 21$

(6) $-36 + 48 + 6 \Rightarrow 54 - 36$
18

(2) $-4 + 16 + 6$

(6, 18) $4\sqrt{3}$ (4, 22)

$(2)^2 + (4)^2$
$\sqrt{4+16}$
20

(i) CSA of cone PVQ $3.14 \times 2 \times \sqrt{20}$ (4.47) $= 28.08$ sq. units

(ii) $f(x) = (x-4)^2$, A be set of all pts inside the reg. range of coordinates

$\frac{-b}{2a}$ (4, 0)

~~16 − 32 + 16~~

$0 \leq y \leq 22$
$[0, 22]$

→ Tutorial - 7

Q. Let curve C represent $y^2 = 4ax$. Is y a func^n of x?

$y = \pm\sqrt{4ax}$
$f(x) = \sqrt{4ax}$
$f(x) = -\sqrt{4ax}$

only 1 image is allowed and we're having 2, so this can't be

y is not a func^n of x

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⇒ Tutorial-8

Q. $4t$ are / s t are is mp. of time interval

(a) total like $l(t) = at^2 + bt + c$, $b = ?$

(b) total likes at end of 60 s

(c) domain of l $[k, \infty]$, $k = ?$

$t, t+1$ $\frac{2t+1}{2}$ $l(t+1) = a(t+1)^2 + b(t+1)$

$l(t+1) - l(t) = 4t + 2$ $l_t = at^2 + bt + c$

(a) $a = 2$ $l(t+1) - l_t = 2at + a + b$

$b = 0$

(b) $2t^2 + c = 2(60)^2 + c$ $l(0) = 0$

$= 7200 + c$

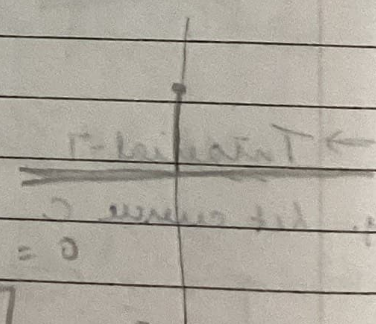
$= 7200$ likes

(c) $k = 0 \longrightarrow \infty$

→ Tutorial-9

Q. $y_1 = a_1 x^2 + b_1 x + c$
$y_2 = a_2 x^2 + b_2 x + c$ } Intersect each other at 2 pts
what are x-coordinates

$\frac{-b_1}{2a_1}$ $\frac{-b_2}{2a_2}$ y_1 : $0, c$
y_2 : $0, c$



$y_1 = y_2$

$a_1 x^2 + b_1 x = a_2 x^2 + b_2 x$

$(a_1 - a_2) x^2 \cancel{+} (b_1 + b_2) x = 0$

$x([(a_1 - a_2) x + (b_1 \cancel{+} b_2)] = 0$

$x = 0$, $x = \frac{(b_1 - b_2)}{-(a_1 - a_2)}$

$a_1 \neq a_2$

⇒ Tutorial 10

Eg. $T(x) = -0.4x^2 + 5x + 25$, $x \to$ no. of hrs.
$T > 40\,°C$ $x \to$ when she'll not go out?

$T(4) = -0.4 \times 16 + 20 + 25$
$= 27 - 6.4$ wrong.
$= 20.6$ curve fit

$T(10) = \cancel{-40} + 50 + 25$
$= 35$

⇒ ~~Tutorial 11~~

Week-4

L4.1 Polynomials

- a kind of mathem. exp. which is sum of several mathematical terms
- it is an algebric exp. in which the only arith. is $+, -, \times$ and natural exponents of variables $t^{1/2} + t \quad x$
  → it becomes a Q
- Greek words Poly (many) + Nomin (terms) Latin
- each term is c/d monomial

poly. in 1 vari $\sum_{m=0}^{n} a_m x^m$

Eg. $x^2 + 4x + 2$ ✓ ; $x + x^{1/2}$ ✗ ; $x + y + xy + x^3$

- Polynomials in 1 var. $x^4 + 1$
  " " 2 " $x^4 + y^5 + xy$
  " " 3 var. $xyz + x^2 z^5$

L4.2 Degree of polynomial

Eg $3x^3 + 4x^2y^2 + 10y + 1 \rightarrow 4$

- exponent on the variable in a term is c/d the degree of that var. in that term
  $4x^2y^2$ $\deg(x) = 2$, $\deg(y) = 2$
- degree of that term is the sum of the degrees of var. in that term
  $\deg(4x^2y^2) = \deg(x) + \deg(y) = 4$
- deg. of polynomial is the largest degree of any one of the terms
- $x = x^1$; $c = cx^0$; 0 polynomial = N.D.
- 0 constant poly. $c, 1, 5$; $c \neq 0$
  1 linear poly. $2x + 4$, $9x + b$
  2 quadratic " $3x^2 + 2$, $4xy + 2x$
  3 cubic " $3x^3$, $4x^2y$, $2y^3 + 1$
  4 Quartic " $10x^4 + y^4$, $x^4 + 10x + 1$

L4.3 Add^n & Subtrac^n

– $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \rightarrow$ gen polynomial in 1 variable
$f: R \rightarrow R$ $\quad a_n \neq 0$

Eg. $p(x) = x^2 + 4x + 4 + q(x) = 10$
$x^2 + 4x + 14$ $\Rightarrow$ 2

Eg. $p(x) = x^2 + 4x + q(x) = x^3 + 1$
$x^3 + x^2 + 4x + 1$

Eg. $p(x) = x^3 + 2x^2 + x + q(x) = x^2 + 2x + 2$
$x^3 + 3x^2 + 3x + 2$

– for degree of polynomial, take max. degree of any polynomial

$\rightarrow$ $p(x) = \sum_{k=0}^{n} a_k x^k$ & $q(x) = \sum_{i=0}^{m} b_i x^i$
$p(x) + q(x) = \sum_{k=0}^{\max. \text{ of } (n \vee m)} (a_k + b_k) x^k$ } Add^n of algorithm

Eg. $p(x) = x^3 + 2x^2 + x \quad (q(x) = x^2 + 2x + 2$
$x^3 + x^2 - x + 2$

L4.4 Multiplica^n of Polynomials

Eg. $p(x) = x^2 + x + 1$ & $q(x) = 2x^3$ $\quad 2x^5 + 2x^4 + 2x^3$
Eg. $p(x) = x^2 + x + 1$ & $q(x) = 2x + 1$
$2x^3 + 2x^2 + 2x + x^2 + x + 1 = 2x^3 + 3x^2 + 3x + 1$

coefficient should be
x^k : $a_j b_{k-j}$

– $p(x) = \sum_{k=0}^{n} a_k x^k$, $q(x) = \sum_{j=0}^{m} b_j x^j$

$$p(x)\,q(x) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} (a_j b_{k-j}) x^k$$

– $p(x) = x^2 + x + 1$ $\quad \cancel{x^4} + \cancel{x^3} + \cancel{x^2} + \cancel{2x^3} + \cancel{2x^2} + \cancel{2x} + \cancel{x^2}$
$q(x) = x^2 + 2x + 1$ $\quad + \cancel{x} + \cancel{1} = x^4 + 3x^3 + 4x^2 + 3x + 1$

L4.5 Division of Polynomial

Eg. $3x^2 + 4x + 3 \div x$

$$3x + 4 + 3/x$$

$$x \overline{\smash{)}\, 3x^2 + 4x + 3}$$

$$\underline{3x^2 + 4x}$$

$$3$$

$$\underline{3}$$

$$0$$

Eg. $\dfrac{3x^2 + 4x + 3}{2x + 1}$

$$1.5x + 1.25$$

$$2x + 1 \overline{\smash{)}\, 3x^2 + 4x + 3}$$

$$\underline{3x^2 + 1.5x}$$

$$2.5x + 3$$

$$\underline{2.5x + 1.25}$$

$$1.75$$

Eg. $\dfrac{3x^2 + 4x + 1}{x + 1}$

$$3x + 1$$

$$x + 1 \overline{\smash{)}\, 3x^2 + 4x + 1}$$

$$\underline{3x^2 + 3x}$$

$$x + 1$$

$$\underline{x + 1}$$

$$0$$

Eg. $\dfrac{x^4 + 2x^2 + 3x + 2}{x^2 + x + 1}$

$$x^2 - x + 2 + \left(\frac{2x + 1}{x^2 + x + 1}\right)$$

$$x^2 + x + 1 \overline{\smash{)}\, x^4 + 2x^2 + 3x + 2}$$

$$\underline{x^4 + x^3 + x^2}$$

$$-x^3 + x^2 + 3x + 1$$

$$\underline{-x^3 - x^2 - x}$$

$$2x^2 + 4x + 2$$

$$\underline{2x^2 + 2x + 2}$$

$$2x + 1$$

L4.6 Division of Algorithms

dividend $\dfrac{p(x)}{q(x)} = \underbrace{x^2 - x + 2}_{\text{quotient}} + \dfrac{2x}{q(x)}$ → remainder; $\dfrac{2x}{q(x)}$ → rational func^n

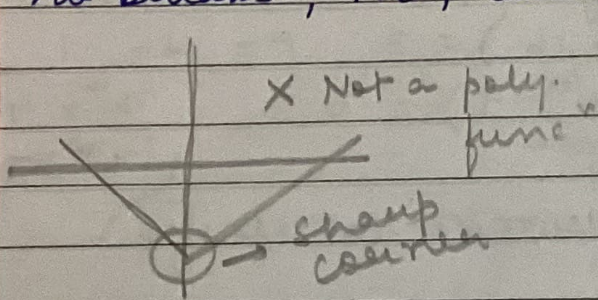
$q(x)$ → divisor

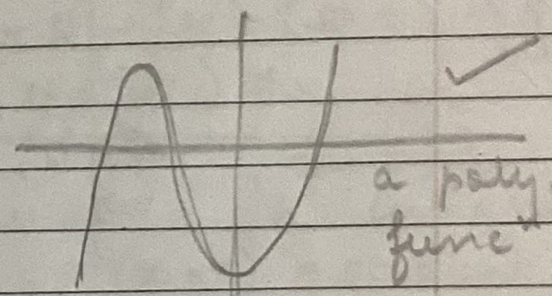
Eg. $2x^3 + 3x^2 + 1 \div 2x + 1$

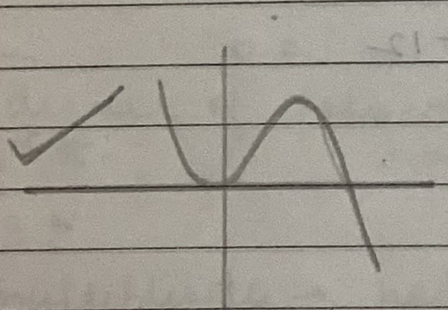
$$\begin{array}{r} x^2 + x - 0.5 + \dfrac{0.5}{2x+1} \\ 2x+1 \overline{) \cancel{2x^3} + 3x^2 + 1} \\ \underline{2x^3 + x^2} \\ \cancel{2x^2} + 1 \\ \underline{2x^2 + x} \\ \cancel{-x} + 1 \\ \underline{-x - 0.5} \\ 0.5 \end{array}$$

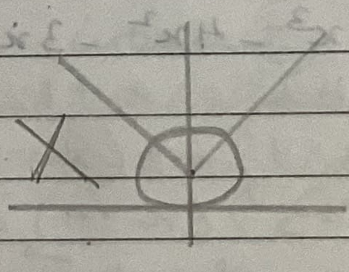
L4.7 **Graphs of Polynomials: Identification**

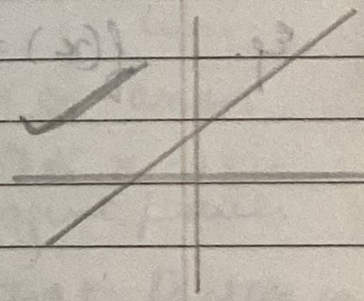
- the polynomial func^n are smooth, no edges or corners allow, draw graphs w/o lifting pen
- the graphs are smooth curves
- no breaks, i.e., continous curves with no breaks

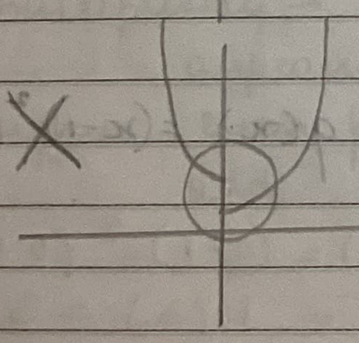












L4.8 **0s of Polynomials**

$f \to$ poly func^n, values of x for which $f(x) = 0$, are c/d 0s of f.

desmos $\Rightarrow$ use to determine intercepts

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- eqn of funcn can be factored, use equal them 0, and solve them
- $x = a$, that is 0 of poly., yields a factor of poly., of the form $(x-a)$
- greatest common factor; factor by grouping; trinomial factoring
- x-intercepts by factoring —
  1. set $f(x) = 0$
  2. if not in any factored form —
     (a) factor out common monomial
     (b) factor — binomials / trinomials
  3. set each to 0.

Eg. $x^6 - 8x^4 + 16x^2 = 0$

$x^2(x^4 - 8x^2 + 16) = 0$

$x^2(x^2-4)(x^2-4) = 0$ $\qquad x^2 = 4$

$x = 0, \pm 2$ $\qquad x = \pm 2$

$\Rightarrow$ use desmos to verify

Eg. $f(x) = x^3 - 4x^2 - 3x + 12 = 0$

$x^3 - 4x^2 - 3x + 12 = 0$

$x^2(x-4) - 3(x-4) = 0$ $\qquad x^2 = 3$

$(x^2-3)(x-4) = 0$ $\qquad x = \pm 3$

$x = \pm\sqrt{3}, +4$

Eg. Find y & x intercept of $g(x) = (x-1)^2(x+3)$

$x \Rightarrow x = \pm 1, -3$ $\qquad g(0) = (0-1)^2(0+3)$

$= 1(3) = 3$

4.8 Multiplicities

Eg. $x^3 + 4x^2 + x + 6$, x-intercept

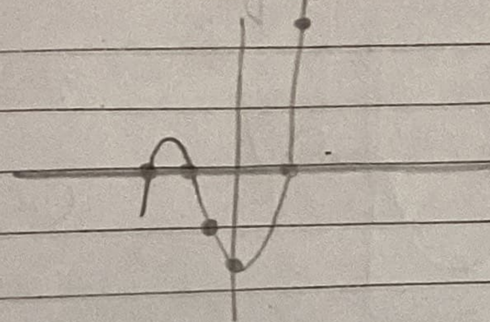
$(-2, 0)$ $(-1, -4)$ $(0, -6)$ $(1, 0)$ $(2, 20)$

$x = -2, 1$ are the x-intercepts

$(x+2)(x-1) \rightarrow x^2 + x - 2) \; f(x) \rightarrow (x+3)$

$x = -3$

$x = -2, 1, -3$

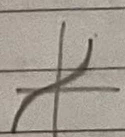


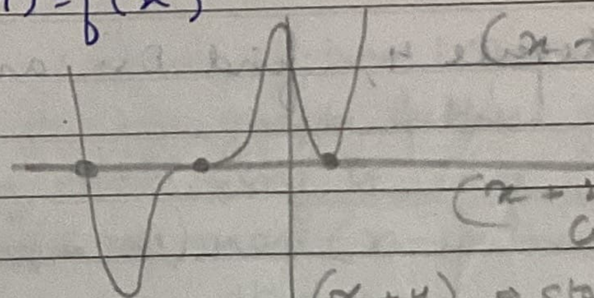
- graphs behave diff. at diff. x-intercepts, sometimes, the graph will cross over the horizontal axis

Eg. $(x-1)^2 (x+2)^3 (x+4) = f(x)$

$1, -2, -4$

$y = x^3$





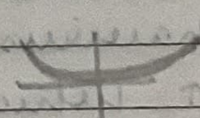
$(x-1)^2 \Rightarrow$ quadratic nature

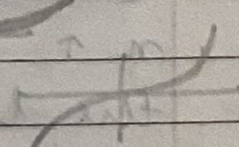
$(x+2)^3 \Rightarrow$ cubic nature

$(x+4) \Rightarrow$ straight line

impt. [degree of poly → odd ⇒ crosses x-axis (↳ intersect)
" " " → even ⇒ bounces x-axis
↳ tangent / touch

- multiplicity → how many times that factor is appearing
- higher even powers → 4, 6 & 8
- " odd powers → 5, 7, 9



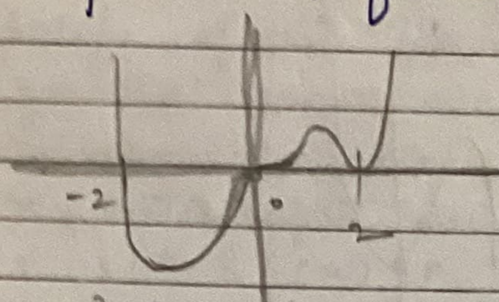


L4.10 Behaviour at x-intercept

- $(x-a)^m$ → multiplicity $\quad x = a$
- sum of multiplicities is not greater $\leq$ than the degree of polynomial func^n

- given the graph of poly. of degree n, identify 0s & their multiplicities

Eg. degree 6, find 0s, their possible mult.



$x = -2, 0, 2$

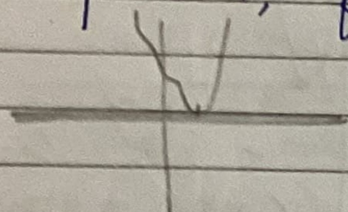
$(x+2)(x)^3(x-2)^2$

$x^3(x+2)(x-2)^2 = x^3(x+2)(x^2-4x+4)$

$(x^4+2x^3)(x^2-4x+4)$

a) $x^3(x-2)^3(x+2)$

Eg. degree 4, find 0s and multiplicity



$x = 2$

$(x-2)^{4x} \Rightarrow (x-2)^2$

$\underbrace{(x^2+1)}_{\text{has no real roots}}(x-2)^2$

L4.11 __End Behaviour__

- behaviour is either ↑ or ↓ as the value of x ↑ which is mainly due to the fact that the leading terms dominate the behaviour of polynomial ↑ This is end-behaviour

$$f(x) = a_n x^n + a_{n-1}x^{n-1} \dots a_0$$

- $a_n x^n$ is an even power func^n & $a_n > 0$, then as x ↑ or ↓, $f(x)$ ↑ & is ∞

$\Rightarrow a_n < 0$, n is even, x ↑ or ↓, $f(x)$ ↓ & ∞

- $a_n x^n$ is an odd power, as $x \uparrow$, $f(x) \uparrow, \infty$
  ↪ $x \downarrow$, $f(x) \downarrow$, ∞

	$n \to$ Even degree	Odd degree
$a_n > 0$	$X \to \infty$ $f(x) \to \infty$ $X \to -\infty$ $f(x) \to \infty$	$X \to \infty$ $f(x) \to \infty$ $X \to -\infty$, $f(x) \to -\infty$
$a_n < 0$	$X \to \infty$ $f(x) \to -\infty$ $X \to -\infty$ $f(x) \to -\infty$	$X \to \infty$, $f(x) \to -\infty$ $X \to -\infty$, $f(x) \to \infty$

L4.12 Turning points

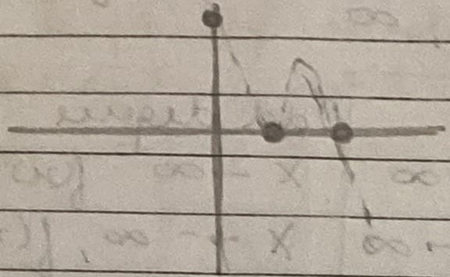
- in quadratic, poly. of degree 2 has 1 turning pt.
- turning pt. – is a pt. of the graph where graph Δs from ↑ing to ↓ing or ↓ing to ↑ing
- poly. of degree n has at most $(n-1)$ turning pt.

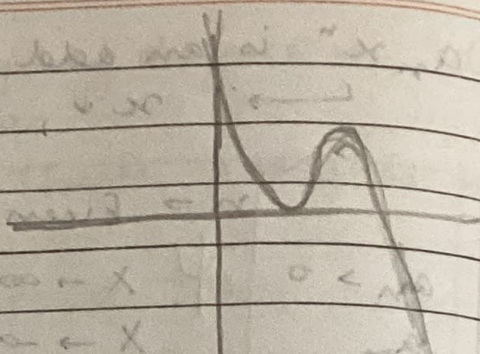
Eg. $f(x) = 1 + x^2 + 4x^5$ max. tp = 4
$f(x) = (x-1)^3 (x+2)$ max tp = 3
$f(x) = (x-2)(x-4)(x+3)$ max tp = 2

L4.13 Graphing a poly. func^n

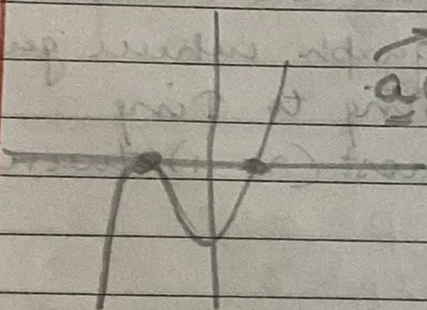
1. find x & y - intercept
2. check for sym.
3. use the multiplicities of 0 for behaviour at x-int
4. determine the end-behaviour by the end term
5. sketch the graph
6. almost no. of tps.
7. ensure tps does not exceed $(n-1)$ n is degree of poly.
8. use graphing tools

Eg. $f(x) = -(x-2)^2 (x-5)$





- Intermediate Value Th. → It states that $f(a)$ & $f(b)$ have opp. sign, thus there exists at least 1 value c btw. a & b for which $f(c) = 0$
- given the graph, find formula of func^n
  stretch factor ⇒ a (unknown value)

Eg. 

$\overset{1/2}{a}(x-1)(x+2)^2$

⇒ Tutorial 1

Eg roots of $p(x)$ $-1, +1$. $q(x) = -5, +6$

$(x+1)(x-1)$ $(x+5)(x-6)$

$p(x) = x^2 - 1$ $q(x) = x^2 - x - 30$

B, C, E

⇒ Tutorial 2

Eg. $\dfrac{3x^4 - 8x^3 + 16x^2 - 10}{x^2 - p}$ → $r(x) = -8x - c$

find p & c ?

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$$\begin{array}{r} 3x^2 - 8x + 13 \\ x^2+1 \overline{) \; 3x^4 - 8x^3 + 16x^2 - 10} \\ \underline{3x^4 \qquad\quad + 3x^2} \qquad\quad \\ -8x^3 + 13x^2 - 10 \\ \underline{-8x^3 \qquad\quad - 8x} \\ 13x^2 + 8x - 10 \\ \underline{13x^2 \qquad\quad + 13} \end{array}$$

$\boxed{p = 1 \quad q = -9}$

→ Tutorial 3

eq. What should be added to $p(x)$ to make it divisible by $x+9$; $p(x) = 2x^3 + 23x^2 + 40x$

$p(x) = q(x) \times$ quotient $+ 0$

$$\begin{array}{r} 2x^2 + 5x \\ x+9 \overline{) \; 2x^3 + 23x^2 + 40x} \\ \underline{2x^3 + 18x^2 \qquad\quad} \\ 5x^2 + 40x \\ \underline{5x^2 + 45x} \\ -5x \end{array}$$

$-5x + 5x = 0$

✓ $\boxed{5x}$

$\Rightarrow -45$

$\Rightarrow x^2 - 126$

→ Tutorial 4

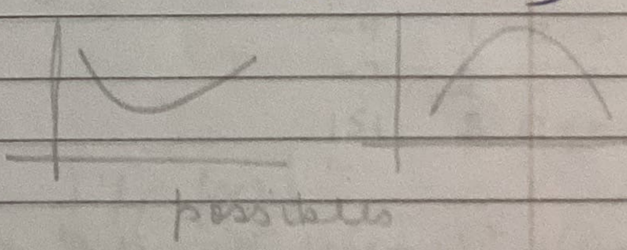
eq. degrees of $p(x) \to 2$, $q(x) \to 3$, $r(x) \to 4$

$h(x) = \dfrac{p(x)\,q(x) - q(x)\,r(x) + r(x)\,p(x)}{p(x) + [p(x) * q(x)]}$

graph?

$(x^5) - (x^{12}) + x^6$

$x^2 \quad x^6$



possibles

→ Tutorial 5

Eq. 6 flat thick sheets $l = (x+4)$, $b = (x+3)$, $h = (x)$

↳ melting solid box ⇒ $\frac{x}{2}, \left(\frac{2x+6}{3}\right) \& \left(\frac{x+4}{5}\right)$

how many solid box can be made -

$6(x+4)(x+3)x \Rightarrow (6x+24)(x^2+3x)$

$(x^2+3x)(6x+24) \Rightarrow 6x^3 + 42x^2 + 72x$

$\cancel{6x(x^2+7x+12)}$

solid box $= \frac{x}{2}\left(\frac{2x+6}{3}\right)\left(\frac{x+4}{5}\right) = \frac{1}{30}(2x^2+6x)(x+4)$

$= \frac{1}{30}(2x^2+6x)(x+4) = \frac{1}{30}(2x^3+14x^2+24)$

$= \frac{1}{30 \cdot 5} \cancel{2x(x^2+7x+12)}$

90 boxes

→ Tutorial -6

eg. $x = 2000$ yr $= 0$ ⇒ total amt. $T(x) = 5x^4 + 3x^3 + x^2 + x$

diff. process cost ⇒

profit?

purchase $x^4 + x^3 + x^2$

transport $x^3 + x^2 + x$

Misc. $0.5x^2 + 0.5x$

$x^4 + 2x^3 + 2.5x^2 + 1.5x$

profit $= 4x^4 + x^3 - 1.5x^2 - 0.5x$

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⇒ Tutorial – 7

Eg. Prod. A

Cost of Prod. | Waste mgmt. cost

$M_1(x) = 100x^3 + 20x^2 + 10$ | $W_1(x) = 0.01x^2 - 0.008x$

$M_2(x) = 20x^4 + 10x^2 - 20$ | $W_2(x) = 0.01x^4 - 0.001x^3 + 0.001x^2$

$M_3(x) = x^3 + 20$ | $W_3(x) = 0.01x^2$

$x \rightarrow$ cost of raw material per kg

(a) effective manufac. cost $\quad E = M + W$

$E_1(x) = 100x^3 + 20.01x^2 - 0.008x + 10$

$E_2(x) = 20.01x^4 - 0.001x^3 + 10.001x^2 - 20$

$E_3(x) = x^3 + 0.01x^2 + 20$

(b) ratio of E_1/E_3 if $x = 1$

$E_1(1) = 100 + 20.01 - 0.008 + 10$ $\quad E_3(1) = 1 + 0.01 + 20$

$= 130.002$ $\quad = 21.01$

$\frac{E_1}{E_3} = 6.187$

(c) which M_1, M_2, M_3 should company use, if $x = 10$

put 10 in E_1, E_2, & E_3 $\Rightarrow M_3$

⇒ Tutorial – 8

Eg. $y = 2x^5 - 4x^4 - 3x + c$ $\quad c = ?$

↳ best fit

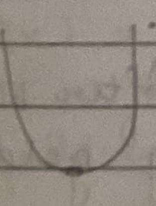
y	x	$f(x)$
0	0	c
-4	1	$c-5$
-7	2	$c-6$
151	3	$c+153$

$-4 = 2 - 4 - 3 + c$

$c = 1$

$[y_i - f(x_i)]^2$

$= (c-0)^2 + (c-1)^2 + (c+1)^2 + (c+2)^2$ SSE



$SSE = 4c^2 + 4c + 5$

$\frac{-b}{2a} = \frac{-4}{2 \times 4} = \boxed{\frac{-1}{2} = c}$

→ Tutorial -9

Eg. (a) no. of tp. 7

(b) no. of roots 5

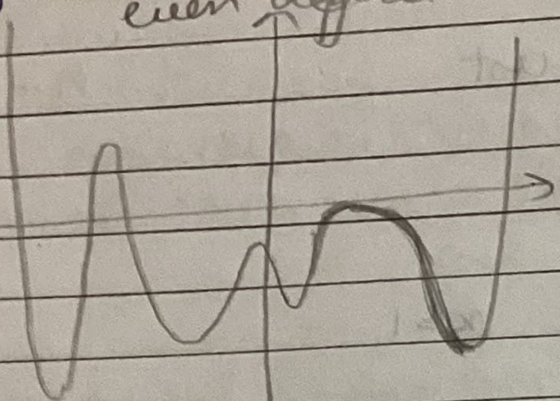
(c) min. possible degree by roots 5

(d) " " by tps. tp + 1 = 8

(e) min. deg. of polynomial 8

(f) end behaviour & coeff. of highest degree positive

even degree



→ Tutorial 10

Eq. $P(x) = (x^4 - 5x^3 + 6x^2 + 4x - 8)(x^2 - 15x + 50)$

$x = -5$ to $x = 20$ ⇒ path of road

& a track is laid along the x-axis

1. how many level crossings are there? 4

2. how many tp on road 3

2, 5, 10, -1

$x^2 - 15x + 50$

$x^2 - 5x - 10x + 50$

$x(x-5) - 10(x-5)$

$(x-5)(x-10)$

$P(-1) = 0$, $P(2) = 0$ } $(x+1)(x-2)$

$x^2 - x - 2$ → divide

$(x^2 - 4x + 4)(x^2 - x - 2)(x^2 - 15x + 50)$

2, 2, 5, 10, -1

→ Tutorial -11

Eq. 8g → ₹ 40,000 on 1st Nov 2020. After 10 months

She sold the chain and new 10 g gold chain

Date / /

by extra 10,000 $G(t) = 0.07t^3 - 1.4t^2 + 7t + 5$

$t = 0$, Now → months

1. rate of gold 1g when she sold her 1st chain

$G(t) = 70 - 140 + 70 + 5 \Rightarrow 5$ ₹/g

2. if she had sold in 6 months, how much she had to pay extra for 10 gms of gold chain

$G(6) = 0.07(216) - 1.4(36) + 47 = 11.72$

= 11720 ₹/gram

$2 \times 11720 = ₹ 23440$ extra

→ Tutorial -12

- plane 3000 m above sea level. → about 500 m parachute → reach 30 m deep in the sea. → she takes a helicopter → then reaches home.
  - Range [-30, 3000]
  - domain → total time take
  - 3 tps
  - degree of poly $1 + 3 + 2 = 6$

→ Tutorial -13

(a) no. of tps 9

(b) min. sum of multiplicity 12

(c) person dies → what polynomial is called in that range of ECG → 0 polynomial

→ Tutorial -14

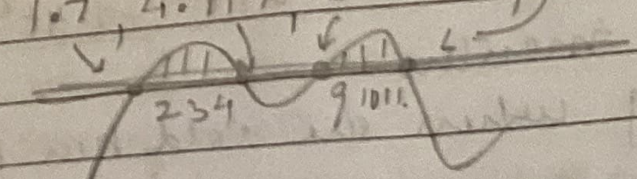
Eg. $P(x) = 5 + 150x - 46.7x^2 + 5.44x^3 - 0.211x^4$

x → months starting from Jan $x = 1$

Golden month if $P(x)$ is => 150,000

find total gold months

$1.7, 4.117, 8.776, 11.18$



2, 3, 4, 9, 10, 11

→ Tutorial - 15

eg $p(x) = (x^2 + kx + 4)(x-5)(x-3)$ & k is set of values of k, find such that p(x) alue 4 real roots

5, 3 $\quad x^2 + kx + 4$

$x^2 - 4x + 4$ $\quad K = \{2 \mid 2 \in (-\infty, -4] \cup [4, \infty)\}$

→ Tutorial - 16

Eg. $p(x) = (x^2 + kx + 4)(x-5)(x-3)$ & K is set of values of k, find correct

$K = \{2 \mid 2 \in (-\infty, -4) \cup (4, \infty)\}$

→ Tutorial - 17

Eg. $d(x)$, $p(x) \rightarrow$ 12 months Jan $x = 0$

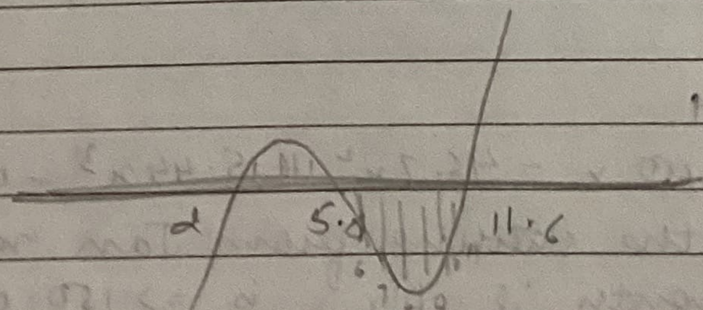
$d(x) - p(x) = a(x^2+1)(x-2)(x-5.8)(x-11.6)$ $\quad a > 0$

find out in which comp. should reduce the period. after Ma

~~$d(x) - p(x) < 0$~~ $\quad 2, 5.8, 11.6$

July → Dec

11.6 → 11



Week 5

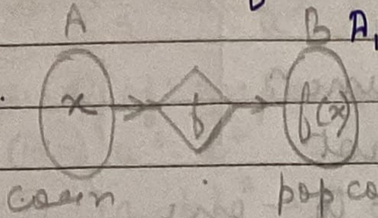
L5.1 **One-to-one func^{ns}: Def^n & Test**

- $y = f(x)$

$f: A \to B$ $A, B \subseteq \mathbb{R}$

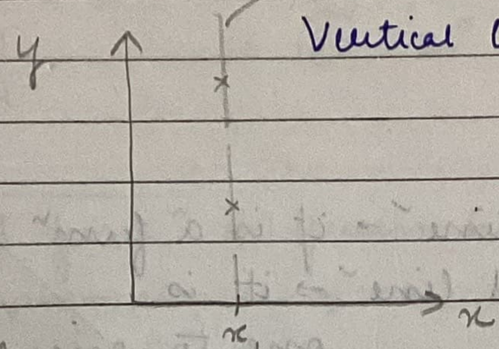
	Domain	Codomain
1. →	X one x	>1 $f(x)$
2. →	✓ >1 x	1 $f(x)$
3. →	✓ 1 x	1 $f(x)$

1.

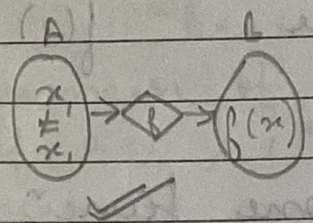


→ 2 pts at 1 line, so, vertical test fails, so, no func^n

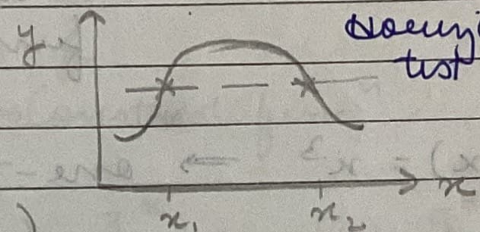
Vertical line Test



2.



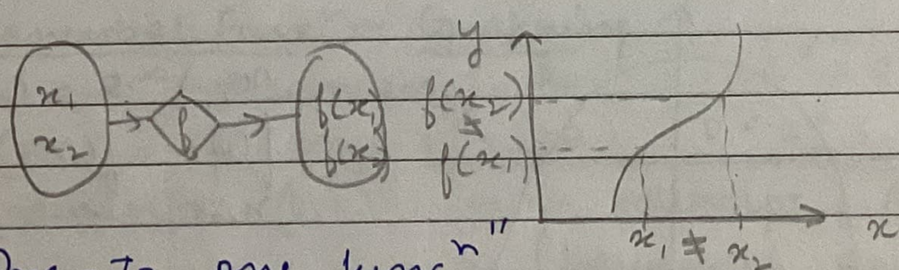
Horizontal line test



$y = f(x) \not\Rightarrow x = f(y)$

It is not reversible. Horizontal line test fails, it is a func^n but not reversible.

3.

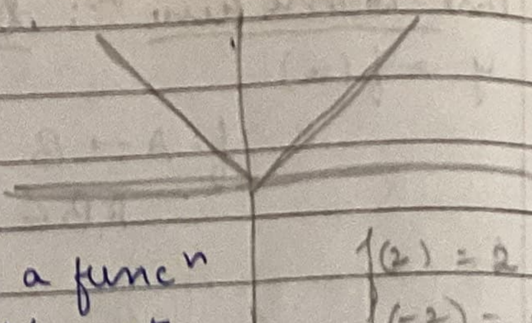


→ "One-to-one func^n"

→ It is reversible

- A func^n $f: A \to B$ is c/d 1-to-1 if, for any $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

1.5.2 Egs. & Theorems

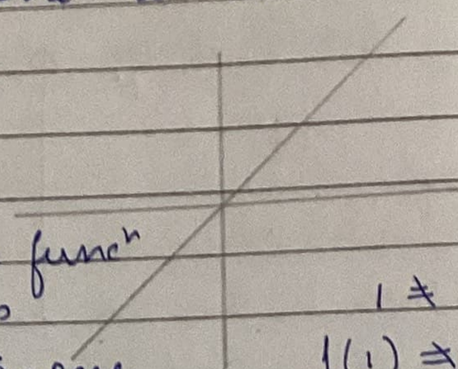


Eg. $f(x) = |x|$

$= \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

vertical line ✓ → it is a func^n
horizontal line ✗ → it's not one-to-one

$f(2) = 2$
$f(-2) = 2$



Eg. $f(x) = x$

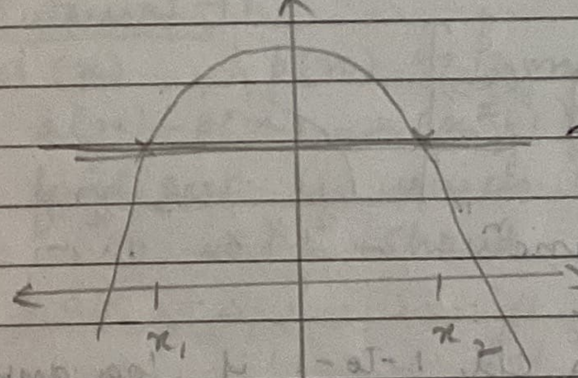
vertical line ✓ → it is a func^n
horizontal line ✓ → it is one-to-one func^n

$1 \neq -2$
$f(1) \neq f(-2)$

Eg. $f(x) = x^3 \Rightarrow$ one-to-one func^n

Th. → If any horizontal line intersects the graph of a func^n f in at most 1 pts., then f is one-to-one.

Proof →



$x_1 \neq x_2$
$f(x_1) = f(x_2)$
not - one - to - one

- $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \Rightarrow$ inc. func^{ns}
- $x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \Rightarrow$ dec. func^n

] one-to-one func^n

→ abundant
→ necessity

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L5.3 Exponential Func^n - Def^n

- exponent a^n, $a > 0$, $n \in Q$
- to avoid complex nos. $\therefore$ $a > 0$
- $2^{\sqrt{2}}$, 5^{π}, a^x $(a > 0)$, $x \in R/Q$ is defined
- Laws of exponents → for $s, t \in R$ & $a, b > 0$

① $a^s \cdot a^t = a^{s+t}$ ② $(a^s)^t = a^{st}$

③ $(a \cdot b)^s = a^s \cdot b^s$ ④ $1^s = 1$

⑤ $a^{-s} = \dfrac{1}{a^s}$ ⑥ $a^0 = 1$, $a > 0$

[0^0 is ND]

- an exponential func^n is given by $f(x) = a^x$, $a > 0$, $a \neq 1$

① domain of f is R

② if $a = 1$, $1^x = 1 \Rightarrow$ Constant func^n, $\therefore$ $a \neq 1$

Exercise →

- 1. (a) 2^x (0,1) (b) 3^x (c) 5^x together
- 2. (a) $(1/2)^x$ (b) $(1/3)^x$ (c) $(1/5)^x$ together

L5.4 Exponential Func^n - Graphing

Eg. $f(x) = 2^x$ → domain = R

→ Range = $(0, \infty)$

→ y-intercept = (0,1) → x-intercept = Nil; $y = 0$ (Horizontal asymptote)

→ End behaviour $x \to \infty$; $x \to -\infty$

→ No roots → increasing func^n (one-to-one)

- every $f(x) = a^x$, $a > 1$ has same prop. as 2^x

Eg. $g(x) = (1/5)^x = 5^{-x}$ → compare with 5^x

→ Domain = R → Range = $(0, \infty)$

→ y-intercept = (0,1) → x-intercept → Nil

→ No roots → End Behaviour $x \to \infty$, $x \to$
→ Decreasing func^n (one-to-one)

- every $f(x) = a^x$, $0 < a < 1$ has same prop. as $(1/5)^x$

L5.5 Natural Exponential Func^n

- $\left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$ [th. of limits]
  ↳ studied in calculus

  e is irrational no ≈ 2.71828

- interest rate calculation
  ↳ Continuous compounding

n	$\left(1 + \frac{1}{n}\right)^n$
1	2
10	2.59
100	2.70
1000	2.716
10,000	2.7181
100,000	2.7182

₹1 10%

$\left(1 + \frac{1}{100}\right)^1 \to$ no. of year

On quaterly basis,

$\left(1 + \frac{0.01}{4}\right)^4 \Rightarrow \left(1 + \frac{0.01}{n}\right)^n \Rightarrow e^{0.01}$

↳ Euler's number

$\left(1 + \frac{x}{n}\right)^{nt} \longrightarrow e^{xt}$

- natural exponential func^n is $f(x) = e^x$

Properties →
→ domain = $\mathbb{R}$ → Range = $(0, \infty)$ → $e > 1$
→ e is the slope of the tangent line to $f(x) = e^x$ at $(1, e)$
→ area under the $f(x) = e^x$ from $(-\infty, 1)$ is e
→ for $f(x) = 1/x$, $x \in (1, e)$, the area under the curve is 1.

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$\to$ min

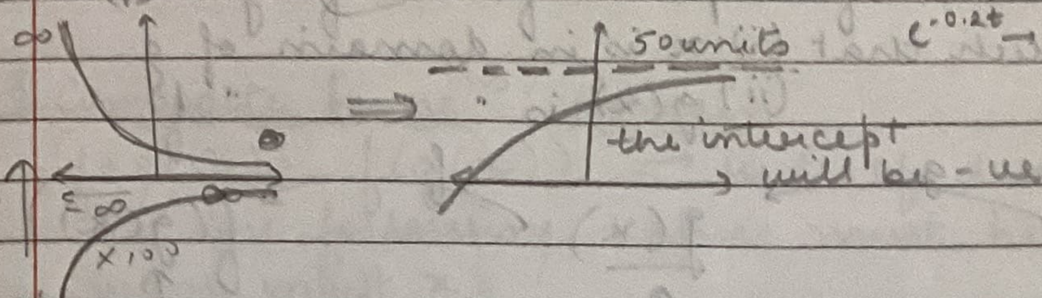
Eg. $R(t) = 50 - 100e^{-0.2t}$

(a) what % of people are responding after 10 min
(b) what is highest percent
(c) how long before $R(t)$ exceeds 30%.

(a) $R(10) = 50 - 100e^{-2} = 36.46\%$

(b) $R(t) = 50 - 100e^{-0.2t}$ $\quad t \to \infty$ $\quad e^{-0.2t} \to 0$

50%.



(c) $30 = 50 - 100e^{-0.2t}$

$100e^{-0.2t} = 20 \Rightarrow \frac{20}{100} = \frac{1}{5} = e^{-0.2t}$

$\boxed{\text{STOP}}$

graphically $\Rightarrow$ 8 min

L5.6 Composite Func^ns

85% of price $-$ ₹ 3000

Let x be the price of computer

$f(x) = 0.85x$ $\qquad g(x) = x - 3000$

$h(x) = 0.85x - 3000 = f(x) - 3000$

$= g(f(x))$

$h(x) = (g \circ f)(x)$ $\qquad$ if $x = 14000$

$(g \circ f)(x) = g(f(x))$.

$= 11900 - 3000$

$= (8900)$

$85 \times 14 = 340, 85, 1190$ $\qquad \frac{85}{100} \times 14000$

- The composiⁿ of function –
  composiⁿ of funcⁿˢ f & g is denoted by $f \circ g$ & is defined by

$$(f \circ g)(x) = f(g(x))$$

- domain of composite funcⁿ $f \circ g$ is set of all x such that (i) x is in domain of g
  (ii) $g(x)$ is " " " f

$$\underset{\text{Dom}(g)}{\overset{}{x}} \xrightarrow{\;g\;} \underset{\text{Dom}(f)}{g(x)} \xrightarrow{\;f\;} \underset{\text{Range}(f)}{f(g(x))}$$

L5.7 Composite Funcⁿˢ: Egs.

Eg. $f(x) = 3x - 4$, $g(x) = x^2$, find (a) $(g \circ f)(x)$
(b) $(f \circ g)(x)$

(a) $(g \circ f)(x) = g(f(x))$
$= (f(x))^2$
$= (3x-4)^2$

$g(\square) = \square^2$

or $(g \circ f)(x) = g(f(x))$
$= g(3x-4) = (3x-4)^2$

(b) $(f \circ g)(x) = f(g(x))$
$= f(x^2)$
$= 3x^2 - 4$

$f(\Delta) = 3\Delta - 4$

or

$f(g(x)) = 3\,g(x) - 4$
$= 3x^2 - 4$

Eg. $f(x) = x+1$, $g(x) = x^2 - 1$, find $(g \circ f)(x)$, $(f \circ g)(x)$

$(g \circ f)(x) = g(f(x))$
$= g(x+1) = (x+1)^2 - 1 = x^2 + 2x$

$(f \circ g)(x) = f(g(x))$
$= f(x^2-1) = x^2 - \cancel{1+1} = x^2$

L5.8 <u>Composite Funcⁿˢ: Domain</u>

- $(f \circ g)(x) = f(g(x))$ → rules of domain —
  ∴, the following values must be excluded from input x.
  (i) $x \notin Dom(g) \Rightarrow x \notin Dom(f \circ g)(x)$
  (ii) $\{x \mid g(x) \notin Dom(f)\}$ must not be included in $Dom(f \circ g)(x)$

Eg. $f(x) = \frac{2}{x-1}$, $g(x) = \frac{3}{x}$, find $(f \circ g)(x)$ & $Dom(f \circ g)$

$f(g(x)) = f\left(\frac{3}{x}\right) = \frac{2}{\frac{3}{x} - 1} = \frac{2}{\frac{3-x}{x}} = \frac{2x}{3-x}$

Rule 1 → $x \notin Dom(g) \Rightarrow x \notin Dom(f \circ g)$
$g(x) = \frac{3}{x}$; $x \notin 0$ ∴, $x = 0 \notin Dom(g)$
$\Rightarrow x = 0 \notin Dom(f \circ g)$

Rule 2 → $g(x) \notin Dom(f)$
$f(x) = \frac{2}{x-1}$, $x \notin 1$ ∴, it should be excluded

$Dom(f \circ g) = \{x \mid x \in R ; x \neq 0$ & $x \neq 3$

LS.9 Inverse Func^n

- $f: R \to R$ $f(x) = x^2$ $f(2) = f(-2) = 4$
  Not 'Reversible' → Inverse func^n not possible

- one-to-one func^n
  $g(x) = 4x$ $h(x) = \frac{x}{4}$
  $x = \frac{y}{4} \Rightarrow h(x) = \frac{x}{4}$.
  $(g \circ h)(x) = I(x) = (h \circ g)(x)$

- The inverse of a func^n f, f^{-1} is a func^n such that
  $(f^{-1} \circ f)(x) = x$
  $(f \circ f^{-1})(x) = x$ $\Big]$ f is one-to-one func^n

  $\Rightarrow f^{-1}$ exists for f $\Rightarrow f^{-1} \neq \frac{1}{f}$

Eg. $g(x) = x^3$ & $g^{-1}(x) = x^{1/3}$ Verify

$$g^{-1}(g(x)) = g^{-1}(x^3) = (x^3)^{1/3} = x$$
$$g(g^{-1}(x)) = g(x^{1/3}) = (x^{1/3})^3 = x$$

Eg. $f(x) = \frac{x-5}{2x+3}$ & $g(x) = \frac{3x+5}{1-2x}$, Verify

$$f(g(x)) = f\left(\frac{3x+5}{1-2x}\right) = \frac{\left(\frac{3x+5}{1-2x} - \frac{5}{1}\right)}{2\left(\frac{3x+5}{1-2x}\right) + \frac{3}{1}}$$

$$= \frac{\frac{3x+\cancel{5}-\cancel{5}+10x}{\cancel{1-2x}}}{\frac{\cancel{6x}+10+3-\cancel{6x}}{\cancel{1-2x}}} = \frac{\cancel{13}x}{\cancel{13}} = x$$

Yes

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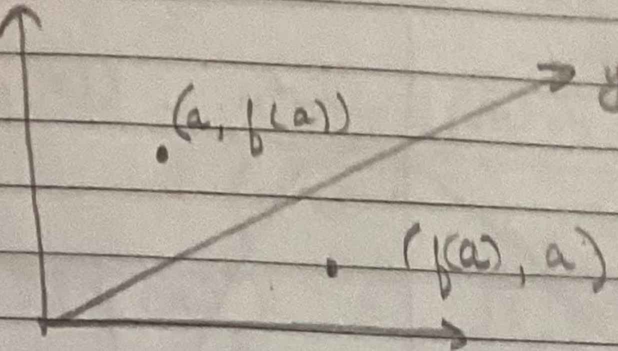
$x = a \xrightarrow{f} f(a) = y$ If $(a, f(a))$ is on the graph of f, then $(f(a), a)$ is on the graph of f^{-1}.

$f^{-1}(f(a)) \xleftarrow{f^{-1}}$

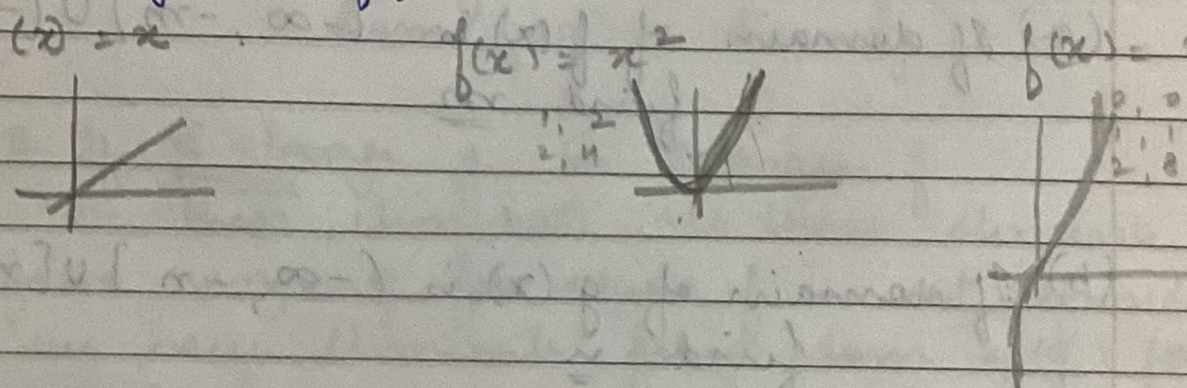


(reflecting the original funcⁿ)

- The graphs of f & f^{-1} are symmetric across $y = x$ line.

⇒ Tutorial-1

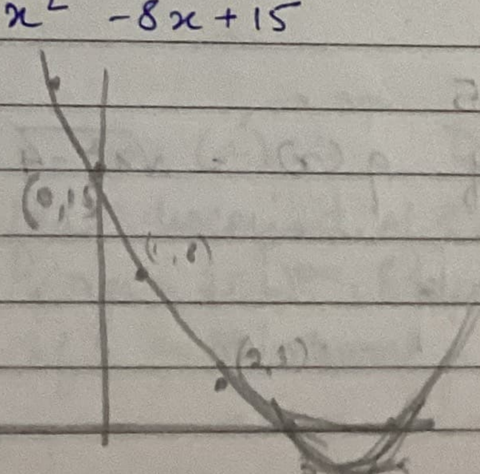
Eg. draw graph of $f(x)$ x, x^2, x^3



⇒ Tutorial-2

Eg. draw $f(x) = x^2 - 8x + 15$

0, 15
1, 8
2, 3
−1, 24



1 − 8 + 15

19 − 16 = 3

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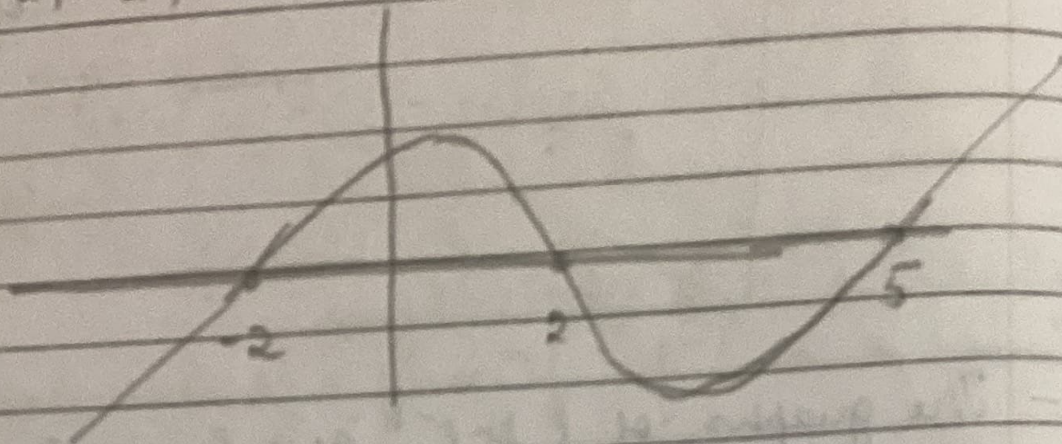
→ Tutorial-3

Eg. draw graph for $f(x) = (x-2)(x+2)(x-5)$

$x = 2, -2, 5$



⇒ Tutorial-4

Eg. $f(x) = \dfrac{x^2-8x+15}{x+3}$, $g(y) = \sqrt{y^2-4}$

(a) If domain of $f(x)$ is $(-\infty, -m) \cup (-m, \infty)$, find m

$m = 3$

(b) If domain of $g(x)$ is $(-\infty, -n] \cup [n, \infty)$, find n

$n = 2$

→ Tutorial-5

Eg. range of $g(x) = \sqrt{x^2-4}$

$(-\infty, -2] \cup [2, \infty]$

Range $= [0, \infty)$

⇒ Tutorial - 6

Eg. domain of $h = (f \circ g)\, x$ $\quad f(x) = \frac{x^2 - 8x + 15}{x + 3}$

$[-\infty, -2] \cup [2, \infty) \rightarrow D(g(x))$ $\quad g(x) = \sqrt{x^2 - 4}$

$$f(g(x)) = \frac{x^2 - 4 - 8(\sqrt{x^2-4}) + 15}{(\sqrt{x^2-4}) + 3} \quad \sqrt{(x+2)(x-2)}$$

$a^2 - b^2$
$(a - b)$
$(a+b)(a-b)$

Range $(g(x)) \Rightarrow [0, \infty]$

domain of $f(x)$: $R \setminus -3$

domain of h = domain of g

* $(f \circ g) = f(g(x))$
Step 1 → calculate domain of g
Step 2 → range of g

Case I If → Range $g \subseteq$ Domain f
Dom$(f \circ g)$ = Dom(g)

Case II If → Range $g \not\subseteq$ Domain f
then, eliminate all those elements from range g for which f is not defined
⇒ we have eliminate from dom of (g) for which range (g) are not in dom (f)

⇒ Tutorial - 7

Eg. Rohan (age = 22). $D(a)$ % , $a \rightarrow$ age, $D(a) = (-a^2 + 50a - 600)$, sunday offer, flat discount of ₹1500, purchase > ₹12000. Friend (age = 25). Express final amt. in terms of a, ₹15000

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$15000 - 1500 = 13500$

$$P_1 = P - P\left(\frac{D(a)}{100}\right) \qquad D(a) = -a^2 + 50a - 600$$

$$= P\left\{\frac{100 - D(a)}{100}\right\} \qquad \frac{-b}{2a} = \frac{-50}{-2} = 25$$

$$P_2 = 15000 - 1500$$

$$P_2 = P - 1500$$

Case I (first apply birthday)

$P = 15000 \qquad P_{12} = P\left(\frac{100 - D(a)}{100}\right) - 1500$

$$= 150\left(90 - D(a)\right)$$

$$P_{12} = 150\{a^2 - 50a + 690\}$$

by $a = 25$ $\quad P_{12} = 9750$ ₹

Case II (apply sunday offer)

$P = 13500 \Rightarrow 13500\left\{\frac{100 - D(a)}{100}\right\}$

$= 135(a^2 - 50a + 700) \qquad [a = 25]$

$= 10,125$ ₹

→ Tutorial - 8

eg. 12 lakhs in bank accnt. , $r = x\%$

1. find total amt. in terms of x, in her accnt after n yrs
2. find a funcⁿ $g(y)$ to calculate the avg. rate based on avg. after n yrs.

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1. $P = A\left(1 + \text{quarterly rate}\right)^{4n}$

$= 12 \text{ lakh} \left(1 + \frac{x}{400}\right)^{4n}$

$f(x) = 12,00,000\left(1 + \frac{x}{400}\right)^{4n}$

$\downarrow$

$x = 12,00,000\left(1 + \frac{y}{400}\right)^{4n}$

$y = 400\left\{\frac{x}{1200000}\right\}^{\frac{1}{4n}} - 400$

$g(y) = 400\left(\left(\frac{y}{12,00,000}\right)^{\frac{1}{4n}} - 1\right)$

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L6.1 Log Funcⁿ

- $f(x) = a^x$ $(a > 0, a \neq 1)$ is one-to-one, it has its inverse
- The log funcⁿ (to the base a) is $y = \log_a(x)$, and is defined to be inverse of $f(x) = a^x$.

$$y = \log_a x \iff x = a^y$$

- $Dom(\log_a) = Range(a^x) = (0, \infty)$
  $Dom(a^x) = Range(\log_a) = \mathbb{R}$

Eg. $f(x) = \log_4(1-x)$. Find dom of f

$dom(\log_4) = (0, \infty) \Rightarrow 1 - x > 0$

$1 > x$

$Dom(f) = (-\infty, 1)$

Eg. $g(x) = \log_3\left(\frac{1+x}{1-x}\right)$, $x \neq 1$ $Dom(g) = ?$

$Dom(\log_3) = (0, \infty)$ $\frac{1+x}{1-x} > 0$

$Dom(g) = (-1, 1)$

Eg. $y = \log_3 x$ $3^y = 3^{\log_3 x} = x$

$x = 3^y$

Eg. $(1.3)^2 = m$ $\log_{1.3}(1.3)^2 = \log_{1.3} m$

$2 = \log_{1.3} m$

$$\boxed{a^{\log_a x} = x}$$

- $a^u = a^w$ $(a > 0, a \neq 1)$ $u = w$

Eg. $\log_3\left(\frac{1}{9}\right)$

$3^{\log_3\left(\frac{1}{9}\right)} = 3^x$

$\frac{1}{9} = 3^x \Rightarrow \log_3(3^{-2}) = -2$

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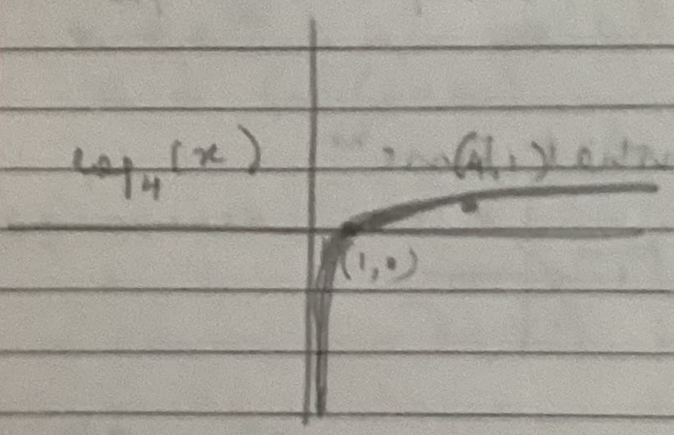
- graph of $f(x) = \log_a x$

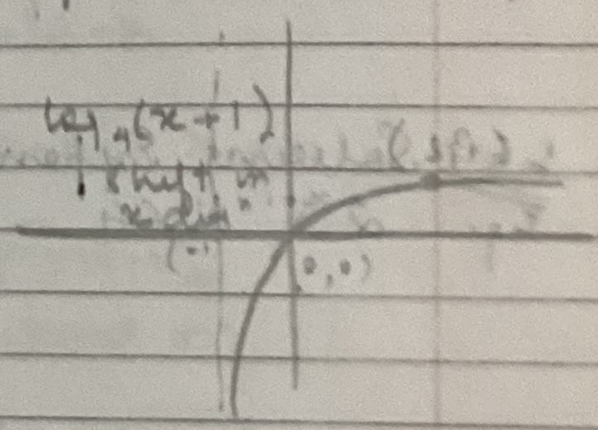
L6.2 Graph of log Func^n

- prop. for $f(x) = \log_a(x)$
  - ⇒ Dom(f) = $(0, \infty)$, Range (f) = R
  - ⇒ x-intercept = (1,0) , y-intercept = Nil
  - ⇒ vertical asymptote at $x = 0$ (y-axis)
  - ⇒ f is 1-to-1 & passes thru (1,0) & (a,1)
    - $0 < a < 1$, f is dec.
    - $a > 1$, f is increasing

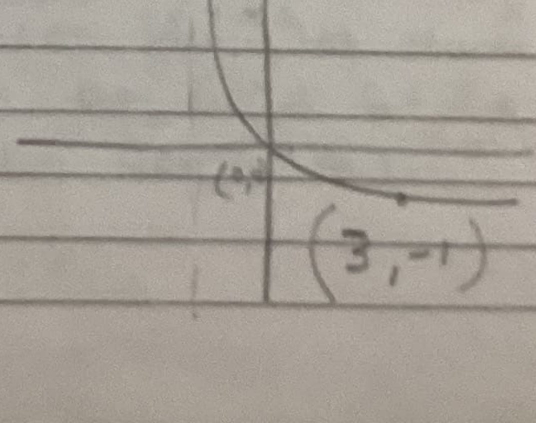
Eg. (i) $f(x) = -\log_4(x+1)$ } → graph
(ii) $g(x) = \log_{\frac{1}{4}}(-x) + 1$

(i) $\log_{(4)}(x)$
↓
$a > 1$ (inc.), should pass (1,0) & (4,1)





$-\log_4(x+1)$



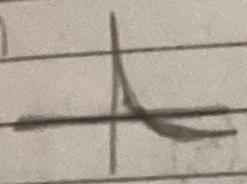
(ii) $\log_{1/4}(-x)+1$

$\log_{1/4}(x)$

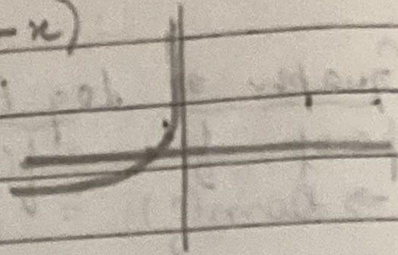
$0<a<1$

$(1,0)$

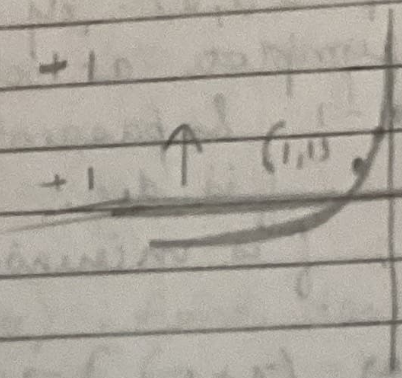
$(\frac{1}{4},1)$



$\log_{1/4}(-x)$



$\log_{1/4}(-x)+1$



- The natural log func^n is $f(x) = \log_e(x)$, where the base is e.
  $f(x) = \ln(x)$
- $\ln(e^x) = x$; $e^{(\ln x)} = x$
- common logarithm ; $\log x = \log_{10} x$

L 6.3 Solving Exponential Func^{ns}

Eg. $2^{x+1} = 64$ $\qquad 2\times2\times2\times2\times2\times2$

$2^{x+1} = 2^6$

$x+1 = 6 \qquad x = 5$

Eg. $e^{-x^2} = (e^x)^2 \frac{1}{e^3} \qquad e^{-x^2} = \frac{e^{2x}}{e^3}$

$e^{-x^2} = e^{2x-3} \Rightarrow -x^2 = 2x-3$

$x^2 + 2x - 3 = 0$

$x^2 + x - 3x - 3 = x(x+1) - 3(x+1)$

$(x+3)(x+1)$

$\Rightarrow (x = -3, +1)$

Eq. $9^x - 2\,3^{x+1} - 27 = 0$ $2 \times 3^{x+1}$ 6^x

$(3^2)^x$ $\downarrow (3^x)^2 - 6(3^x) - 27 = 0$ $-9, 3$

$(3^x)^2$

$t^2 - 6t - 27 = t^2 + 3t - 9t - 27 = 0$

$t(t+3) - 9(t+3) = (t-9)(t+3) = 0$

$(t = 9, -3)$

$3^x = (3)^2$ $\boxed{3^x = -3}$ Not possible

$x = 2$

Eq. $5^{x-2} = 3^{3x+2}$

$\ln(5^{x-2}) = \ln(3^{3x+2})$

$(x-2)\ln 5 = (3x+2)\ln 3$

$-2(\ln 5 + \ln 3) = [3x \ln 3] - [x \ln 5]$

$-2(\ln 15) = x[3\ln(3) - \ln 5]$

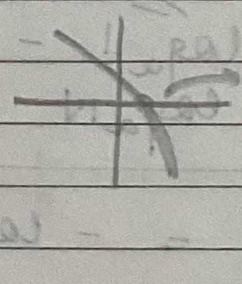
$$x = \frac{-2\ln(15)}{\ln(27) - \ln(5)} = \frac{\ln(1/225)}{\ln(27/5)}$$

Eq. $x + e^x = 2 \Rightarrow e^x = (2-x)$

$x = \ln_e(2-x)$

$\Rightarrow \ln(2-x) - x = 0$

$\left\{ \begin{matrix} (1,0) \\ (a,1) \end{matrix} \right.$



$(0.443, 0)$

$x = 0.443$

L6.4 Log. Func^{ns} - Prop. 1

- $a \in (0,1)$ or $a > 1$ for log Func^n

$\Rightarrow a^{\log_a x} = x = \log_a(a^x) = x$

$\Rightarrow \log_a 1 = 0$ & $\log_a a = 1$ $\left\{ \begin{matrix} (1,0) \\ (a,1) \end{matrix} \right.$

Eq. $3^{\log_3(\pi/2)} = \frac{\pi}{2}$

- Laws of Log (by Napier)

Let $r \in R$, $a > 1$ or $0 < a < 1$; $M, N > 0$

① $\log_a(M \times N) = \log_a M + \log_a N$

② $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

③ $\log_a(1/N) = -\log_a N$

④ $\log_a(M^r) = r \log_a M$

- Proof of 1st law ⇒

$A = \log_a M$ $\quad A + B = \log_a M + \log_a N$

$B = \log_a N$ $\quad a^{A+B} = a^{\log_a M + \log_a N}$

$= a^{\log_a M} \times a^{\log_a N}$

$= M \times N$

$\log_a(MN) = \log_a(a^{A+B}) = A + B$

$$\boxed{\log_a(MN) = \log_a M + \log_a N}$$

- for the proof of 2nd law use, A − B

$$\boxed{\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N}$$

- 3rd law,

$\log_a\left(\frac{1}{N}\right) = \overset{=0}{\log_a 1} - \log_a N$

$= 0 - \log_a N$

$$\boxed{\log_a\left(\frac{1}{N}\right) = -\log_a N}$$

L6.5 Log Funcⁿ : Applications

eg. $\log_a\left[\frac{x^3\sqrt{x^2+1}}{(x+3)^4}\right] = \log_a\left(x^3\sqrt{x^2+1}\right) - \log_a(x+3)^4$

$= \log_a(x^3) + \log_a\left[(x^2+1)^{1/2}\right] - 4\log_a(x+3)$

$= 3\log_a x + \frac{1}{2}\log_a(x^2+1) - 4\log_a(x+3)$

Eg. $2\log_a x + \log_a 9 + \log_a (x^2+1) - \log_a 5$

$\log_a(9x^2) + \log_a\left(\frac{x^2+1}{5}\right) = \log_a\left(\frac{9x^2(x^2+1)}{5}\right)$

- $\log_a(M+N) = X$; $\log_a(M-N) = X$

BEWARE

L6.6 Log Func^ns : Prop: 2

- $M = N \iff \log_a M = \log_a N$ — Log func^ns are 1-to-1
- impt. values of a are e & 10
  (e: ln, 10: log)
- if $0<a<1$ or $a>1$ (old base) & $0<b<1$ or $b>1$ (new base)

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$$

Eg. $\log_5 89 \quad \frac{\ln 89}{\ln 5} = 2.78$

Eg. $\log_{\sqrt{2}} \sqrt{5} \quad \frac{\ln \sqrt{5}}{\ln \sqrt{2}} = 2.32$

L6.7 Log Eq^ns

Eg. graph $\log_2 x = f(x) = \frac{\ln(x)}{\ln(2)} = \left(\frac{1}{\ln 2}\right)\ln x$

Eg. prove that $\frac{1}{\log_2 \pi} + \frac{1}{\log_6 \pi} > 2$

LHS $= \frac{1}{\log_2 \pi} + \frac{1}{\log_6 \pi} = \frac{\ln 2}{\ln \pi} + \frac{\ln 6}{\ln \pi}$

$= \frac{\ln 12}{\ln \pi} > 2 \Rightarrow \ln(12) > \ln(\pi^2) \Rightarrow 12 > \pi^2$

True, thus proved

Eg. $2\log_{0.5} x = \log_{0.5} 4$ find x

$\log_{0.5}(x^2) = \log_{0.5} 4 \Rightarrow \frac{\ln x^2}{\ln 0.5} = \frac{\ln 4}{\ln 0.5}$

$x^2 = 4 \Rightarrow x = \pm 2 \quad \cancel{-2}, +2$

log func^n are defined on +ve nos

Eg. $\log_8(x+1) + \log_8(x-1) = 1$

$\log_8(x^2-1) = 1 \quad x^2 - 1 = 8$

$x^2 = 9 \quad = \pm 3$

$x = +3$

Eg. $\log_3 x + \log_4 x = 4 \Rightarrow \frac{\ln x}{\ln 3} + \frac{\ln x}{\ln 4} = 4$

$\ln x \left(\frac{1}{\ln 3} + \frac{1}{\ln 4}\right) = 4 \Rightarrow \ln x = \frac{4}{\left[\frac{1}{\ln 3} + \frac{1}{\ln 4}\right]}$

$\ln x = 4 \left[\frac{\ln 3 \cdot \ln 4}{\ln(12)}\right]$

$x = e^{\frac{\ln 3^4 \cdot \ln 4}{\ln(12)}}$

Eg. $\ln(x^2) = (\ln x)^2$

$\ln x^2 = 2(\ln x) = (\ln x)^2$

$2t = t^2$

$t = 2, 0$

$\ln x = 0 \qquad \ln x = 2$

$x = e^0 \qquad x = e^2$

$x = 1$

$\Rightarrow$ Tutorial - 1

Eg. $3\log_p 9 - 2\log_q 27 = 2\log_{p-q} 81$

$3\log_p 3^2 - 2\log_q 3^3 = 2\log_{p-q} 3^4$

$6\log_p 3 - 6\log_q 3 = 8\log_{p-q} 3$

$$\cancel{6}^{3}\left[\frac{1}{\log_3 P}-\frac{1}{\log_3 q}\right] = \cancel{8}^{4}\left[\frac{1}{\log_3 P/q}\right]$$

$$= 4\left[\frac{1}{\log q - \log p}\right]$$

$\log_3 P = m$

$\log_3 q = n$

$$3\left[\frac{1}{m}-\frac{1}{n}\right] = 4\left[\frac{1}{n-m}\right]$$

$$3(n-m)^2 = 4mn \Rightarrow 3m^2 - 10mn + 3n^2 = 0$$

$$(m-3n)(3m-n) = 0$$

$$\log_3 P = \cancel{3}\log_3 q^3$$

$$P = q^3 \Rightarrow \text{also}, \quad q = p^3$$

⇒ Tutorial - 2

Eq. $\log_5(x^2+x+5) + \log_4(x) = 3$

$$\log_5(x^2+x+5) = 3 - \log_4(x)$$

$$\underbrace{x^2+x+5}_{>1} = \underbrace{5^{3-\log_4(x)}}_{>1}$$

$\Rightarrow \log_4(x) > 0$

$3 > \log_4(x)$

$\log_4 x = \{0, 1, 2, -1, -2 \ldots\}$

I $7 \neq 5^{3-0}$ ✗

I $\log_4 x = 0$

$x = 1$

II $25 = 5^2$ ✓

II $\log_4 x = 1$

$x = 4$

III $16^2 \neq 5$ ✗

III $\log_4 x = 2$

$x = 16$

→ Tutorial - 3

Eq. $A = A_0 e^{-E/RT}$, $A_0 = e^{35}$

$A(T)$	0.00018	0.0027	0.0030	0.2
T	750 (T_1)	796 (T_2)	850 (T_3)	89 (T_4)

what is most likely Actn energy?

$\ln A = \ln A_0 - \frac{E_a}{RT}$ $\quad x_1 = \frac{1}{T_1} = 0.0013$

$\ln A = -\left(\frac{E_a}{R}\right)\frac{1}{T} + \ln A_0$ $\quad x_2 = \frac{1}{T_2}, \; x_3 = \frac{1}{T_3}$

$y = m \quad x + c$

$c = \ln e^{35} = 35$ $\quad SSE = \sum_{i=1}^{4}(y_i - mx_i - c)^2$

$m_1 = -\frac{E_a}{R} = -32716$

$E_a = 272\ kJ$

$m_2 = 32716$

$m_3 = -52682$

$m_4 = 52682$

→ Tutorial - 4

Point	x	y	Grp
a	1	0.5	G1
b	5	0.5	G2
c	4	0.2	
d	4	0.8	
e	3	0.8	
f	7	0.8	

sigmoidal curve (with $y = 1$ as max)

$f(x) = \frac{1}{1 + e^{-2x+9}}$

(a) Find the logic

(b) groupings for c, d, e, f

$a = (1, 0.5)$ $\quad f(1) = \frac{1}{1 + e^{7}} = 0.000911$

$y = 0.5$

$y > f(1) \longrightarrow G1$

$b = (5, 0.5)$ $\quad f(5) = \frac{1}{1 + e^{-1}} = 0.731$

$y < f(1) \longrightarrow G2$

(c) 4, 0.20 $\quad f(4) = \frac{1}{1+e} = 0.27$ $\quad$ G2

(d) 4, 0.8 $\quad f(4) = \frac{1}{1+e} = 0.27$ $\quad$ G1

(e) 3, 0.8 $\quad f(3) = \frac{1}{1+e^{3}} = 0.47$ $\quad$ G1

(f) 7, 0.8 $\quad f(7) = \frac{1}{1+e^{-5}} = 0.99$ $\quad$ G2

→ Tutorial - 5

Eg. $q = p+1$ $\quad f(x) = \log_p (q^x - 1)$

$g(x) = \log_q (p^x + 1)$

Find the coordinates f & g will intersect

$f(x) = g(x)$

$\log_p (q^x - 1) = t$ $\quad | \quad p^x + 1 = q^t$

$q^x - 1 = p^t$

$p^x + q^x = p^t + q^t$ $\quad$ $p^x + q^x$

$x = t$

$q^x - 1 = p^x \Rightarrow \left(\frac{q}{p}\right)^x - \left(\frac{1}{p}\right)^x = 1$

$q = p+1$ $\quad \left(\frac{q}{p}\right)^x - 1 \quad \left(\frac{1}{p}\right)^x$ // dec func^n

inc. func^n $\quad q > p$

$x = 1$

$\Rightarrow q = p+1$

$f(1) = \log_p (q-1)$

$1, \log_p (q-1)$

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⇒ Tutorial-6

Eg. Find roots of $x^{\left(\frac{3}{4}(\log_3 x)^2 + \frac{5}{4}\log_3 x - 4\right)} = 3^{\sqrt{3}}$

$$\log_3\left(x^{\quad}\right) = \log_3 3^{3} = 3$$

$$\left[\frac{3}{4}(\log_3 x)^2 + \frac{5}{4}\log_3 x - 4\right]\log_3 x = 3$$

$$\log_3 x = t$$

$$\left(\frac{3}{4}t^2 + \frac{5}{4}t - 4\right)t = 3$$

$$\Rightarrow 3t^3 + 5t^2 - 16t - 12 = 0$$

$$\Rightarrow (t-2)(3t^2 + 11t + 6) = 0$$

$$\Rightarrow (t-2)(t+3)(3t+2) = 0$$

$$t = 2, -3, -\frac{2}{3} \qquad \log_3 x = 2$$

$$x = 3^2$$

$$x = 3^{-3}$$

$$x = 3^{-2/3}$$

$$\Rightarrow \boxed{\log_x b = \frac{1}{\log_b x}}$$

$$\Rightarrow \boxed{\log_{a^b} c = \frac{1}{b}\log_a c}$$

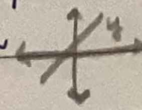
Week-7

L7.1 Some topics from maths-1

- func^n - rela^n from input to output, where each input is related to 1 output $f: X \to Y$

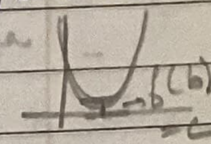
  Dom: X   Codom: Y   Range: $\{f(x) \mid x \in X\}$

- linear func^{ns} - $f(x) = mx + c$ $m, c \in R$



- quad. func^n - $f(x) = a(x-b)^2 + c$ ($a > 0$) $a, b, c \in R$

  Parabola

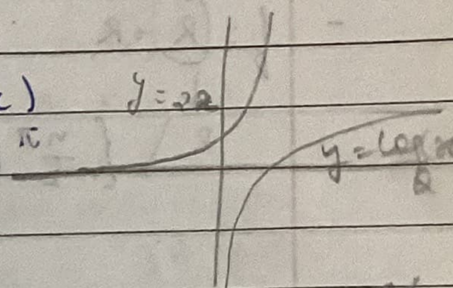


- poly. func^n - $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

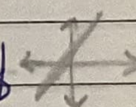
  $a_i \in R$

- the exponential & log. func^{ns}

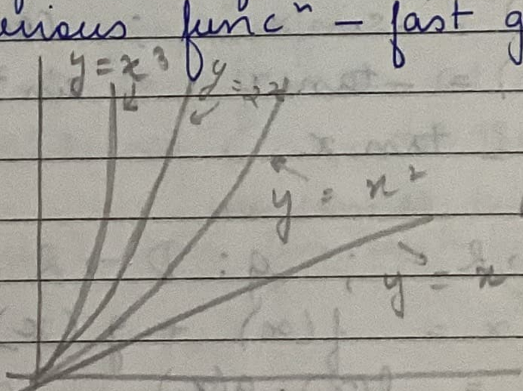
  $f(x) = a^x$ $g(x) = \log_a(x)$



- A func^n $f: R \to R$ is said to be monotone inc. if $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2)$ & vice-versa

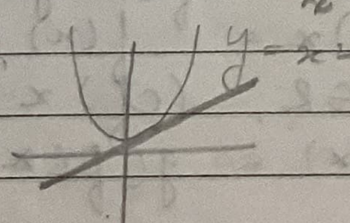


- comparing various func^n - fast growth, close to 0



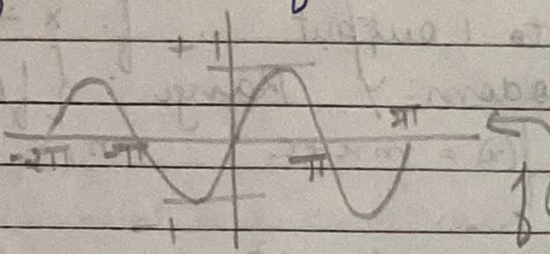
but, after some time, exponential func is bigger than poly func. always

- Tangent lines



17.2 Func^n of 1 Var

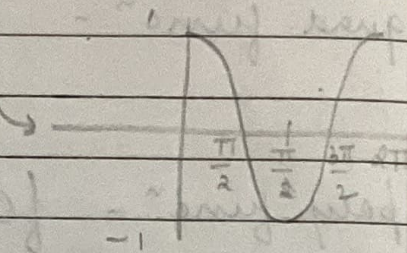
- $f: R \to R$ $\quad f(x) = \sin x \Rightarrow$ trigonometric func^n



$f(x) = \sin x$

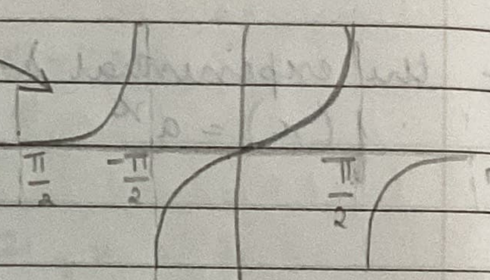
Also c/d periodic func^n

- $f: R \to R$ $\quad f(x) = \cos x$



- $f: R \to R$ $\quad f(x) = \tan x$

$R \setminus \{\frac{n\pi}{2}$ (where n are odd integers)$\}$



- $\cot(x) = \frac{1}{\tan x}$ $\quad \sec(x) = \frac{1}{\cos x}$ $\quad \operatorname{cosec}(x) = \frac{1}{\sin x}$
- $\sin(-x) = -\sin x$; $\cos(-x) = \cos x$
- $\tan(-x) = -\tan x$; $\sin^2 x + \cos^2 x = 1$
- $\frac{\sin x}{\cos x} = \tan x$

- $f: D \to R$; $g: D \to R$ $\quad (D \subset R)$

1. $(f+g)x = f(x) + g(x) \quad x \in D$
2. $(fg)x = f(x) \cdot g(x) \quad x \in D$
3. $(f/g)x = f(x)/g(x), \; x \in D, \; g(x) \neq 0$
4. Let $c \in R$, $(cf)x = c \times f(x), \; x \in D$
5. $g \circ f(x) = g(f(x))$
6. $(f/g)x = \frac{f(x)}{g(x)} \neq 0$

Eg. $f(x) = x^2 + 1$, $g(x) = \sqrt{x}$

$g(f(x)) = \sqrt{x^2+1}$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

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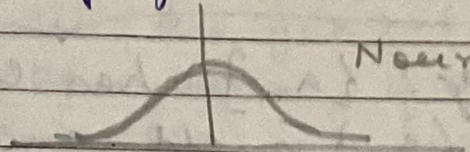
L7.3 Graphs and Tangents

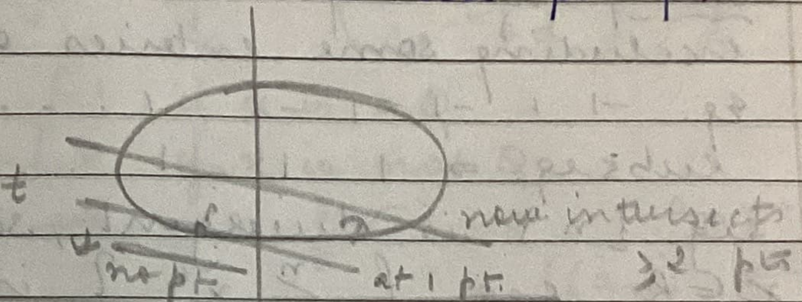
- $\lambda(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$

  $\{(x, y) \mid y = f(x)\}$

- $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$



- curve is a fig. that is obtained as the path of a moving point
- a tangent line to a curve C at a pt p is a line which represents the instantaneous direcⁿ in which curve C moves at the pt. p.



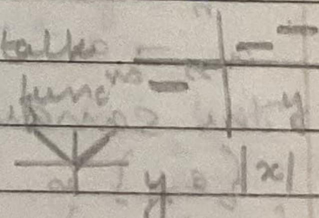
- tangent (line) to a funcⁿ $f: D \to R$, $D \subset R$

  $\gamma(f)$ - the graph of

  then, a tangent to f at x at the pt. $(x, f(x$

- floor of $6.289 \Rightarrow 6$, $-2.986 \Rightarrow 3$

  we cannot talk tangent line for such funcⁿ



L7.4 Limits for sequences

$2 - \frac{1}{n^2}$ as $n \uparrow$; $2 - \frac{1}{n^2}$ as $n\uparrow$; $2 - \frac{1}{(1+ \log$

$\to 2$

- Let $\{a_n\}$ be a seq. of real nos. We say that $\{a_n\}$ … limit $a \in R$ if as $n \uparrow$, the no. $a_n \to a$

* $\left\{n\left(\frac{\sqrt{2\pi n}}{n!}\right)^{\frac{1}{n}}\right\} \rightsquigarrow$ cvr $\left\{\frac{n}{\sqrt[n]{n!}}\right\}$

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- $\{a_n\}$ tends to a / $\{a_n\}$ converges to a / $a_n \to a$ / $\lim_{n\to\infty} a_n = a$ / $\lim a_n = a$ / $a_n \xrightarrow{n\to\infty} a$ / $\lim \{a_n\} = a$ / $\lim_{n\to\infty} \{a_n\} = a$

  seq. $\{a_n\}$ has limit a
- seq $\{a_n\}$ is c/d convergent if it converges to some limit. Eg. $\left\{\frac{1}{n}\right\} \to 0$
- seq. $\{a_n\}$ is c/d divergent if not convergent
  Eg. $\{(-1)^n\} \to$ this oscillates
- subseq. of a seq. is a new seq. formed by excluding some enteries of a seq.
  Eg. -1 1 -1 1 -1 1 . . .
  subseq. ⇒ 1 1 1 . . .
- seq. $\{n\}$ is divergent, same for $\{-n\}$

series $x \in \mathbb{R}$, seq. $\left\{\sum_{k=0}^{n} \frac{x^k}{k!}\right\}$ is convergent to e^x
- $x \in \mathbb{R}$, then $\left\{\left(1+\frac{x}{n}\right)^n\right\}$ converges to e^x *
- if $a_n \to a$, then every subseq. of $\{a_n\} \to a$
- $a_n \to a$ & $b_n \to b$, then $a_n + b_n \to a+b$
- $a_n \to a$ & $c \in \mathbb{R}$, $ca_n \to ca$
- The sandwitch princ. :→ $a_n \to a$, $b_n \to a$
  $\{c_n\}$ is such $a_n \le c_n \le b_n$, then $c_n \to a$
- $\left\{\frac{(-1)^n}{n}\right\} \to 0$ $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, -\frac{1}{5}$ - - -
  $\frac{1}{n} \to 0$, $\frac{-1}{n} \to 0$ $c_n = (-1)^n \left[\frac{1}{n}\right]$ $c_n \to 0$
- $a_n \to a$, $b_n \to b$ then $a_n - b_n \to a-b$
- $a_n \to a$, $b_n \to b$, $a_n b_n \to ab$
- $a_n \to a$, $c \in \mathbb{R}$ then $c^{a_n} \to c^a$
- $a_n \to a$, $c \in \mathbb{R}$ $a_n > 0\ \forall n$ $\log_c(a_n) \to \log_c(a)$

$\left\{\frac{\frac{1}{\ln(1+n)} + \frac{5n^2}{1+n^2}}{\left(1+\frac{1}{n}\right)^{2n}}\right\}$ $\ln(1+n) \to \infty$ $\frac{1}{\ln(1+n)} \to 0$ $\frac{5n^2}{1+n^2} \to 5$ $\frac{1+\frac{1}{n^2}}{} \to 1$ $((1+1/n)^n)^2 \to e^2$ $\frac{5}{e^2}$

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2.7.5 Limits for func^{ns} of 1 variable

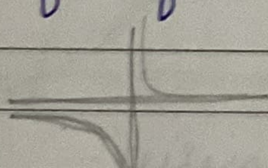
- $a_n \to a$, then $a_n^2 \to a^2$
- $f(x) = x^2$, $f(a_n) \longrightarrow f(a)$, whenever $a_n \to a$
- in contrast, consider the $g(x) = \lfloor x \rfloor$
  ↳ $g(x)$ happens at integer value, if a is non-integer value $g(a_n) \to g(a)$, $a_n \to a$
- $\lim_{x\to a} x^k$; $k \geq 0$ $\quad a^k$, $\lim_{x\to a} e^x \quad e^a$, $\lim_{x\to a} x^k$; $k<0$ $a \neq 0$ $\quad a^k$
- $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ $\qquad \lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$ $\quad \log_e(x)$

$$\lim_{x\to\infty} \frac{a+be^x}{c+de^x} = \frac{b}{d} \qquad \left(\frac{ae^{-x}+b}{ce^{-x}+d}\right) \sim \frac{b}{d}$$

- $f(x) = \begin{cases} 1 & \text{if } x \text{ is a } \mathbb{Q} \\ 0 & \text{if } x \text{ is not } \mathbb{Q} \end{cases}$

$\sqrt{2}$ is a irrat. no. $\qquad f(\sqrt{2}) = 0 \qquad a_n \to \sqrt{2}$
$f(a_n) \not\to f(\sqrt{2})$

- limit of a func^n at a pt. from left
  $\lim_{x\to a^-} (f(x)) = L$
- f, a be a pt. $a_n \to a$ where a_n belongs the domain of def^n of f. $\lim_{x\to a} f(x)$
- $f(x) = \frac{1}{x}$

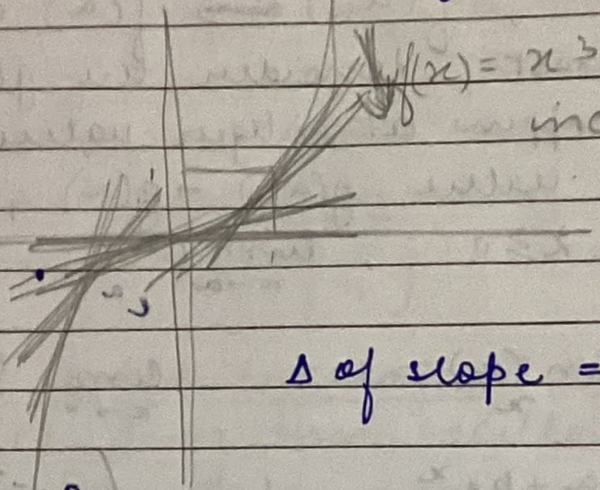


$\lim_{x\to\infty,\ x\to-\infty} \frac{1}{x} = 0 \qquad \lim_{x\to 0^-,\ x\to 0^+} \frac{1}{x} = \text{DNE}$

- $\lim_{x\to a} f(x)$. Sometimes, we can substitute the value of 'a' in the expression for $f(x)$ and obtain the limit
- f is continous at the pt. a if $\lim_{x\to a} f(x) = f(a)$
  ↳ continuity → the limit at a can be obtained by evaluating the func^n at a.

⇒ Tutorial - 1

Eg. $f(x) = x^3$ polynomial func^n → degree 3

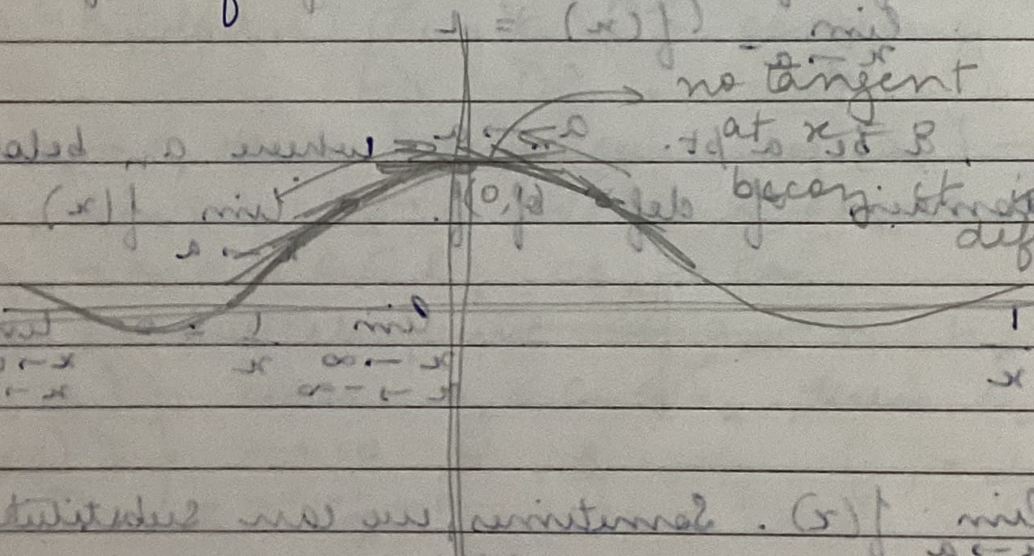


Δ of slope = Δ of $f(x)$

⇒ Tutorial - 2

Eg. $f(x) = \frac{\sin(x)}{x}$ $\lim_{x \to 0} f(x) = 1$

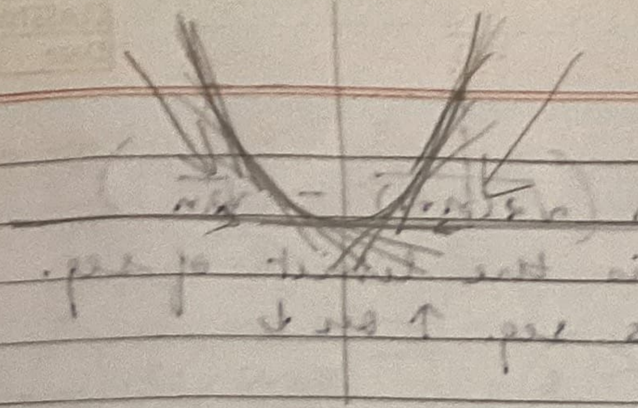
Dom $f = \mathbb{R} \setminus \{0\}$



⇒ Tutorial - 3

Eg. $f(x) = x^2$ quad. func^n ⇒ polynomial of degree 2

Dom $f = \mathbb{R}$ $f(0) = 0$ $\lim_{x \to 0} f(x) = 0$



→ Tutorial-4

Eq. $f(x) = \lfloor x \rfloor$ floor of x

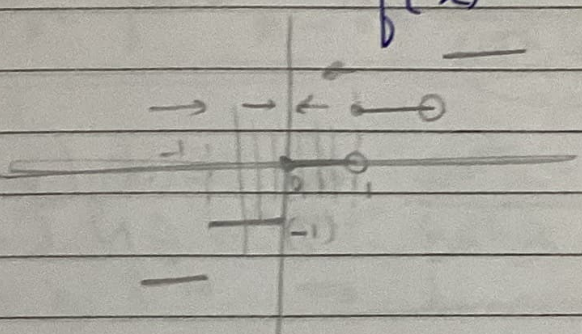
$f(x) = a$ if $a \leq x < a+1$ a is an Z

greatest integer less than or equal to x

$f(2.7) = 2$ $f(3.2) = 3$ $[2,3)$

$f(3) = 3$ $f(x) = 2$

$f(x) = a$ $[a, a+1)$



has no limit for any integer

→ Tutorial-5

Eq. $f(x) = \lceil x \rceil$ ceiling of x

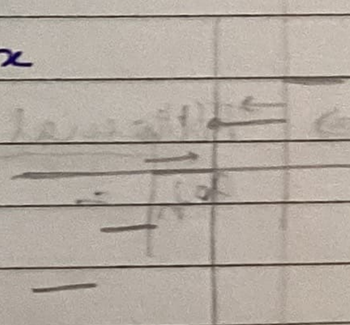
$f(x) = a+1$ if $a < x \leq a+1$, where a is an Z

$= a+1$ $x \in (a, a+1]$

least integer $\geq x$

$f(2.3) = 3$ $f(3) = 3$

$f(2.5) = 3$ $f(3.2) = 4$



tangent doesn't exist at $x = a$

⇒ Tutorial-6

$a_n = \sqrt{2}n\left(\sqrt{2(n+1)} - \sqrt{2n}\right)$

→ what is the limit of seq.

→ is this seq. ↑ or ↓

$$a_n = \frac{\sqrt{2n}\left(\sqrt{2(n+1)} - \sqrt{2n}\right)\cdot\left(\sqrt{2(n+1)} + \sqrt{2n}\right)}{\left(\sqrt{2(n+1)} + \sqrt{2n}\right)}$$

$$= \frac{\sqrt{2n}\left(2(n+1) - 2n\right)}{\sqrt{2(n+1)} - \sqrt{2n}}$$

$$a_n = \frac{2\sqrt{2n}}{\sqrt{2(n+1)} + \sqrt{2n}} \qquad \lim_{n\to\infty} a_n = \frac{2\sqrt{2n}}{\sqrt{2n}\left(\sqrt{1+\frac{2}{n}}+1\right)}$$

$\frac{2}{n} \to \lim = 0$

$$= \frac{2}{2} = 1$$

$$\boxed{\begin{array}{l} a_{(n+1)} \geq a_n \ \forall\, n \in N \text{ (inc)} \\ a_{(n+1)} \leq a_n \ \forall\, n \in N \text{ (dec)} \end{array}}$$

$$a_n = \frac{2\sqrt{2n}}{\sqrt{2(n+1)} + \sqrt{2n}} \qquad a_{n+1} = \frac{2\sqrt{2(n+1)}}{\sqrt{2n+2} + \sqrt{2(n+1)}}$$

$- a_n \geq 0$ ∴ inc.

⇒ Tutorial -7

$a_n = \frac{\cos n\pi}{2n^2}$ limit of seq?

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$\cos n\pi \Rightarrow \cos\pi = -1$ $\quad -1 \le \cos n\pi \le 1 \quad \forall\, n \in N$

$\cos 2\pi = 1$

$\cos 3\pi = -1$

$$\frac{-1}{2n^2} \le \frac{\cos n\pi}{2n^2} \le \frac{1}{2n^2}$$

$0 \leftarrow \frac{-1}{2n^2}$, $\frac{1}{2n^2} \to 0$, $\frac{\cos n\pi}{2n^2} \to 0$

$\therefore$, limit is 0

→ Tutorial-8

$$f(x) = \frac{e^{1/2x} - 1}{e^{1/2x} + 1} \qquad \text{Dom } (f(x)) \{R/0\}$$

$\lim_{x \to 0}$ exist? If yes, what is limit

L.H $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} \frac{e^{-\frac{1}{2h}} - 1}{e^{-\frac{1}{2h}} + 1} = -1$

R.H $\lim_{x \to 0^+} f(x) = \lim_{h \to 0^+} \frac{e^{\frac{1}{2h}} - 1}{e^{\frac{1}{2h}} + 1} = 1$

L.H $\ne$ R.H $\Rightarrow$ hence, limit exist but not equal

⇒ Tutorial-9

$f(x) = \frac{\lfloor x \rfloor}{x}$, find the value of $\lim_{x \to \infty} f(x)$

$\lfloor x \rfloor \Rightarrow$ greatest integer value $\le x$ $\qquad [3.5] = 3$

$\lfloor x \rfloor \le x \qquad \frac{x-1}{x} < \frac{\lfloor x \rfloor}{x} \le \frac{x}{x}$

$$1 - \frac{1}{x} < \frac{\lfloor x \rfloor}{x} < 1$$

$1 \leftarrow 1 - \frac{1}{x}$, $1 \to 1$, $\frac{\lfloor x \rfloor}{x} \to 1$

→ Tutorial-10

$f(x) = \frac{x}{2}\left(\sqrt{x^2+4} - x\right)$ $\lim_{x\to\infty} f(x), \lim_{x\to-\infty} f(x)$

$\dfrac{\frac{x}{2}\left(\sqrt{x^2+4} - x\right)\left(\sqrt{x^2+4} + x\right)}{\sqrt{x^2+4} + x} = \dfrac{2x}{\left(\sqrt{x^2+4}\right)+4}$

$= \dfrac{2x}{|x|\sqrt{1+\frac{4}{x^2}}+1} = \dfrac{2}{\sqrt{1+\frac{4}{x^2}}+1} = \frac{2}{2} = 1$

$\dfrac{2x}{-(x)\sqrt{1+\frac{4}{x^2}}+x}$ $\lim_{x\to-\infty}$

Week – 8

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18.1 Limits & continuity

- limit $a_n = a$ or $a_n \to a$
- limit of f at a from the left (resp. right) exists and equals m

left $\Rightarrow$ $\lim_{x \to a^-} f(x) = M$
right $\quad x \to a^+$

- f is continous at a if the limit of f at a exists and $\lim_{x \to a} = f(a)$

$f(a_n) \to f(a)$, when $a_n \to a$

1. $\lim_{x \to a} f(x) = F$, $\lim_{x \to a} g(x) = G$ $\quad \lim_{x \to a} (f+g)(x) = F+G$
2. F, $c \in R$ $\quad \lim (cf)(x) = cF$
3. F $\quad G$ $\quad \lim_{x \to a} (fg)(x) = FG$
4. $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = L$

$f(x) \leq h(x) \leq g(x)$ $\quad \lim_{x \to a} h(x) = L$

Eg. 1. $f(x) = 5x^3 + 0.45x^2 - 2x + 100$

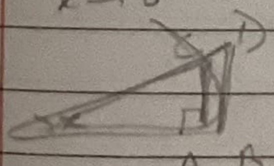
2. $\lim_{x \to 0} \frac{100}{x^2 - 5x + 6}$ $\quad$ 3. $\lim_{x \to 0} \frac{\sin(x)}{x}$

1. $\lim_{x \to a} f(x) = 5 \lim_{x \to a} x^3 + 0.45 \ldots$ $\quad \lim_{x \to a} x^n = a^n$

$= f(a)$

$= 5a^3 + 0.45a^2 - 2a + 100 = f(a)$

2. $\lim_{x \to 0} x^2 - 5x + 6 = 6 \neq 0$ $\quad = \frac{100}{6} = 16.67$

3. 

$OB = 1$ $\quad BD = \tan x$

$AC = \sin x$

$|Arc\ BC| = x$

& $\sin x \leq x$ $\quad \lim_{x \to 0} \frac{\sin(x)}{x} = 1$

Eg. $\lim_{x \to 0} \frac{\tan(x)}{x}$ $\lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1$

Eg. $\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$1 - \cos 2\theta = 2\sin^2\theta$

$= 2\sin^2 \frac{x}{2}$

$= \frac{2}{4} \lim_{x \to 0} \frac{\sin^2 \frac{x}{2}}{\frac{x^2}{4}} = \frac{1}{2}$

Eg. $f(x) = \lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$

$\frac{\frac{x}{1+x}}{x} < \frac{\ln(1+x)}{x} < \frac{x}{x} \to 1$

- func f is conti. if it is continous at all pts. in its domain, for all pts. a for which f(a) is defined
Eg. Poly., e^x, $\log(x)$, $\sin(x)$, $\cos(x)$
- tangents are meant for those functions which are at least continious functions
- for step func^n, at integers we can't talk about tangents

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L8.2 Differentiability and the derivative

Eg. 2900 km in abt. 72 hrs. ; what is speed?

260 km of Maharashtra in 4 hrs after food
in 1 hr

$\frac{260}{3}$ 86.6 km / hr

- avg. and instantaneous speed are diff. concepts
- an infinitesimal time, we obtain a limit!

$$\text{Inst. speed} = \lim_{\Delta t \to 0} \frac{\text{dist. in } \Delta t}{\Delta t}$$

- Let f on an open interval around a. Then f is differentiable at a if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists

- If f is differentiable at a, then it is continuous at a.

- for a func^n $f(x)$ its derivative func^n

$$f'(x) = \frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$$

6.8.3 Computing Derivatives & L'Hopital's Rule

- Linearity - $f(x)$ & $g(x)$ are differentiable at pt a, so is $(f+g)(x)$ & $(f+g)'(a) = f'(a) + g'(a)$
- Prod. rule - $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
- Quotient rule - $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$
- composition: the chain rule:
  $(f(g))'(x) = f'(g(x))\,g'(x)$

Eg $f(x) = x^n$ $\quad n \in \mathbb{N}$

$= nx^{n-1}$

Eg. $5x^3 - 12x^2 + \pi x - 0.5$ $\Rightarrow f(x) = a_n x^n + \dots + a_0$

$f'x = n a_n x^{n-1} + \dots + a_1$

Eg. $x^7 \sin(x)$

$f'(x) = 7x^6 \sin(x) + x^7 \cos(x)$

Eg. $f(x) = \tan(x)$

$f'x = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos(x)^2}$

$= \frac{1}{\cos^2(x)} = \sec^2(x)$

Eg. $f(x) = \frac{1}{x^n}$; $n > 0$

$f'(x) = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}$

Eg. $\tan(2x) = \dfrac{\sin(2x)}{\cos(2x)} = \dfrac{2\tan(x)}{1-\tan^2(x)}$

$g(x) = 2x$ $\quad f(x) = \tan(g(x))$

$f'(x) = h'(g(x))\, g'(x)$

$= 2\sec^2(2x)$

- Indeterminate limits - $f(x)$ & $g(x)$ are defined on an interval around pt. a

$\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$ or diverge to $\infty / -\infty$

and $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ $\Rightarrow$ use L'Hopital's rule

1. $f'(x)$ & $g'(x)$ exist on this interval
2. $g(x) \neq 0$
3. $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = L$

$\therefore \lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = L$

Eg. $\lim\limits_{x \to 0} \dfrac{\log_e(1+x)}{x} = \dfrac{\frac{1}{1+x}}{1} = \dfrac{1}{1+x}$

Eg. $\lim\limits_{x \to 2} \dfrac{x^2-5x+6}{x-2} = \dfrac{2x-5}{1} = -1$

Eg. $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = \dfrac{\cos(x)}{1} = \cos(0) = 1$

App → GeoGebra

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L 8.4 Derivatives, tangents & linear approx.

- instantaneous direcⁿ in which graph $\Gamma(f)$ moves at $(a, f(a))$ ⇒ tangent
  it is limit of secants

$$y - f(a) = \frac{f(a+h) - f(a)}{a+h-a}(x-a)$$

↳ eqⁿ of secant'

$$\boxed{y = f'(a)(x-a) + f(a)}$$

↳ tangent to f at a exists

Eg. $f(x) = 5x^3 - 17x^2 + \pi x - 0.5$, $a = 0$

$f'(x) = 15x^2 - 34\pi + \pi$

$f'(0) = 0$

$\therefore y = \pi(x - 0) + f(0)$

$= \pi x - 0.5$

Eg. $f = \cos(x)$; $a = \frac{\pi}{3}$

$y = -\sin\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$

Eg. $f(x) = x^{1/3}$ $x \neq 0$ ⇒ $f'x = \frac{1}{3x^{2/3}}$

- linear approx. → $L(x) = c + dx$

$L(x) = f(a) + m(x-a)$

Best approx. $f(x) \approx L(x)$ ∀ x close to a

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eg. $f(x) = x^3$; $a = 1$

$f'(1) = 3$

$L(x) = 3(x-1) + 1 = 3x - 2$

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

Week-9

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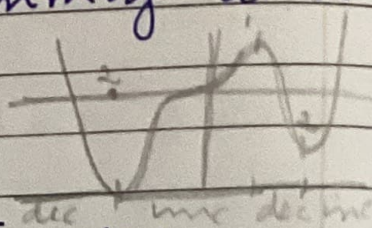
L9.1 Critical pts: local maxima & minima

- an interval of inc. is an int. on the func^n $f(x)$ is inc. if $x_1 < x_2$ then $f(x_1) < f(x_2)$
- pt. x is a t.p. of x if
  (i) either there is an int. of inc. ending at x and int. of dec. beginning at x
  (ii) there is an int. of dec. ending at x & an int. of inc. beginning at x



local extrema $\begin{cases} \text{(i)} \longrightarrow \text{local max} \\ \text{(ii)} \longrightarrow \text{local min} \end{cases}$

- tangents at t.ps are horizontal

$$y = f'(a)(x-a) + f(a)$$

Thus, tangent is horizontal at a t.p. shows that $f'(a) = 0$

- a pt a is a critical pt. of a func^n $f(x)$ if either f is not differentiable at a or $f'(a) = 0$. Thus, every t.p. is a critical pt.
- saddle pt. → critical pt. which is not a local max. / local min

f' checks montonacity of f.
f'' " " f'

- if f is twice differentiable, we check f'' at all critical pts
  (i) if a is critical pt. & $f''(a) > 0$, $a \to$ local min
  (ii) " " " " & $f''(a) < 0$, $a \to$ local max
  (iii) " " " " & $f''(a) = 0$, its inconclusive

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L9.2 Egs: Computing maxima & minima

Eg. $f(x) = x^3 + 12x$ $f'(x) = 3x^2 - 12 = 0$

$3x^2 = 12 \Rightarrow x = \pm 2$

critical pts are $+2, -2$ $f''(x) = 6x$

$f''(2) = 12 > 0$ $f''(-2) = -12 < 0$

↳ local min ↳ local max

Eg. $f(x) = \cos x$

$f'(x) = \sin x = 0 \rightarrow$ critical pts $\{k\pi \mid k \in \mathbb{Z}\}$

$f''(x) = -\cos x$

$f''(k\pi) = -\cos(k\pi) = \begin{cases} -1 & k \text{ is even } \mathbb{Z} \\ 1 & k \text{ is odd } \mathbb{Z} \end{cases}$

k is even $\Rightarrow$ local max

k is odd $\Rightarrow$ local min

Eg. $f(x) = x^3 + x^2 - x + 5$

$f'(x) = 3x^2 + 2x - 1 = 0 \Rightarrow -1, \frac{1}{3}$

$f''(x) = 6x + 2$

$f''(-1) = -6 + 2 = -4$ $f''(\frac{1}{3}) = 4$

↳ local max ↳ local min

- local extrema of a func^n f on a closed interval $I = [a, b]$

Eg. $f(x) = x^2$ $\rightarrow [-1, 1]$

$f'(x) = 2x = 0$

$f''(x) = 2$ $f''(0) = 2 > 0$

↳ local min / global min

-1 & 1, are local (maxima)

- If the int. I is closed & bounded & f is conti, then max & min must exist, max & min are local, unless the boundary pts.

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Q. $f(x) = \begin{cases} x^3 + x^2 - x + 5 & \text{if } 0 \leq x \leq 100 \\ x^3 + 2x^2 + x - 5 & \text{if } -100 \leq x < 0 \end{cases}$

Bdry pts = 100, -100, 0

$f'(x) = \begin{cases} 3x^2 + 2x - 1 & \text{if } 0 < x < 100 \quad \rightarrow -1, 1/3 \\ 3x^2 + 4x + 1 & \text{if } -100 < x < 0 \end{cases}$

$\hookrightarrow -1, -1/3$

critical pts : $1/3, -1, -1/3$

$f(1/3) = \frac{130}{27}$ $\qquad f(-1/3) = -\frac{139}{27}$

$f(0) = 5$ $\qquad f(-1) = -5$

$f(100) = \text{big}$ $\qquad f(-100) = (-100)^3$

100 ⇒ global max

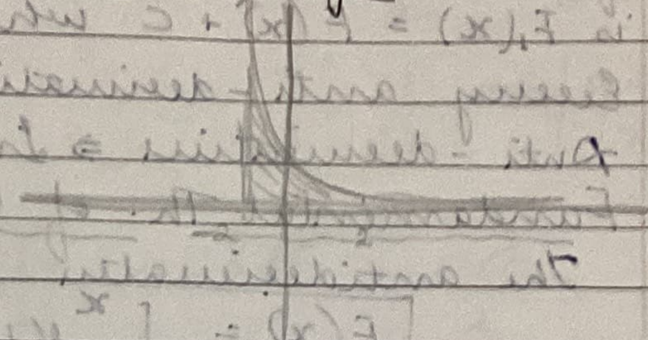
-100 ⇒ " min

L9.3 Computing areas

- area of rec. $= l \times b \Rightarrow A(l, b)$
  $A(2l, b) = 2A(l, b)$
  $A(nl, b) = nA(l, b)$; $n \in \mathbb{N}$
  $A(cl, db) = cd\, A(l, b)$
- area of parallelogram = base × height
- $\text{area}(\Delta) = \frac{1}{2}$ base × height
- area (trapezium) $= \frac{1}{2} \times (l_1 + l_2) \times h$
- for any quad. & polygons ↳ divide & conquer rule

L9.4 Reimann Sums and the integral

- graph of $y = e^{-x}$



- Reimann Sums

f be a func^n from $D \to R$, $[a, b]$ is in D

$$a = x_0 < x_1 < x_2 \dots < x_n = b$$

a choice of $x_i^* \in [x_{i-1}, x_i]$

$\Delta x_i = x_i - x_{i-1}$ & $\|P\| = \max_i \{\Delta x_i\}$

Sum $\to$ $$\boxed{S(P) = \sum_{i=1}^{n} f(x_i^*) \Delta x_i}$$

- the definite integral of f from $a \to b$

$$\lim_{\|P\| \to 0} S(P) = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

- Area under the graph of the func^n above the int. $[a, b]$

$$\int_a^b f(x)\, dx = \lim_{\|P\| \to 0} S(P)$$
$$= \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

L9.5 Integrals as anti-derivatives (Indefinite integral)

- An anti-derivative of the func^n f is a func F is such that $F'(x) = f(x)$ for all $x \in D$

Eg. anti-derivative of $x^7 + 2x^6$

$$\frac{x^8}{8} + \frac{2x^7}{7}$$

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- If F is an anti-derivative of f, then so is $F_1(x) = F(x) + C$ where c is any constant. Every anti-derivative has this form.
- Anti-derivative ⇒ Indefinite ⇒ $\int f(x)\,dx$
- Fundamental Th. of Calculus

The antiderivative of f is —

$$\boxed{F(x) = \int_a^x f(t)\,dt}$$ t is used to avoid confusion

Also,

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Func^n	Derivative	Integral
⇒ 1	0	x
⇒ x^n	nx^{n-1}	$\frac{x^{n+1}}{n+1}$
		$\ln\lvert x\rvert$ $n=-1$
⇒ $\sin(x)$	$\cos(x)$	$-\cos(x)$
⇒ $\cos(x)$	$-\sin(x)$	$\sin(x)$
⇒ $\tan(x)$	$\sec^2(x)$	$\ln\lvert\sec(x)\rvert$
⇒ $\sec(x)$	$\sec(x)\tan(x)$	$\ln\lvert\sec(x)+\tan(x)\rvert$
⇒ $\cot(x)$	$\sec^2(x)$	$\ln\lvert\sin(x)\rvert$
⇒ $\operatorname{cosec}(x)$	$\sec^2(x)$	$\ln\lvert\operatorname{cosec}(x)-\cot(x)\rvert$
$e^{\lambda x}$	$\lambda e^{\lambda x}$	$\frac{e^{\lambda x}}{\lambda}$
⇒ a^x	$a^x(\ln(a))$	$\frac{a^x}{\ln(a)}$
⇒ $\ln(x)$	$\frac{1}{x}$	
⇒ $\frac{1}{\sqrt{a^2-x^2}}$		$\sin^{-1}\left(\frac{x}{a}\right)$

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$\Rightarrow \frac{1}{a^2+x^2}$		$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\Rightarrow \frac{1}{a^2-x^2}$		$\frac{1}{2a}\ln\left\|\frac{x+a}{x-a}\right\|$

Eg. $\int_1^2 x\,dx \Rightarrow \left[\frac{x^2}{2}\right]_1^2 \quad \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$

Eg. $\int_0^{\pi} \sin(x)\,dx \Rightarrow [-\cos(x)]_0^{\pi}$

$-\underbrace{\cos(\pi)}_{(-1)} - \cos(0) \quad -1-1 = 2$

Eg. $\int_0^{\infty} e^{-2x}\,dx \Rightarrow \left[-\frac{e^{-2x}}{2}\right]_0^{\infty} \Rightarrow \frac{-e^{-2(\infty)}}{2} + \frac{1}{2}$

$\frac{-e^{-2(\infty)}}{20} + \frac{1}{2} \quad \frac{1}{2} \quad \frac{\cancel{e^{-\infty}}}{2} \quad \frac{1}{2}$

19.6 Computing areas using integrals

- $\int c\,f(x)\,dx = c\int f(x)\,dx$
- $\int (f+g)(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
- Integraⁿ by parts $\rightarrow$

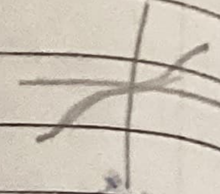
  $\int (fg')(x)\,dx = fg(x) - \int (f'g)(x)\,dx$
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \Rightarrow$ for definite int.
- For $c \in \mathbb{R}$, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- for $f(x) \geq g(x)$ for int $[a,b]$ $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Eg. $\int_1^3 (x^2 - 4x + 2)\,dx \Rightarrow \left[\frac{x^3}{3}\right]_1^3 - \left[2x^2\right]_1^3 + \left[2x\right]_1^3$

$\left[9 - \frac{1}{3}\right] - \underbrace{\cancel{[18-2]}}_{16} + \underbrace{\cancel{[6-2]}}_{4} \qquad 9 - \frac{1}{3} = \frac{26}{3}$

$\frac{26}{3} - \cancel{16+4} - \frac{12}{1} = \frac{26-36}{3} = \frac{-10}{3}$

Eg. $\int_{-2}^{2} x^2 \sin(x)\,dx$ $\quad \left(+\frac{x^3}{3}\right)\left(-\cos x\right)\Big|_{-2}^{2}$



$\left(\cancel{\frac{8}{3}+\frac{8}{3}}\right)\left(-\cos 2 - \cos(-2)\right)$

$\frac{16}{3}$

$= \int_{-2}^{0} x^2 \sin x\,dx + \int_0^2 x^2 \sin(x)\,dx$

$= \cancel{-\int_0^2 x^2 \sin x\,dx + \int_0^2 x^2 \sin x\,dx}$

$= 0$

Eg. $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 3-x & 1 < x \leq 2 \end{cases}$ What is $\int_0^2 f(x)\,dx$

$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$= \int_0^1 x\,dx + \int_1^2 (3-x)\,dx$

$= \left[\frac{x^2}{2}\right] + \left[3x - \frac{x^2}{2}\right]$

$\frac{1}{2}$ $\quad 4 - 2.5 = 1.5$

$\int_a^b (fg')(x)\,dx = (fg)(b) - (fg)(a) - \int_a^b (f'g)(x)\,dx$

Eg. $\int_0^\infty 3xe^{-3x}\,dx \Rightarrow \int_0^b 3x\,e^{-3x}\,dx$

$= \left[-xe^{-3x}\right]_0^b - \int_0^b -e^{-3x}\,dx$

$f(x) = x$

$g'(x) = 3e^{-3x}$

$g(x) = -e^{-3x}$

$= -be^{-3b} - \left(\frac{e^{-3b}}{3} - \frac{1}{3}\right)$

$= \frac{1}{3}$

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- $\int_a^b (f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

Eg. $\int_0^a \sqrt{a^2-u^2}\,du$

$u = a\sin x$

limits are

$0 \longrightarrow \frac{\pi}{2}$

$f(u) = \sqrt{a^2-u^2}$

$f(g(x)) = \sqrt{a^2-a^2\sin^2 x}$

$$= \int_0^{\pi/2} \left(\sqrt{a^2-a^2\sin^2(x)}\right)(a\cos(x))\,dx$$

$$= a^2 \int_0^{\pi/2} \cos^2(x)\,dx$$

$$= \frac{a^2}{2}\left[x + \frac{\sin(2x)}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi a^2}{4}$$

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L10.1 Introducⁿ to graphs
- cartesian prod. $A \times B$ $\{(a,b) \; a \in A, b \in B\}$
- relaⁿ is a subset of $A \times B$
- Graph $G := (V, E)$ $\underbrace{E \subseteq V \times V}_{\text{Binary Relaⁿ}}$
  (set of vertices / nodes) → set of edges
- Directed graph ⇒ $(v, v') \in E \not\Rightarrow (v', v) \in E$
  ↳ Teacher-course graph is directed.
- Undirected graph ⇒ $(v, v') \in E$ iff $(v', v) \in E$
  ↳ Eg. Friendship ↳ they both are same edge
- graphs help in finding paths (seq. of connecⁿ)
- Path - seq. of vertices $v, v_2, \ldots v_k$ connected by edges
- The path does not visit a vertex twice, if 2 times occur → it is called walk.
- $(n-1)$ edges (max.) in a path
- paths in directed graphs, vertex v is reachable from vertex u, if there is a path from $u \to v$.

L10.2 Some gen. graph problems
- map colouring → each state is a vertex, edges must be if they share a border
- assign colours to nodes so that endpts. of an edge have diff. colours, now, we need the underlying graph
- abstracⁿ → if we distort the graph, problem is unchanged
- graph $G = (V, E)$, set of colours C
  $c : V \to C$, such that $(u, v) \in E \Rightarrow c(u) \neq c(v)$
- 4 colour theorem → for graphs from geographical maps, 4 colours suffice

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- classroom scheduling → map colour problem
- vertex cover → marking a v covers all edges from v, mark smallest subset of V to cover all edges
- dance example
- independent set → subset of vertices such that no 2 are connected by an edge
- matching - project example

L10.3 Representation of Graphs
- let $|V| = n$, assume $V = \{0, 1, \ldots, n-1\}$
- edges are now pairs (i, j), where $0 \leq i, j < n$
  ↳ usually, $i \neq j$, no self loops
- adjacency matrix
  $A[i, j] = 1$, if $(i, j) \in E$
- undirected graph, $A[i, j] = 1$ iff $A[j, i] = 1$
  ↳ sym. across main diagonal
- neighbours of j - column j with entry 1
  ↳ scan row i to identify neighbours of i
- [rows represent outgoing edges
  → columns represent incoming edges
- No. of edges incident of vertex i degree(i) = _
  $0 \leq deg. \leq n-1$
- for directed graphs, there are indegree and outdegree
- 2 1° strategy
  [BF - propagate marks in layers
  → DF - explore a path till it dies out, then backtrack

- Adjacency matrix has many 0's
  - size is n^2, regardless of no. of edges
  - undirected graph : $|E| \leq \frac{n(n-1)}{2}$
  - directed graph : $|E| \leq n(n-1)$
  - $|E|$ is less than n^2
- adjacency list - list of neighbours for each vertex

L10.4 BFS

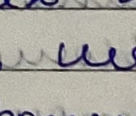
- explore the graph lvl. by lvl.
  - 1st visit vertices 1 step away
  - then 2 steps away ...
- Each visited vertex has to explored
- maintain info. about vertices during exploraⁿ
  - visited
  - explored
- Assume $V = \{0, 1, \ldots, n-1\}$
  visited : $V \to \{True, False\}$ tells us whether $u \in V$ has been visited, (initially all False)
- maintain a seq. of visited vertices yet explored by a queue (first in, first out)
- remove and explore vertex i at head of queue; stop when queue is empty.
- enhance BFS to record paths $i \to j$
  - record parent$(j) = k$
  - from j, follow parent links to reach i
- BFS explores neighbours lvl. by lvl., lvl. gives a notion of distance, with visited we can maintain lvl(j)
- edges are labelled with cost (dist., time, price)

L10.5 **DFS**

- start from i, visit an unexplored neighbour j; suspend i work till ...
- suspended vertices are stored in a <u>Stack</u> — last in, first out, most recent suspended is checked first
- Eg. Google chrome browsing
- paths discovered by DFS are not shortest paths, unlike BFS
- DFS numbering — maintain a counter
  - ↳ increment & record counter value each time u start & finish exploring a vertex
  - ↳ it helps for find cut vertices (deleting vertex disconnects graph)
  - ↳ finds bridges (deleting edge disconnects graph)

L10.6 **Applica^{n} of BFS & DFS**

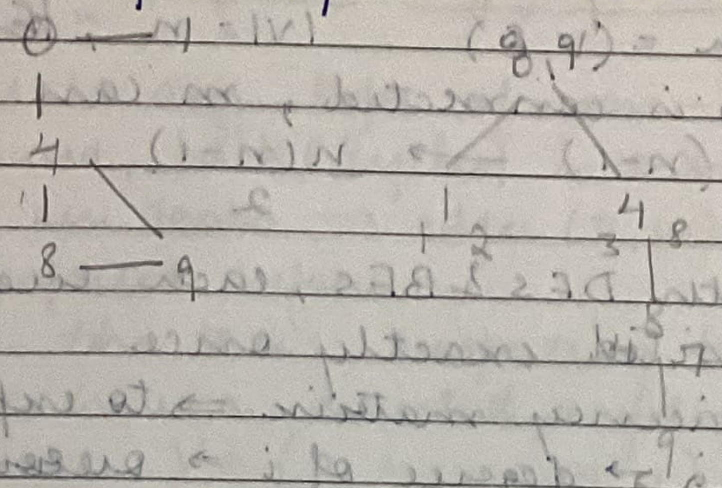
- <u>connectivity</u> - an undirected graph is connected if every vertex is reachable from every other vertex
- assign each vertex a <u>component no.</u>
- a <u>cycle</u> is a path (walk) that starts & ends at the same vertex



  - ↳ cycle shouldn't repeat edges
- <u>simple cycle</u> - only repeated vertices are start & end
- a graph is <u>acyclic</u> if it has no cycles
- a <u>tree</u> is a minimally connected graph

- edges explored by BFS form a tree. One tree / component, collecⁿ of trees is a forest.
- Tree facts
  → a tree on n vertices has $(n-1)$ edges
  → a tree is acyclic
  → any non-tree edge creates a cycle, detect cycles by searching for non-tree edges
- DFS tree - each vertex is assigned an entry no. (pre) & exit no. (post)

eg.



- as before non-tree edges gen. cycles

L10.7 Application - 2

- in a directed graph, a cycle must follow the same direcⁿ

DFS ⇒ n vertices → $(2n-1)$ nos.

- diff. type of non-tree edges → Forward, Back, Cross edges

only back edges correspond to cycles

- Forward tree edge (u, v) ⇒ Int. [pre(u), post(u)] contains [pre(v), post(v)]

→ and vice-versa give Back edge

→ for cross edge, they are disjoint

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- vertex i & j are strongly connected if there is a path from $i \to j$ & $j \to i$
- directed graph can be decomposed into strongly connected components (SCC)
  - ↳ within SCC, each pair of vertices are strongly connected

L10.8 Complexity of DFS & BFS

- Graph = (V, E) $|V| = n$, $|E| = m$
  - ↳ If G is connected, m can vary from $(n-1) \longrightarrow \frac{n(n-1)}{2}$
- In both DFS & BFS, each reachable vertex is visited exactly once
- In adjacency matrix ⇒ to explore i, scan row i → degree of i ⇒ overall n^2 steps
- In adjacency list ⇒ to explore i, scan list of neighbours of i
  - ↳ total time is the sum of degrees
  - ↳ $= 2m$
  - ↳ overall time proportional to $(n+m)$
- degree of a vertex i is
  - → no. of 1s in row of adjacency matrix
  - → 1s in column
  - → length of adjacency list
- sum of degrees is $2m$
  - ↳ sum is an even no.
- degree of vertex btw: $0 \to (n-1)$
  - ↳ degree 0 → disconnected graph
  - ↳ degree $(n-1)$ → connected to all

- comp. graph - every vertex has degree $(n-1)$
- if all degrees are bounded by k, at most $\frac{kn}{2}$ edges

- for directed graphs, indegree & out degree

L10.9 Directed Acyclic Graphs
- In a directed graph, a cycle must follow same direcⁿ
- DFS reveals diff. type of non-tree edges
  ↳ forward, back (cycles) & cross edges
- Vertices are tasks → edge (t, u), if task t is to be computed before u.
- schedule the task's respecting the dependancy
- Now, we have a directed acyclic graph (DAG)
  ↳ Enumerate $V = \{0, 1, \ldots n-1)$, i appears before j
  ↳ Topological Sorting
  ↳ longest paths

L10.10 Topological Sorting
- it represents a feasible schedule
- a graph with directed cycles cannot be sorted topologically
- Path $i \rightsquigarrow j$ means i must be listed before j
- Every DAG can be topologically sorted.
- first list vertices with no dependancies
- as we proceed, list vertices whose dependancies have already been listed
- a vertex with no dependencies has no

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incoming edges, indegree (v) = 0
- every DAG has a vertex with indegree 0

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L11.1 Longest Paths in DAGs

Eg. Find the longest path in DAG

- If indegree (i) = 0,
  longest-path-to (i) = 0
- If indeg (i) > 0, longest is 1 more than longest path to incoming neighbours
  longest-path-to (i) = 1 + max {longest-path-to(j) | (j,i) ∈ E}
- no repeated vertices in a path of graph with cycles, so path has at most (n-1) edges

L11.2 Transitive Closure

- can compute the ancestor relaⁿ from parent relaⁿ
- $p = u_0$ --- $q = u_n$
- This is c/d transitive closure of $R \Rightarrow R^+$
  - $R^+ \subseteq S \times S$ is a relaⁿ
  - R^+ comes from $R \subseteq S \times S$
- Perform BFS/DFS on all vertices to compute R^+
- another strategy, $A[i,j] = 1$ — path of length 1 from $i \rightarrow j$
- We want $A^+[i,j] = 1 \rightarrow$ path of length ≥ 1 from $i \rightarrow j$
- $A^2[i,j] = 1$, such that $A[i,k]$ & $A[k,j] = 1$
- $A^3[i,j] = 1$, $A^2[i,k] = 1$ & $A[k,j] = 1$
- we can stop, by checking path with length $(n-1)$
- $A^+[i,j] = \max\{A^l[i,j] \mid 1 \leq l < n\}$
  - each $A^l[i,j]$ is either 0/1

- this calculaⁿ → matrix multiplicaⁿ
- $A^2 = A \times A$, $A^3 = A^2 \times A \ldots$, $A^{L+1} = A^L \times A$

L11.3 Matrix Multiplicaⁿ

- a matrix is a 2d → $r \times c$ (r → row, c → column)
- graphs with n nodes, $n \times n$ adjacency matrix
- just add up the value of 2 graphs
- for a matrix M, $M[i,j]$ is the entry in row i, column j
  - $C[i,j] = A[i,j] + B[i,j]$
  - $C = A + B$ → matrix addition
- simple multiplicaⁿ is not useful for matrix multiplicaⁿ
- $r = c = n$
- $C[i,j] = A[i,0] \cdot B[0,j] + A[i,1] \cdot B[1,j] + A[i,n-1] \cdot B[n-1,j]$
- matrix product $C = A \times B$

$$\boxed{C[i,j] = \sum_{k=0}^{n-1} A[i,k] \cdot B[k,j]}$$

- $\left.\begin{array}{l} A \to m \times n \\ B \to n \times p \end{array}\right\}$ $A \times B = m \times p$
- Algebra of boolean values   True → 1, False → 0

$$\left.\begin{array}{l} 0 + 0 = 0 \\ 0 + 1 = 1 \\ 1 + 1 = 1 \\ 1 + 0 = 1 \end{array}\right\} \text{or}, +$$

$$\text{and}, \times \left\{\begin{array}{l} 1 \times 1 = 1 \\ 0 \times 0 = 0 \\ 1 \times 0 = 0 \\ 0 \times 1 = 0 \end{array}\right.$$

- $A^2[i,k] = (A[i,0] \text{ and } A[0,j])$ or $(A[i,1] \text{ and } A[1,j])$

$$\boxed{A^2[i,j] = \sum_{k=0}^{n-1} A[i,k] \times A[k,j]}.$$

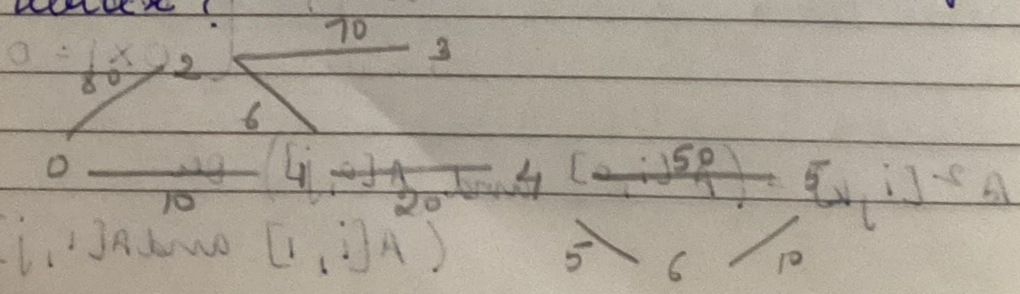
- $A^+ = A + A^2 + \dots + A^{n-1}$

L11.4 Shortest Paths in Wtd. graphs

- we assign values to edges → cost, time, distance ⟹ weighted graphs
- $G = (V, E)$, $W : E \to R$
- adj. matrix - record the wt. whenever there was a edge
- in weighted graph, adds up the wts.. in path
- single source shortest paths → find the shortest paths from a fixed vertex to every other vertex
- all pair shortest paths → find shortest paths btw. every pair of vertices i & j
- -ve edge weights - ola / uber example → going towards home
- The -ve cycle is one whose wt. is negative.

L11.5 Single source shortest paths

- $G = (V, E)$ $W : E \to R$
- find shortest path fixed to every other vertex?



- t = 86 everything burns in prev. diag
- compute expected burn time for each vertex
- algorithm due to Edsger W. Dikjstra
- Dikjstra's algorithm ⟹ very efficient
- each new shortest path we discover extends an earlier one.
- by induction, assume we have found shortest paths to all vertices already burnt.
- here we see, that, this algorithm can't be used with negative edge weights.

L11.6 Single source shortest path with -ve wts.
- burning pipeline analogy [vertices burnt, expected burn time

- the alogrithm's correctness req. non-negative edge weights
- the difficulty with -ve edge wts. is that we stop updating the burn time once a vertex is burnt
- every prefix of this path must itself be a minimum wt. path
- Bellman-Ford Algorithm

  $D(j)$: min. dist. known so far vertex j

for each edge $(j,k) \in E$

$D(k) = \min(D(k), D(j) + W(j,k))$

- update the graph till $(n-1)$ times
- if the algorithm doesn't converge after $(n-1)$ iterations, there is a -ve cycle.

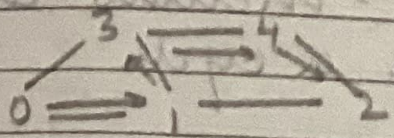
L11.7 All-pair shortest paths

- find shortest paths btw. every pair of vertices i & j
- Eg. Travel agency
- $B^k[i,j] = 1$, if there is a path from $i \to j$ via vertices $\{0, 1, \ldots, k-1\}$
- $B^0[i,j] = 1$, if there is a direct edge
  $B^0 = A$ (transitive closure algorithm)
- Warshall's Algorithm → it computes transitive closure also
- $B^n[i,j] = 1$, if there is a path from $i \to j$ with intermediate vertices in $\{1, 2, \ldots, n-1\}$
- We adapt Warshall's algorithm to compute all pair shortest paths
- $SP^k[i,j]$ be the shortest length from $i \to j$
  ↳ $SP^0[i,j] = W[i,j]$
  (no intermediate vertices, shortest path is wt. of direct edge)
- This is Floyd-Warshall Algorithm

L11.6 Min. Cost Spanning Trees

Eg. Roads damaged so that people start moving as fast as possible
- retain min. set of edges so that graph remains connected → tree

- add the edge, it becomes a loop
- want a tree connects all the vertices — spanning tree



we can have multiple such trees

- now, spanning tree with lowest cost
- A tree is a connected acyclic graph, a tree on n vertices has exactly (n-1) edges
- Add an edge to a tree will create a cycle
- In a tree, every pair of vertices is connected by a unique path.

Facts abt. trees

Tree
G is connected
G is acyclic
G is (n-1) edge

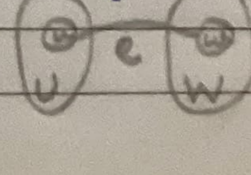
- Two strategies

① Prim's Algorithm - start with smallest edge & grow

② Kruskal's Algorithm - connect the unique vertices

L11.9 Prim's Algorithm

- grow with min cost → take the smallest wt edge overall → extend the current tree by add smallest edge which is not in the tree
- for correctness of Prim's algorithm → main. separator lemma



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L11.10 Kruskal's Algorithm
- start with n components, each a single vertex → process edges in ascending order of cost → include edge, if it does not create a cycle