CG – Assignment 05

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5.1) Normalization of the Phong BRDF:

a) **Assume that $k_s = 0$ (this means $\rho_s = 0$ and $\rho_d \in [0, 1]$). Derive the expression of Cd so that f is energy conserving.**

Energy conserving implies:

$$\int_{\Omega+} f(\omega_i, \omega_o) \cos\theta_0 \, d\omega_o \leq 1$$

$$\int_{\Omega+} k_d \cos\theta_0 \, d\omega_o \leq 1$$

$$\int_{\phi=0}^{2\pi} \int_{\theta_0=0}^{\pi/2} k_d \cos\theta_0 \sin\theta_0 \, d\theta_0 d\phi \leq 1$$

And solving the integral:

$$\int_{\phi=0}^{2\pi} \frac{-k_d d\phi}{4} (\cos\pi - \cos 0) \leq 1$$

$$\int_{\phi=0}^{2\pi} \frac{k_d}{2} d\phi \leq 1; \;\; k_d \pi \leq 1$$

Now, knowing that ρ_d=1 in the worst case we can conclude:

$$\frac{\rho_d}{C_d}\pi \leq 1 \; \rightarrow \; C_d \geq \pi$$

b) **Assume that $k_d = 0$ (this means $\rho_d = 0$ and $\rho_s \in [0, 1]$). Derive the expression of Cs so that f is energy conserving.**

Similarly, we start from the definition of energy conserving:

$$\int_{\Omega+} f(\omega_i, \omega_o) \cos\theta_0 \, d\omega_o \leq 1$$

$$\int_{\Omega+} k_s \cos^n(\omega_i, \omega_o) \cos\theta_0 \, d\omega_o \leq 1$$

$$\int_{\phi=0}^{2\pi} \int_{\theta_0=0}^{\pi/2} k_s \cos^n(\omega_i, \omega_o) \cos\theta_0 \sin\theta_0 \, d\theta_0 d\phi \leq 1$$

We are looking for the maximum energy, so we can use the provided hint. We have:

$$\int_{\phi=0}^{2\pi} \int_{\theta_0=0}^{\pi/2} k_s \cos^n(\omega_i, N) \cos\theta_0 \sin\theta_0 \, d\theta_0 d\phi \leq 1$$

In these conditions we can define the angle between ω_i and N as $\theta_0/2$:

$$\int_{\phi=0}^{2\pi}\int_{\theta_0=0}^{\pi/2} k_s\left(\cos\frac{\theta_0}{2}\right)^n \cos\theta_0 \sin\theta_0\, d\theta_0 d\phi \le 1$$

To solve it we first need to transform the half-angle cosine and then we will substitute $t=\cos\theta_0$, which will make $dt=-\sin\theta_0 d\theta_0$:

$$\int_{\phi=0}^{2\pi}\int_{\theta_0=0}^{\pi/2} k_s\left(\sqrt{\frac{1+\cos\theta_0}{2}}\right)^n \cos\theta_0 \sin\theta_0\, d\theta_0 d\phi \le 1$$

$$\int_{\phi=0}^{2\pi}\int_{\theta_0=0}^{\pi/2} k_s\left(\frac{1+\mathrm{t}}{2}\right)^{n/2} \mathrm{t}\, dt d\phi \le 1$$

$$-2\pi\int_{1}^{0} k_s\left(\frac{1+\mathrm{t}}{2}\right)^{n/2} \mathrm{t}\, dt = 2\pi\int_{0}^{1} k_s\left(\frac{1+\mathrm{t}}{2}\right)^{n/2} \mathrm{t}\, dt \le 1$$

$$2\pi\int_{0}^{1} k_s\left(\frac{1+\mathrm{t}}{2}\right)^{n/2} \mathrm{t}\, dt \le 1$$

Integrating by parts using f=1 and g' = $((1+t)/2)^{n/2}$:

$$2\pi k_s\left(\left[\frac{4}{n+2}t\left(\frac{1+t}{2}\right)^{(n+2)/2}\right]_{t=0}^{t=1} - \frac{4}{n+2}\int_0^1\left(\frac{1+t}{2}\right)^{(n+2)/2} dt\right) \le 1$$

$$\frac{8\pi k_s}{n+2}\left(\left[t\left(\frac{1+t}{2}\right)^{(n+2)/2}\right]_{t=0}^{t=1} - \frac{4}{n+4}\left[\left(\frac{1+t}{2}\right)^{(n+4)/2}\right]_{t=0}^{t=1}\right) \le 1$$

And going back to $t=\cos\theta_0$:

$$\frac{8\pi k_s}{n+2}\left[\cos\theta_0\left(\cos\frac{\theta_0}{2}\right)^{n+2} - \frac{4}{n+4}\left(\cos\frac{\theta_0}{2}\right)^{n+4}\right]_{\theta_0=\pi/2}^{\theta_0=0} \le 1$$

$$\frac{8\pi k_s}{(n+2)(n+4)}\left[\left(\cos\frac{\theta_0}{2}\right)^{n+2}\left(\left(\cos\frac{\theta_0}{2}\right)^2 - (n+4)\cos\theta_0\right)\right]_{\theta_0=0}^{\theta_0=\pi/2} \le 1$$

$$\frac{8\pi k_s}{(n+2)(n+4)}\left[\left(\cos\frac{\theta_0}{2}\right)^{n+2}(2(1+\cos\theta_0) - (n+4)\cos\theta_0)\right]_{\theta_0=0}^{\theta_0=\pi/2} \le 1$$

$$\frac{8\pi k_s}{(n+2)(n+4)}\left[\left(\cos\frac{\theta_0}{2}\right)^{n+2}(2 - (n+2)\cos\theta_0)\right]_{\theta_0=0}^{\theta_0=\pi/2} \le 1$$

$$\frac{8\pi k_s}{(n+2)(n+4)}\left(2\left(1/\sqrt{2}\right)^{n+2} - 2(1)^{n+2}(2-(n+2)1)\right) \le 1$$

$$\frac{8\pi k_s}{(n+2)(n+4)}\left(2^{1-(n+2)/2} - (2-(n+2))\right) \le 1$$

Finally arriving at:

$$\frac{8\pi k_s\left(2^{n/2}+n\right)}{(n+2)(n+4)} \leq 1$$

And knowing that ρ_s=1 in the worst case, we can conclude:

$$\frac{\rho_s}{C_s}\frac{8\pi\left(2^{n/2}+n\right)}{(n+2)(n+4)} \leq 1 \;\rightarrow\; C_s \geq \frac{8\pi\left(2^{n/2}+n\right)}{(n+2)(n+4)}$$

5.2) Analytical solution of the rendering equation in 2D:

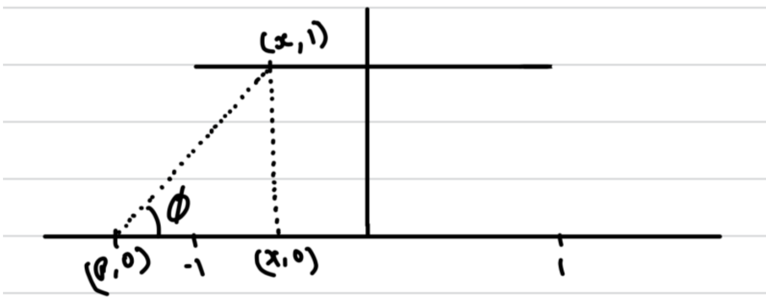
a) Solve the rendering equation analytically for each point x = (p, 0) and direction ω_o.

$$L(x,\omega_o) = L_e(x,\omega_o) + \int_0^{\pi} f_r\ (\omega, x, \omega_o) L(x,\omega)\cos\phi\, d\omega$$

We are told that f_r=1/π, the surface is not emitting any light so L_e=0 and for 2D we have that dω=dϕ. The equation can be then re-written as:

$$L(x,\omega_o) = \frac{1}{\pi}\int_0^{\pi} L(x,\omega)\cos\phi\, d\phi$$

We know that L=1 for every direction that intersects with the light source. We need to find the limit angles where these intersections occur so we can integrate over those.



If we define the triangle above, we can obtain the angle where we integrate as:

$$\sin\phi = \frac{1}{\sqrt{(p-x)^2+1^2}}$$

The light source goes from (-1,1) to (1,1). If we substitute the formula for these points, we obtain the limit angles:

$$\phi_0 = arcsin\frac{1}{\sqrt{(p-1)^2+1}}$$
$$\phi_1 = arcsin\frac{1}{\sqrt{(p+1)^2+1}}$$

Now we can use it in our expression and solve it:

$$L(x,\omega_o) = \frac{1}{\pi}\int_{arcsin\frac{1}{\sqrt{(p-1)^2+1}}}^{arcsin\frac{1}{\sqrt{(p+1)^2+1}}} \cos\phi \, d\phi$$

$$L(x,\omega_o) = \frac{-1}{\pi}[\sin\phi]_{arcsin\frac{1}{\sqrt{(p-1)^2+1}}}^{arcsin\frac{1}{\sqrt{(p+1)^2+1}}} = \frac{1}{\pi}\left[\frac{1}{\sqrt{(p-1)^2+1}} - \frac{1}{\sqrt{(p+1)^2+1}}\right]$$

b) Let S be the set of all points on the light source (the y = 0 plane can be ignored here) then the point form of the rendering equation in 2D is given by the expression. Again, solve the equation analytically, but now by integrating over the light source.

$$L(x,\omega_o) = L_e(x,\omega_o) + \int_{y\in S} f_r\ (\omega,x,\omega_o)L\big(y,-\omega_i(x,y)\big)\frac{\cos\phi_i\cos\phi_y}{|x-y|}dA_y$$

Since the light source is aligned with x axis, we will have dAy = dx and we will integrate from -1 to 1 keeping all variables except 'x' as constants.

$$L(x,\omega_o) = L_e(x,\omega_o) + \int_{-1}^{1}\frac{1}{\pi}\frac{\cos\phi_i\cos\phi_y}{|x-y|}dx$$

$$let\ x-y=t$$

$$= L_e(x,\omega_o) + \int_{-1-y}^{1-y}\frac{1}{\pi}\frac{\cos\phi_i\cos\phi_y}{|t|}dt$$

$$= L_e(x,\omega_o) + \int_{-1-y}^{0}\frac{1}{\pi}\frac{\cos\phi_i\cos\phi_y}{-t}dt + \int_{0}^{1-y}\frac{1}{\pi}\frac{\cos\phi_i\cos\phi_y}{t}dt$$

$$= L_e(x,\omega_o) - \frac{\cos\phi_i\cos\phi_y}{\pi}[\log(t)]_{-1-y}^{0} + \frac{\cos\phi_i\cos\phi_y}{\pi}[\log(t)]_{0}^{1-y}$$

$$= L_e(x,\omega_o) - \frac{\cos\phi_i\cos\phi_y}{\pi}[\log(t)]_{-1-y}^{0} + \frac{\cos\phi_i\cos\phi_y}{\pi}[\log(t)]_{0}^{1-y}$$

$$= L_e(x,\omega_o) - \frac{\cos\phi_i\cos\phi_y}{\pi}[\log(x-y)]_{-1}^{0} + \frac{\cos\phi_i\cos\phi_y}{\pi}[\log(x-y)]_{0}^{1}$$

$$= L_e(x,\omega_o) - \frac{\cos\phi_i\cos\phi_y}{\pi}[[\log(y)-\log(-1-y)] - [\log(1-y)-\log(-y)]]$$

$$= L_e(x,\omega_o) - \frac{\cos\phi_i\cos\phi_y}{\pi}\left[\left[\log\left(\frac{y}{-1-y}\right)\right] - \left[\log\left(\frac{1-y}{-y}\right)\right]\right]$$

$$= L_e(x,\omega_o) - \frac{\cos\phi_i\cos\phi_y}{\pi}\left[\log\left(\frac{-y^2}{(-1-y)(1-y)}\right)\right]$$

$$= L_e(x,\omega_o) - \frac{\cos\phi_i\cos\phi_y}{\pi}\left[\log\left(\frac{-y^2}{y^2-1}\right)\right]$$