# I4.0 Week 11 Notes

## L1: Pricing

### Multiple classes of service and prices

customers

WTP

willingness to pay

- Consider a hotel where the rooms are rented for a particular time period. For the purpose of this analysis, we will assume that the time window is the same for all the renters.
- Multiple types of customer may want to book a room in this hotel. For now, let us assume that there are only two classes: budget and business; or economy and business. (in the airline industry the two classes could be discount and full-fare).
- Typically, the room on economy fare are going to be sold *before* the rooms are rented to the business customers – one of the reasons being the time at which these customers take decisions. (Business customers typically book last minute.)

Time period
↳ same

Two types of customer

Economy (earlier book) → $p_e$

Business (late book) → $p_b$

$0 < p_e < p_b$

### Two classes of customers

100

WTP

- The price charged to the economy customers is $p_e$, and that for business travellers is $p_b$. Of course, $0 < p_e < p_b$.
- Hotel opens the room booking for economy class first and then in the last phase of booking (possibly when the economy class customers have been exhausted), the rooms are sold to the business customers.
- The capacity allocation problem: How many rooms should be sold at the economy price $p_e$? Alternately, the hotel should protect how many rooms for the business customers?
- The tradeoff in the context of revenue and probabilities!

### Capacity allocation problem

- In the theory of yield management (or revenue management), this problem is about finding "discount booking limit" (Maximum number of low-priced booking allowed).
- Alternately, the problem is to find "protection level" for the business bookings. If $C$ is the total capacity of the hotel (# of rooms), and $k$ is the booking limit, then, protection level, $y = C - k$.

(Here, we assume that the room are reserved for the full day. Hence, on any given day, the rooms are available or not. Also, we neglect cancellations and no-shows.)

$C \rightarrow$ total capacity

$k \rightarrow$ limit

$y = C - k$
↳ protecⁿ level

### Tradeoff explained

- Setting booking limit too low – Possibility of more economy customers. With no rooms reserved at low price, they may not book. If enough business customers don't show up last minute, then we have empty rooms at hand. This is called spoilage.
- Setting booking limit too high – We sell a lot of rooms at low price. Some of these customers could actually be business customers with higher WTP. Potential loss of revenue. This is dilution (of revenue).
- Let $F_b(x)$ be the cumulative distribution function of the business category demand. Similarly, for the economy category ($F_e(x)$).
- Therefore, let the probability that the demand in business category is more than x is $\bar{F}_b(x) = 1 - F_b(x)$. Similarly, $\bar{F}_e(x) = 1 - F_e(x)$.
- Start with a minimum value of the booking limit $k$.
- The marginal change in revenue by adding one more seat to $k$ (that is, $k+1$).

WTP → willingness to pay

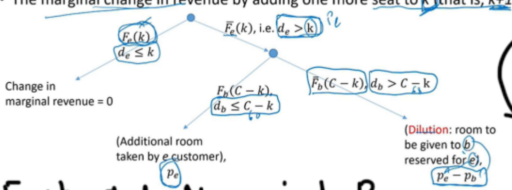
$F_b(x) \rightarrow$ CDF of B

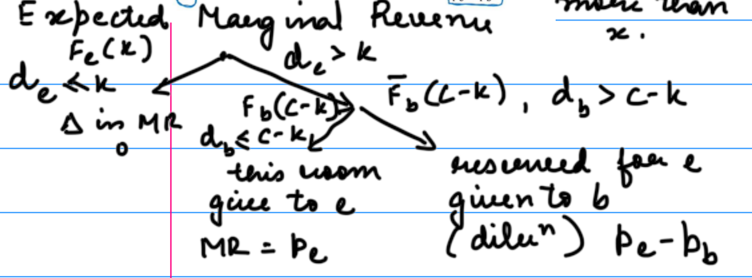
$F_e(x) \rightarrow$ CDF of e

$\bar{F}_b(x) = 1 - F_b(x)$

$\bar{F}_e(x) = 1 - F_e(x)$

demand in B category is more than $x$.




$$E[m(k)] = \bar{F}_e(k)[p_e - \bar{F}_b(C-k) * p_b]$$

$E[m(k)] = \bar{F}_e(k)[p_e - \bar{F}_b(C-k) * p_b]$

- As long as this expected marginal revenue is positive, $k$ can be increased.
- Particularly, as long as $p_e - \bar{F}_e(C-k) * p_b \geq 0$, $k$ should be increased.
- Of course, $k$ can be increased only up to $C$ ($k \leq C$).
- At $k = C$, the entire hotel is a budget hotel!

If $E[m(k)] \geq 0$
$k \uparrow$

$k \leq C$

if $k = C$
↳ budget hotel → an economy hotel

if $k = 0$
↳ business hotel

- Since $F_b(C-k)$ is decreasing in $k$, $\bar{F}_b(C-k)$ will be increasing in $k$.
- Taking $k = 0$, if $p_e < \bar{F}_b(C) * p_b$, then the hotel is better off not reserving any room for the economy customers.
- Otherwise, at least one room should be reserved for the economy customers.

1. Set $k = 0$.
2. If $k = C$, set $k^* = C$ and stop. Else go to 3.
3. Calculate $E[m(k)]$.
4. If $E[m(k)] \leq 0$, or $F_e(k+1) = 1$ then set $k^* = k$ and stop.
5. If $E[m(k)] > 0$ and $F_e(k+1) < 1$, then $k \leftarrow k + 1$ and go to 2.

$k$ y: $C-k$

(The algorithm can be for fixing the booking limit (for economy customers); or can also be considered to be for removing rooms from the protection level (for the business customers)).

→ y

## L2: Capacity Allocation

### Littlewood's approximation

calculates protecⁿ level.

- Kenneth Littlewood (Analyst with British Overseas Airways Company, BOAC). 1972.
- BOAC later became British Airways.
- The value of $k^*$ can be approximated by solving:

$\bar{F}_b(y^*) = \frac{p_e}{p_b}$ or alternately $F_b(y^*) = 1 - \frac{p_e}{p_b}$, where $y^* = C - k^*$.

$F_b(y^*) = \frac{p_e}{p_b}$

$$E[m(k)] = \bar{F}_e(k)[p_e - \bar{F}_b(C-k) * p_b]$$

$F_b(y^*) = \frac{p_b - p_e}{p_b}$

- As long as this expected marginal revenue is positive, $k$ can be increased.

To maximize expected revenue, the probability that business demand will exceed the protection level should be equal to the price ratio of economy price to the business price.

### Newsvendor problem (Remember)

- Single season inventory optimization.
- $Q$ is the quantity to be ordered.
- Demand is stochastic (with a probability density $f$ and distribution function $F$).
- Shortage cost (Cost of under-stocking) is $C_u$; Salvage value (cost of over-stocking) is $C_o$.
  ↳ $C_s$ ↳ $C_o$
- The inventory order quantity should be such that:

$$\Pr(D \leq Q) = F(Q) = \frac{C_u}{C_u + C_o}$$

$\frac{C_s}{C_s + C_o}$ } critical ratio

- Hotel needs to decide how many rooms to protect for the business customers (order quantity). However, the # of business customers is stochastic.
- Shortage cost = protecting one less business room and hence a business customer paying an economy rent = $p_b - p_e$
- Salvage cost = cost of protecting one extra business rooms and therefore turning away a economy customer = $p_e$
- Optimal protection level should be such that:

$$F(y^*) = \frac{C_u}{C_u + C_o} = \frac{p_b - p_e}{p_b - p_e + p_e} = 1 - \frac{p_e}{p_b}$$

$p_b - p_e$

$p_e$

$F_b(y^*) = \frac{p_b - p_e}{p_b}$

- Let demand for each customer class $i$ be independent r.v. Class $i$ demand have characteristics, density: $f_i$, distribution: $F_i$; "over" probability $\bar{F}_i \equiv \Pr(d_i > x)$. There are $n$ such customer classes.
- Product has $n$ price classes corresponding to each customer class. Price classes are numbered in the descending order: $p_1 > p_2 > \cdots > p_n$.
- Demand books in the increasing order of the price. The lowest price is offered first and customers pay that price for a certain time period. Then, the second-lowest price is offered (that time, the purchase at the lowest price is closed). This goes on till the last time period when only the highest price is available for purchase at.

- During the first period, the hotel sees booking from the customers who pay the lowest fare ($p_n$). We denote the accepted number of bookings by $x_n (\geq 0)$.
- The hotel reservation manager's problem, at the start of the booking period $j$, is to determine this $x_j$. So, for each class, she needs to find the booking limit $b_j$. The manager can wait till the beginning of the period to decide this (no need to take this decision apriori for all the time periods).
- The manager already knows the booking that have been accepted in the previous periods (at lower prices), viz. $\sum_{i=j+1}^{n} x_i$.
- At the start of the booking period $j$, unbooked capacity is $C_j = C - \sum_{i=j+1}^{n} x_i$.
- The manager knows about the probability distributions of the future booking periods (i.e. $F_i's$).

Complexity of the problem –

- Consider the problem of finding $b_3$. Decision is whether to increase the limit by one more room.
- If there is no demand for this extra room available at this price, there is no change in the revenue.
- If there is demand, that customer pays $p_3$. But, now we have one room less to sell at a higher price (either $p_2$ or $p_1$).
- Do did we displace a customer from price point $p_2$ or from price point $p_1$ is difficult to determine. But we need those probabilities to calculate the optimal value of $b_3$.

# Expected Marginal Seat Revenue (EMSR) Heuristics

$P_4 < P_3 < P_2 < P_1$

- Designed by Peter Belobaba (1987) for the airline industry (hence the name).
- Supposed to be one of the best-known algorithms.
- Based on the Littlewood's approximation of calculating the protection levels.

$\bar{F}(y) = 1 - \frac{P_e}{P_b}$

Consider class 4 and its protection level.

Step-wise calculation of protection level for class 4 against class 1 (using Littlewood's formula)

$$\bar{F}_1(y_{41}) = \frac{p_4}{p_1} \Rightarrow y_{41} = \bar{F}_1^{-1}\left(\frac{p_4}{p_1}\right)$$

- Similarly, protection level for class 4 against class 2 and 3:

$y_{42} = \bar{F}_2^{-1}\left(\frac{p_4}{p_2}\right)$ and $y_{43} = \bar{F}_3^{-1}\left(\frac{p_4}{p_3}\right)$.

- So, the protection level of class 4 is $y_4 = \min(y_{41} + y_{42} + y_{43}, C)$.
- And hence the booking limit, $b_4 = Max(C_4 - y_4, 0)$.
- Generalizing, for price classes $j \geq 2$, the EMSR protection level is given by:

$$y_j = Min\left[\sum_{i=1}^{j-1} \bar{F}_i^{-1}\left(\frac{p_j}{p_i}\right), C\right]$$

- Assume that we have historical data for demands by booking class for each of the $T$ preceding days (same type of rooms, same services, etc.)
- We have $n$ booking classes $p_1 > p_2 > \cdots > p_n$.
- We need to calculate the booking limits ($b = (b_1, b_2 \dots, b_n)$) for the period $T+1$.
- Data driven approach would try to find the booking limits that would have maximized the revenue when these limits are applied for the time periods $1,\dots,T$.
- The same booking limits can then be applied to the period $T+1$.
- Logic – if the demand distribution for each class is the stationary, then the booking limits would also be similar.

- Let $d_i^t$ be the demand for the class $i$, at time $0 \leq t \leq T$, when the booking limits were $b_1, b_2 \dots, b_n$.
- The realized booking limits would have been:

$$x_n^t = \min[d_n^t, b_n]$$

$$x_{n-1}^t = \min[d_{n-1}^t, b_{n-1} + b_n - x_n^t]$$

$$x_i^t = \min[d_i^t, b_i + b_{i+1} + \cdots + b_n - x_{i+1}^t - x_{i+2}^t - \cdots - x_n^t]$$

- Optimal data driven booking limit is the one that maximizes the revenue:

$$b^* = argmax \sum_{t=1}^{T}\sum_{i=1}^{n} p_i * x_i^t$$

Subject to $b_1 = C, b \geq 0$.

where $b$ is a vector: $b = (b_1, b_2 \dots, b_n)$, and $x_i^t = f(d_i^t, b)$

# L3: LittleWood's Approximation

$C \rightarrow$ Business $(y^*)$

$C \rightarrow$ Economy $(k^*)$

$k + y = C$ $\qquad \bar{F}_b(y) = \frac{p_e}{p_b}$

$C = 100,\ p_e = 250,\ p_b = 750,\ D_b \in U[10, 20]$

$F_b(y) = Pr(D \leq y)$

$\bar{F}_b(y) = 1 - Pr(D \leq y)$

$1 - \frac{y-10}{20-10} = \frac{\cancel{250}}{\cancel{750}} \frac{1}{3} \Rightarrow \frac{2}{3} = \frac{y-10}{10}$

$20 = 3y - 30$ $\quad \boxed{y = 17}$ $\quad k = 83$

If $D \in U[10, 40]$

$1 - \frac{y-10}{40-10} = \frac{1}{3} \Rightarrow \frac{2}{3} = \frac{y-10}{30}$

$60 = 3y - 30 \Rightarrow 3y = 90$ $\quad y = 30$

$k = 70$