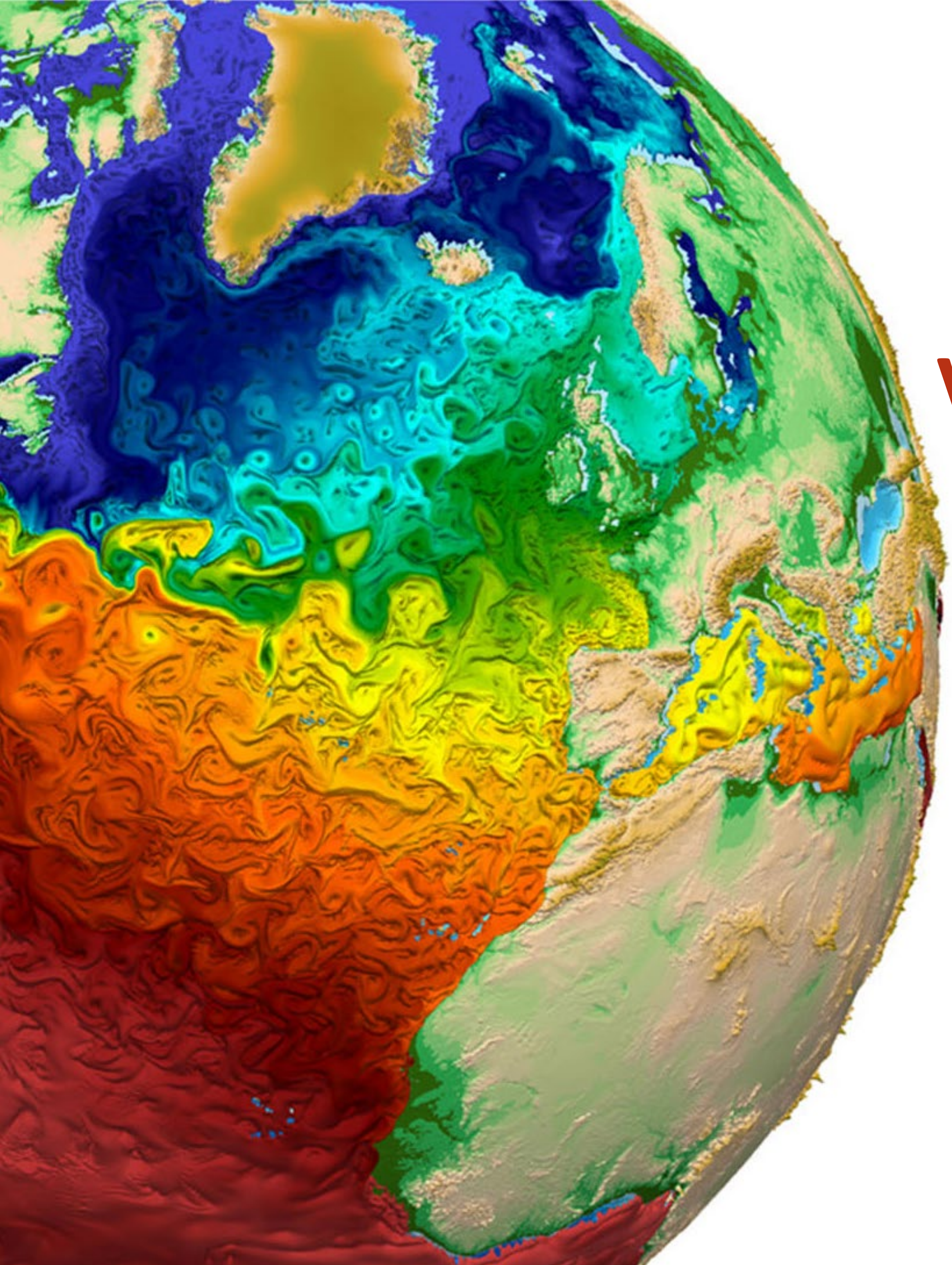

Tensor Visualization

Visualizing the Diffusion Tensor

Scientific Visualization

Professor Eric Shaffer

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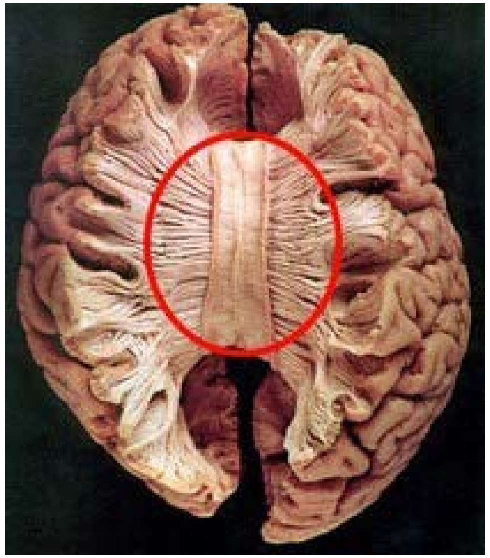
The Diffusion Tensor

- consider an anisotropic material (e.g. tissue in the human brain)
- water diffuses in this tissue
  - strongly along neural fibers
  - weakly across fibers

Actual image of a dissected human brain



speed of water motion in tissue

Diffusion tensor

$$D(x,s) = \frac{\partial^2 f(x)}{\partial s^2}$$

diffusivity at a point x in a direction s

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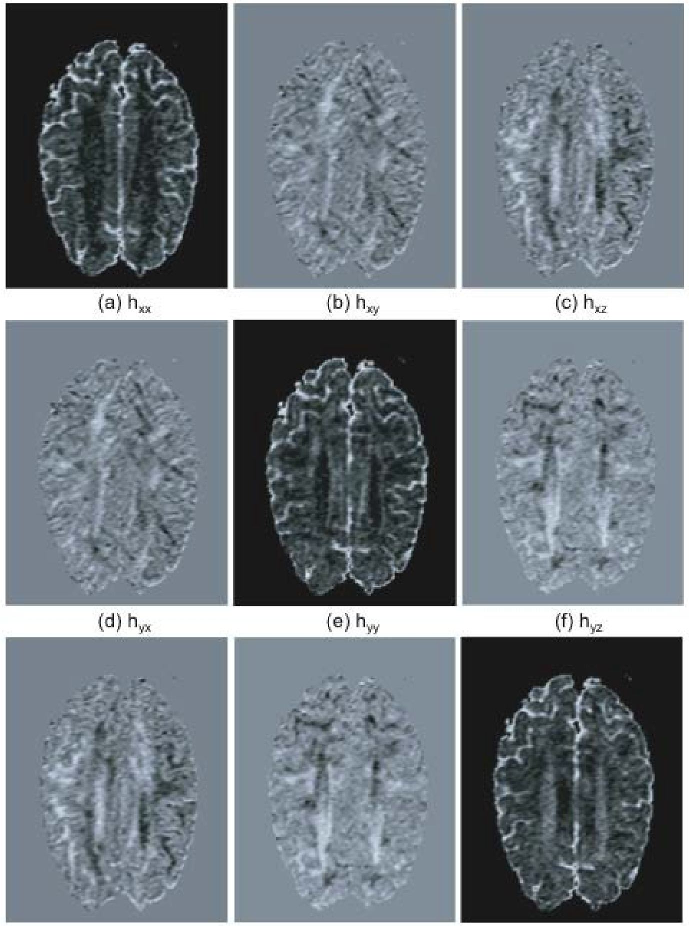
Diffusion tensor: measured by a technique called **DT-MRI** (diffusion tensor magnetic resonance imaging)

The Diffusion Tensor



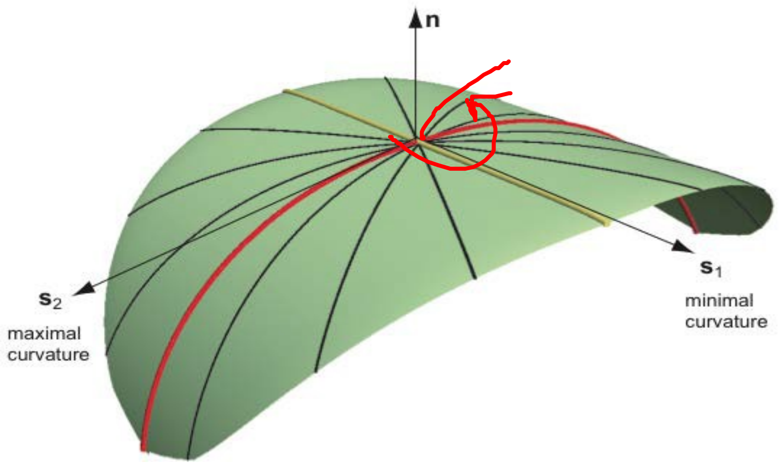
First visualization try

- compute hessian $H = \{h_{ij}\}$ in $\mathbf{R}^3$
- select some slice of interest
- visualize all components h_{ij} using e.g. color mapping

Simple, but not very useful

- we get a lot of images (9)…
- we see the tensor is symmetric…
- …but we don't really care about diffusion along x, y, z axes!

Principal Component Analysis



$$C(x,s) = \frac{\partial^2 f(x)}{\partial s^2}$$

- fix some point x_0 on the surface
- compute $C(x_0,s)$ for all possible tangent directions s at x_0
- denote α = angle of s with local coordinate axis x_0

So we have

$$\frac{\partial^2 f}{\partial s^2} = s^T H s = h_{11} \cos^2 \alpha + (h_{12} + h_{21}) \sin \alpha \cos \alpha + h_{22} \cos^2 \alpha$$

Now, let's look for the values of α for which this function is extremal!

Principal Component Analysis

Our curvature (as function of α) is extremal when $\frac{\partial C}{\partial \alpha} = 0$

This is equivalent to a system of equations

$$\begin{cases} h_{11}\cos\alpha + h_{12}\sin\alpha & = \lambda\cos\alpha \\ h_{21}\cos\alpha + h_{22}\sin\alpha & = \lambda\sin\alpha, \end{cases}$$

which in matrix form is $H\mathbf{s} = \lambda\mathbf{s}$ or $(H - \lambda I)\mathbf{s} = 0$

Since we're looking for the non-trivial solution $\mathbf{s} \neq \mathbf{0}$ this means

$$\det(H - \lambda I) = (h_{11} - \lambda)(h_{22} - \lambda) - h_{12}h_{21} = 0$$

Solving the above 2nd order equation in λ yields

- two real values λ_1, λ_2 eigenvalues (principal values) of tensor

Plugging λ_1, λ_2 into $H\mathbf{s} = \lambda\mathbf{s}$ yields

- two direction vectors $\mathbf{s}_1, \mathbf{s}_2$ eigenvectors (principal directions) of tensor

Summarizing

- Given a 2x2 tensor, we can compute its principal directions and values
- directions: those in which tensor has extremal (minimal, maximal) values
- can be shown that eigendirections are orthogonal to each other
- eigenvalues: the actual minimal and maximal values

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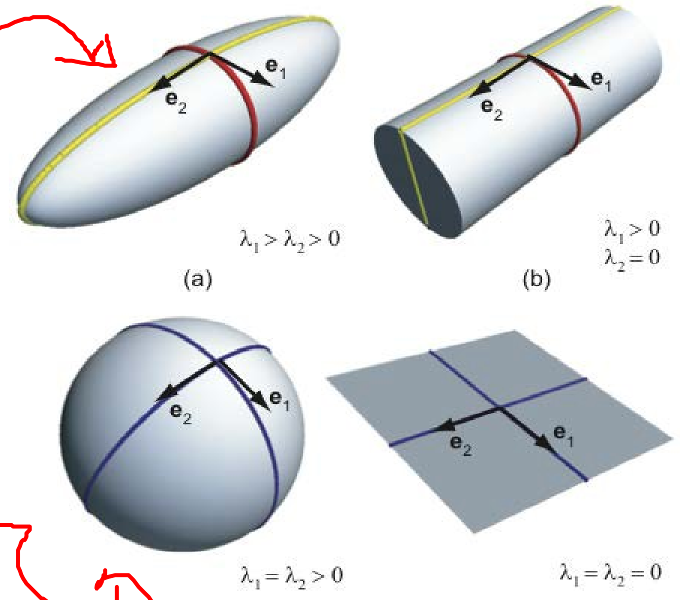
Principal Component Analysis

How about a 3x3 tensor, like the diffusion tensor?

•3 eigenvalues, 3 eigenvectors (computed similarly, see Sec. 7.1)
Say we order eigenvalues (and their vectors) as $\lambda_1 > \lambda_2 > \lambda_3$

$\lambda_1, \mathbf{s}_1$ **major** eigenvector i.e. direction of strongest diffusion
$\lambda_2, \mathbf{s}_2$ **medium** eigenvector (no particular meaning)
$\lambda_3, \mathbf{s}_3$ **minor** eigenvector i.e. direction of weakest diffusion

What if two or more eigenvalues are equal (so we cannot fully order them all)?



a,b) all values ordered: unique eigendirections

c,d) equal eigenvalues: eigendirections not determined (any two orthogonal vectors tangent to surface are valid eigendirections)

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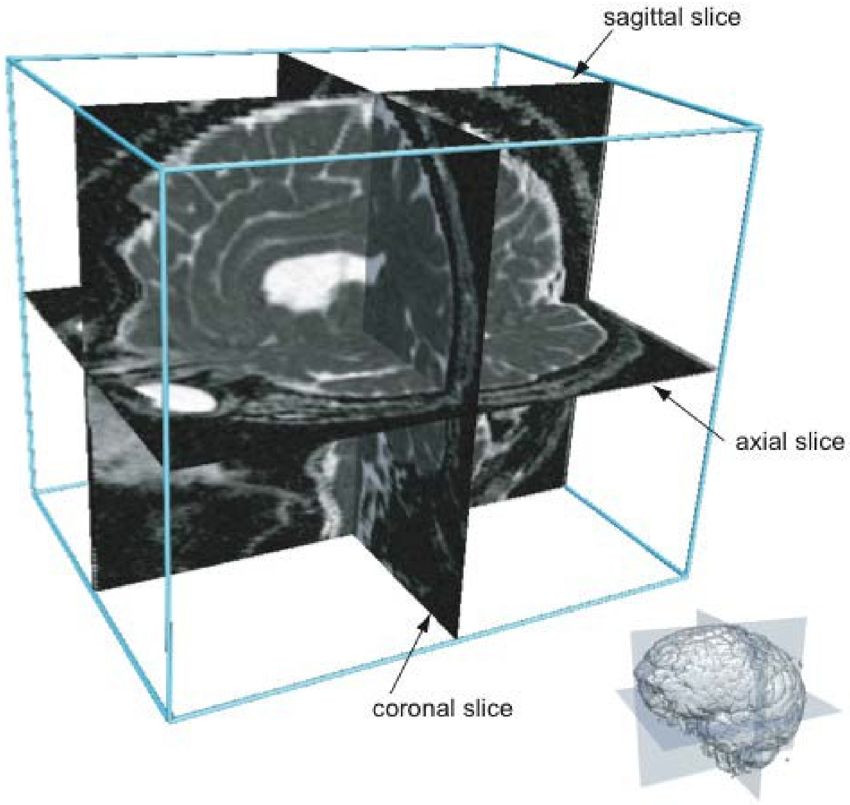
Principal Component Analysis

How to use PCA for visualization?

Visualize mean diffusivity $\mu = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)$



white: strong mean diffusivity
black: weak mean diffusivity

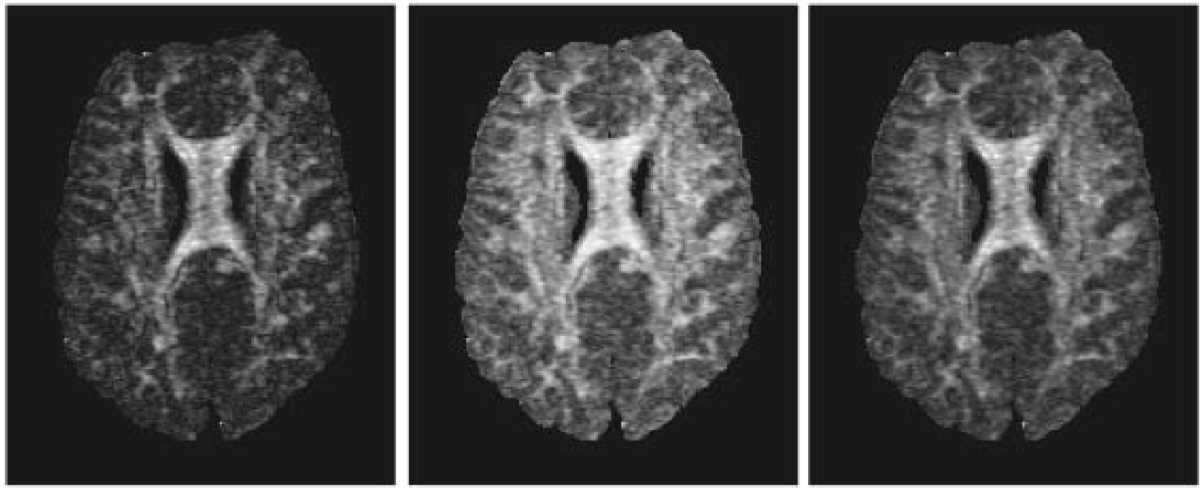
Principal Component Analysis

Linear diffusivity $$c_l = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}$$

Fractional anisotropy $$FA = \sqrt{\frac{3}{2}} \frac{\sqrt{\sum_{i=1}^{3} (\lambda_i - \mu)^2}}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$$ where $$\mu = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)$$

Relative anisotropy $$RA = \sqrt{\frac{3}{2}} \frac{\sqrt{\sum_{i=1}^{3} (\lambda_i - \mu)^2}}{\lambda_1 + \lambda_2 + \lambda_3}$$

All above measures estimate how much 'fiber-like' is the current point



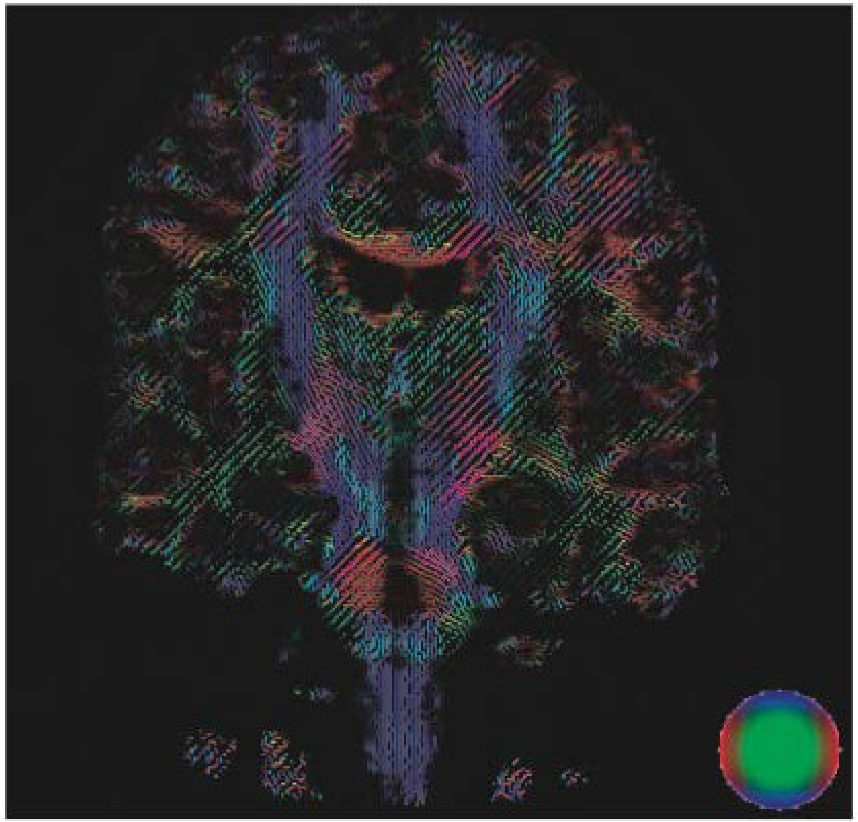
(a) c_l linear estimator (b) fractional anisotropy (c) relative anisotropy

white: strong fibers

Principal Component Analysis

Exploit the directional information in the eigenvectors

- major eigenvector $\mathbf{e}_1$: along the **strongest** diffusion direction
- for DTI tensors, it thus indicates fiber directions



Directional color coding

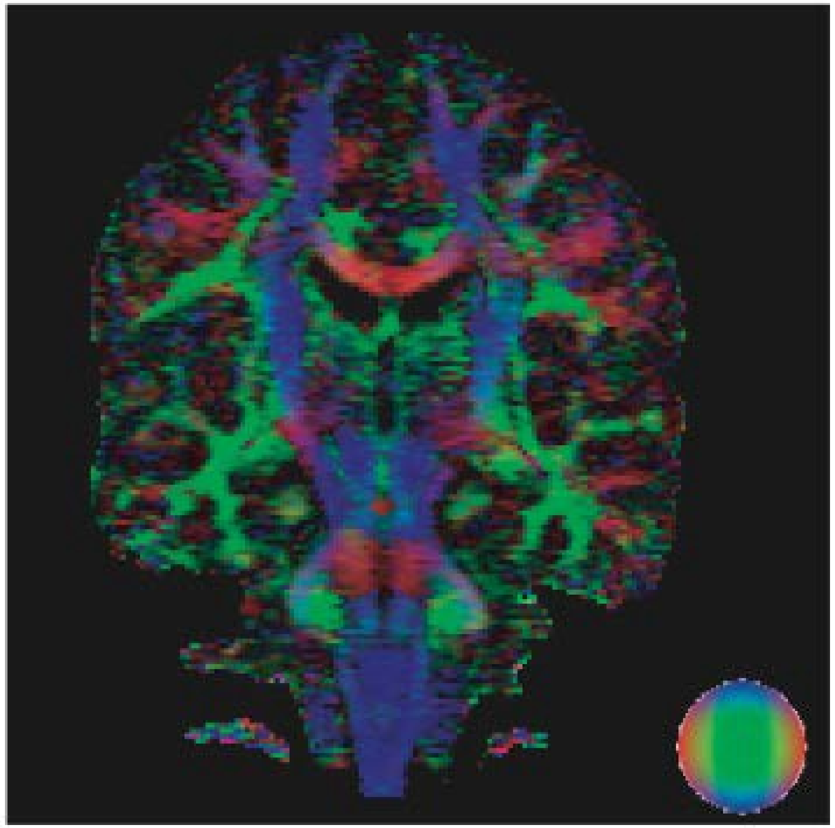
- like for vectors (see Module 4)
- use simple colormap

$$\begin{aligned} R &= |\mathbf{e}_1 \cdot \mathbf{x}|, \\ G &= |\mathbf{e}_1 \cdot \mathbf{y}|, \\ B &= |\mathbf{e}_1 \cdot \mathbf{z}|. \end{aligned}$$

- use vector glyphs / hedgehogs
- seed only points where c_1, FA or RA are large enough (other points don't cover fibers)

- OK, but takes training to grasp

Vector PCA



- Directional color coding
  - like before, but simply color points by direction
  - no glyphs drawn
  - no occlusion/clutter
  - direction coded only by color – less intuitive images

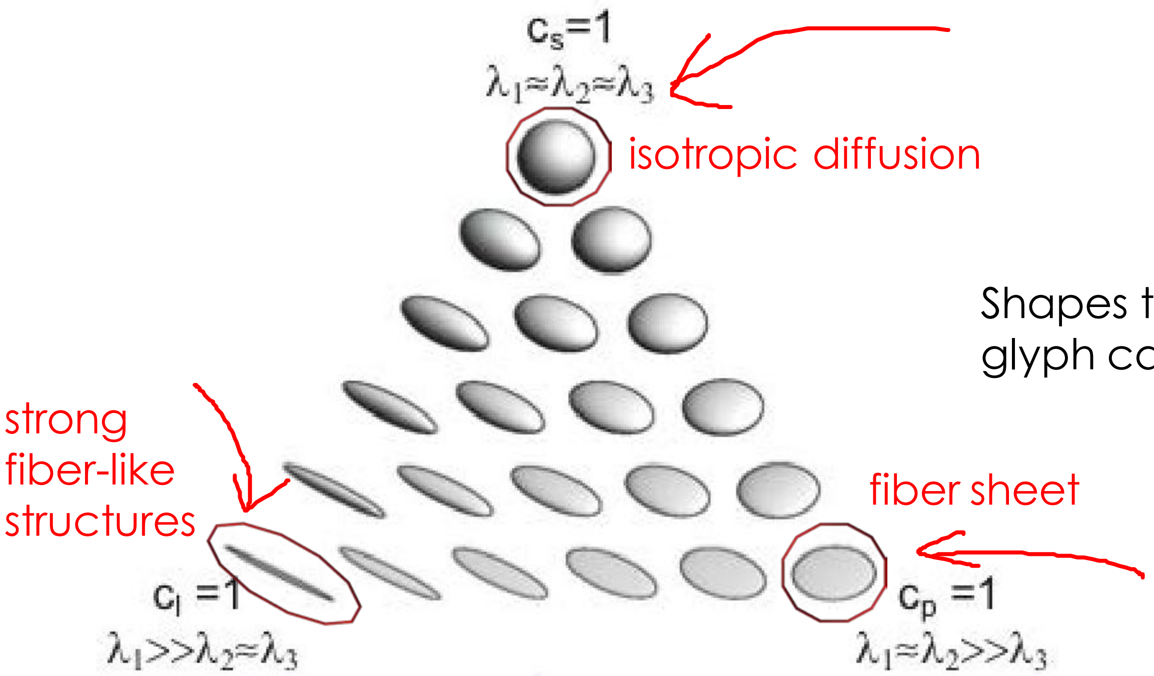
Tensor Glyphs

So far, we only visualized the major eigenvector $\mathbf{e}_1$

- so we reduced a tensor field to a vector field
- we **threw away** existing information (medium+minor eigenvectors $\mathbf{e}_2,\mathbf{e}_3$)

Ellipsoid glyph: Use all eigenvalues + eigenvectors

- orient glyph along eigensystem $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$
- scale it by eigenvalues $(\lambda_1,\lambda_2,\lambda_3)$



Shapes that an ellipsoid glyph can assume

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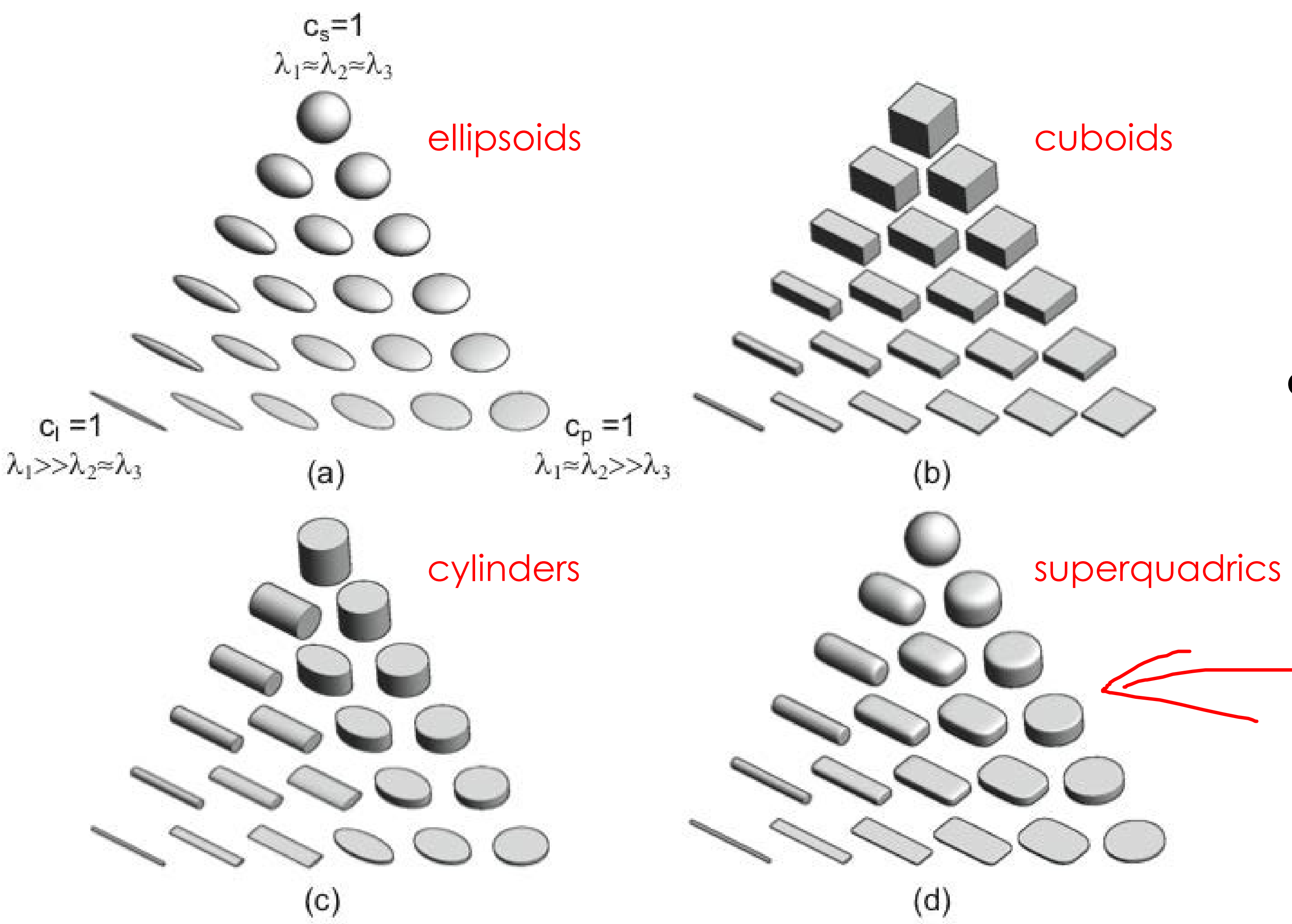
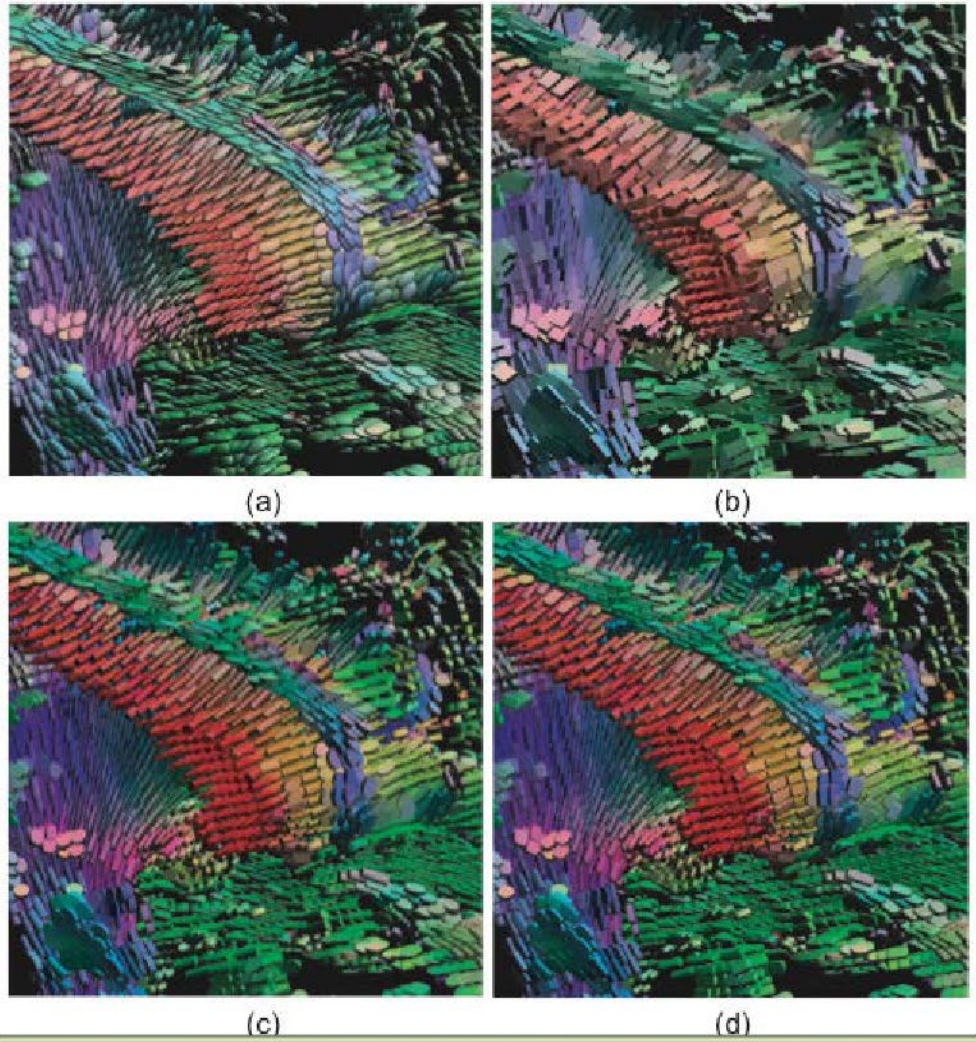
Tensor Glyphs



Can use other glyph shapes besides ellipsoids

Tensor Glyphs

Zoom-in on brain DT-MRI dataset



a) ellipsoids
b) cuboids
c) cylinders
d) superquadrics

Superquadrics look arguably most ‘natural’