# Logistic Regression

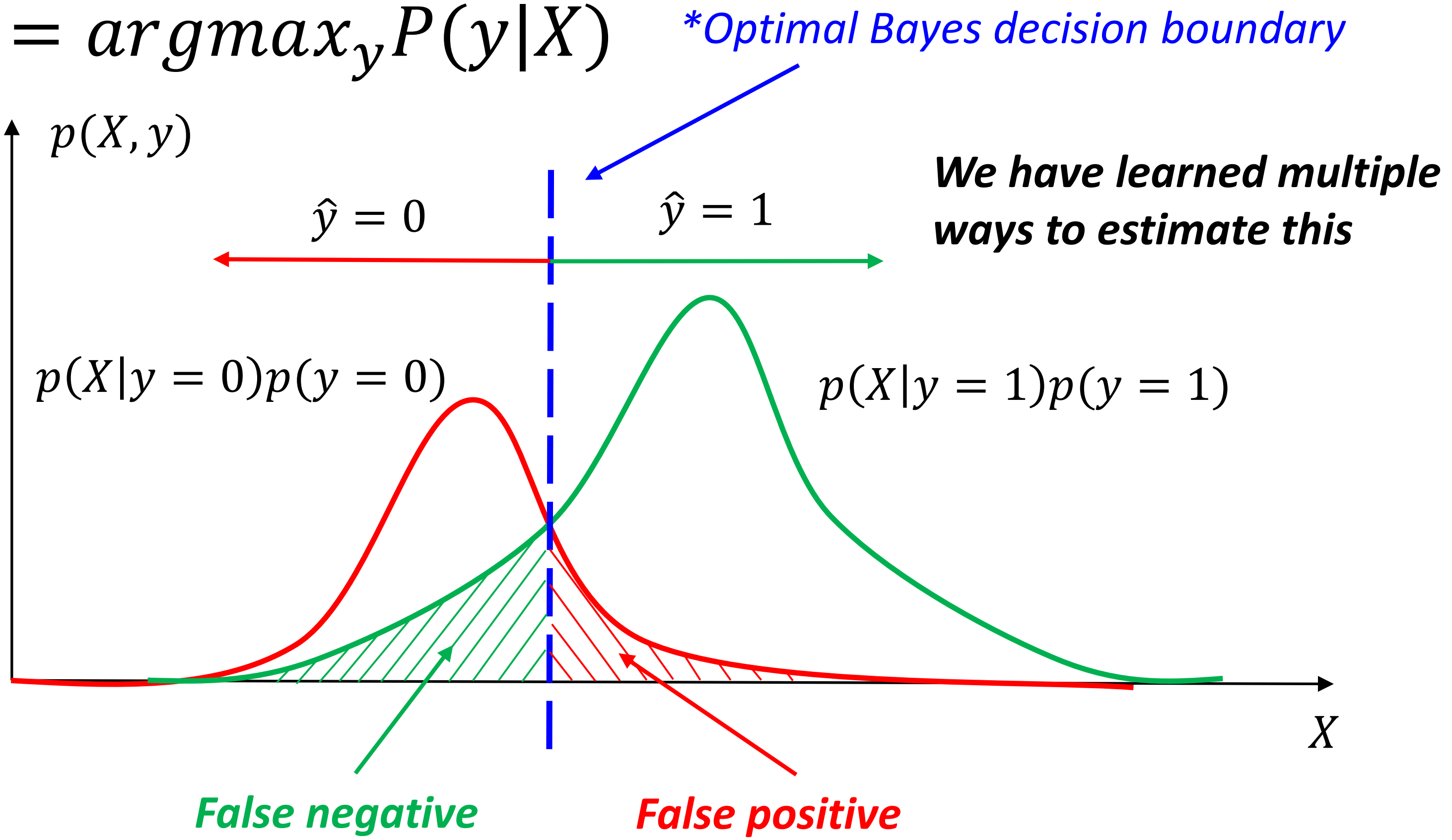
Hongning Wang
CS@UVa

# Today's lecture

- Logistic regression model
  – A discriminative classification model
  – Two different perspectives to derive the model
  – Parameter estimation

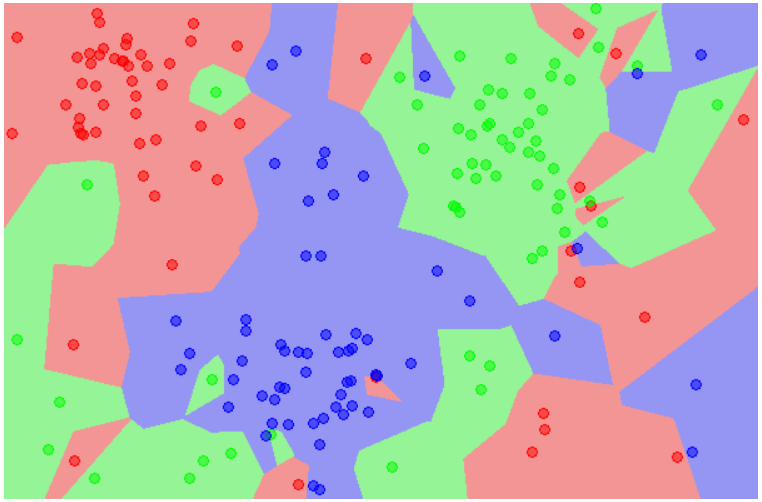
# Review: Bayes risk minimization

- Risk – assign instance to a wrong class
  - $y^* = argmax_y P(y|X)$



# Instance-based solution

- k nearest neighbors
  - Approximate Bayes decision rule in a subset of data around the testing point



# Instance-based solution

- k nearest neighbors
  - Approximate Bayes decision rule in a subset of data around the testing point
  - Let $V$ be the volume of the $m$ dimensional ball around $x$ containing the $k$ nearest neighbors for $x$, we have

$$p(x)V = \frac{k}{N} \quad => \quad p(x) = \frac{k}{NV} \qquad p(x|y=1) = \frac{k_1}{N_1 V} \qquad p(y=1) = \frac{N_1}{N}$$

*Total number of instances*

*Total number of instances in class 1*

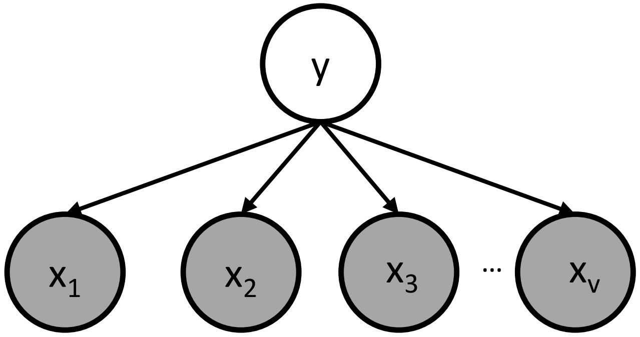
With Bayes rule:

$$p(y=1|x) = \frac{\frac{N_1}{N} \times \frac{k_1}{N_1 V}}{\frac{k}{NV}} = \frac{k_1}{k}$$

*Counting the nearest neighbors from class1*

# Generative solution

- Naïve Bayes classifier
  - $y^* = argmax_y P(y|X)$
    $= argmax_y P(X|y)P(y)$ **By Bayes rule**
    $= argmax_y \prod_{i=1}^{|d|} P(x_i|y)\, P(y)$ **By independence assumption**



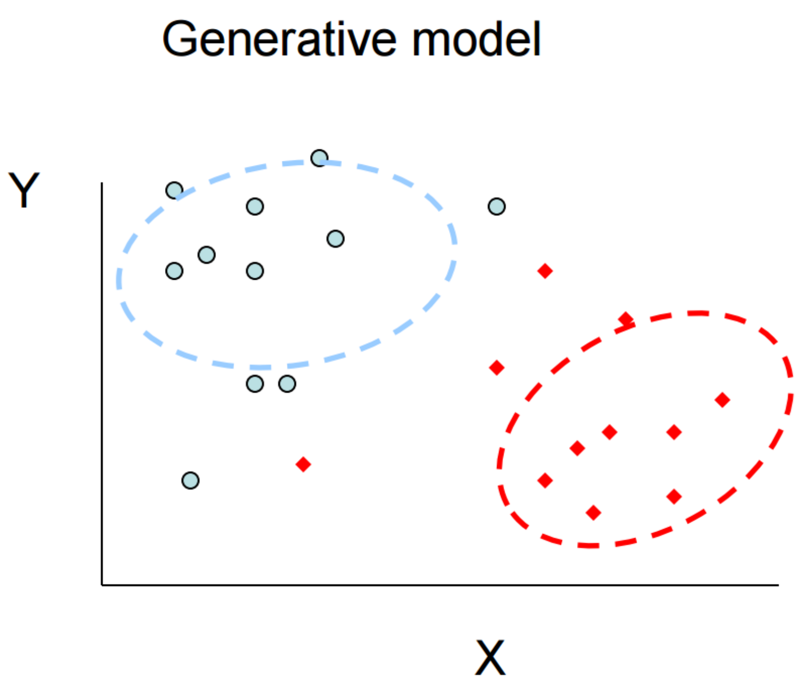
# Estimating parameters

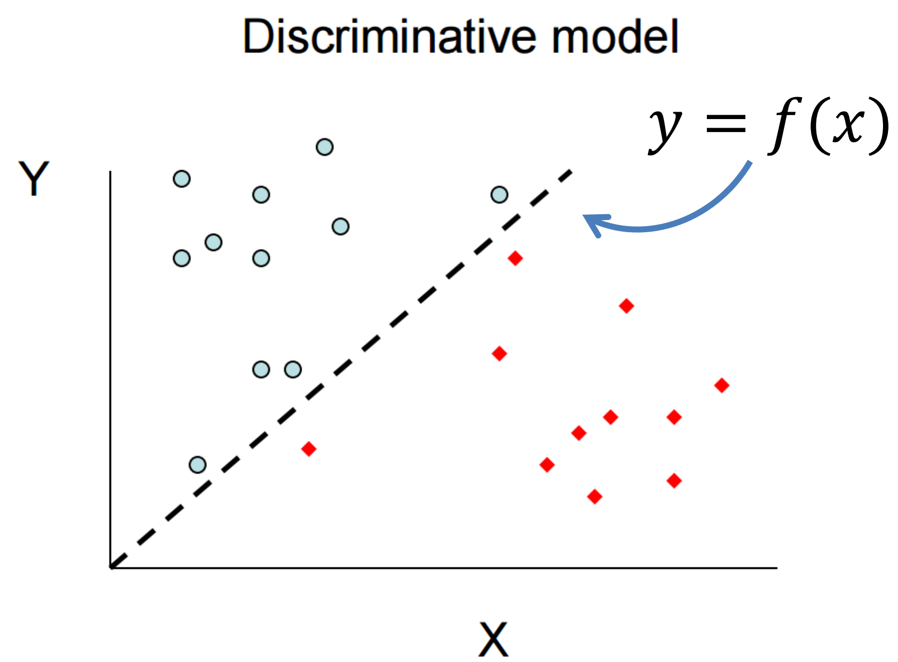
- Maximial likelihood estimator
  - $P(x_i|y) = \frac{\sum_d \sum_j \delta(x_d^j = x_i, y_d = y)}{\sum_d \delta(y_d = y)}$
  - $P(y) = \frac{\sum_d \delta(y_d = y)}{\sum_d 1}$

| | text | information | identify | mining | mined | is | useful | to | from | apple | delicious | **Y** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| D1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | **1** |
| D2 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | **1** |
| D3 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | **0** |

# Discriminative v.s. generative models

<u>All instances</u> are considered for probability density estimation




More attention will be put onto the <u>boundary points</u>

# Parametric form of decision boundary in Naïve Bayes

- For binary case
  - $f(X) = sgn(\log P(y=1|X) - \log P(y=0|X))$

$$= sgn\left(\log\frac{P(y=1)}{P(y=0)} + \sum_{i=1}^{|d|} c(x_i,d)\log\frac{P(x_i|y=1)}{P(x_i|y=0)}\right)$$

$$= sgn\boxed{(w^T\bar{X})}$$

where

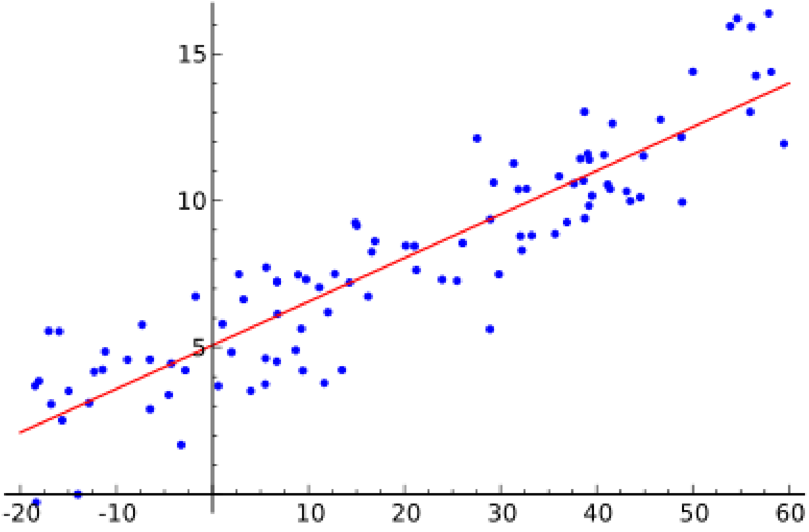
*Linear regression?*

$$w = \left(\log\frac{P(y=1)}{P(y=0)}, \log\frac{P(x_1|y=1)}{P(x_1|y=0)}, \dots, \log\frac{P(x_v|y=1)}{P(x_v|y=0)}\right)$$

$$\bar{X} = (1, c(x_1,d), \dots, c(x_v,d))$$

# Regression for classification?

- Linear regression
  - $y \leftarrow w^T X$
  - Relationship between a <u>scalar</u> dependent variable $y$ and one or more explanatory variables



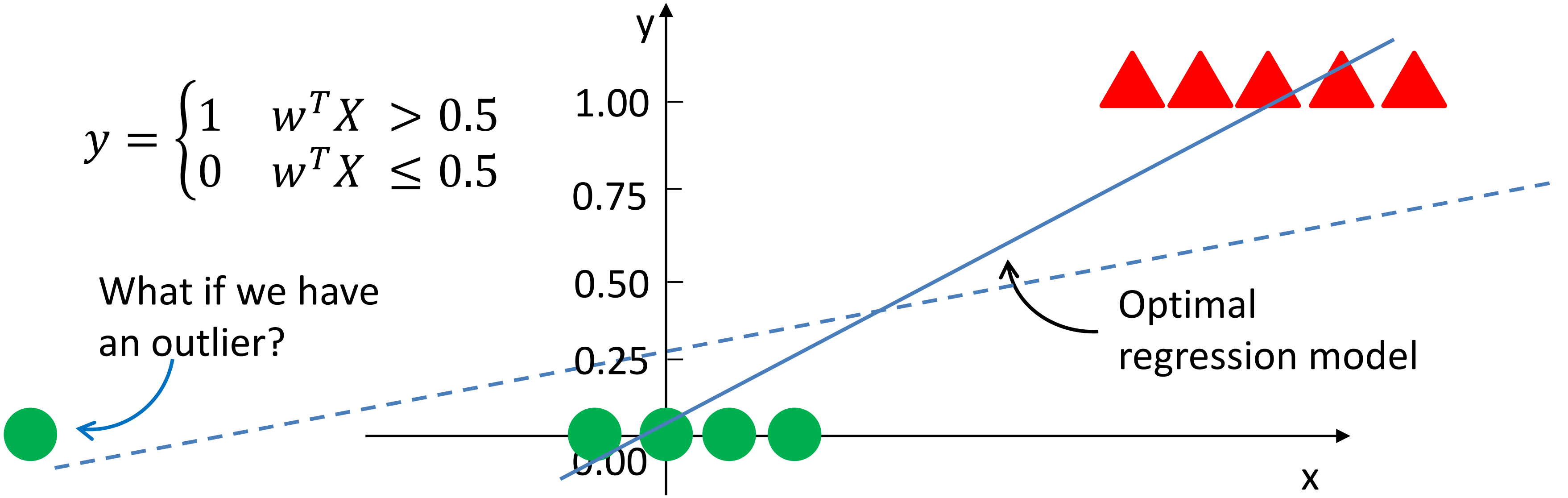
# Regression for classification?

- Linear regression
  - $y \leftarrow w^T X$
  - Relationship between a <u>scalar</u> dependent variable $y$ and one or more explanatory variables

***Y is discrete in a classification problem!***

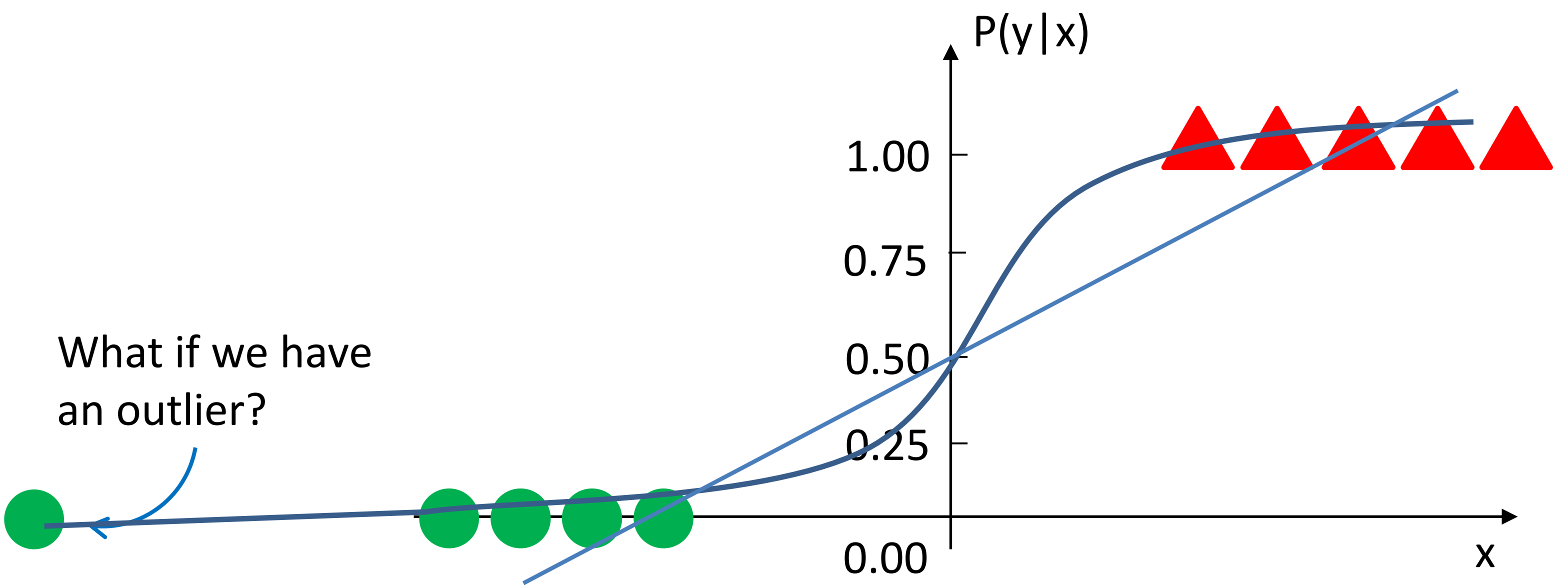
$$y = \begin{cases} 1 & w^T X > 0.5 \\ 0 & w^T X \leq 0.5 \end{cases}$$

What if we have an outlier?



# Regression for classification?

- Logistic regression
  - $p(y|x) = \sigma(w^T X) = \frac{1}{1+\exp(-w^T X)}$ Sigmoid function
  - Directly modeling of class posterior



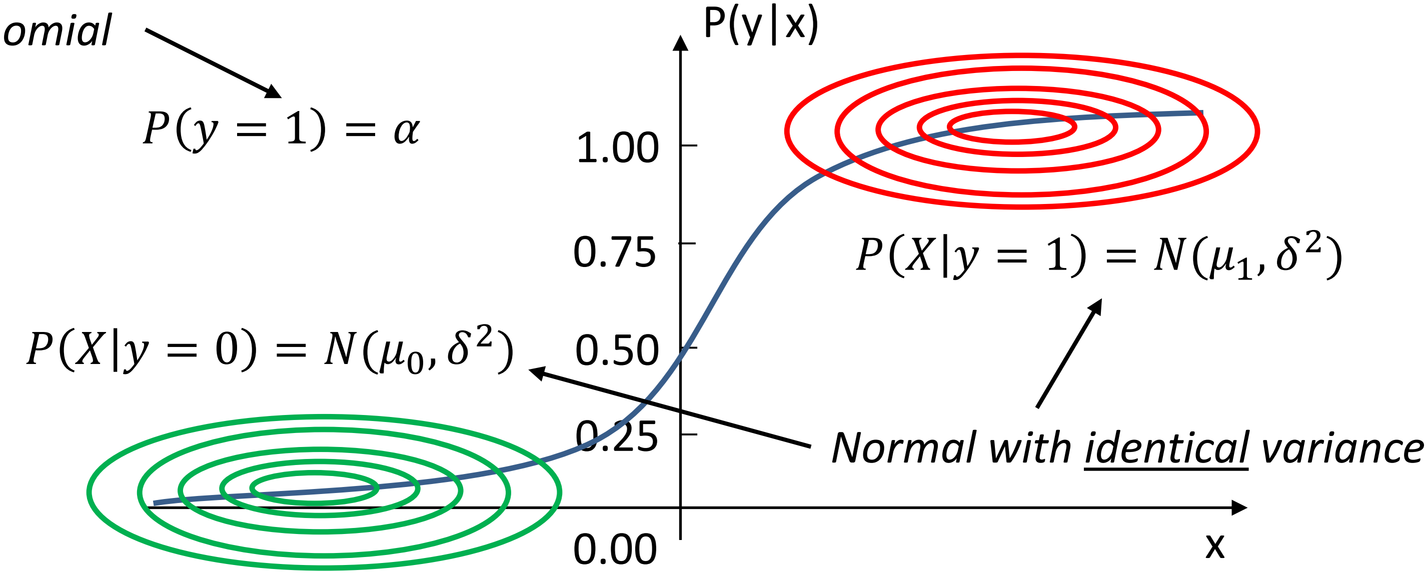
# Logistic regression for classification

- Why sigmoid function?
  - $P(y=1|X) = \frac{P(X|y=1)P(y=1)}{P(X|y=1)P(y=1)+P(X|y=0)P(y=0)}$

$$= \frac{1}{1+\boxed{\frac{P(X|y=0)P(y=0)}{P(X|y=1)P(y=1)}}}$$

*Binomial*

$P(y=1) = \alpha$



# Logistic regression for classification

- Why sigmoid function?
  - $P(y=1|X) = \frac{P(X|y=1)P(y=1)}{P(X|y=1)P(y=1)+P(X|y=0)P(y=0)}$

$$= \frac{1}{1+\boxed{\frac{P(X|y=0)P(y=0)}{P(X|y=1)P(y=1)}}}$$

$$= \frac{1}{1+\exp\left(-\ln\underline{\frac{P(X|y=1)P(y=1)}{P(X|y=0)P(y=0)}}\right)}$$

# Logistic regression for classification

- Why sigmoid function?

$$P(x|y) = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\delta^2}}$$

  - $\ln \frac{P(X|y=1)P(y=1)}{P(X|y=0)P(y=0)} = \ln \frac{P(y=1)}{P(y=0)} + \sum_{i=1}^{V} \ln \frac{P(x_i|y=1)}{P(x_i|y=0)}$

$$= \boxed{\ln \frac{\alpha}{1-\alpha}} + \sum_{i=1}^{V} \left( \boxed{\frac{\mu_{1i} - \mu_{0i}}{\delta_i^2}} x_i - \boxed{\frac{\mu_{1i}^2 - \mu_{0i}^2}{2\delta_i^2}} \right)$$

$$= \boxed{w_0} + \sum_{i=1}^{V} \boxed{\frac{\mu_{1i} - \mu_{0i}}{\delta_i^2}} x_i$$

$$= w_0 + w^T X$$

$$= \bar{w}^T \bar{X}$$

Origin of the name: logit function

# Logistic regression for classification

- Why sigmoid function?
  - $P(y=1|X) = \frac{P(X|y=1)P(y=1)}{P(X|y=1)P(y=1)+P(X|y=0)P(y=0)}$

$$= \frac{1}{1+\boxed{\frac{P(X|y=0)P(y=0)}{P(X|y=1)P(y=1)}}}$$

$$= \frac{1}{1+\exp\left(-\underline{\ln\frac{P(X|y=1)P(y=1)}{P(X|y=0)P(y=0)}}\right)}$$

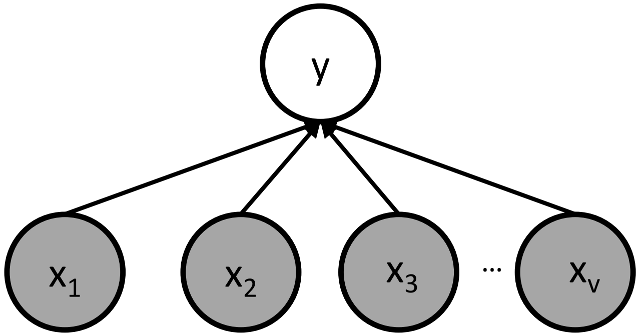
$$= \frac{1}{1+\exp(-\bar{w}^T\bar{X})}$$

**Generalized Linear Model**

*Note: it is still a linear relation among the features!*

# Logistic regression for classification

- For multi-class categorization
  - $P(y=k|X)=\frac{\exp(w_k^T X)}{\sum_{j=1}^{K}\exp(w_j^T X)}$
  - $P(y=k|X) \propto \exp(w_k^T X)$




*Warning: redundancy in model parameters,*

When $K$=2,

$$P(y=1|X)=\frac{\exp(w_1^T X)}{\exp(w_1^T X)+\exp(w_0^T X)}$$
$$=\frac{1}{1+\exp(-\underbrace{(w_1-w_0)}_{\bar{w}}{}^T X)}$$

# Logistic regression for classification

- Decision boundary for binary case
  - $\hat{y} = \begin{cases} 1, p(y=1|X) > 0.5 \\ 0, \quad\quad otherwise \end{cases}$

$$p(y=1|X) = \frac{1}{1+\exp(-w^T X)} > 0.5$$

**i.f.f.**

$$\exp(-w^T X) < 1$$

**i.f.f.**

$$w^T X > 0$$

  - $\hat{y} = \begin{cases} 1, \quad w^T x > 0 \\ 0, otherwise \end{cases}$ ← ***A linear model!***

# Logistic regression for classification

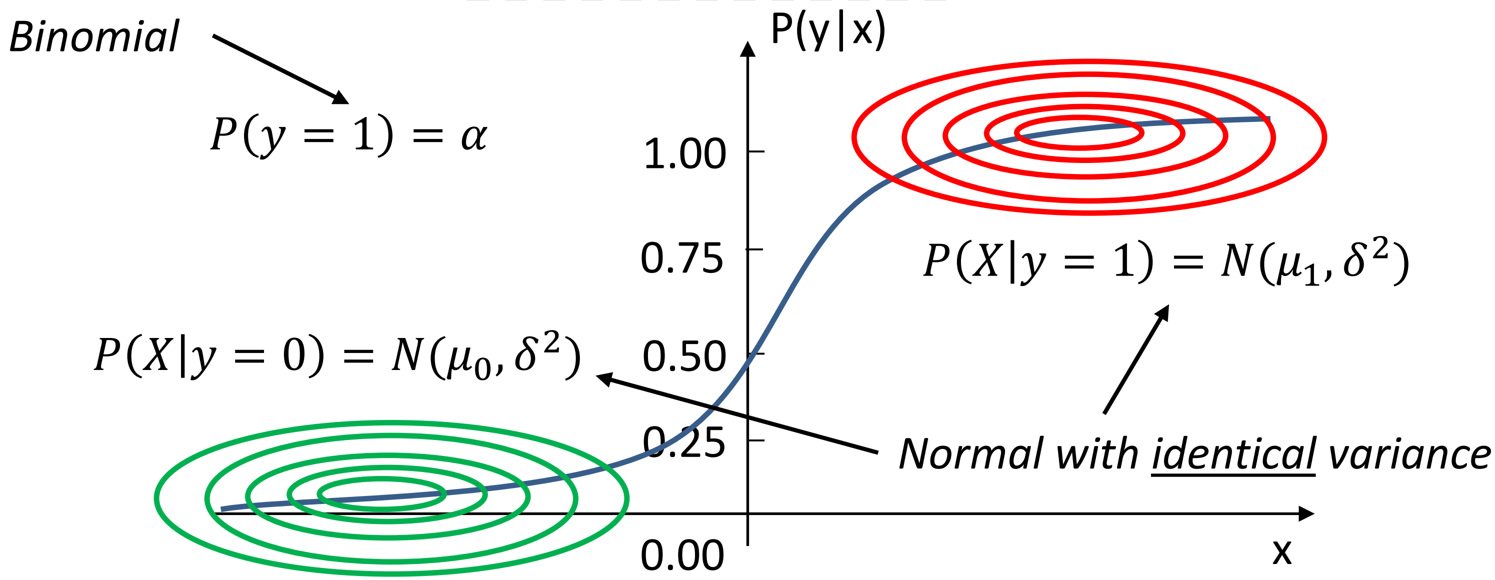
- Decision boundary in general
  - $\hat{y} = argmax_y p(y|X)$
    $= argmax_y \exp(w_y^T X)$
    $= argmax_y w_y^T X$

***A linear model!***

# Logistic regression for classification

- Summary
  - $P(y=1|X) = \frac{P(X|y=1)P(y=1)}{P(X|y=1)P(y=1)+P(X|y=0)P(y=0)}$

$$= \frac{1}{1+\boxed{\frac{P(X|y=0)P(y=0)}{P(X|y=1)P(y=1)}}}$$



# A different perspective

- Imagine we have the following

| Documents | Sentiment |
|---|---|
| *“happy”, “good”, “purchase”, “item”, “indeed”* | positive |

$$p(x = \text{"happy"}, y = 1) + p(x = \text{"good"}, y = 1) + p(x = \text{"}purchase\text{"}, y = 1) \\ + p(x = \text{"item"}, y = 1) + p(x = \text{"}indeed\text{"}, y = 1) = 1$$

Question: find a distribution $p(x, y)$ that satisfies this observation.

Answer1: $p(x = \text{"item"}, y = 1) = 1$, and all the others 0

Answer2: $p(x = \text{"indeed"}, y = 1) = 0.5$, $p(x = \text{"good"}, y = 1) = 0.5$, and all the others 0

*We have too little information to favor either one of them.*

# Occam's razor

- A problem-solving principle
  - “among competing hypotheses that predict equally well, the one with the fewest assumptions should be selected.”
    - William of Ockham (1287–1347)
  - Principle of Insufficient Reason: "when one has no information to distinguish between the probability of two events, the best strategy is to consider them equally likely”
    - Pierre-Simon Laplace (1749–1827)

# A different perspective

- Imagine we have the following

| Documents | Sentiment |
| --- | --- |
| *“happy”, “good”, “purchase”, “item”, “indeed”* | positive |

$$p(x = \text{"happy"}, y = 1) + p(x = \text{"good"}, y = 1) + p(x = \text{"}purchase\text{"}, y = 1)$$
$$+p(x = \text{"item"}, y = 1) + p(x = \text{"}indeed\text{"}, y = 1) = 1$$

Question: find a distribution $p(x, y)$ that satisfies this observation.

As a result, a ***safer*** choice would be:

$$p(x = \text{"}\cdot\text{"}, y = 1) = 0.2$$

Equally favor every possibility

# A different perspective

- Imagine we have the following

| Observations | Sentiment |
|---|---|
| *"happy", "good", "purchase", "item", "indeed"* | positive |
| *30% of time "good", "item"* | positive |

$$p(x = \text{"happy"}, y = 1) + p(x = \text{"good"}, y = 1) + p(x = \text{"}purchase\text{"}, y = 1)$$
$$+p(x = \text{"item"}, y = 1) + p(x = \text{"}indeed\text{"}, y = 1) = 1$$

$$p(x = \text{"good"}, y = 1) + p(x = \text{"item"}, y = 1) = 0.3$$

Question: find a distribution $p(x, y)$ that satisfies this observation.

Again, a ***safer*** choice would be:

$p(x = \text{"}good\text{"}, y = 1) = p(x = \text{"}item\text{"}, y = 1) = 0.15,$ and all the others $\frac{7}{30}$

Equally favor every possibility

# A different perspective

- Imagine we have the following

| Observations | Sentiment |
|---|---|
| *"happy", "good", "purchase", "item", "indeed"* | positive |
| *30% of time "good", "item"* | positive |
| *50% of time "good", "happy"* | positive |

$$p(x = \text{"happy"}, y = 1) + p(x = \text{"good"}, y = 1) + p(x = \text{"}purchase\text{"}, y = 1)$$
$$+p(x = \text{"item"}, y = 1) + p(x = \text{"}indeed\text{"}, y = 1) = 1$$

$$p(x = \text{"good"}, y = 1) + p(x = \text{"item"}, y = 1) = 0.3$$

$$p(x = \text{"good"}, y = 1) + p(x = \text{"happy"}, y = 1) = 0.5$$

Question: find a distribution $p(x, y)$ that satisfies this observation.
Time to think about:

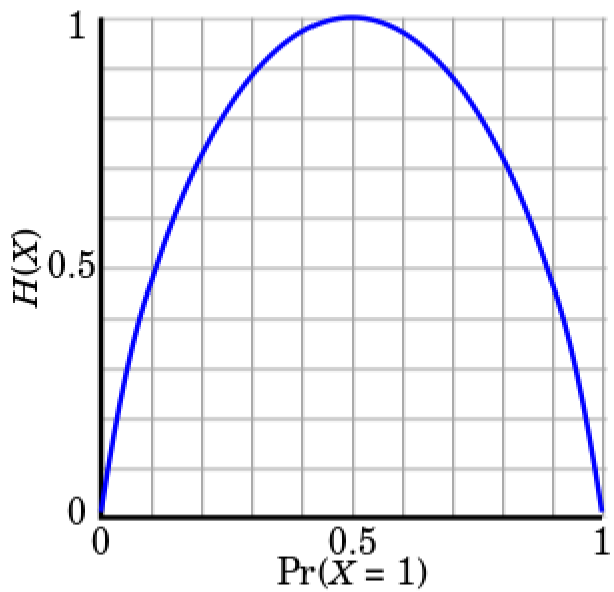
*1) what do we mean by equally/uniformly favoring the models?*

*2) given all these constraints, how could we find the most preferred model?*

# Maximum entropy modeling

- A measure of uncertainty of random events
  - $H(X) = E[I(X)] = -\sum_{x \in X} P(x) \log P(x)$

Maximized when P(X) is uniform distribution




*Question 1 is answered, then how about question 2?*

# Represent the constraints

- Indicator function
  - E.g., to express the observation that word 'good' occurs in a positive document
    - $f(x, y) = \begin{cases} 1 & \text{if } y = 1 \text{ and } x = \text{‘}good\text{’} \\ 0 & \text{otherwise} \end{cases}$
  - Usually referred as feature function

# Represent the constraints

- Empirical expectation of feature function over a corpus
  - $E[\tilde{p}(f)] = \sum_{x,y} \tilde{p}(x,y) f(x,y)$

    where $\tilde{p}(x,y) = \frac{c(f(x,y))}{N}$ *i.e., frequency of observing $f(x,y)$ in a given collection.*
- Expectation of feature function under a given statistical model
  - $E[p(f)] = \sum_{x,y} \tilde{p}(x) p(y|x) f(x,y)$

*Empirical distribution of $x$ in the same collection.*

*Model's estimation of conditional distribution.*

# Represent the constraints

- When a feature is important, we require our preferred statistical model to accord with it
  - $C := \{p \in P | E[p(f_i)] = E[\tilde{p}(f_i)], \forall i \in \{1,2,\dots,n\}\}$
  - $E[p(f_i)] = E[\tilde{p}(f_i)]$

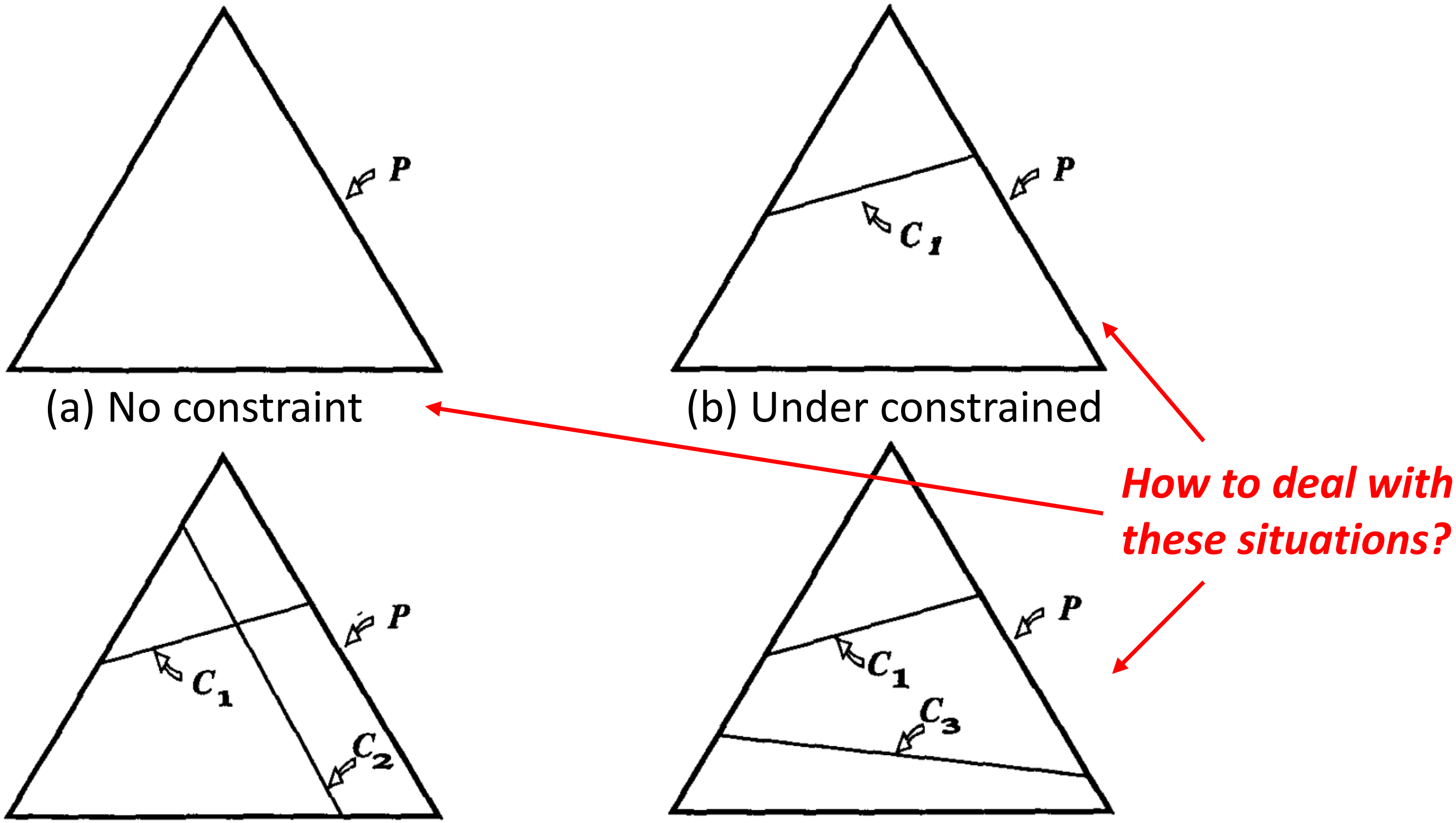
$$\Rightarrow \quad \sum_{x,y} \tilde{p}(x,y)\, f_i(x,y) = \sum_{x,y} \tilde{p}(x) \boxed{p(y|x)} f_i(x,y)$$

*We only need to specify this in our preferred model!*

*Is Question 2 answered?*

# Represent the constraints

- Let's visualize this




(a) No constraint
(b) Under constrained
(c) Feasible constraint
(d) Over constrained

# Maximum entropy principle

- To select a model from a set $C$ of allowed probability distributions, choose the model $p^* \in C$ with maximum entropy $H(p)$

$$p^* = argmax_{p \in C} H(p)$$

$p(y|x)$

*Both questions are answered!*

# Maximum entropy principle

- Let's solve this constrained optimization problem with Lagrange multipliers

Primal:

$$p^* = argmax_{p \in C} H(p)$$

Lagrangian:

a strategy for finding the local maxima and minima of a function subject to equality constraints

$$L(p, \lambda) = H(p) + \sum_i \lambda_i (p(f_i) - \tilde{p}(f_i))$$

# Maximum entropy principle

- Let’s solve this constrained optimization problem with Lagrange multipliers

Lagrangian:

$$L(p,\lambda) = H(p) + \sum_i \lambda_i (p(f_i) - \tilde{p}(f_i))$$

Dual:

$$p_\lambda(y|x) = \frac{1}{Z_\lambda(x)} \exp\left(\sum_i \lambda_i f_i(x,y)\right)$$

$$\Psi(\lambda) = -\sum_x \tilde{p}(x) \log Z_\lambda(x) + \sum_i \lambda_i \, \tilde{p}(f_i)$$

# Maximum entropy principle

- Let's solve this constrained optimization problem with Lagrange multipliers

Dual:

$$\Psi(\lambda) = -\sum_{x} \tilde{p}(x) \log Z_{\lambda}(x) + \sum_{i} \lambda_i \, \tilde{p}(f_i)$$

where

$$\mathrm{Z}_{\lambda} = \sum\nolimits_{y} \exp\left(\sum_{i} \lambda_i f_i(x, y)\right)$$

# Maximum entropy principle

- Primal: maximum entropy
  - $p^* = argmax_{p \in C} H(p)$
- Dual: logistic regression
  - $p_\lambda(y|x) = \frac{1}{Z_\lambda(x)} \exp(\sum_i \lambda_i f_i(x, y))$

where $$Z_\lambda = \sum_y \exp\left(\sum_i \lambda_i f_i(x, y)\right)$$

$\lambda^*$ is determined by $\Psi(\lambda)$

# Maximum entropy principle

- Let's take a close look at the dual function

$$\Psi(\lambda) = -\sum_{x} \tilde{p}(x) \log Z_{\lambda}(x) + \sum_{i} \lambda_i \, \tilde{p}(f_i)$$

where

$$\mathrm{Z}_{\lambda} = \sum\nolimits_{y} \exp\left(\sum_{i} \lambda_i f_i(x, y)\right)$$

# Maximum entropy principle

- Let's take a close look at the dual function

$$\Psi(\lambda) = -\sum_x \tilde{p}(x) \boxed{\log Z_\lambda(x)} + \sum_x \tilde{p}(x) \boxed{\sum_i \lambda_i \tilde{p}(f_i)}$$

$$= \sum_x \tilde{p}(x) \log \frac{\exp(\sum_i \lambda_i \tilde{p}(f_i))}{Z_\lambda(x)}$$

$$= \sum_x \tilde{p}(x) \log p(y|x)$$

***Maximum likelihood estimator!***

# Questions haven't been answered

- Class conditional density
  - Why it should be Gaussian with equal variance?
- Model parameters
  - What is the relationship between $w$ and $\lambda$?
  - How to estimate them?

# Maximum entropy principle

- Let's take a close look at the dual function

$$\Psi(\lambda) = -\sum_x \tilde{p}(x) \boxed{\log Z_\lambda(x)} + \sum_x \tilde{p}(x) \boxed{\sum_i \lambda_i \tilde{p}(f_i)}$$

$$= \sum_x \tilde{p}(x) \log \frac{\exp(\sum_i \lambda_i \tilde{p}(f_i))}{Z_\lambda(x)}$$

$$= \sum_x \tilde{p}(x) \log p(y|x)$$

***Maximum likelihood estimator!***

# Maximum entropy principle

- The maximum entropy model subject to the constraints $C$ has a parametric solution $p_{\lambda}^{*}(y|x)$ where the parameters $\lambda^{*}$ can be determined by maximizing the likelihood function of $p_{\lambda}(y|x)$ over a training set

*With a Gaussian distribution, differential entropy is maximized for a <u>given variance</u>.*

Features follow Gaussian distribution ⇔ Maximum entropy model

Logistic regression

# Recall: logistic regression

- Why sigmoid function?

$$P(x|y) = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\delta^2}}$$

  - $\ln \frac{P(X|y=1)P(y=1)}{P(X|y=0)P(y=0)} = \ln \frac{P(y=1)}{P(y=0)} + \sum_{i=1}^{V} \ln \frac{P(x_i|y=1)}{P(x_i|y=0)}$

$$= \boxed{\ln \frac{\alpha}{1-\alpha}} + \sum_{i=1}^{V} \left( \boxed{\frac{\mu_{1i} - \mu_{0i}}{\delta_i^2}} x_i - \boxed{\frac{\mu_{1i}^2 - \mu_{0i}^2}{2\delta_i^2}} \right)$$

$$= \boxed{w_0} + \sum_{i=1}^{V} \boxed{\frac{\mu_{1i} - \mu_{0i}}{\delta_i^2}} x_i$$

$$= w_0 + w^T X$$

$$= \bar{w}^T \bar{X}$$

Origin of the name:
logit function

# Maximum entropy principle

- Let's solve this constrained optimization problem with Lagrange multipliers

Primal:

$$p^* = argmax_{p \in C} H(p)$$

a strategy for finding the local maxima and minima of a function subject to equality constraints

Lagrangian:

$$L(p, \lambda) = H(p) + \sum_i \lambda_i (p(f_i) - \tilde{p}(f_i))$$

# Parameter estimation

- Maximum likelihood estimation
  - $L(w) = \sum_{d \in D} y_d \log p(y_d = 1|X_d) + (1 - y_d) \log p(y_d = 0|X_d)$
  - Take gradient of $L(w)$ with respect to $w$

$$\frac{\partial L(w)}{\partial w} = \sum_{d \in D} y_d \boxed{\frac{\partial \log p(y_d = 1|X_d)}{\partial w}} + (1 - y_d) \boxed{\frac{\partial \log p(y_d = 0|X_d)}{\partial w}}$$

# Parameter estimation

- Maximum likelihood estimation
  - $\frac{\partial \log p(y_d=1|X_d)}{\partial w} = -\frac{\partial \log\left(1+\exp(-w^T X_d)\right)}{\partial w}$
    $$= \frac{\exp(-w^T X_d)}{1+\exp(-w^T X_d)} X_d$$
    $$= (1 - p(y_d = 1|X_d))X_d$$
  - $\frac{\partial \log p(y_d=0|X_d)}{\partial w} = (0 - p(y_d = 1|X_d))X_d$

# Parameter estimation

- Maximum likelihood estimation
  - $L(w) = \sum_{d \in D} y_d \log p(y_d = 1|X_d) + (1 - y_d) \log p(y_d = 0|X_d)$
  - Take gradient of $L(w)$ with respect to $w$

$$\frac{\partial L(w)}{\partial w} = \sum_{d \in D} y_d \boxed{\frac{\partial \log p(y_d = 1|X_d)}{\partial w}} + (1 - y_d) \boxed{\frac{\partial \log p(y_d = 0|X_d)}{\partial w}}$$

$$= \sum_{d \in D} y_d \big(1 - p(y_d = 1|X_d)\big) X_d + (1 - y_d)\big(0 - p(y_d = 1|X_d)\big) X_d$$

$$= \sum_{d \in D} \underline{\big(y_d - p(y = 1|X_d)\big) X_d}$$

Can be easily generalized to multi-class case

***Good news: neat format, concave function for w***

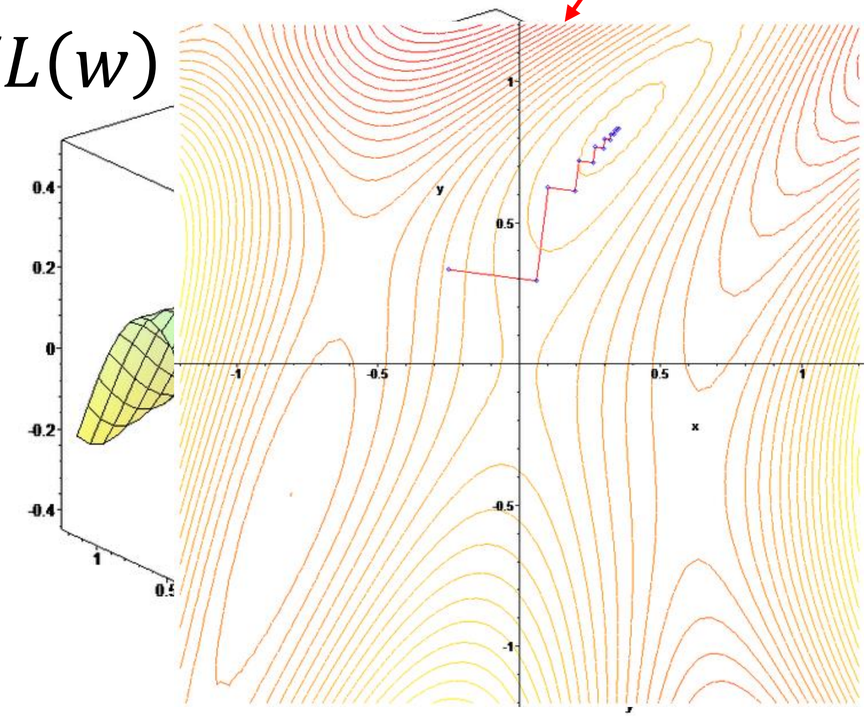
***Bad news: no close form solution***

# Gradient-based optimization

- Gradient descent
  - $\nabla L(w) = [\frac{\partial L(w)}{\partial w_0}, \frac{\partial L(w)}{\partial w_1}, \dots, \frac{\partial L(w)}{\partial w_V}]$
  - $w^{(t+1)} = w^{(t)} - \eta^{(t)} \nabla L(w)$

Step-size, affects convergence

Iterative updating



# Parameter estimation

- Stochastic gradient descent

while not converge

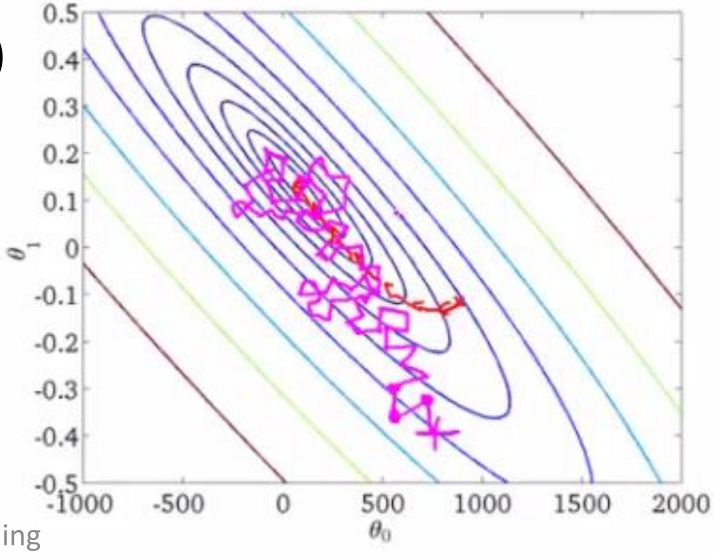
randomly choose $d \in D$

$$\nabla L_d(w) = [\frac{\partial L_d(w)}{\partial w_0}, \frac{\partial L_d(w)}{\partial w_1}, \dots, \frac{\partial L_d(w)}{\partial w_V}]$$

$$w^{(t+1)} = w^{(t)} - \eta^{(t)} \nabla L_d(w)$$

$$\eta^{(t+1)} = a\eta^{(t)}$$

Gradually shrink the step-size



# Parameter estimation

- Batch gradient descent

while not converge

Compute gradient w.r.t. all training instances

$$\nabla L_D(w) = [\frac{\partial L_D(w)}{\partial w_0}, \frac{\partial L_D(w)}{\partial w_1}, \dots, \frac{\partial L_D(w)}{\partial w_V}]$$

Compute step size $\eta^{(t)}$

$$w^{(t+1)} = w^{(t)} - \eta^{(t)} \nabla L_d(w)$$

Line search is required to ensure sufficient decent

First order method

*Second order methods, e.g., quasi-Newton method and conjugate gradient, provide faster convergence*

# Model regularization

- Avoid over-fitting
  - We may not have enough samples to well estimate model parameters for logistic regression
  - Regularization
    - Impose additional constraints over the model parameters
    - E.g., sparsity constraint – enforce the model to have more zero parameters

# Model regularization

- L2 regularized logistic regression
  - Assume the model parameter $w$ is drawn from Gaussian: w $\sim N(0, \sigma^2)$
  - $p(y_d, w|X_d) \propto p(y_d|X_d, w)p(w)$
  - $L(w) = \sum_{d \in D}[y_d \log p(y_d = 1|X_d)$
    $+(1 - y_d) \log p(y_d = 0|X_d)] - \boxed{\frac{w^T w}{2\sigma^2}}$

*L2-norm of w*

# Generative V.S. discriminative models

**Generative**

- Specifying joint distribution
  - Full probabilistic specification for all the random variables
- Dependence assumption has to be specified for $p(X|y)$ and $p(y)$
- Flexible, can be used in unsupervised learning

**Discriminative**

- Specifying conditional distribution
  - Only explain the target variable
- Arbitrary features can be incorporated for modeling $p(y|X)$
- Need labeled data, only suitable for (semi-) supervised learning

# Naïve Bayes V.S. Logistic regression

**Naive Bayes**

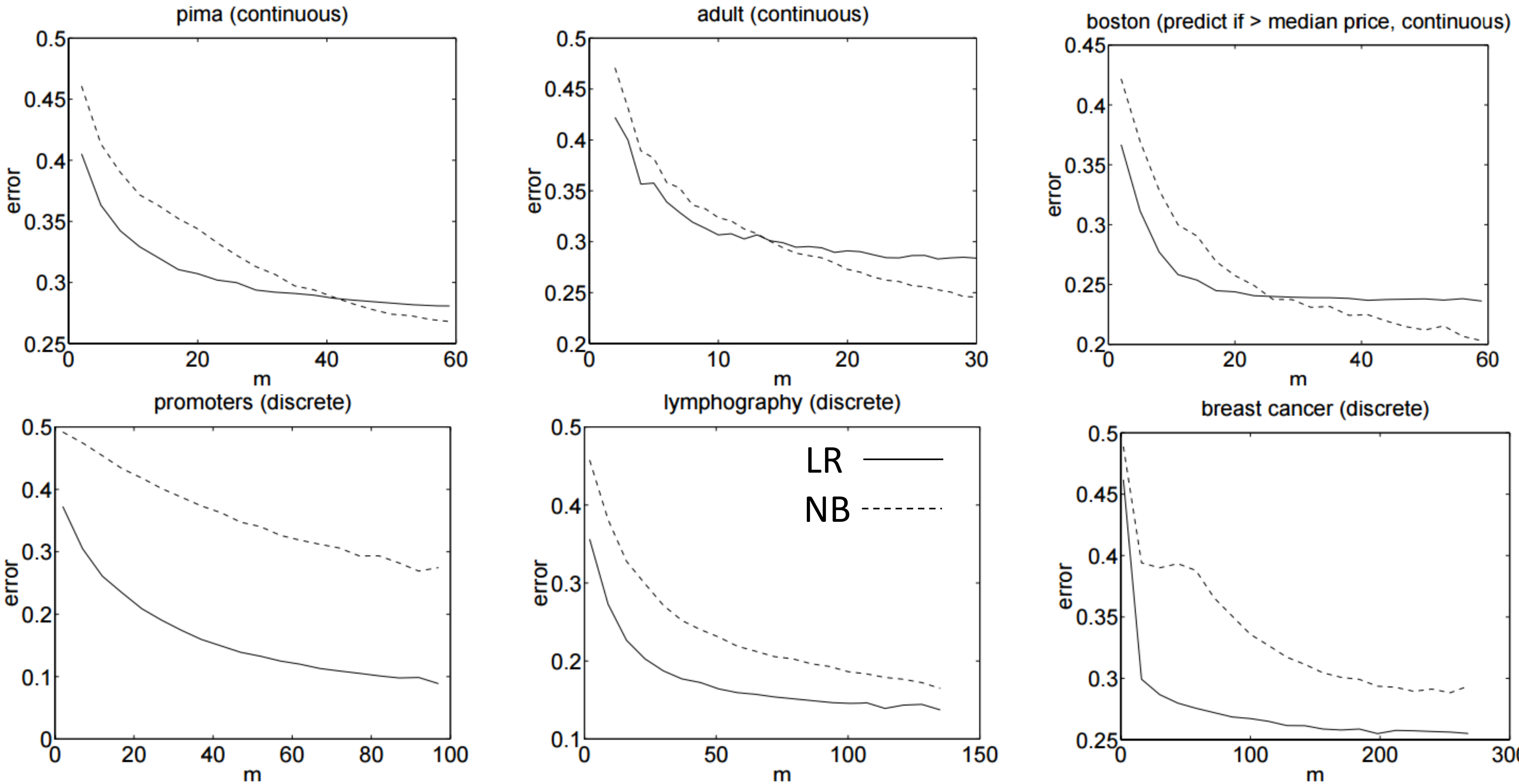
- Conditional independence
  - $p(X|y) = \prod_i p(x_i|y)$
- Distribution assumption of $p(x_i|y)$
- # parameters
  - $k(V+1)$
- Model estimation
  - Closed form MLE
- Asymptotic convergence rate
  - $\epsilon_{NB,n} \leq \epsilon_{NB,\infty} + O(\sqrt{\frac{\log V}{n}})$

**Logistic Regression**

- No independence assumption
- Functional form assumption of $p(y|X) \propto \exp(w_y^T X)$
- # parameters
  - $(k-1)(V+1)$
- Model estimation
  - Gradient-based MLE
- Asymptotic convergence rate
  - $\epsilon_{LR,n} \leq \epsilon_{LR,\infty} + O(\sqrt{\frac{V}{n}})$

Need more training data

# Naïve Bayes V.S. Logistic regression




*"On discriminative vs. generative classifiers: A comparison of logistic regression and naive bayes." – Ng, Jordan NIPS 2002, UCI Data set*

# What you should know

- Two different derivations of logistic regression
  - Functional form from Naïve Bayes assumptions
    - $p(X|y)$ follows equal variance Gaussian
    - Sigmoid function
  - Maximum entropy principle
    - Primal/dual optimization
  - Generalization to multi-class
- Parameter estimation
  - Gradient-based optimization
  - Regularization
- Comparison with Naïve Bayes

# Today's reading

- Speech and Language Processing
  - Chapter 6: Hidden Markov and Maximum Entropy Models
    - 6.6 Maximum entropy models: background
    - 6.7 Maximum entropy modeling