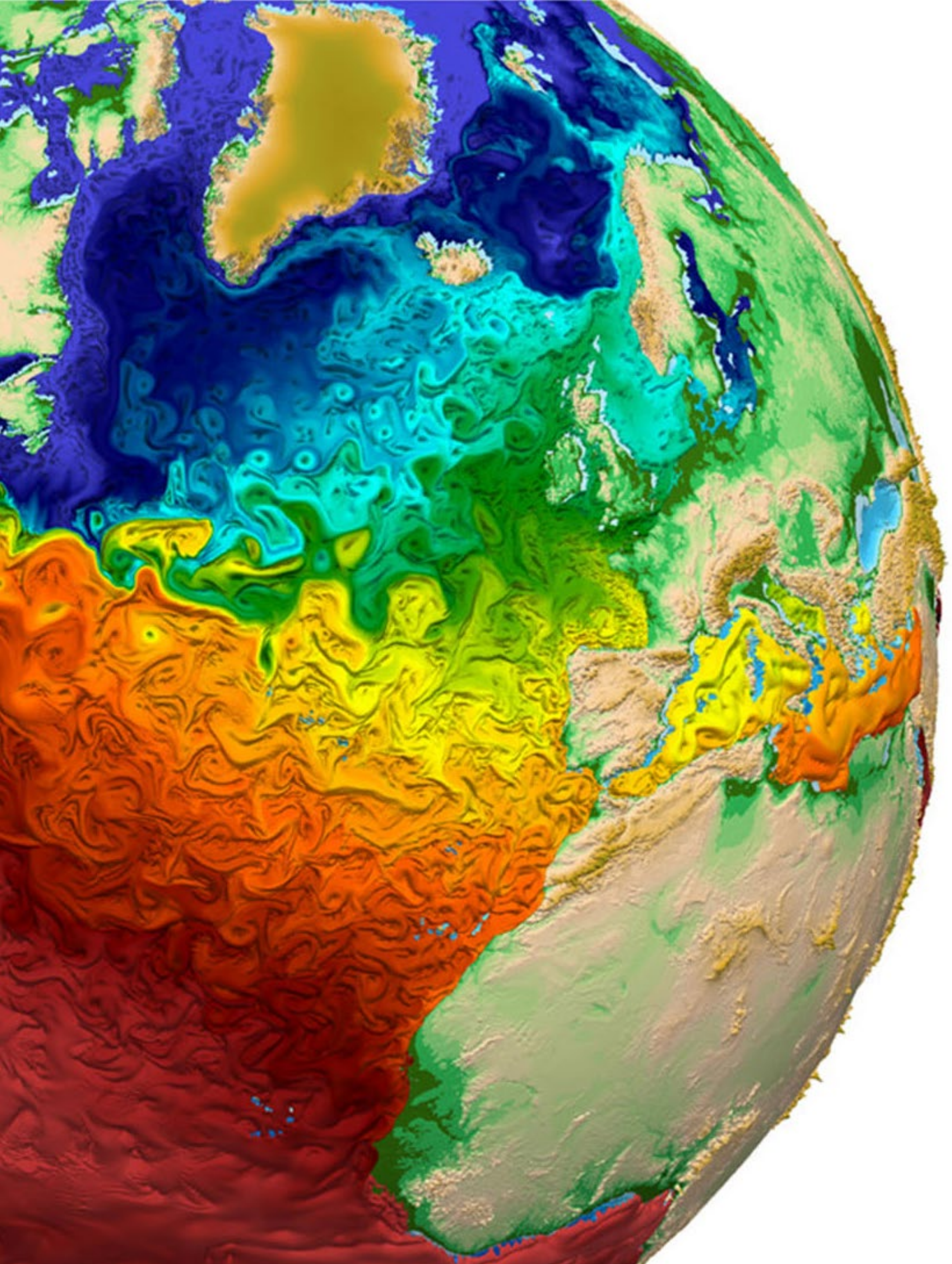


Tensor Visualization

Curvature Tensor

Scientific Visualization

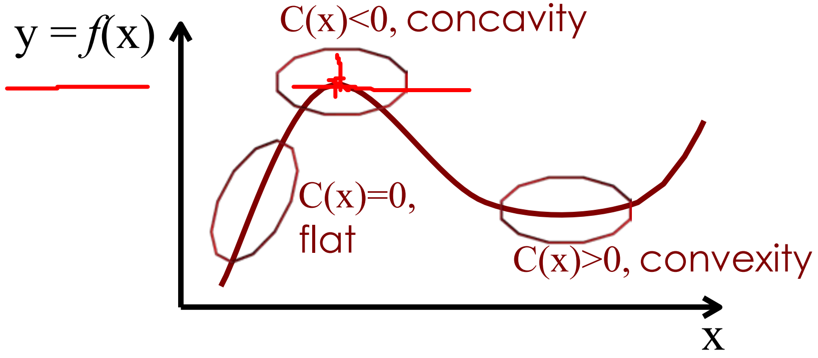
Professor Eric Shaffer

ILLINOIS

Curvature

Curvature in 1D

- take a curve c
- locally, c can be described as a function $y = f(x)$



- curvature of f $C(x) = \dfrac{\partial^2 f}{\partial x^2}$ (2nd derivative of f)

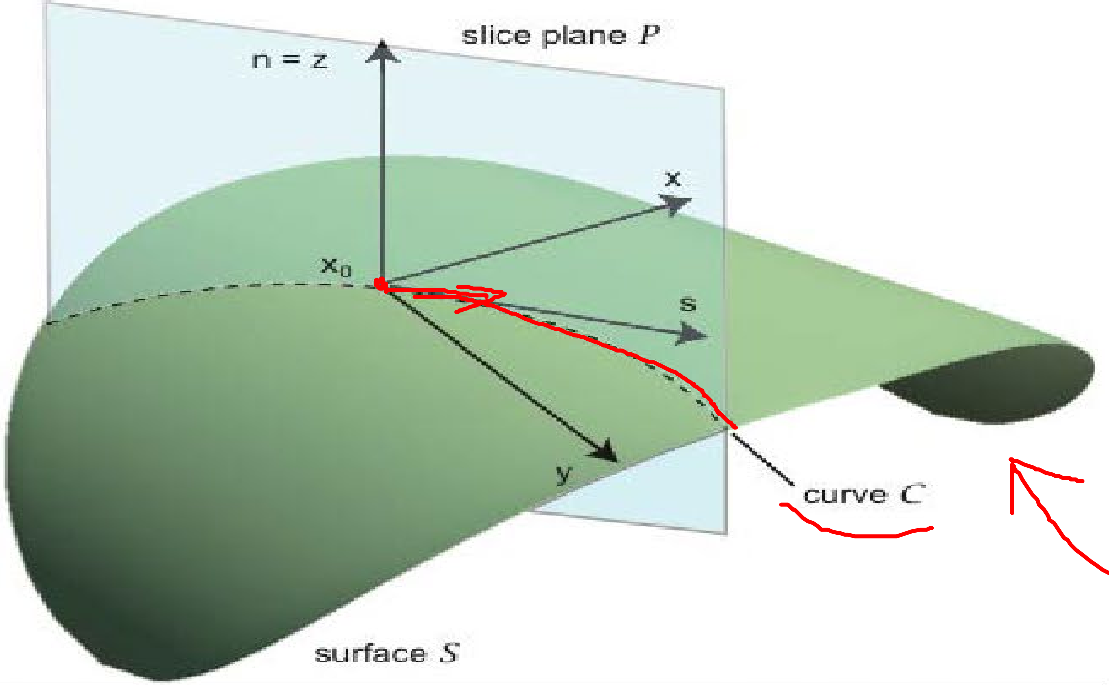
- analytically: $C(x)$ = how quickly the normal $\mathbf{n}_c$ changes around x (why? Because the tangent to c is $\partial f / \partial x$ and its change is $\partial^2 f / \partial x^2$)

Curvature in 2D

- take a surface $S \subset \mathbf{R}^3$
- at each $x_0 \in S$
  - take a coordinate system xyz with x,y tangent to S and z along $\mathbf{n}_S$
  - locally, S can be described as a function $z = f(x,y)$

How to describe 2D curvature?

- 1D analogy: how quickly the normal $\mathbf{n}_S$ changes around x_0
- problem: we have a surface – in which direction to look for change?



We must compute

$$C(x,s) = \frac{\partial^2 f(x)}{\partial s^2}$$

for any direction s

Curvature Tensor

$$C(x,s) = \frac{\partial^2 f(x)}{\partial s^2}$$

- recall our definition of a tensor $\mathbf{T} : \mathbf{R}^3 \times \mathbf{R}^3 \rightarrow \mathbf{R}$? The above is precisely that

Also note that

$$\frac{\partial^2 f}{\partial s^2}(x_0) = \mathbf{s}^T H \mathbf{s}$$

where H is the so-called Hessian of f

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

In other words, if we have H, we can compute the curvature tensor

- at any point x_0
- in any direction s

The Curvature Tensor

However, there's a problem with the previous definition

- we need to construct local coordinate systems at every point on S
- not obvious how to do that….

General solution:

Describe S as an implicit function (i.e. the zero-level isosurface of a function)

$$S = \{x \in \mathbf{R}^3 \mid f(x) = 0\} \text{ for a given } f : \mathbf{R}^3 \to \mathbf{R}$$

Then, we still have

$$\frac{\partial^2 f}{\partial s^2}(x_0) = \mathbf{s}^T H \mathbf{s}$$ where H is the 3x3 Hessian matrix $$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

Conclusion

- A curvature tensor is fully described by a 3x3 matrix of 2nd order derivatives

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