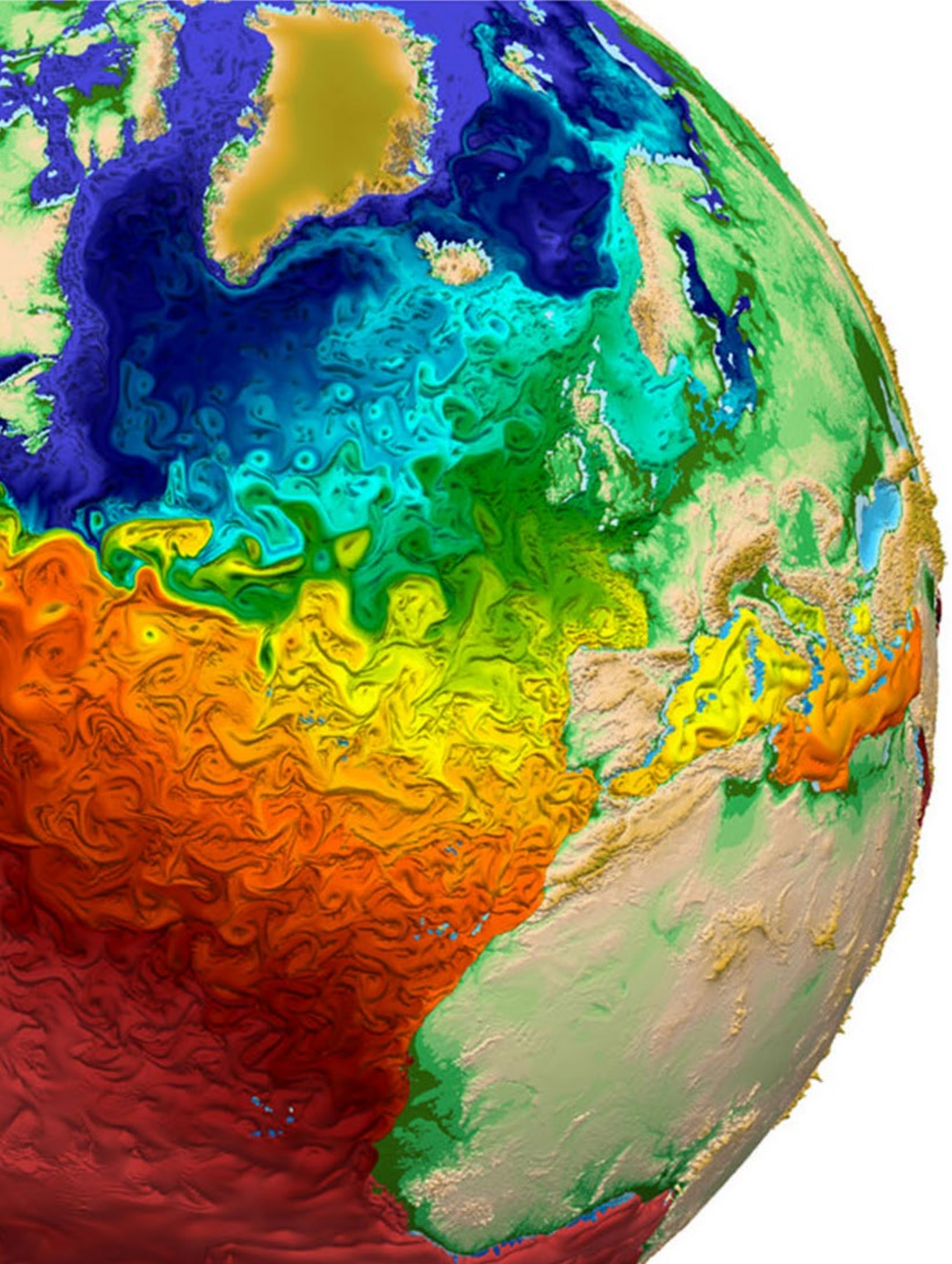


Data Science for People in a Hurry

Scattered Data Interpolation

Scientific Visualization

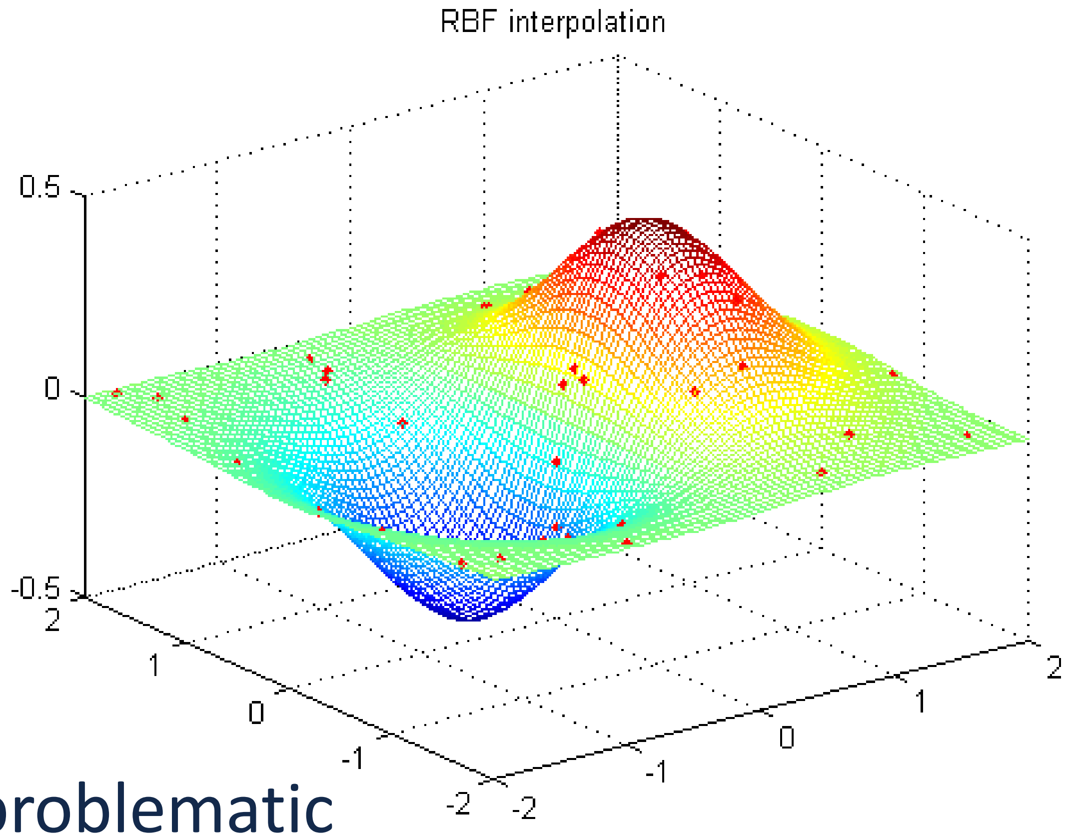
Professor Eric Shaffer

ILLINOIS

Scattered Data

Scattered data is irregularly sampled

No spatial structure



Using bilinear or trilinear interpolation problematic

Shepard's Method

- Simplest scattered data interpolation method

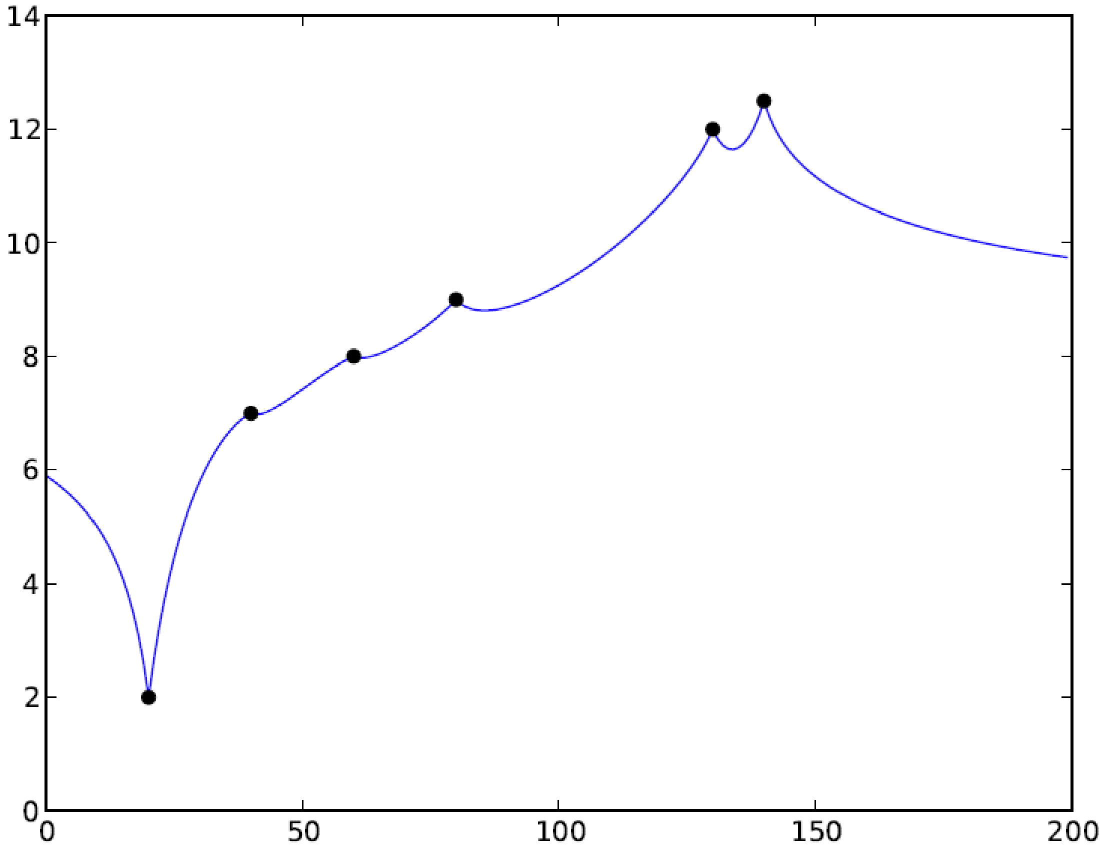
$$\tilde{f}(\mathbf{x}) = \sum_{k}^{N} \frac{w_k(\mathbf{x})}{\sum_j w_j(\mathbf{x})} f(\mathbf{x}_k)$$

$$w_j(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_i\|^{-p}$$

- x_k are the locations in space with known function values
- x is the query point
- w is a weight function inversely dependent on distance to x
- p is a positive real number
  - larger p is, the greater influence points close to x will have

Shepard's Method Issues

For p ≤ 1 the interpolant has peaks…not ideal for smooth interpolation

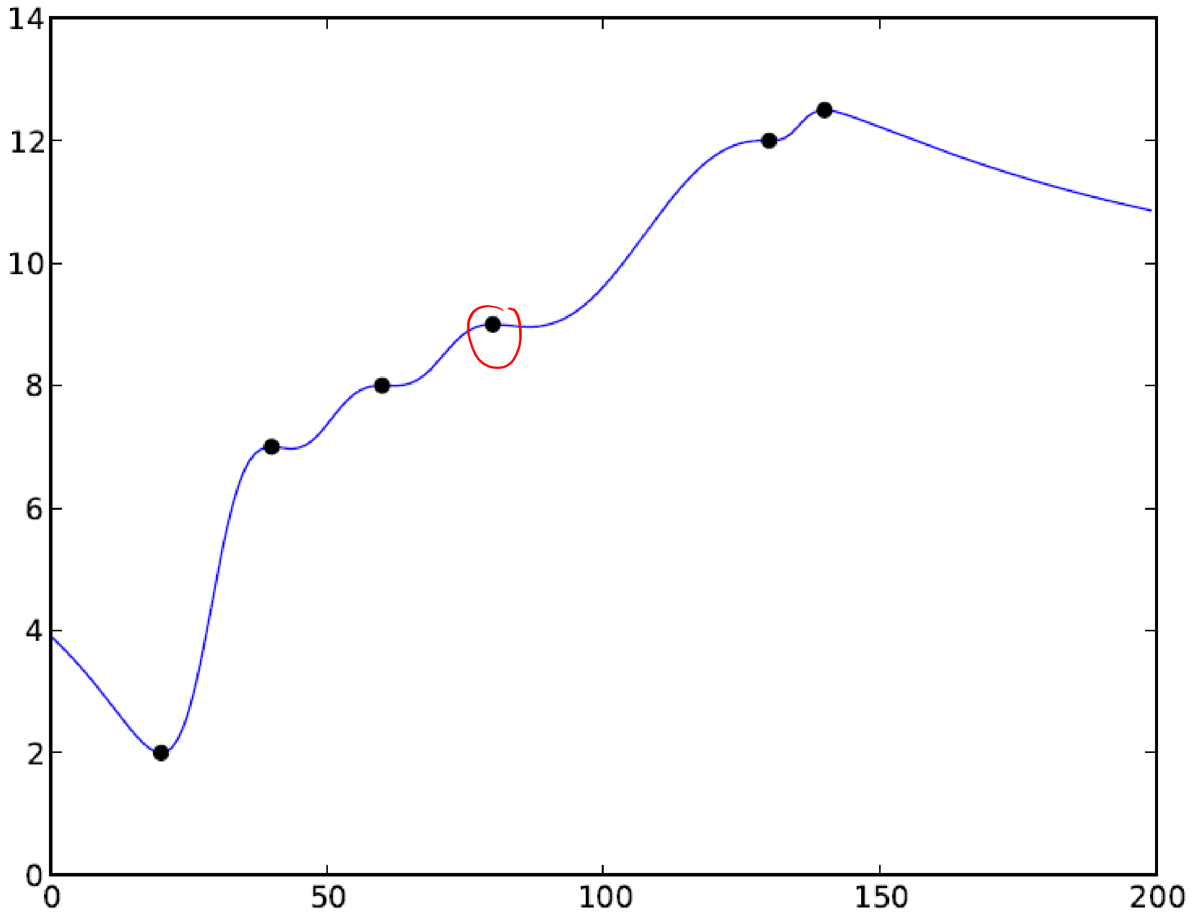


Example with p=1

Shepard's Method Issues

For $p > 1$ the interpolant is smooth…

But, first derivative is equal to 0 at data points…again not usually a desired behavior



Example with p=2

Modified Shepard's Method

One other issue is lack of scalability…ALL points in a data set used at each query x

Modified Shepard's Method uses only points within a radius of r around x

For those points, the weight function is

$$w_j(\mathbf{x}) = \left[\frac{r - d(\mathbf{x}, \mathbf{x}_i)}{r d(\mathbf{x}, \mathbf{x}_i)} \right]^2$$

Requires use of a spatial data structure such as kd-tree or quadtree/octree

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Radial Basis Functions

- Any function dependent on distance from a center is *radial*
- We can compute an interpolating function as a weighted sum…

$$\boxed{\phi(x,p) = \phi(\|x-p\|)}$$

$$\boxed{f(x) \approx \sum_{i=1}^{N} w_i \phi(x,p_i)}$$

- Some popular kernel functions

$r = \|x - p\|$

$$\phi(r) = e^{-\lambda r^2}$$ Gaussian

$$\phi(r) = \frac{1}{1+r^2}$$ Inverse distance

RBFs Computing Weights

- Need to compute weights
- Constraint is function interpolates data points
- For scalability use a kernel function with width
  - Can lead to non-smooth interpolant

$$f(p_j) = \sum_{i=1}^{N} w_i \phi(p_j, p_i)$$

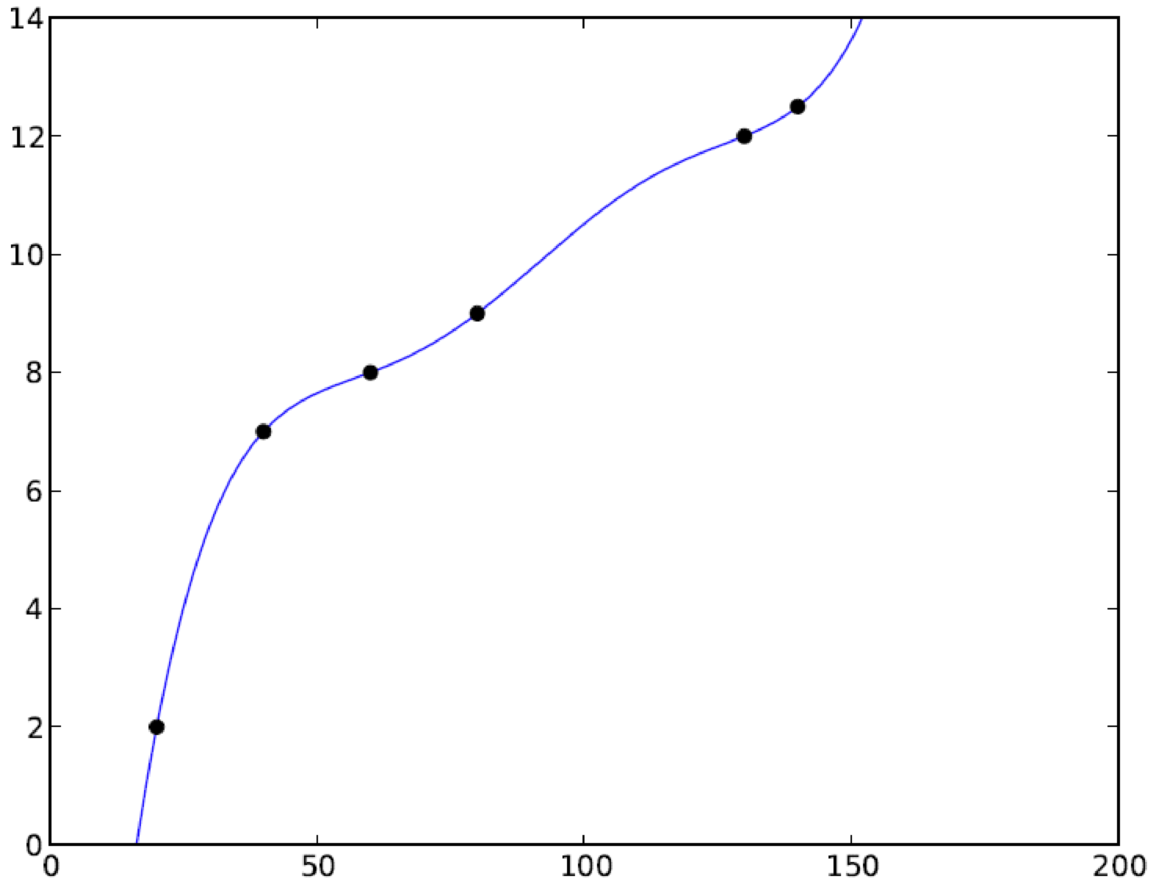
$$Aw = p$$

$$A = \begin{bmatrix} \phi(p_1, p_1) & \dots & \phi(p_1, p_N) \\ \dots & \dots & \dots \\ \phi(p_N, p_1) & \dots & \phi(p_N, p_N) \end{bmatrix}$$

$$w = \begin{bmatrix} w_1 \\ \dots \\ w_N \end{bmatrix}$$

$$p = \begin{bmatrix} f(p_1) \\ \dots \\ f(p_N) \end{bmatrix}$$

RBF Interpolation Example



Example radial basis interpolation with a Gaussian kernel

Kernel Function Choice

Positive definite functions will result in non-singular A for any data points

One definition of positive definite function: matrix A has all positive eigenvalues

Some useful kernels are not positive definite

Other Interpolation Method Options

- Moving Least Squares
- Natural Neighbor Interpolation
- …many more