CG – Assignment 01

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1.1) Let u, v, w be vectors in R³. Prove or disprove the following statements:

a) (**u**·**u**) is the square of the length of **u**:

- Let **u** = (a, b)

$$(\boldsymbol{u} \cdot \mathbf{u}) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a^2 + b^2 = \sqrt{(a^2+b^2)^2} = \|\boldsymbol{u}\|^2$$

b) (**u**·**v**)² + (**u**×**v**)² =|**u**|²|**v**|²:

- Let **u** = (a, b) and **v** = (c, d)

$$(\boldsymbol{u} \cdot \boldsymbol{v})^2 = \left(\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right)^2 = (ac+bd)^2$$

$$(\boldsymbol{u} \times \boldsymbol{v}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - dc)\boldsymbol{k}$$

- With **k** being the unitary vector perpendicular to the plane containing **u** and **v**.

$$(\boldsymbol{u} \times \boldsymbol{v})^2 = [(ad-dc)\boldsymbol{k}]^2 = (ad-dc)^2$$

- Now:

$$\begin{aligned}(\boldsymbol{u} \cdot \boldsymbol{v})^2 + (\boldsymbol{u} \times \boldsymbol{v})^2 &= (ac+bd)^2 + (ac+bd)^2 \\ &= a^2c^2 + b^2d^2 + 2acbd + a^2d^2 + b^2c^2 - 2acbd \\ &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 \\ &= a^2(c^2+d^2) + b^2(c^2+d^2) \\ &= (a^2+b^2) + (c^2+d^2)\end{aligned}$$

$$\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 = \sqrt{(a^2+b^2)^2} + \sqrt{(c^2+d^2)^2} = (a^2+b^2) + (c^2+d^2)$$

- Effectively proving (**u**·**v**)² + (**u**×**v**)² =|**u**|²|**v**|².

c) |**u**+**v**|² =|**u**|² +|**v**|² +2(**u**·**v**):

- Let **u** = (a, b) and **v** = (c, d), so **u** + **v** = (a + c, b + d)

$$\|\boldsymbol{u}\|^2 = (a^2+b^2)$$
$$\|\boldsymbol{v}\|^2 = (c^2+d^2)$$
$$(\boldsymbol{u} \cdot \boldsymbol{v}) = (ac+bd)$$

- Now:

$$\begin{aligned}\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= \left[\sqrt{(a+c)^2+(b+d)^2}\right]^2 = (a+c)^2+(b+d)^2 \\ &= a^2 + c^2 + 2ac + b^2 + d^2 + 2bd \\ &= a^2 + b^2 + c^2 + d^2 + 2(ac+bd) \\ &= \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + 2(\boldsymbol{u} \cdot \mathbf{v})\end{aligned}$$

d) $(\mathbf{u}\cdot\mathbf{v})^2 = \mathbf{u}^2\cdot\mathbf{v}^2$, where $\mathbf{t}^2$, for vector $\mathbf{t}$, is defined as $\mathbf{t}\cdot\mathbf{t}$:

- Let $\mathbf{u}$ = (a, b) and $\mathbf{v}$ = (c, d)

$$(\boldsymbol{u}\cdot\boldsymbol{v})^2 = (ac+bd)^2 = a^2c^2 + \ b^2d^2 + 2acbd$$

$$(\boldsymbol{u}\cdot\mathbf{u}) = a^2 + b^2 \ and \ (\boldsymbol{v}\cdot\mathbf{v}) = c^2 + d^2$$
$$(\boldsymbol{u}^2\cdot\boldsymbol{v}^2) = (a^2+b^2)(c^2+d^2) = a^2c^2 + \ a^2d^2 + b^2c^2 + \ b^2d^2$$

- Comparing both results we can see that the condition is **not** met. Except for some special cases like a=b=c=d; or a=c=1 and b=d=0; or a=b=1 and b=c=0.

e) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$:

- Let $\mathbf{u} = u_1\hat{\mathbf{i}} + u_2\mathbf{j} + u_3\mathbf{k}$; $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\mathbf{j} + v_3\mathbf{k}$; and $\mathbf{w} = w_1\hat{\mathbf{i}} + w_2\mathbf{j} + w_3\mathbf{k}$;
  With $\hat{\mathbf{i}}$, $\mathbf{j}$ and $\mathbf{k}$ being the unitary vectors for x, y, z in R^3.

$$\begin{aligned}
\boldsymbol{u}\,\mathrm{x}\,(\boldsymbol{u}\,\mathrm{x}\,\mathbf{w}) &= \boldsymbol{u}\ x \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\
&= \boldsymbol{u}\ x\ [\boldsymbol{i}(v_2w_3 - v_3w_2) - \boldsymbol{j}(v_1w_3 - v_3w_1) + \boldsymbol{k}(v_1w_2 - v_2w_1)] \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ (v_2w_3 - v_3w_2) & (v_1w_3 - v_3w_1) & (v_1w_2 - v_2w_1) \end{vmatrix} \\
&= \boldsymbol{i}\big[\big(u_2(v_1w_2 - v_2w_1) + u_3(v_1w_3 - v_3w_1)\big)\big] \\
&\quad + \boldsymbol{j}\big[\big(u_1(v_1w_2 - v_2w_1) - u_3(v_2w_3 - v_3w_2)\big)\big] \\
&\quad + \boldsymbol{k}\big[-\big(u_1(v_1w_3 - v_3w_1) - u_2(v_2w_3 - v_3w_2)\big)\big]
\end{aligned}$$

- This way, we obtain:

$$\begin{aligned}
\boldsymbol{u}\,\mathrm{x}\,(\boldsymbol{u}\,\mathrm{x}\,\mathbf{w}) = \boldsymbol{i}&[(u_2v_1w_2 - u_2v_2w_1 + u_3v_1w_3 - u_3v_3w_1)] \\
+ \boldsymbol{j}&[(-u_1v_1w_2 + u_1v_2w_1 + u_3v_2w_3 - u_3v_3w_2)] \\
+ \boldsymbol{k}&[(-u_1v_1w_3 + u_1v_3w_1 - u_2v_2w_3 + u_2v_3w_2)]
\end{aligned}$$

- Now, using the same notation:

$$\begin{aligned}
&(\boldsymbol{u}\cdot\boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{w} \\
&\qquad = [(u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k})\cdot(w_1\boldsymbol{i} + w_2\boldsymbol{j} + w_3\boldsymbol{k})]\boldsymbol{v} \\
&\qquad - [(u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k})\cdot(v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k})]\boldsymbol{w} \\
&\qquad = \boldsymbol{i}[(u_1w_1v_1 + u_2w_2v_1 + u_3w_3v_1)] \\
&\qquad + \boldsymbol{j}[(u_1w_1v_2 + u_2w_2v_2 + u_3w_3v_2)] \\
&\qquad + \boldsymbol{k}[(u_1w_1v_3 + u_2w_2v_3 + u_3w_3v_3)] \\
&\qquad - \boldsymbol{i}[(u_1v_1w_1 + u_2v_2w_1 + u_3v_3w_1)] \\
&\qquad + \boldsymbol{j}[(u_1v_1w_2 + u_2v_2w_2 + u_3v_3w_2)] \\
&\qquad + \boldsymbol{k}[(u_1v_1w_3 + u_2v_2w_3 + u_3v_3w_3)]
\end{aligned}$$

- We finally obtain:

$$(\boldsymbol{u}\cdot\boldsymbol{w})\boldsymbol{v}-(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{w}$$
$$= \boldsymbol{i}[(u_2v_1w_2+u_3v_1w_3-u_2v_2w_1-u_3v_3w_1)]$$
$$+\boldsymbol{j}[(u_1v_2w_1+u_3v_2w_3-u_1v_1w_2-u_3v_3w_2)]$$
$$+\boldsymbol{k}[(u_1v_1w_3+u_2v_2w_3-u_1v_3w_1-u_2v_3w_2)]$$

- Equivalent to the expression obtained previously and proving that **u×(v×w) = (u·w)v−(u·v)w.**

f) (**u**·**v**)·**w** = **u**·(**v**·**w**):
- Let **u** = (a, b); **v** = (c, d) and **w**=(e, f)

$$(\boldsymbol{u}\cdot\boldsymbol{v})\cdot\boldsymbol{w}=(ac+bd)\cdot(e,f)=(ace+bde,acf+bdf)$$

$$\boldsymbol{u}\cdot(\boldsymbol{v}\cdot\boldsymbol{w})=(a,b)\cdot(ce+df)=(ace+adf,bce+bdf)$$

- Comparing both results we can see that the condition is **not** met. This was expectable, knowing that the dot product is **not** associative.

1.2) How are the values of pos, dir and up changing when the user decides to:
- **Rotate left (yaw) by an angle α.**
- **Look up (pitch) by an angle β.**
- **Move forward by a distance d.**

a) Space movement:

In this case, the relevant direction is **dir**, and as **up** might not be perpendicular to it, so we create an orthonormal basis from v.

$$\boldsymbol{u}=\frac{(\boldsymbol{dir}\ \mathrm{x}\ \mathbf{up})}{\|(\boldsymbol{dir}\ \mathrm{x}\ \mathbf{up})\|}$$
$$\boldsymbol{v}=\frac{(\boldsymbol{u}\ \mathrm{x}\ \mathbf{dir})}{\|(\boldsymbol{u}\ \mathrm{x}\ \mathbf{dir})\|}$$
$$\boldsymbol{w}=\frac{-(\boldsymbol{dir})}{\|\boldsymbol{dir}\|}$$

- Rotate left by an angle α: **dir** rotates in the direction of vector **u**, **up** follows that rotation, **pos** does not change as it is a point. We first normalize the vectors, and then multiply them by their original length.

$$\boldsymbol{dir}'=\left(\frac{(\boldsymbol{dir})}{\|(\boldsymbol{dir})\|}-tan(\alpha)\boldsymbol{u}\right)\|(\boldsymbol{dir})\|$$
$$\boldsymbol{up}'=\left(\frac{(\boldsymbol{up})}{\|(\boldsymbol{up})\|}-tan(\alpha)\boldsymbol{u}\right)\|(\boldsymbol{up})\|$$
$$\boldsymbol{pos}'=\boldsymbol{pos}$$

- Look up by an angle β: Same procedure, but in the direction of **v**.

$$\boldsymbol{dir}' = \left(\frac{(\boldsymbol{dir})}{\|(\boldsymbol{dir})\|} - tan(\beta)\boldsymbol{v}\right)\|(\boldsymbol{dir})\|$$
$$\boldsymbol{up}' = \left(\frac{(\boldsymbol{up})}{\|(\boldsymbol{up})\|} - tan(\beta)\boldsymbol{v}\right)\|(\boldsymbol{up})\|$$
$$\boldsymbol{pos}' = \boldsymbol{pos}$$

- Move forward by a distance d: Vectors do not change with translations, only **pos** will move in the direction of **dir**, escalated by d

$$\boldsymbol{pos}' = \boldsymbol{pos} + d\frac{(\boldsymbol{dir})}{\|(\boldsymbol{dir})\|}$$
$$\boldsymbol{dir}' = \boldsymbol{dir}$$
$$\boldsymbol{up}' = \boldsymbol{up}$$

b) Ground Movement:

Now we have the restriction of the floor, this means that **up** will be our relevant vector, and we will build our orthonormal base using it.

$$\boldsymbol{u} = \frac{(\boldsymbol{dir}\ x\ \mathbf{up})}{\|(\boldsymbol{dir}\ x\ \mathbf{up})\|}$$
$$\boldsymbol{v} = \frac{(\boldsymbol{up})}{\|(\boldsymbol{up})\|}$$
$$\boldsymbol{w} = \frac{(\boldsymbol{u}\ x\ \mathbf{up})}{\|\boldsymbol{u}\ x\ \boldsymbol{up}\|}$$

- Rotate left by an angle α: similar to last case, but now **up** won't change as it is fixed with the floor.

$$\boldsymbol{dir}' = \left(\frac{(\boldsymbol{dir})}{\|(\boldsymbol{dir})\|} - tan(\alpha)\boldsymbol{u}\right)\|(\boldsymbol{dir})\|$$
$$\boldsymbol{up}' = \boldsymbol{up}$$
$$\boldsymbol{pos}' = \boldsymbol{pos}$$

- Look up by an angle β: Again, same procedure, but in the direction of **v**.

$$\boldsymbol{dir}' = \left(\frac{(\boldsymbol{dir})}{\|(\boldsymbol{dir})\|} - tan(\beta)\boldsymbol{v}\right)\|(\boldsymbol{dir})\|$$
$$\boldsymbol{up}' = \boldsymbol{up}$$
$$\boldsymbol{pos}' = \boldsymbol{pos}$$

- Move forward by a distance d: Now we are restricted to move in the floor plain, so to find our direction we will need to project **dir** into that ground plain → $\boldsymbol{p} = \boldsymbol{dir} - \|\boldsymbol{dir}\|\, tan(\gamma)\, \boldsymbol{v}$; with γ being the angle that **dir** forms this the floor on **up** direction:

$$\boldsymbol{pos}' = \boldsymbol{pos} + d\,\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}$$
$$\boldsymbol{dir}' = \boldsymbol{dir}$$
$$\boldsymbol{up}' = \boldsymbol{up}$$