Growth curves for multiple-output response variables via Bayesian quantile regression models

Joint work with Agatha Rodrigues and Thomas Kneib

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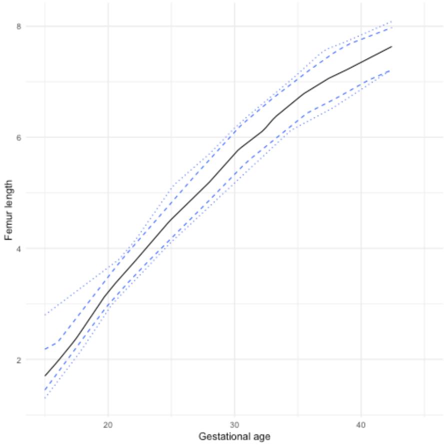
Motivation

- Growth curves - Fetus example



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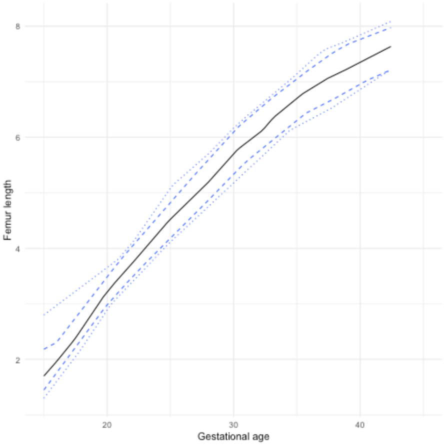
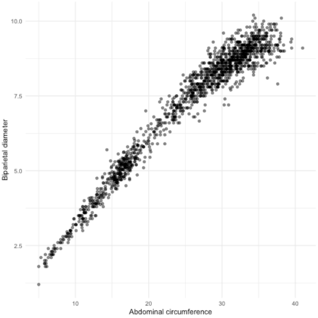
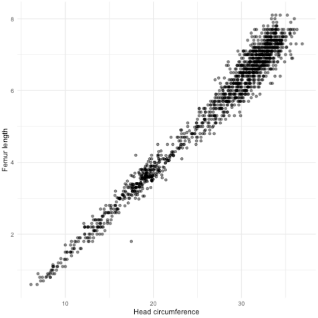


- How to build those curves when the dimension of the response variable is larger than 1?

Data set

- Data on four fetus biometric measurements:
  - Y_1: femur length (F);
  - Y_2: head circumference (HC);
  - Y_3: abdominal circumference (AC);
  - Y_4: biparietal diameter (BPD).



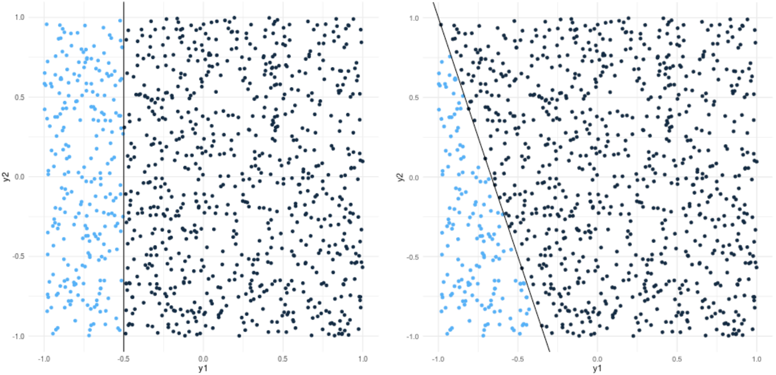


Outline

1. Bayesian quantile regression models for multiple-output response variables
   - Main definitions
   - Selection of directions
2. Rearrangement of quantile hyperplanes for coverage improvement
3. Application
   - Results for our 4D data set with fetus measurements
4. Final remarks

Tukey depth (Halfspace depth)

- Halfspace example



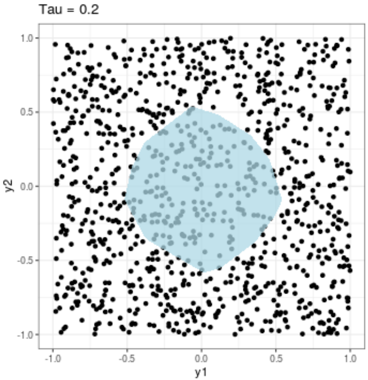
- Tukey depth:

$$HD(z, P) := \inf\{P(H) : H \text{ is a closed halfspace containing } z\}.$$

Depth region

- Definition:

$$D(\tau) := \{z \in \mathbb{R}^k : HD(z, P) \geq \tau\}$$



Directional quantile regression model

- Response variable is defined as $\boldsymbol{Y} \in \mathbb{R}^k$.
- Directional index can be defined by $\boldsymbol{\tau} \in \mathcal{B}^k := \{\boldsymbol{v} \in \mathbb{R}^k : 0 < ||\boldsymbol{v}|| < 1.\}$.
  - $\boldsymbol{\tau} = \tau \boldsymbol{u}$.
  - Direction: $\boldsymbol{u} \in \mathcal{S}^{k-1} := \{z \in \mathbb{R}^k : ||z|| = 1\}$;
  - Magnitude: $\tau \in (0, 1)$.
- Define $\boldsymbol{\Gamma}_u$, an arbitrary $k \times (k-1)$ matrix of unit vectors.
  - $(u \vdots \Gamma_u)$ is an orthonormal basis of $\mathbb{R}^k$.

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DEFINITION:

The $\boldsymbol{\tau}$th quantile of $\boldsymbol{Y}$ is the τth quantile hyperplane obtained from the regression of $\boldsymbol{Y}_u := \boldsymbol{u}^{'}\boldsymbol{Y}$ on the marginals of $\boldsymbol{Y}^{\perp} := \boldsymbol{\Gamma}_u{}^{'}\boldsymbol{Y}$ with an intercept term.

Estimation setup

The $\boldsymbol{\tau}$th quantile of $\boldsymbol{Y}$ is any element of the collection Λ_τ of hyperplanes

$$\lambda_\tau := \{\boldsymbol{y} \in \mathbb{R}^k : \boldsymbol{u}^{'}\boldsymbol{y} = \hat{\boldsymbol{b}}_\tau \boldsymbol{\Gamma}^{'}_{u}\boldsymbol{y} + \hat{a}_\tau\},$$

such that $(\hat{a}_\tau, \hat{\boldsymbol{b}}_\tau)$ are the solutions of the minimization problem

$$\min_{(a_\tau, \boldsymbol{b}_\tau) \in \mathbb{R}^k} E[\rho_\tau(\boldsymbol{u}^{'}\boldsymbol{y} - \boldsymbol{b}_\tau \boldsymbol{\Gamma}^{'}_{u}\boldsymbol{y} - a_\tau)].$$

where $\rho_\tau(u)$ is a known loss function in the quantile regression literature defined as

$$\rho_\tau(u) = u(\tau - \mathbb{I}(u < 0)), \quad 0 < \tau < 1.$$

Bayesian directional quantile regression model

- Consider the mixture representation of the asymmetric Laplace distribution

$$\begin{aligned} Y_i|w_i &\sim N(\mu + \theta w_i, \psi^2 \sigma w_i) \\ w_i &\sim \mathrm{Exp}(\sigma) \qquad\qquad \Rightarrow Y \sim AL(\mu, \sigma, \tau) \end{aligned}$$

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- Then one can consider that, for each direction u,

$$Y_u | \boldsymbol{b}_\tau, \boldsymbol{\beta}_\tau, \sigma, w \sim N(Y^\perp b_\tau + \boldsymbol{x}' \boldsymbol{\beta}_\tau + \theta w_i, \psi^2 \sigma w_i),$$

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- That result makes it possible:
  - to use interesting developments of the univariate in the multivariate case.

Upper and lower halfspaces

With predictor variables, we have

$$\lambda_\tau(\boldsymbol{X}) = \{\boldsymbol{u}^{'}\boldsymbol{y} = \hat{\boldsymbol{b}}_\tau \boldsymbol{\Gamma}_u^{'}\boldsymbol{y} + \boldsymbol{x}^{'}\hat{\beta}_\tau + \hat{a}_\tau\},$$

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We can say that each element $(\hat{a}_\tau, \hat{\boldsymbol{b}}_\tau, \hat{\beta}_\tau)$ define an upper closed quantile halfspace

$$H^{+}_{\tau\boldsymbol{u}} = H^{+}_{\tau\boldsymbol{u}}(\hat{a}_\tau, \hat{\boldsymbol{b}}_\tau, \hat{\beta}_\tau) = \{\boldsymbol{y} \in \mathbb{R}^k : \boldsymbol{u}^{'}\boldsymbol{y} \geq \hat{\boldsymbol{b}}_\tau \boldsymbol{\Gamma}^{'}_u \boldsymbol{y} + \boldsymbol{x}^{'}\hat{\beta}_\tau + \hat{a}_\tau\}$$

and an analogous lower open quantile halfspace switching $\geq$ for $<$.

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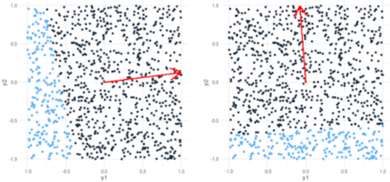
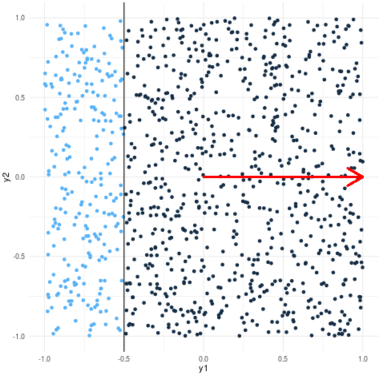


Properties

- Probabilistic nature of quantiles:

$$P(\boldsymbol{Y} \in H_{\tau\boldsymbol{u}}^{-}) = \tau,$$



Quantile region

Moreover, fixing τ we are able to define the τ quantile region $R(\tau)$ as

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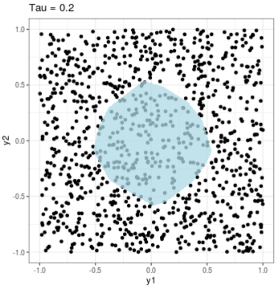

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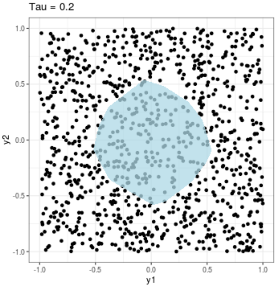

Depth and quantile regions

$$R(\tau) = D(\tau)$$

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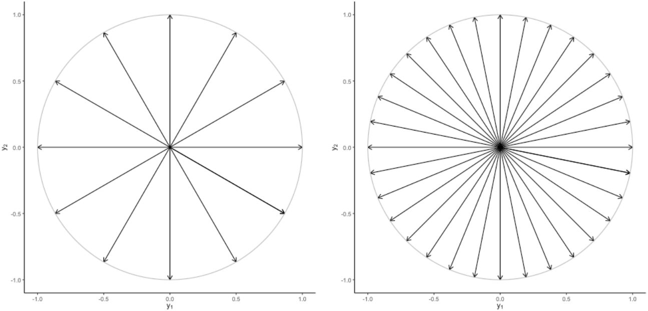

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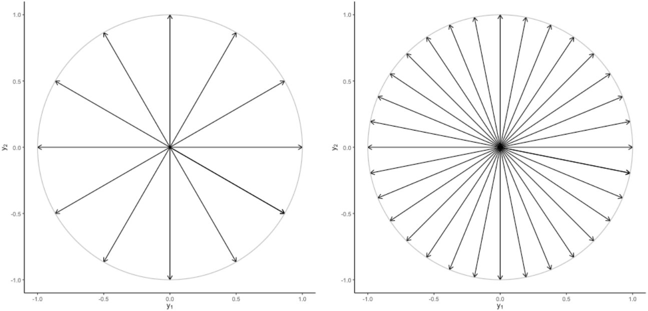
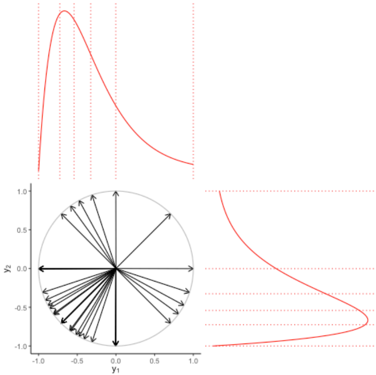
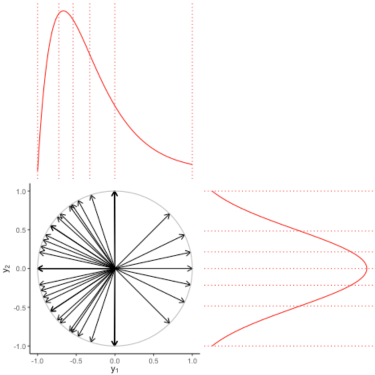


- The same idea becomes more difficult with 3 or more dimensions.
  - And possibly it is not efficient.

Selection of directions based on marginal quantiles

- We propose selecting the directions based on marginal quantiles.
  - Example in 2 dimensions:





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Readjustment of intercept for all directions

$$H^{+}_{\tau \boldsymbol{u}_a} = \{\boldsymbol{y} \in \mathbb{R}^k : \boldsymbol{u}^{'}\boldsymbol{y} \geq \hat{\boldsymbol{b}}_{\tau}\boldsymbol{\Gamma}^{'}_{u}\boldsymbol{y} + \boldsymbol{x}^{'}\hat{\boldsymbol{\beta}}_{\tau} + \alpha^{\lambda}_{\tau}\},$$

where

$$\alpha^{\lambda}_{\tau} = \hat{a}_{\tau} - \lambda, \quad \lambda > 0,$$

such that

$$P(Y \in \bigcap_{\boldsymbol{u} \in \mathcal{S}^{k-1}} H^{+}_{\tau \boldsymbol{u}_a}) = 1 - \tau$$

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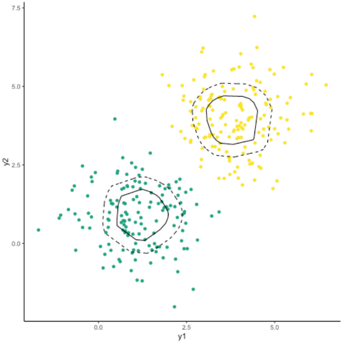

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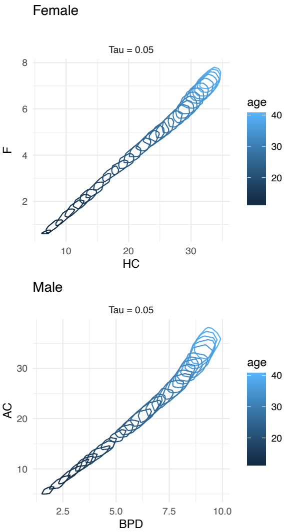
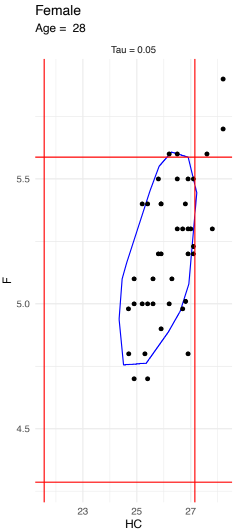
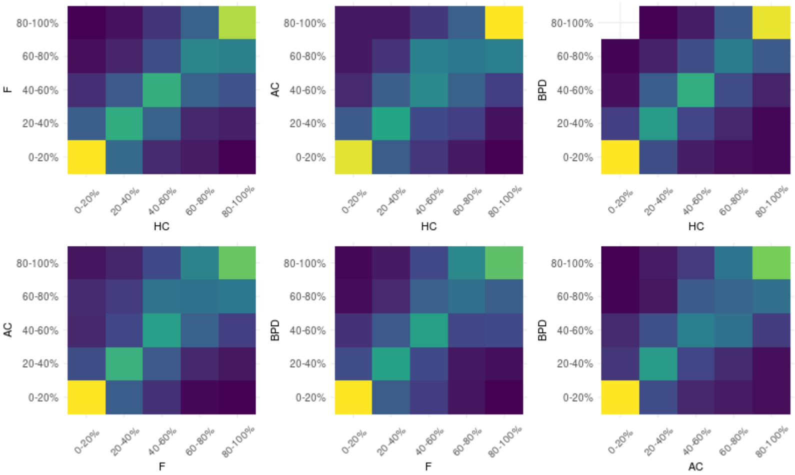
- Covariates:
  - gestational age (nonlinear effect);
  - sex;
  - mother BMI;

Examples of results





Distributions of observations outside quantile region



Final remarks

- We consider Bayesian quantile regression models for multiple-outputs to build growth curves for fetus measurements.
- We propose a method to select directions for higher dimensions based on marginal quantiles.
- We consider an adjustment in the quantile hyperplanes in order to improve the coverage of the conditional quantile regions.
- Next steps:
  - Add random effects to this framework.
  - Study the effects of the intercept adjustment for higher dimensions.

Thank you for your attention!

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