Homework 1

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1

(Please grade this one) **If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.**
Say $r = \frac{a}{b}$, and that $r + x$ is rational. Therefore:

$$\frac{a}{b} + x = \frac{m}{n}$$
$$x = \frac{m}{n} - \frac{a}{b}$$
$$x = \frac{m-a}{n-b}$$

This would mean x is irrational, which is a contradiction
Say $r = \frac{a}{b}$, and that rx is rational. Therefore:

$$\frac{ax}{b} = \frac{m}{n}$$
$$x = \frac{bm}{an}$$

Again, this means x is rational

2

Prove that there is no rational number whose square is 12
Assume $\sqrt{12}$ is rational. Then there exists integers a and b share no common factors and $\frac{a}{b} = \sqrt{12}$

$$\frac{a^2}{b^2} = 12$$

Because a and b share no common factors, b^2 must be 1 in order for the fraction to produce an integer. Therefore

$$a^2 = 12$$

There are no integers whose square equal 12, so we have reached a contradiciton.

3

Suppose that $z = a + bi$, $w = u + iv$, and

$$a = (\tfrac{|w|+u}{2})^{1/2},\ b = (\tfrac{|w|-u}{2})^{1/2}$$

Prove that $z^2 = w$ if $v \geq 0$ and that $\overline{z}^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

$$\begin{aligned}
z^2 &= a^2 + 2abi - b^2 \\
&= (\frac{|w|+u}{2}) + 2abi - (\frac{|w|-u}{2}) \\
&= u + 2abi \\
&= u + 2\sqrt{\frac{1}{4}(|w|^2 - u^2)}i \\
&= u + \sqrt{|w|^2 - u^2}i \\
&= u + \sqrt{u^2 + v^2 - u^2}i \\
z^2 &= u + vi
\end{aligned}$$

4

If z_1,..., z_n are complex, prove that:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

$$\begin{aligned}
\sqrt{(a_1+b_1)^2 + (a_2+b_2)^2} &\leq \sqrt{(a_1+b_1)^2} + \sqrt{(a_2+b_2)^2} \\
(a_1+b_1)^2 + (a_2+b_2)^2 &\leq a_1^2 + a_2^2 + b_1^2 + b2^2 + 2\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)} \\
a_1^2 + 2a_1b_1 + b_1^2 + a_2^2 + 2a_2b_2 + b_2^2 &\leq a_1^2 + a_2^2 + b_1^2 + b2^2 + 2\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)} \\
a_1b_1 + a_2b_2 &\leq \sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)} \\
(a_1b_1 + a_2b_2)^2 &\leq (a_1^2+a_2^2)(b_1^2+b_2^2) \\
a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2b_2^2 &\leq a_1^2b_1^2 + a_2^2b_2^2 + a_1^2b_2^2 + a_2^2b_1^2 \\
2a_1b_1a_2b_2 &\leq a_1^2b_2^2 + a_2^2b_1^2 \\
0 &\leq a_1^2b_2^2 - 2a_1b_1a_2b_2 + a_2^2b_1^2 \\
0 &\leq (a_1b_2 - a_2b_1)^2
\end{aligned}$$

This will always be true, and since the above inequality can be simplified down into this, that will also always be true

5

If x,y are complex, prove that:

$$||x| - |y|| \leq |x - y|$$

Let $x = a + bi$ and $y = c + di$ (subscripts were tedious and not very neat last time):

$$\begin{aligned}
|\sqrt{a^2+b^2} - \sqrt{c^2+d^2}| &\leq \sqrt{(a-c)^2 + (b-d)^2} \\
a^2 + b^2 - 2\sqrt{(a^2+b^2)(c^2+d^2)} + c^2 + d^2 &\leq (a-c)^2 + (b-d)^2 \\
a^2 + b^2 - 2\sqrt{(a^2+b^2)(c^2+d^2)} + c^2 + d^2 &\leq a^2 - 2ac + c^2 + b^2 - 2bd + d^2 \\
\sqrt{(a^2+b^2)(c^2+d^2)} &\leq ac + bd \\
(a^2+b^2)(c^2+d^2) &\leq a^2c^2 + 2abcd + b^2d^2 \\
a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 &\leq a^2c^2 + 2abcd + b^2d^2 \\
a^2d^2 + b^2c^2 &\leq 2abcd \\
0 &\leq (ad - bc)^2
\end{aligned}$$

6

If z is a complex number such that $|z| = 1$, that is, such that $z\overline{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2$$

Say $z = a + bi$

$$\begin{aligned}
|(a+1) + bi|^2 + |(a-1) + bi| \\
(a+1)^2 + b^2 + (1-a)^2 + b^2 \\
a^2 + 2a + 1 + b^2 + 1 - 2a + a^2 + b^2 \\
2a^2 + 2b^2 + 2 \\
2(a^2 + b^2) + 2 \\
2|z| + 2 = 4
\end{aligned}$$

7

(Please grade this one) **Under what conditions does equality hold in the Schawrz inequality** The Schwarz inequality is equal when the two vectors are pointed in the same direction/one is a scalar multiple of the other or *vis versa*