Maths: Week-1

L1 Polynomials

- kind of a mathematical expression which is a sum of math. expression
- each term is called a monomial.

  Eg. $x^2 + 4y^2 + z + 10$.
- It can be variables, constant or product of several variables.
- Polynomial → an algebraic expression in which only arithmatic would +, −, × & natural exponents of the variables.

  Eg. $t^{1/2} + t$ is not a polynomial.

Greek → Poly + Nomen ← Latin

Poly → many, Nomen → terms

- each term is monomial, binomial (2) and trinomial (3) etc.
- A polynomial in 1 variable will be –

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 = \sum_{m=0}^{n} a_m x^m$$

a_m → Coeff., x → var., m → exponent

- Check polynomial –

Eg. $x^2 + 4x + 2$ ⇒ yes

$x + x^{1/2}$ ⇒ no

$x + y + xy + x^3$ ⇒ yes

- we are handling → polynomials with real coefficients
- Poly. in 1 var. ⇒ x^4+1
  " " 2 var. ⇒ x^4+y^5+xy
  " " >2 var. ⇒ $xyz+y^4+z^5$

L2 Degree of Polynomials

- first find degree of each terms
  $3x^3 \Rightarrow 3$, $4x^2y^2 \Rightarrow 4$

degree of that term is defined as the exponent on the variable in a term.

$10y \Rightarrow 1$, $1 \Rightarrow 0$

→ Also, it is the sum of the degrees of the variables in that term.

- finally, degree of the polynomial of any one of the terms with non-0 coefficients.

Eg. $x = x^1$ & $c = c.x^0$.

What about 0?

$$0 = 0x + 0x^2 \dots$$

There is no degree (undefined) for the 0 polynomial.

Degree		Name of Poly.	
0	———	constant	$\to$ c, 1, etc
1	———	linear	$\to 2x+4$
2	———	quadratic	$\to 3x^2 + 2$
3	———	cubic	$\to 4x^2y$
4	———	quartic	$\to x^4 + 10x + 1$

L3 Algebra: Addition & Subtracⁿ

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0, \text{ where } a_n \neq 0$$

Domain is $\mathbb{R}$

Range depends on the funcⁿ.

- How to add polynomials?

$$\begin{array}{r} x^2 + 4x + 4 \\ + \qquad\qquad 10 \\ \hline x^2 + 4x + 14 \end{array} \qquad \begin{array}{r} x^4 + 4x \qquad\quad \\ + \qquad\qquad x^3 + 1 \\ \hline x^4 + x^3 + 4x + 1 \end{array}$$

$$\begin{array}{r} x^3 + 2x^2 + x \qquad \\ + \qquad x^2 + 2x + 2 \\ \hline x^3 + 3x^2 + 3x + 2 \end{array}$$

the final answer's degree should be equal to the max degree of each of the polynomial involved!

- $p(x) = \sum_{k=0}^{n} a_k x^k$, $q(x) = \sum_{j=0}^{m} b_j x_j$

$$p(x) + q(x) = \sum_{k=0}^{m \vee m} (a_k + b_k) x^k .$$

- The same procedure goes for subtracn.

eg. $x^3 + 2x^2 + x$

$- \quad (x^2 + 2x + 2)$

$\overline{x^3 - x^2 - x - 2}$

- $$p(x) - q(x) = \sum_{k=0}^{m \vee n} (a_k - b_k) x^k .$$

L4 Multiplican

eg. $(x^2 + x + 1)(2x^3) = 2x^5 + 2x^4 + 2x^3$

↳ rules of exponent will apply here.

eg. $(x^2 + x + 1)(2x + 1)$

$= 2x(x^2 + x + 1) + (x^2 + x + 1)$

$= 2x^3 + 2x^2 + 2x + x^2 + x + 1$

$= 2x^3 + 3x^2 + 3x + 1$

- $p(x) = a_2 x^2 + a_1 x + a_0 \Rightarrow \sum_{k=0}^{n} a_k x^k$

$q(x) = b_1 x + b_0 \qquad \Rightarrow \sum_{j=0}^{m} b_j x^j$

$$p(x)\,q(x) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} \left(a_j\, b_{k-j}\right) x^k.$$

Eg. $(x^2 + x + 1)(x^2 + 2x + 1)$

$= \cancel{x^4} + \cancel{2x^3} + \cancel{x^2} + \cancel{x^3} + \cancel{2x^2} + \cancel{x} +$
$\cancel{x^2} + \cancel{2x} + 1$

$= x^4 + 3x^3 + 4x^2 + 3x + 1$

L5 Division

Eg. $\dfrac{3x^2 + 4x + 3}{x} = 3x + 4 + 3/x$

- Remember, the degree of numerator should be greater than or equal to the degree of denominator.

Eg. $\dfrac{3x^2 + 4x + 1}{x+1}$

$$\begin{array}{r}
3x + 1 \\
x+1 \,\big)\, 3x^2 + 4x + 1 \\
-\; 3x^2 + 3x \\
\hline
x + 1 \\
-\; x + 1 \\
\hline
0
\end{array}$$

$= 3x + 1$

eg. $\dfrac{x^4+2x^2+3x+2}{x^2+x+1} = x^2 - x + 2 + \dfrac{2x}{x^2+x+1}$

$$
\begin{array}{r|l}
 & x^2 - x + 2 \\
\hline
x^2+x+1 & \cancel{x^4} + 2x^2 + 3x + 2 \\
 & \underline{-\cancel{x^4} \mp x^2 \mp x^3} \\
 & \cancel{-x^3} + x^2 + 3x + 2 \\
 & \underline{\overset{+}{-}\cancel{x^3} \overset{+}{-} x^2 \overset{+}{-} x} \\
 & \cancel{2x^2} + 4x + \cancel{2} \\
 & \underline{\underset{-}{}2x^2 \underset{-}{+} 2x \underset{-}{+} 2} \\
 & 2x
\end{array}
$$

L6 Division Algorithms

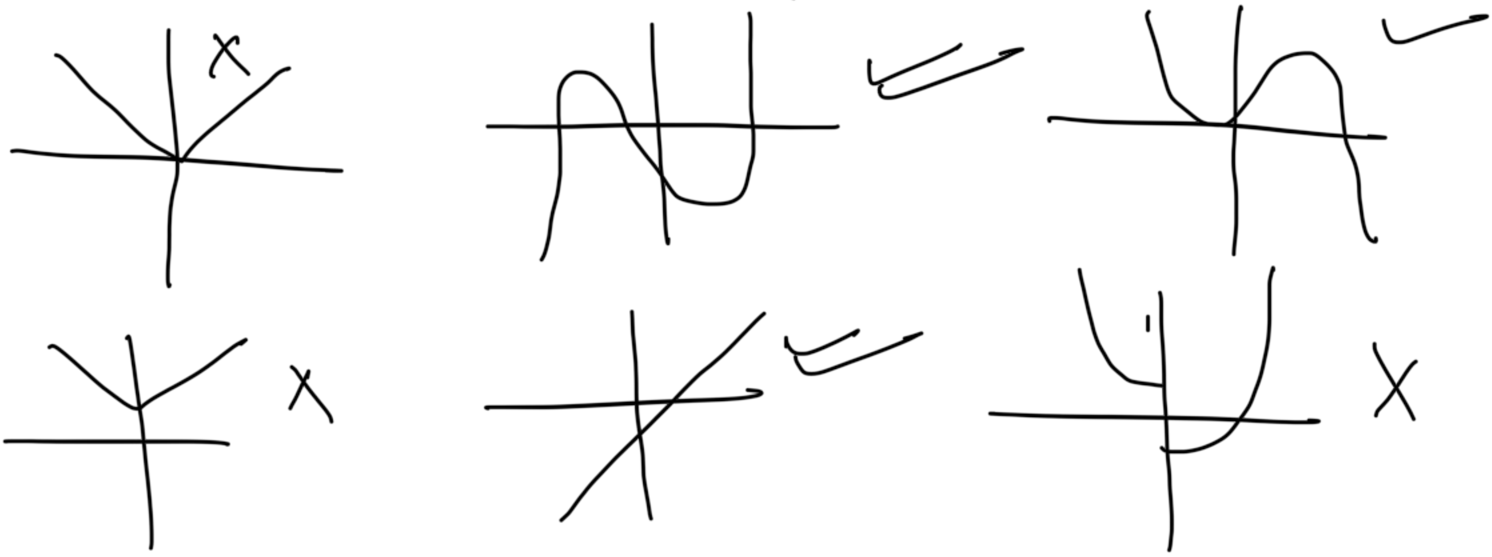
$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \overbrace{\frac{\text{remainder}}{\text{divisor}}}^{\text{rational func}^n}$$

1. arrange the terms in desc. order.
2. start dividing
3. when degree of remainder becomes strictly less than the degree of divisor, then stop.

$$\frac{\cancel{2x^3} + 3\cancel{x^2} + 1}{2x+1} = x^2 + x - \frac{1}{2} + \frac{3/2}{2x+1}$$

L7 Identifying & Charac. of Poly.

- there are no abrupt jerks. the curve should be smooth.
- graphs do not have sharp corners.
- func^n is always continuous, there shouldn't be any breaks.



- these is how we represent the polynomial functions.

L8 Zeroes of Polynomial Functions

- if f is a poly. func^n, values of x for which $f(x) = 0$ are c/d 0s of f.
- if the func^n can be factored, we can set factor $= 0$, and get the 0s.
- the methods —
  a. greatest common factor
  b. factor by grouping

c. trinomial factoring
d. or use tools like desmos.

- basically, they are x-intercepts.

Eg. $f(x) = x^6 - 8x^4 - 16x^2$

$\Rightarrow x^2(x^4 - 8x^2 - 16)$

$\Rightarrow x^2(x^2 - 4)^2 = 0$

$\Rightarrow x^2 = 4$

$x = \pm 2$

$\Rightarrow x = 0, +2, -2$

Eg. $f(x) = x^3 - 4x^2 - 3x + 12$

$\Rightarrow x^2(x-4) - 3(x-4) = 0$

$\Rightarrow (x^2-3)(x-4) = 0$

$\Rightarrow x = \cancel{4}, \sqrt{3}, -\sqrt{3}$

Eg. $g(x) = (x-1)^2(x+3)$

x intercepts $= 1, -3$

y intercept $= 3$

L9 Multiplicities

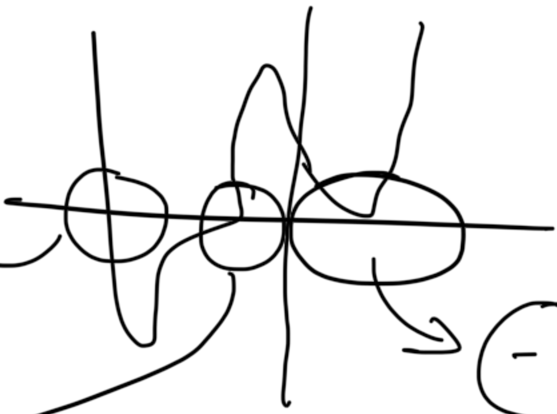
Eg. x intercept of $x^3 + 4x^2 + x - 6$.

find the 4-5 pairs (ordered) of x, y and just plot them.

- Now we need to understand the multiplicities of the zeroes of the polynomial.

Eg. $(x-1)^2 (x+2)^3 (x+4)$

$x = +1, -2, -4$

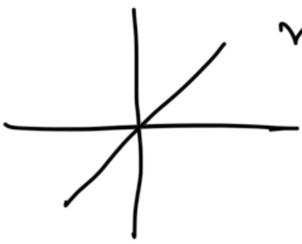


-4 it is linear, therefore, acts like a straight line.

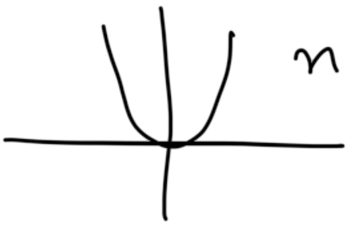
-2, it is a cube, it cross x-axis

-1 it is a sq., it bounces off

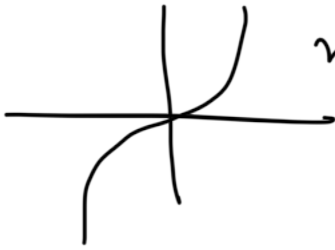
$$\boxed{\begin{array}{l}\text{Even degree} \Rightarrow \text{bounce off} \\ \text{Odd degree} \Rightarrow \text{cross } x\text{-axis}\end{array}}$$



$n = 1$



$n = 2$



$n = 3$

$\Downarrow$

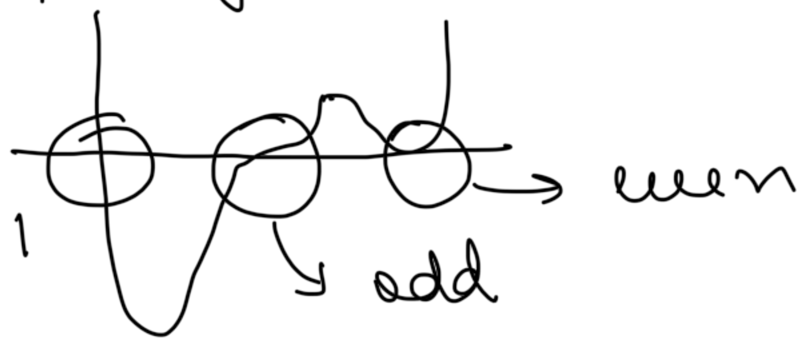
as n increas the graph becomes flatter at the point of bounce

as $n \uparrow$ the graph becomes flatter as it approaches and crosses x-axis.

L10 Behaviour at x-intercepts

- $(x-a)^m \Rightarrow x=a$ is a zero of multiplicity m.
- remember, multiplicities sum can be less than or equal to the degree of polynomial.

Eg. degree 6.



$1+(3 \text{ or } 5)+(2 \text{ or } 4) \leq 6$

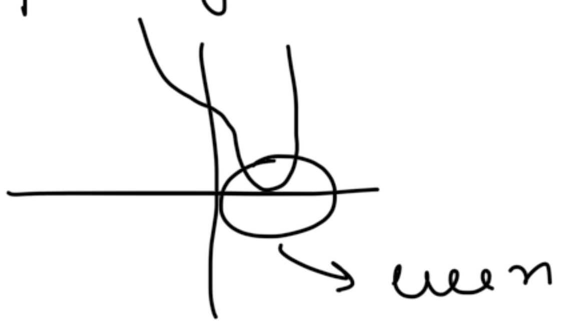
$1+3+2 \leq 6$ ✓

$x = -2, 0, 2$

$ax^3(x+2)(x-2)^2 = 0$ ✓

a → constant by fitting the polynomial.

Eg. degree 4.



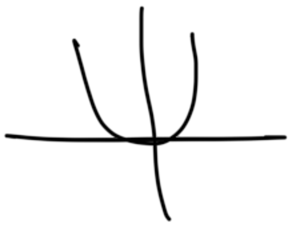
$2 \text{ or } 4 \leq 4$

$(x-2)^2(x^2+1)$ gives the kink.

L11 End Behaviour

	even degree	odd degree
$a_n > 0$	1 $x \to \infty$ $f(x) \to \infty$ $x \to -\infty$ $f(x) \to \infty$	2 $x \to \infty$ $f(x) \to \infty$ $x \to -\infty$ $f(x) \to -\infty$
$a_n < 0$	3 $x \to \infty$ $f(x) \to -\infty$ $x \to -\infty$ $f(x) \to -\infty$	4 $x \to \infty$ $f(x) \to -\infty$ $x \to -\infty$ $f(x) \to \infty$

1



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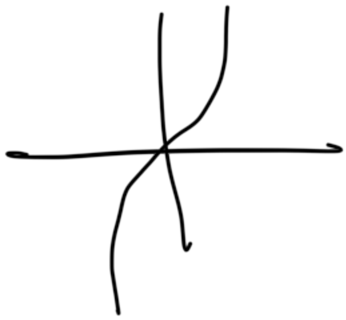

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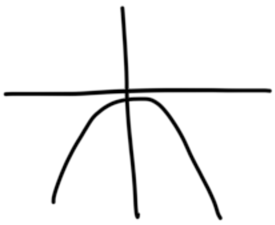

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