Homework 3

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September 21, 2020

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(Grade This One) **Let $A_1, A_2, A_3, ...$ be subsets of a metric space.**

(a) **If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\bar{B_n} = \bigcup_{i=1}^{n} \bar{A}_1$, for $n = 1, 2, 3, ...$**

(b) **If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^{\infty} \bar{A}_1$.**

Show, by example, that this inclusion can be proper
Answer:

(a) Let p be a limit point of $\bar{A}_j \cup \bar{A}_k$ where $j, k < n$. Let $q \in \bar{A}_j \cup \bar{A}_k$ and $q \in N_\epsilon(p)$. Because all closures of a set are closed, the point q must be in either $\bar{A}_j$, $\bar{A}_k$, or both, by the rules of a set union. This means no new limit points are introduced during the union, therefore $\bar{B_n} = \bigcup_{i=1}^{n} \bar{A}_1$

(b) The above proof handles the case where $B = \bigcup_{i=1}^{\infty} \bar{A}_1$. Prove the rest, I will provide an example of an inclusion which is proper. Let A_j contain $\frac{1}{2^j}$ and nothing else. $0 \in \bar{B}$ and $0 \notin \bigcup_{i=1}^{\infty} \bar{A}_1$, because each set A_j has no limit point, and one only emerges after the union of every set.

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(Grade This One) **Is every point of every open set $E \in R^2$ a limit point of E? Answer the same question for closed sets in R^2** Every point of the open set E is a limit point of E. Because E is open, every point is an interior point, meaning that there is a neighborhood based around that point that is a subset of E. If that is the case, then for every point $q \in E$, $N_q(\epsilon)$ contains a point of E. Closed sets do not satisfy this condition, which I will prove with an example: set $A = 2, 3$. This set contains all of its limit points, which is none, so it is closed. 2 is not a limit point of A.

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Let E^o denote the set of all interior points in E

(a) **Prove that E^o is always open**

(b) **Prove that E is open if and only if $E^o = E$**

(c) **If $G \subset E$ and G is open, prove that $G \subset E^o$**

(d) **Prove that the complement of E^o is the closure of the complement of E**

(e) **Do E and $\bar{E}$ always have the same interiors?**

(f) **Do E and E^o always have the same closures?**

Answer:

(a) E^o is the set of all interior points of E, and a set is open if every point is an interior point, therefore E^o is open

(b) If $E^o = E$, then every point in E is an interior point, and therefore E is open. If there is a point in E and not in E^o, then that point is not an interior point, and E is not open

(c) If G is open, it must contain only interior points. If $G \subset E$, then it must contain only points of E. If both of these are true, G must contain only interior points of E, and therefore $G \subset E^o$

(d) Any points in E but not in E^o are not interior points, meaning you cannot find a neighborhood around those points that is a subset of E. Because of this, they would be a limit point of E^c, as you can get arbitrarily close to the point without being in E. This means they would be included in $\bar{E^c}$. All other points not in either would be included in $\bar{E^c}$ because $E^o \subset E$

(e) Not always, consider the sets $A = \frac{1}{n}$ for all $n \in \mathbb{N}$, $B = \frac{-1}{n}$ for all $n \in \mathbb{N}$, and $C = A \cup B$. Set C has a limit point at 0, which is not in the set, but is in $\bar{C}$, and is an interior point as well

(f) No, any isolated points in E will be in $\bar{E}$ but not $\bar{E^o}$

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For $x \in R^1$ and $y \in R^1$, define

$$\begin{aligned}
d_1(x,y) &= (x-y)^2, \\
d_2(x,y) &= \sqrt{|x-y|}, \\
d_3(x,y) &= |x^2 - y^2|, \\
d_4(x,y) &= |x - 2y|, \\
d_5(x,y) &= \frac{|x-y|}{1+|x-y|}
\end{aligned}$$

Determine, for each of these, whether it is metric or not

1. $d_1(-1,1) \leq d_1(-1,0) + d_1(0,1)$
   $4 \leq 1+1$
   Violates the triangle inequality, not metric

2. d_2 will always be greater or equal to zero as you cannot get a negative number out of a square root, and $\sqrt{|x-x|} = 0$
   $|x-y| = |y-x|$
   $\sqrt{|x-y|} \leq \sqrt{|x-r|} + \sqrt{|r-y|}$
   $|x-y| \leq |x-r| + 2\sqrt{|x-r|}\sqrt{|r-y|} + |r-y|$
   Ignoring the middle term leaves: $|x-y| \leq |x-r| + |r-y|$. This is the triangle inequality. Since $2\sqrt{|x-r|}\sqrt{|r-y|}$ will always be positive, the triangle inquality holds
   d_2 is metric

3. $d_3(-1,1) = 0$, d_3 is not metric

4. $d_4(-2,1) = 0$, d_4 is not metric

5. Let $a = d_5(x,y), b = d_5(x,r), c = d_5(r,y)$
   $\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$
   $a + ab + ac + abc \leq b + c + ab + ac + 2bc + 2abc$
   $a \leq b + c + 2bc + abc$

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Let $K \in R^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, ...$ Prove that K is compact directly from the definition (without using the Heine-Borel theorem) Let G be an open cover of K. Let G_n be a subcover of k. Because $0 \in K$, 0 must be in at one of G_n, call it G_0. Since G is open, and $K \subset G$, find the smallest r such that $N_r(0)$ that has an element of K. Let that element of K be q. Let G_n be an open set that contains $1/n$. $G = \cup_{n=0}^{\frac{1}{q}} G_n$

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Construct a compact set of real numbers whose limit points form a countable set. Let $G = 1, 3$. G is compact as the open cover $\mathbb{R}$ has a finite subcover $(0, 4)$. G has no limit points, so they form the empty set, which is countable

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Give an example of an open cover of the segment (0,1) which has no finite subcover Let G be an open cover and G_j be its subcovers for $j =$

$1, 2, 3, 4, ...$ Let $G_j = (0 + \frac{1}{j}, 1 - \frac{1}{j})$. G has finite subcover, because if it did, you would take the largest value of j, and add 1 one to it, which wouldn't be covered

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Show that Theorem 2.36 and its Corollary become false (in R^1, for example) if the word "compact" is replaced by "closed" or by "bounded"
Let K_n be the bounded set $(\frac{1}{n}, 1)$. Any intersection of these sets will be non-empty, but take the whole of them and the intersection will be empty.
Let K_n be the closed set $[n, \inf)$. Any intersection of these sets will be non-empty, but the intersection of the whole of them will be empty.