Maths1 Week-6

L1 Logarithmic Functions

- inverse of exponential funcns

$f(x) = a^x \quad (a > 0,\ a \neq 1)$

is 1-to-1, it has its reverse.

- The logarithmic funcn (to the base a) in std. form as — $\boxed{y = \log_a(x)}$

and it is the inverse of $f(x) = a^x$.

Remember,

$$y = \log_a x \iff x = a^y$$

(7-rule)

Further,

$\boxed{a^{\log_a x} = x}$ $\qquad$ $\boxed{\log_a a^x = x}$

$\Rightarrow f(f^{-1}(x)) = x \qquad f^{-1}(f(x)) = x$

Domain $(a^x) = R$ $\Rightarrow$ range for $\log_a$

Range $(a^x) = (0, \infty)$ $\Rightarrow$ dom for $\log_a$

Dom. $(\log_a) = (0, \infty)$

Range $(\log_a) = R$

$\therefore$, log funcns are inverse exp. funcn.

Eg. $f(x) = \log_4(1-x)$

Domain of $(\log_4) = (0, \infty)$

$1 - x > 0 \Rightarrow 1 > x$

Dom. $(f) = (-\infty, 1)$

Eg. $g(x) = \log_3\left(\frac{1+x}{1-x}\right), \quad x \neq 1$

Dom$(\log_3) = (0, \infty)$ Dom$(g) = (-1, 1)$

$\frac{1+x}{1-x} > 0 \qquad x \neq 1$

Eg. $y = \log_3 x \Rightarrow 3^y = 3^{\log_3 x} = x$

$\Rightarrow x = 3^y$

Eg. $(1.3)^2 = m \Rightarrow \log_{1.3}(1.3)^2 = \log(1.3)^m$

$\Rightarrow 2 = \log_{1.3} m$

Or $a^u = a^v \quad (a > 0, \; a \neq 1)$

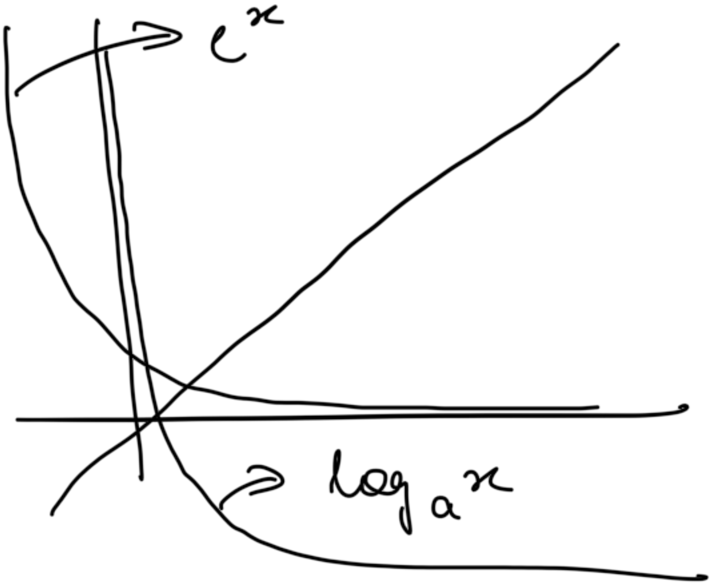
$u = v$

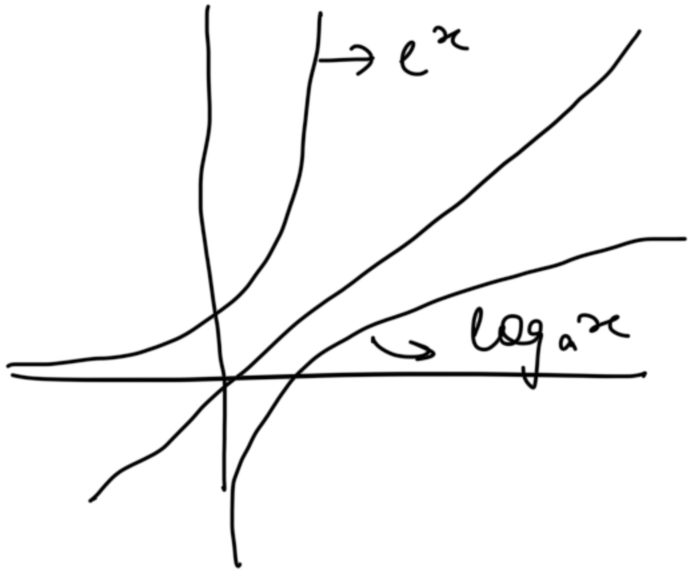
Eg. $\log_3\left(\frac{1}{9}\right) = \log_3(3^{-2}) = -2$

- Graphs of $f(x) = \log_a x$.

For exp.

for $0 < a < 1$ $\Downarrow$ for $a > 1$ $\Downarrow$





L2 More on Graphs

- Prop. of $\log_a x$

$Dom(f) = (0, \infty)$, Range $= (\mathbb{R})$

x-intercept $\Rightarrow (1, 0)$

y-intercept $\Rightarrow$ Nil

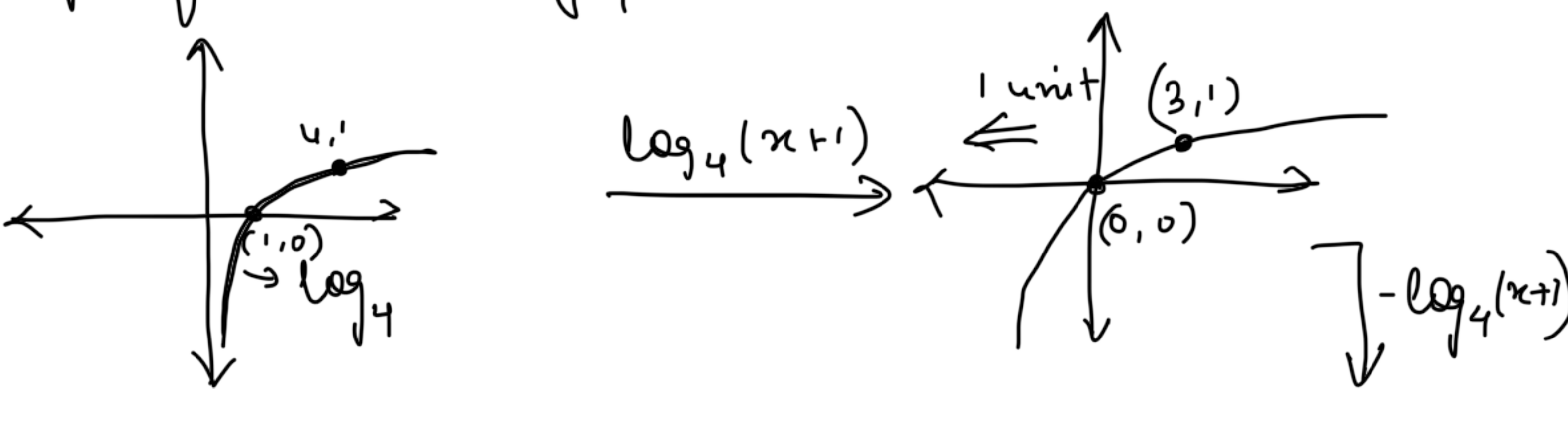
It has vertical asymptote

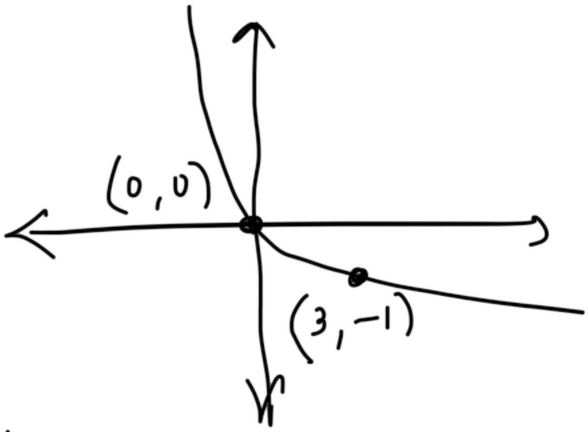
f is 1-to-1 & passes thru $(1, 0)$ & $(a, 1)$

$0 < a < 1 \Rightarrow f$ decreases

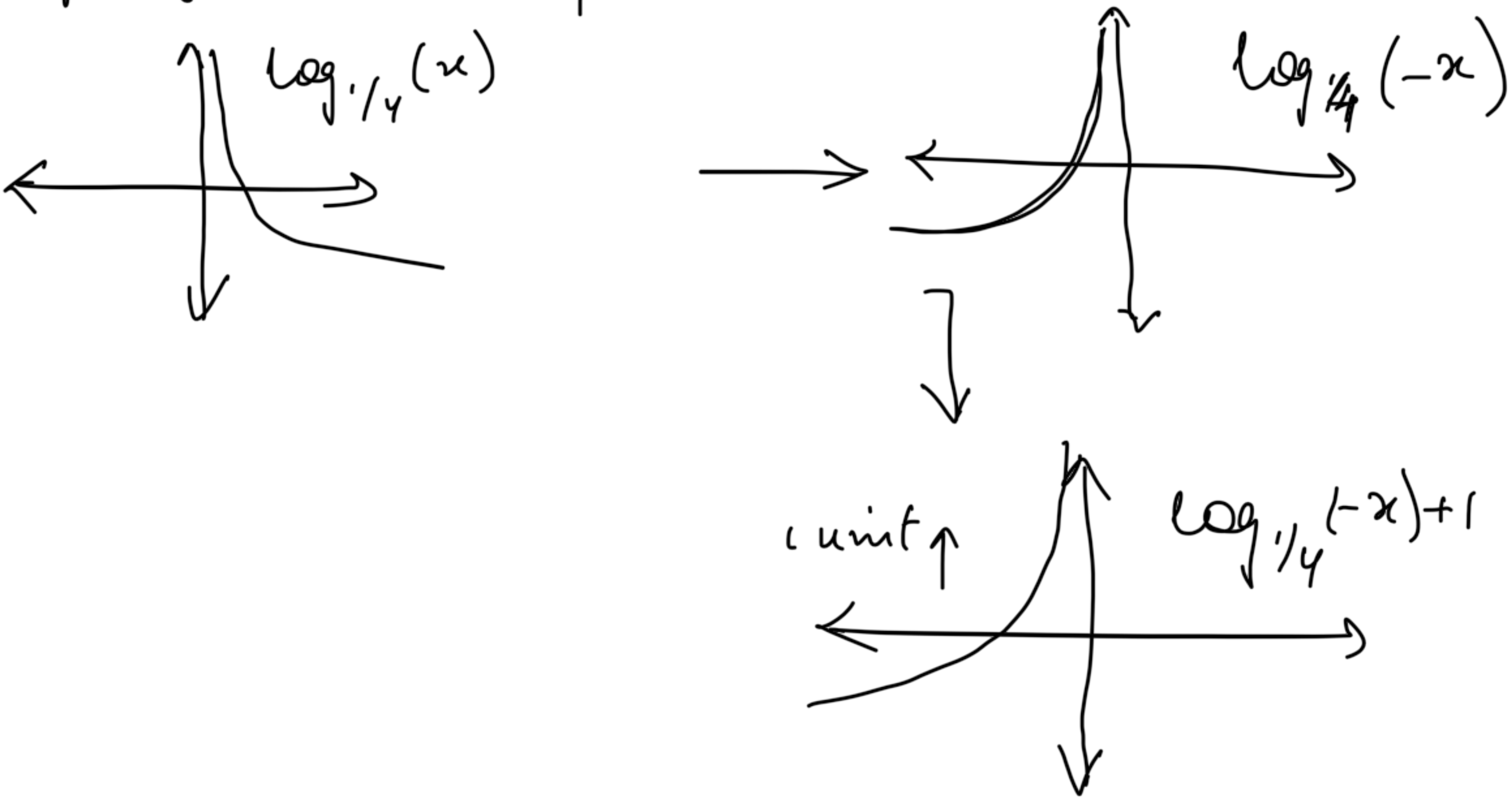
$a > 1 \Rightarrow f$ increases

Eg. $f(x) = \log_4 (x+1)$





Eg. $g(x) = \log_{1/4}(-x) + 1$



- The natural log. func^n is $f(x) = \log_e(x)$ where the base is e. It is denoted by $\ln(x)$.

$$\ln(e^x) = x \qquad \forall x \in \mathbb{R} = \text{Dom}(e^x)$$

$$e^{\ln(x)} = x \qquad \forall x \in (0, \infty) = \text{Dom}(\ln x)$$

$$\boxed{\log x = \log_{10} x} \qquad \boxed{\ln = \log_e x}$$

L3 Solving Exponential Equations

Eg. $2^{x+1} = 64 \Rightarrow 2^{x+1} = 2^6$

$\log_2 2^{x+1} = \log_2 2^6$

$x+1 = 6 \Rightarrow x = 5$

$$\boxed{\log_a(a^x) = x}$$

Eg. $e^{-x^2} = (e^x)^2 \dfrac{1}{e^3}$

$\Rightarrow e^{-x^2} = e^{2x-3} \Rightarrow$ take ln

$-x^2 = 2x - 3 \Rightarrow x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3, 1$

Eg. $9^x - 2 \times 3^{x+1} - 27 = 0$

$(3^2)^x - 6(3^x) - 27 = 0$

$(3^x)^2 - 6(3^x) - 27 = 0$

$3^x = t \Rightarrow t^2 - 6t - 27 = t^2 + 3t - 9t - 27 =$

$t(t+3) - 9(t+3) = 0$

$(t-9)(t+3) = 0$

$t = 9, -3$

$3^x = 9$ or $3^x = -3$

$\Downarrow$ not possible

$3^x = 3^2$ $x = 2$

Eg. $5^{x-2} = 3^{3x+2}$

$\ln(5^{x-2}) = \ln(3^{3x+2})$

$(x-2)\ln(5) = (3x+2)\ln 3$

$-2(\ln(5) + \ln(3)) = 3x(\ln 3) - x\ln 5$

$-2\ln(15) = x(3\ln(3) - \ln(5))$

$$x = \frac{-2\ln(15)}{\ln(27) - \ln(5)} = \frac{\ln(1/225)}{\ln(27/5)}$$

Eg. $x + e^x = 2$

$e^x = 2 - x \Rightarrow x = \ln(2-x)$

$x = 0.433$ (with the help of graph)

L4 Properties : Part - 1

- $\log_a 1 = 0$ for $a \in (0,1)$ or $a > 1$
- $\log_a a = 1$
- $a^{\log_a(x)} = x = \log_a(a^x) = x$

Eg. $3^{\log_3(\pi/2)} = \frac{\pi}{2}$ (simple)

Let $x \in R$, $a < a < 1$ or $a > 1$

$M, N \geq 0$

Then, laws of Logarithm.

1. $\log_a(M.N) = \log_a(M) + \log(N)$
2. $\log_a(M/N) = \log_a(M) - \log_a(N)$
3. $\log_a(1/N) = -\log_a(N)$
4. $\log_a(M^x) = x \log_a(M)$

1.5 Applications

Eg. $\log_a\left[\dfrac{x^3\sqrt{x^2+1}}{(x+3)^4}\right]$

$= \log_a(x^3(x^2+1)^{1/2}) - \log_a(x+3)^4$

$= \log_a(x^3) + \frac{1}{2}\log(x^2+1) - 4\log_a(x+3)$

Eg. $\underbrace{2\log_a x + \log_a 9}$ $+$ $\underbrace{\log_a(x^2+1) - \log_a(5)}$

$= \log_a 9x^2 + \log_a\left(\dfrac{x^2+1}{5}\right)$

$= \log_a\left(\dfrac{9x^2(x^2+1)}{5}\right)$

L6 Properties - 2

- Th. Let $0<a<1$ & $a>1$ & $M, N > 0$.

$$M = N \Leftrightarrow \log_a M = \log_a N.$$

The impt. value of a are e (ln) and 10 (log).

- Change of base rule.

Th. - If $\underbrace{0<a<1 \text{ or } a>1}_{\text{old base}}$ & $\underbrace{0<b<1 \text{ or } b>1}_{\text{new base}}$

then for $x>0$,

$$\boxed{\log_a x = \frac{\log_b(x)}{\log_b(a)}}$$

Eg. $\log_5 89 = \dfrac{\ln 89}{\ln 5} = 2.78$

Eg. $\log_{\sqrt{2}} \sqrt{5} = \dfrac{\ln\sqrt{5}}{\ln\sqrt{2}} = 2.32$

L7 Logarithmic Equations

- Graph the $\log_2 x$

$$f(x) = \log_2 x = \frac{\ln x}{\ln 2} = \left(\frac{1}{\ln 2}\right) \ln x$$

Eg. $\frac{1}{\log_2 \pi} + \frac{1}{\log_6 \pi} > 2$

$= \frac{\ln(2)}{\ln \pi} + \frac{\ln(6)}{\ln \pi} > 2 \Rightarrow \frac{\ln(12)}{\ln(\pi)} > 2$

$\ln(12) > 2 \ln(\pi) \Rightarrow \ln(12) \geqslant \ln(\pi)^2$

$12 > \pi^2 \Rightarrow$ Yes.

Eg. $2 \log_{0.5} x = \log_{0.5} 4$

$\log_{0.5}(x)^2 = \log_{0.5}(4)$

$x^2 = 4 \Rightarrow x = +2, -2$
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Eg. $\log_8 (x+1) + \log_8 (x-1) = 1$

$\log_8 (x^2 - 1) = 1$

$x^2 - 1 = 8 \Rightarrow x^2 - 9 = 0 \Rightarrow x = \pm 3$

but -3 won't work, $\therefore$ $x = 3$.

Eg. $\log_3 x + \log_4 x = 4$

$\frac{\ln x}{\ln 3} + \frac{\ln x}{\ln 4} = 4$

$\ln x \left(\frac{1}{\ln 3} + \frac{1}{\ln 4} \right) = 4$

$\ln x = 4 \left[\frac{\ln 3 \cdot \ln 4}{\ln(3) + \ln(4)} \right]$

$$\ln x = 4\left(\frac{\ln 3 \cdot \ln 4}{\ln 12}\right)$$

$$x = e^{4\left(\frac{\ln 3 \cdot \ln 4}{\ln 12}\right)}$$

Eg. $\ln(x^2) = (\ln x)^2$

$$2\cancel{\ln(x)} = \cancel{\ln(x)} \quad \ln(x)$$

$$\ln(x) = 2 \quad \text{or} \quad 2t = t^2$$

$$x = e^2 \qquad t = 0 \text{ or } t = 2$$

$$\ln(x) = 0 \quad \text{or} \quad \ln(x) = 2$$

$$x = 1 \quad \text{and} \quad x = e^2$$