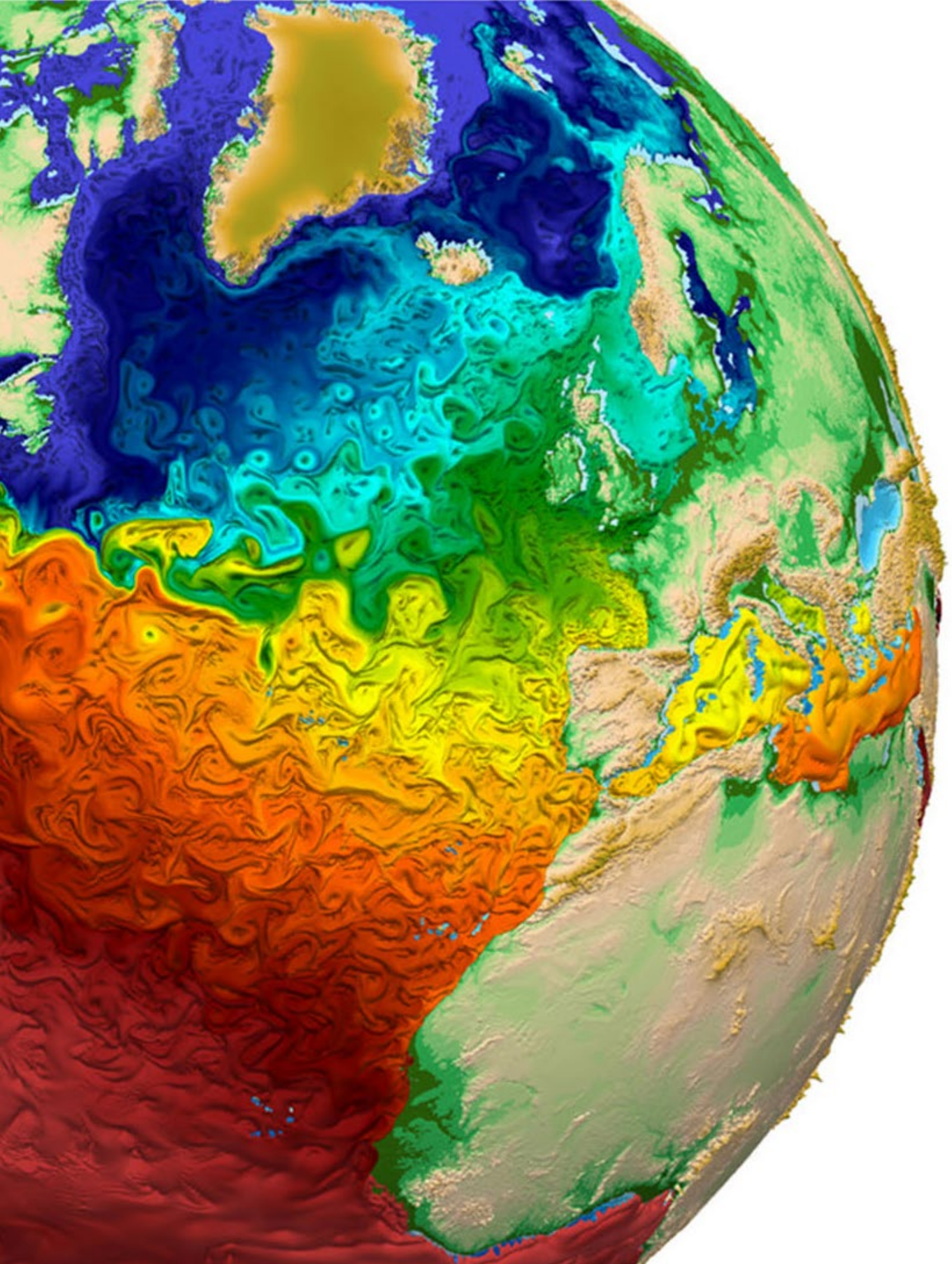

Data Science for People in a Hurry

Barycentric Coordinates and Interpolation

Scientific Visualization

Professor Eric Shaffer

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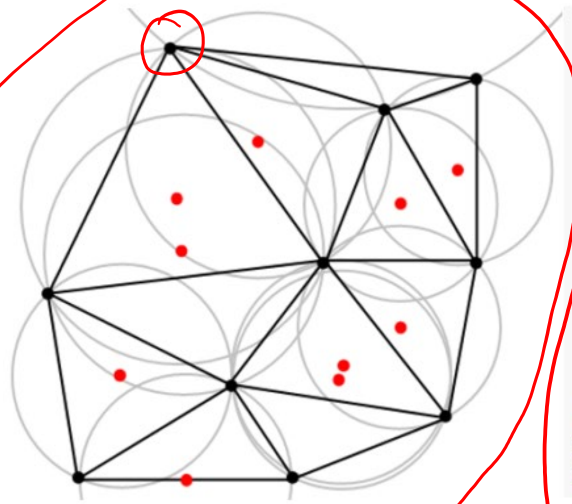
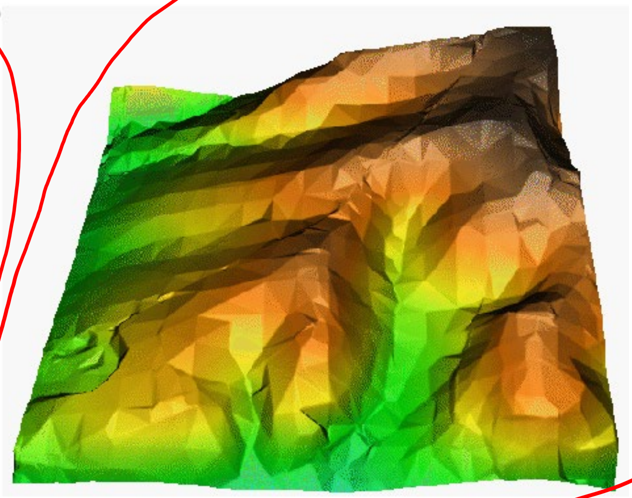
Barycentric Interpolation

How can we linearly interpolate a function over triangles?



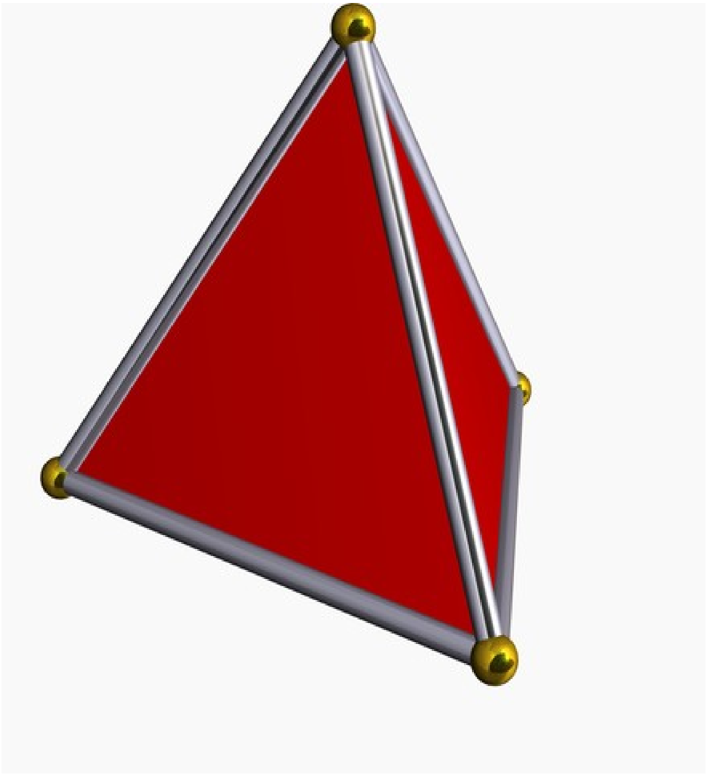


Can be useful:

- Scattered data can be triangulated and then the function interpolated
- Many datasets involve triangulated domains already

Barycentric Interpolation

- Barycentric coordinates apply to more than just triangles
- Used on any simplex
- A simplex is a convex hull of k+1 points in a k-dimensional space
  - Simplest convex "polygon" in a k-dimensional space
  - A 3-simplex is a triangle
- Barycentric coordinates provide a way to interpolate over simplices



Barycentric Coordinates for Triangles

Describe location of a point in relation to the vertices of a given triangle

Express point p in barycentric coordinates $p=(\lambda_1, \lambda_2, \lambda_3)$

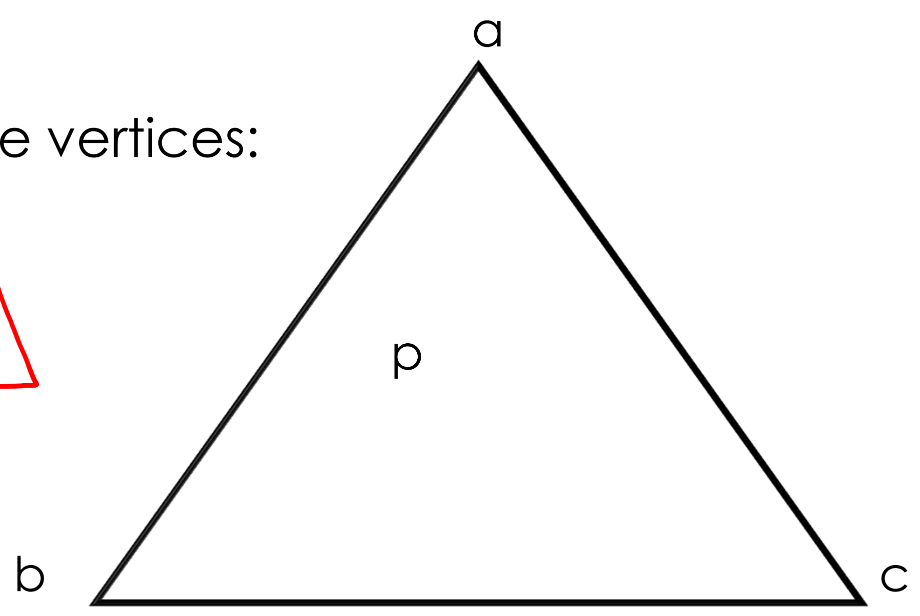
The following must be true

$p=\lambda_1 a + \lambda_2 b + \lambda_3 c$

$\lambda_1+\lambda_2+\lambda_3= 1$

To interpolate a function sampled at the vertices:

$$f(p)=\lambda_1 f(a) + \lambda_2 f(b) + \lambda_3 f(c)$$



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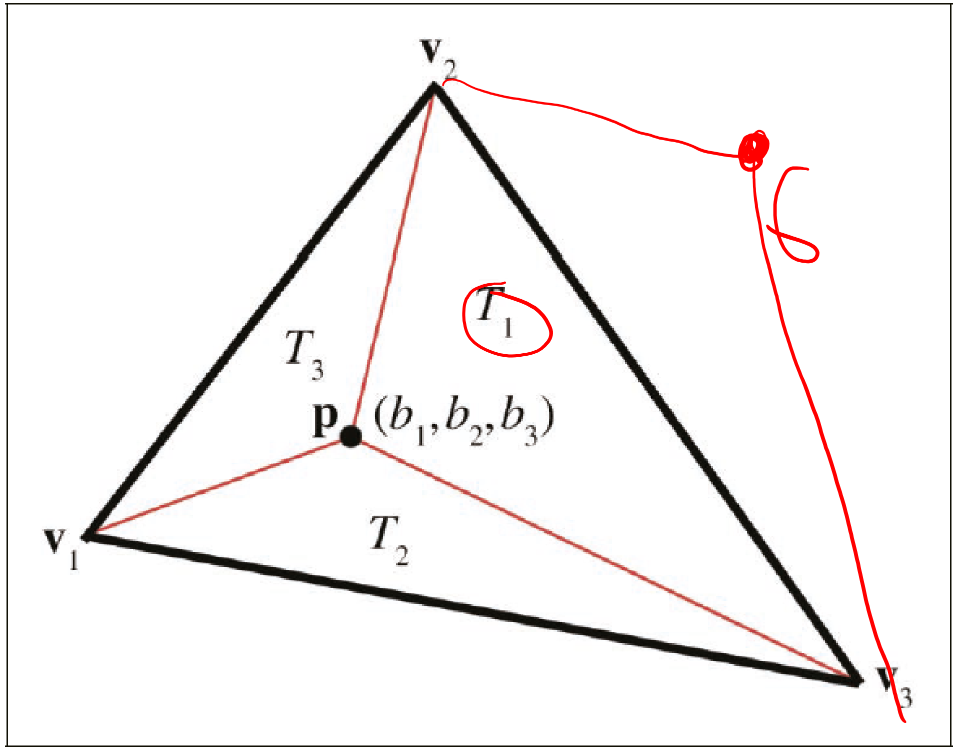
Computing Barycentric Coordinates for Triangles

Coordinates are the signed area of the opposite subtriangle divided by area of the triangle



$$
\begin{aligned}
b_1x_1 + b_2x_2 + b_3x_3 &= p_x, \\
b_1y_1 + b_2y_2 + b_3y_3 &= p_y, \\
b_1 + b_2 + b_3 &= 1.
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \frac{(p_y - y_3)(x_2 - x_3) + (y_2 - y_3)(x_3 - p_x)}{(y_1 - y_3)(x_2 - x_3) + (y_2 - y_3)(x_3 - x_1)}, \\
b_2 &= \frac{(p_y - y_1)(x_3 - x_1) + (y_3 - y_1)(x_1 - p_x)}{(y_1 - y_3)(x_2 - x_3) + (y_2 - y_3)(x_3 - x_1)}, \\
b_3 &= \frac{(p_y - y_2)(x_1 - x_2) + (y_1 - y_2)(x_2 - p_x)}{(y_1 - y_3)(x_2 - x_3) + (y_2 - y_3)(x_3 - x_1)}.
\end{aligned}
$$

$$b_1 = A(T_1)/A(T), \qquad b_2 = A(T_2)/A(T), \qquad b_3 = A(T_3)/A(T)$$

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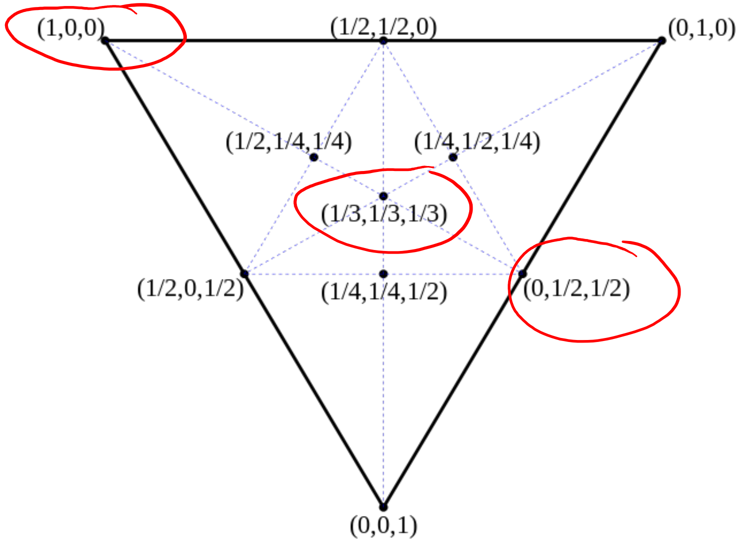
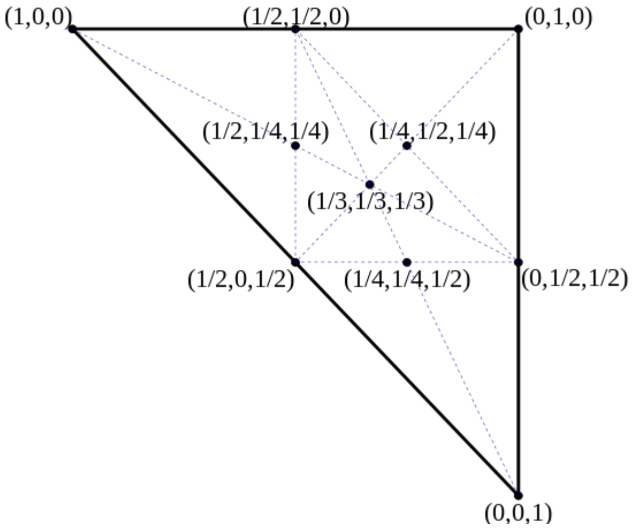
Some Important Points





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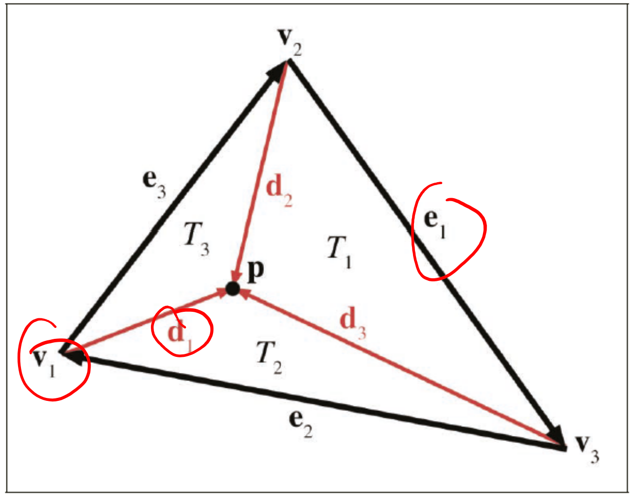
Computing Coordinates for 3D Triangles



$$\mathbf{e}_1 = \mathbf{v}_3 - \mathbf{v}_2, \qquad \mathbf{e}_2 = \mathbf{v}_1 - \mathbf{v}_3, \qquad \mathbf{e}_3 = \mathbf{v}_2 - \mathbf{v}_1,$$

$$\mathbf{d}_1 = \mathbf{p} - \mathbf{v}_1, \qquad \mathbf{d}_2 = \mathbf{p} - \mathbf{v}_2, \qquad \mathbf{d}_3 = \mathbf{p} - \mathbf{v}_3.$$

$$\hat{\mathbf{n}} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\|\mathbf{e}_1 \times \mathbf{e}_2\|}.$$

$$A(T) = ((\mathbf{e}_1 \times \mathbf{e}_2) \cdot \hat{\mathbf{n}})/2,$$
$$A(T_1) = ((\mathbf{e}_1 \times \mathbf{d}_3) \cdot \hat{\mathbf{n}})/2,$$
$$A(T_2) = ((\mathbf{e}_2 \times \mathbf{d}_1) \cdot \hat{\mathbf{n}})/2,$$
$$A(T_3) = ((\mathbf{e}_3 \times \mathbf{d}_2) \cdot \hat{\mathbf{n}})/2.$$

$$b_1 = A(T_1)/A(T) = \frac{(\mathbf{e}_1 \times \mathbf{d}_3) \cdot \hat{\mathbf{n}}}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \hat{\mathbf{n}}},$$

$$b_2 = A(T_2)/A(T) = \frac{(\mathbf{e}_2 \times \mathbf{d}_1) \cdot \hat{\mathbf{n}}}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \hat{\mathbf{n}}},$$

$$b_3 = A(T_3)/A(T) = \frac{(\mathbf{e}_3 \times \mathbf{d}_2) \cdot \hat{\mathbf{n}}}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \hat{\mathbf{n}}}.$$

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Barycentric Coordinates for Tetrahedra

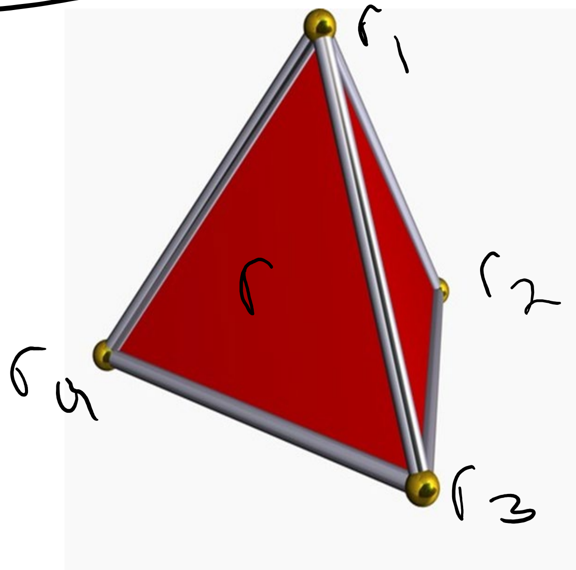
We have 4 vertices of a tetrahedron r_1, r_2, r_3, and r_4

To find the coordinates of a point r we can compute

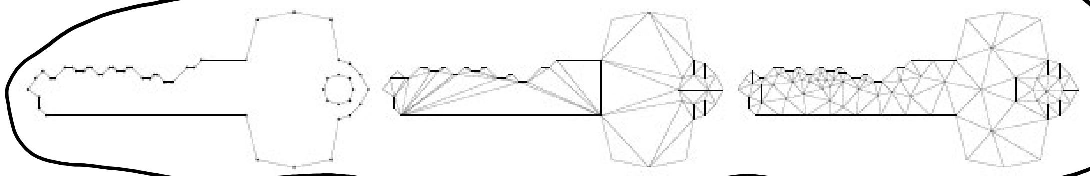
$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \mathbf{T}^{-1}(\mathbf{r} - \mathbf{r}_4)$$

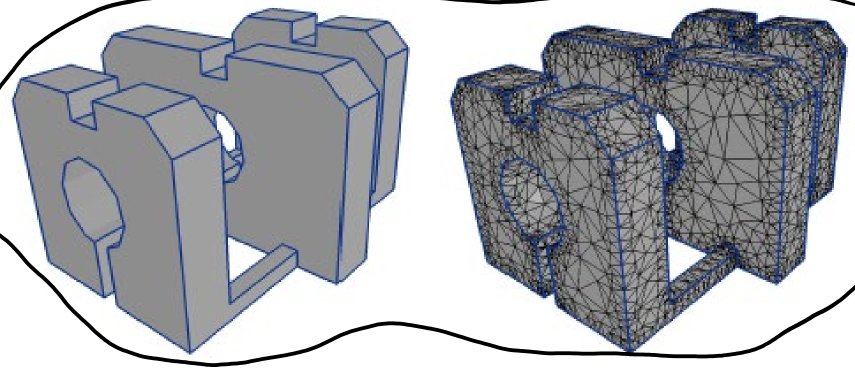
$$\mathbf{T} = \begin{pmatrix} x_1 - x_4 & x_2 - x_4 & x_3 - x_4 \\ y_1 - y_4 & y_2 - y_4 & y_3 - y_4 \\ z_1 - z_4 & z_2 - z_4 & z_3 - z_4 \end{pmatrix}$$

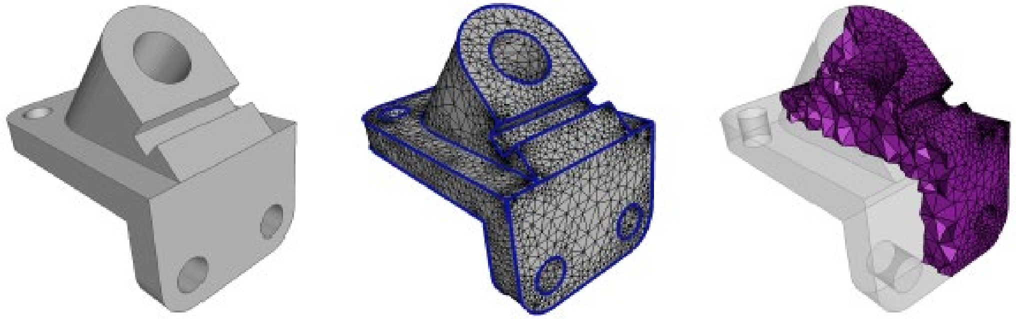
and $\lambda_4 = 1 - \lambda_1 - \lambda_2 - \lambda_3$



Triangle and Tetrahedral Mesh Generation







Lots of packages…
Ansys
Gridgen
Maya
Autocad
CGAL
Many more…