CG – Assignment 06

Group 5: Basavaraj Rajendra Sunagad - 7015644 and Javier Usón Peirón - 7024337

5.1) Interpolation:

a) **Show that the linear interpolated value of a function can be computed as shown:**

To perform an interpolation, we multiply the value of the known points to their distance to the interpolated point but keeping in mind the contribution is bigger the closer it is to the known point. we can write this as:

$$f(x) \approx [1-(x-x_0)]f(x_0) + [1-(x_1-x)]f(x_1)$$

But this only works if the difference between x_0 and x_1 is 1. If this is not the case, we need to normalize these distances:

$$f(x) \approx \left(1-\frac{x-x_0}{x_1-x_0}\right)f(x_0) + \left(1-\frac{x_1-x}{x_1-x_0}\right)f(x_1)$$

Now that the distance is normalized, we can rewrite as:

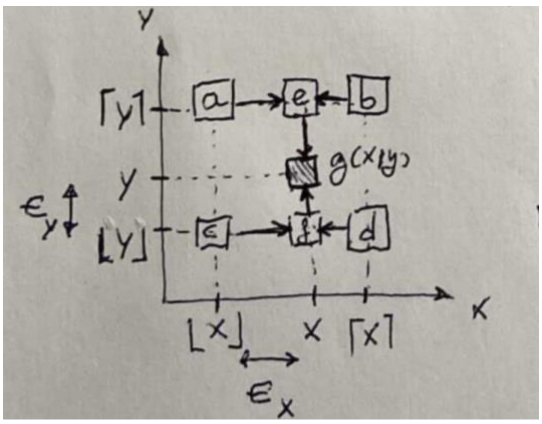
$$f(x) \approx \left(1-\frac{x-x_0}{x_1-x_0}\right)f(x_0) + \left(\frac{x-x_0}{x_1-x_0}\right)f(x_1)$$

And reordering we can conclude:

$$f(x) \approx f(x_0) + (x-x_0)\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

b) **Derive an equation for computing bilinear interpolated value of f(x,y). You can assume that the values of f at the four points (x0, y0), (x0, y1), (x1, y0) and (x1, y1) are given.**

As it has been explained in the assignment sheet, to perform a bilinear interpolation we perform 3 individual interpolations as shown in the figure:



This way we calculate e and f from the figure (first two interpolations in x direction)

$$\begin{cases} e \approx f(x_0, y_1) + (x - x_0)\dfrac{f(x_1, y_1) - f(x_0, y_1)}{x_1 - x_0} \\ f \approx f(x_0, y_0) + (x - x_0)\dfrac{f(x_1, y_0) - f(x_0, y_0)}{x_1 - x_0} \end{cases}$$

And now we get the interpolated value by interpolating e and f in y direction:

$$f(x, y) \approx f + (y - y_0)\frac{e - f}{y_1 - y_0}$$

If we substitute values, we obtain:

$$f(x, y) \approx f(x_0, y_0) + (x - x_0)\frac{f(x_1, y_0) - f(x_0, y_0)}{x_1 - x_0}$$
$$+(y - y_0)\frac{f(x_0, y_1) + (x - x_0)\frac{f(x_1, y_1) - f(x_0, y_1)}{x_1 - x_0} - f(x_0, y_0) - (x - x_0)\frac{f(x_1, y_0) - f(x_0, y_0)}{x_1 - x_0}}{y_1 - y_0}$$

Now we need to reorder it so the value for each know point only appears once:

$$\begin{aligned} f(x, y) \approx f(x_0, y_0) &+ \frac{x - x_0}{x_1 - x_0} f(x_1, y_0) - \frac{x - x_0}{x_1 - x_0} f(x_0, y_0) + \frac{y - y_0}{y_1 - y_0} f(x_0, y_1) \\ &+ \frac{y - y_0}{y_1 - y_0}\frac{x - x_0}{x_1 - x_0} f(x_1, y_1) - \frac{y - y_0}{y_1 - y_0}\frac{x - x_0}{x_1 - x_0} f(x_0, y_1) \\ &- \frac{y - y_0}{y_1 - y_0} f(x_0, y_0) - \frac{y - y_0}{y_1 - y_0}\frac{x - x_0}{x_1 - x_0} f(x_1, y_0) \\ &+ \frac{y - y_0}{y_1 - y_0}\frac{x - x_0}{x_1 - x_0} f(x_0, y_0) \end{aligned}$$

The final result is:

$$\begin{aligned} f(x, y) \approx &\left(1 - \frac{x - x_0}{x_1 - x_0} - \frac{y - y_0}{y_1 - y_0} + \frac{y - y_0}{y_1 - y_0}\frac{x - x_0}{x_1 - x_0}\right) f(x_0, y_0) \\ &+ \left(\frac{x - x_0}{x_1 - x_0}\left(1 - \frac{y - y_0}{y_1 - y_0}\right)\right) f(x_1, y_0) \\ &+ \left(\frac{y - y_0}{y_1 - y_0}\left(1 - \frac{x - x_0}{x_1 - x_0}\right)\right) f(x_0, y_1) \\ &+ \frac{y - y_0}{y_1 - y_0}\frac{x - x_0}{x_1 - x_0} f(x_1, y_1) \end{aligned}$$

As an additional note, this expression can be simplified if we define the normalized distances as:

$$Let\ \epsilon_x = \frac{x - x_0}{x_1 - x_0} and\ \epsilon_y = \frac{y - y_0}{y_1 - y_0}$$

$$f(x,y) \approx (1-\epsilon_x-\epsilon_y+\epsilon_x\epsilon_y)f(x_0,y_0)+\left(\epsilon_x(1-\epsilon_y)\right)f(x_1,y_0)$$
$$+\left(\epsilon_y(1-\epsilon_x)\right)f(x_0,y_1)+\epsilon_x\epsilon_y f(x_1,y_1)$$

5.2) Fourier Transformation: Show that the Fourier transformation of the box function B(x) is a sinc type function:

Let's use the Fourier transform expression:

$$F(k) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi ikx}dx$$
$$F(k) = \int_{-1}^{+1} 1 * e^{-2\pi ikx}dx$$
$$F(k) = \left[\frac{-1}{2\pi ik}e^{-2\pi ikx}\right]_{-1}^{+1}$$
$$F(k) = \frac{1}{2\pi ik}\left(e^{2\pi ik}-e^{-2\pi ik}\right)$$
$$F(k) = \frac{1}{2\pi ik}2i\sin(2\pi k)$$

And thus, arriving at the sinc definition:

$$F(k) = 2\frac{\sin(2\pi k)}{2\pi k}$$

5.3) Sampling Theory

a) Is an exact signal reconstruction of f(x) possible? If so, why?

When a signal is sampled, in the frequency (Fourier) domain we observe replicas of its spectrum appear centered on integer multiples of the sampling frequency. If the signal fulfills the Nyquist property, these replicas will NOT overlap, and the reconstruction of the original signal is theoretically possible.

b) What mathematical operations in Fourier space need to be performed to reconstruct the image?

In the Fourier domain we can perform a filtering of the replica centered in the 0 frequency. This will eliminate the rest of the replicas and yield the original signal. To perform an exact reconstruction, we would have to use a perfect box filter (and defined on 5.2).

c) What mathematical operations in image space need to be performed to reconstruct the image?

As proved on 5.2, the act of filtering using a box function is equal to a convolution with a sinc function in the image domain. This can be seen as an actual interpolation, and it can be argued that using the sinc interpolator is not the optimal procedure when working on real life implementations.