CG – Assignment 04

Group 5: Basavaraj Rajendra Sunagad - 7015644 and Javier Usón Peirón - 7024337

4.1) The left house should be transformed into the house on the right by a linear transformation. The point M is at (4,5) and lines that seem to be parallel are indeed parallel. Specify the complete transformation matrix as a sequence of primitive transformations

Following the plot that we are presented with, we deduce that 3 transformations were performed: first a scale in both coordinates, then a shear, then a rotation by some angle and then a translation. We will use a 2D space, so 3D homogeneous coordinates.

For these operations, we need to know the angle of rotation, from that we will calculate any other parameters needed. If the longer side of the result figure is indeed parallel to the y axis, and the upper left corner of the dotted lined figure, the upper left corner of the final figure and **o** form an isosceles and rectangle triangle we can assume the rotation was by an angle of 45 degrees.

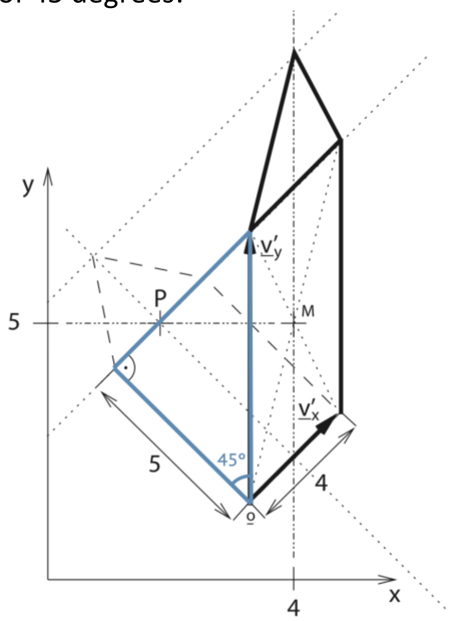

Figure 1: mentioned triangle

For the first transformation, the **scale**, we can see in the dotted lined figure that the width of the side goes from 1 to 4 and for the heigh it goes from 1 to 5, so the transformation can be expressed by the following matrix:

$$M_s = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next, we will talk about the **shear** operation. We apply this before rotation because this way the calculations are much simpler, but this means we won't get dotted lined figure as an intermediate step.

We can see that, from the dotted lined figure, an xy shear was performed (only x values change). The magnitude of this can be calculated getting the difference between upper left corner of the dotted lined figure and this same corner on the resulting figure. As the rotation is 45 degrees and the resulting figure is parallel to the y axis, this distance is equal to the distance between upper left and lower left corners, 5. To get the shear we divide by the y value of the upper left corner, 5 again, obtaining 1.

$$M_H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we perform a **rotation**. We already know the angle is 45 degrees so the matrix then will be:

$$M_R = \begin{bmatrix} \cos 45° & -\sin 45° & 0 \\ \sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, we **translate** the resulting figure to the point **o**. We know that the figure center (0.5,0.5) ends up in **m**=(4,5). Doing some basic trigonometry using the 45 degrees, we can obtain **o** from the position of **m**:

$$t_x = 4 - \sqrt{2}$$

$$t_y = 5 - \frac{\sqrt{2} * 5}{2} - \sqrt{2}$$

$$M_R = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Now, to obtain the points for the resulting figure, we just have to multiply the original points by this matrix:

$$M = M_T M_R M_H M_S = \begin{bmatrix} 2.8284 & 0 & 2.5858 \\ 2.8284 & 7.0711 & 0.0503 \\ 0 & 0 & 1 \end{bmatrix}$$

Additionally, if we wanted to obtain the dotted lined figure we would use:

$$D = M_T M_R M_S = \begin{bmatrix} 2.8284 & -3.5355 & 2.5858 \\ 2.8284 & 3.5355 & 0.0503 \\ 0 & 0 & 1 \end{bmatrix}$$

All these results were computed and tested using the Matlab script submitted alongside this document. It can also be found in this link. Here we tested what the result would be when applying the computed transformations to some key points, and we confirm that they end up in the expected coordinates.

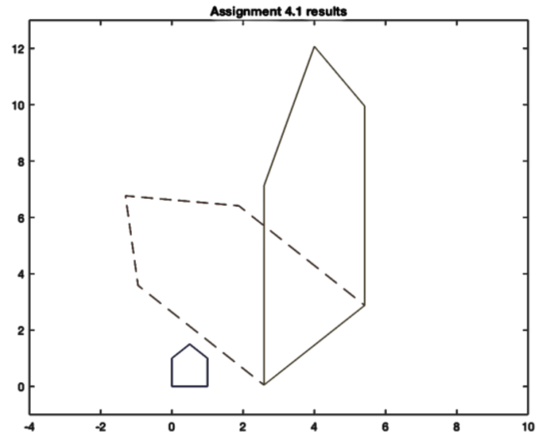

Figure 2: Reconstruction of both figures with matlab

4.2) Homogeneous coordinates:

a) **Show that multiplying the homogeneous point (x, y, z, w=/= 0) with an arbitrary scalar α=/= 0 yields an equivalent homogeneous point again.**

Let **p**=(x,y,z), we can define the homogeneous coordinates as **p**=(x*w,y*w,z*w,w)^T. Then:

$$\boldsymbol{p}' = \alpha \begin{bmatrix} xw \\ yw \\ zw \\ w \end{bmatrix} = \begin{bmatrix} \alpha xw \\ \alpha yw \\ \alpha zw \\ \alpha w \end{bmatrix}$$

To go back to the original representation, we divide the 3 first coordinates by the fourth coordinate:

$$\boldsymbol{p}' = \left(\frac{\alpha xw}{\alpha w}, \frac{\alpha yw}{\alpha w}, \frac{\alpha zw}{\alpha w}\right);\ \boldsymbol{p}' = (x, y, z)$$

b) **Show that the component wise addition of three homogeneous points (a_0,b_0,c_0,1), (a_1,b_1, c_1,1), and (a_2,b_2,c_2,1) yields the center between that points.**

Let's perform the sum:

$$\boldsymbol{p} = \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ 1 \end{bmatrix} + \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ 1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_0 + a_1 + a_2 \\ b_0 + b_1 + b_2 \\ c_0 + c_1 + c_2 \\ 3 \end{bmatrix}$$

Going back to the original point we obtain:

$$\boldsymbol{p} = \left(\frac{a_0 + a_1 + a_2}{3}, \frac{b_0 + b_1 + b_2}{3}, \frac{c_0 + c_1 + c_2}{3}\right)$$

Which can be recognized as the coordinates for the center of mass (centroid) of the triangle that the three proposed points form.

4.3) Homogeneous Lines in 2D: Prove that the cross product between two homogeneous points p = (p_x, p_y, p_w) and q = (q_x, q_y, q_w) yields the homogeneous coordinates of the connecting line.

The Euclidean coordinates for the points would be:

$$\boldsymbol{p} = \left({}^{p_x}/_{p_w}, {}^{p_y}/_{p_w}\right) \; and \; \boldsymbol{q} = \left({}^{q_x}/_{q_w}, {}^{q_y}/_{q_w}\right)$$

First, we will discuss how to get the representation of a line using the Euclidean coordinates of both points. We know that to define a line we can use a point and sum to it the direction vector multiplied by a scalar, this would yield:

$$\boldsymbol{t} = \boldsymbol{p} + \lambda(\boldsymbol{q} - \boldsymbol{p}) = \begin{cases} t_x = {}^{p_x}/_{p_w} + \lambda\left({}^{q_x}/_{q_w} - {}^{p_x}/_{p_w}\right) \\ t_y = {}^{p_y}/_{p_w} + \lambda\left({}^{q_y}/_{q_w} - {}^{p_y}/_{p_w}\right) \end{cases}$$

We now get lambda on both equations and make them equal:

$$\frac{t_x - {}^{p_x}/_{p_w}}{\left({}^{q_x}/_{q_w} - {}^{p_x}/_{p_w}\right)} = \frac{t_y - {}^{p_y}/_{p_w}}{\left({}^{q_y}/_{q_w} - {}^{p_y}/_{p_w}\right)}$$

And we try to make this expression equal to 0:

$$t_y = \frac{t_x\left({}^{q_y}/_{q_w} - {}^{p_y}/_{p_w}\right)}{\left({}^{q_x}/_{q_w} - {}^{p_x}/_{p_w}\right)} + \frac{-{}^{p_x}/_{p_w}\left({}^{q_y}/_{q_w} - {}^{p_y}/_{p_w}\right) + {}^{p_y}/_{p_w}\left({}^{q_x}/_{q_w} - {}^{p_x}/_{p_w}\right)}{\left({}^{q_x}/_{q_w} - {}^{p_x}/_{p_w}\right)}$$

$$t_y = \frac{t_x(q_y p_w - p_y q_w)}{(q_x p_w - p_x q_w)} + \frac{p_y(q_x p_w - p_x q_w) - p_x(q_y p_w - p_y q_w)}{p_w(q_x p_w - p_x q_w)}$$

$$t_x(p_y q_w - q_y p_w) + t_y(q_x p_w - p_x q_w) + \frac{p_x(q_y p_w - p_y q_w) - p_y(q_x p_w - p_x q_w)}{p_w} = 0$$

$$t_x(p_y q_w - q_y p_w) + t_y(q_x p_w - p_x q_w) + p_x q_y - p_y q_x + \frac{p_x q_w p_y - p_y q_w p_x}{p_w} = 0$$

And finally, we arrive at:

$$t_x(p_y q_w - q_y p_w) + t_y(q_x p_w - p_x q_w) + (p_x q_y - p_y q_x) = 0$$

On the other hand, let's calculate the cross product:

$$\begin{bmatrix} p_x \\ p_y \\ p_w \end{bmatrix} \mathrm{x} \begin{bmatrix} q_x \\ q_y \\ q_w \end{bmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ p_x & p_y & p_w \\ q_x & q_y & q_w \end{vmatrix} = \begin{bmatrix} p_y q_w - q_y p_w \\ q_x p_w - p_x q_w \\ p_x q_y - p_y q_x \end{bmatrix}$$

We can see now that if we express this as $x_x t_x + x_y t_y + x_w$ we obtain:

$$t_x(p_y q_w - q_y p_w) + t_y(q_x p_w - p_x q_w) + (p_x q_y - p_y q_x) = 0$$

And so, we demonstrate that both processes are equivalent, but doing the cross product is significantly simpler.