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CS 536 HW2

1. State

Task 1.1

Given: Let $\sigma = \{x = 5, y = 2, z = 1, a = [8; 2; 5]\}$.
For questions where the answer is a state, please give it as a set of variable-value pairs (e.g., $\{x = 5, y = 2, z = 1, a = [8; 2; 5]\}$, not as a state update).

a) What is $\sigma[x \mapsto 3][x \mapsto 5]$?

Ans. $\sigma[x \mapsto 3][x \mapsto 5] = \{x = 5, y = 2, z = 1, a = [8; 2; 5]\}$

b) What is $\sigma[w \mapsto 4](w)$?

Ans. $\sigma[w \mapsto 4](w) = 4$

c) What is $\sigma[y \mapsto 7][w \mapsto 8]$?

Ans. $\sigma[y \mapsto 7][w \mapsto 8] = \{x = 5, y = 7, z = 1, a = [8; 2; 5], w = 8\}$

d) What is $|\sigma(a)|$?

Ans. $|\sigma(a)| = 3$
(Note: $|\sigma(a)|$ means the size/length of the array)

Task 1.2

For each of the following, say whether the satisfaction holds or not. If not, why?

a) $\{x = 0\} \vDash \forall y \in \mathbb{Z}. x \le y^2$

Ans. As per my understanding, x in the $\forall y$ is a free variable and we have been provided its value in the state which is $x = 0$.
So, considering any value of y, say y = any positive number (example: 3), $0 \le 3^2$.
Similarly, if we consider y = any negative number (example: -3), $0 \le (-3)^2$
This tells us that the given satisfaction holds.

b) $\{x = 2, y = 4\} \vDash \exists x \in \mathbb{Z}. x > y$

Ans. As per my understanding, y in the $\exists x$ is a free variable and we have been provided its value in the state which is $y = 4$. Also, by the concept of shadowing, we understand that x given in the state is different from x in the $\exists x$ bound. So, we can just use the value of y.

Considering a value of x, say x = 5, 5 > 4. Similarly, we can think of another value as well, say x = 10, we just have to show that there exists one such value.
Based on all these things, we say that the given satisfaction holds.

c) $\{x = 1, y = 2\} \models \forall z \in \mathbb{Z}. z > x \rightarrow y \cdot z > 0$

Ans. As per my understanding, x and y in the ∀z are free variables and we have been provided their values in the state which is x = 1, y = 2.
Considering any positive value of z, say z = 5, $5 > 1 \rightarrow 2 \cdot 5 > 0$ (i.e., T → T, which is T).
Considering any negative value of z, say z = -10, $-10 > 1 \rightarrow 2 \cdot (-10) > 0$ (i.e., F → F, which is T).
Based on all these things, we say that the given satisfaction holds.

d) $\{x = 5\} \models \exists y \in \mathbb{Z}. 2 \cdot y = x$

Ans. As per my understanding, x in the ∃y is a free variable and we have been provided its value in the state which is x = 5.
A basic logical idea is that given value of x is an odd number and multiplying 2 with any value of $y \in \mathbb{Z}$ will never result in an odd number. This gives us an idea that such a number does not exists.
Based on this we say that the given satisfaction does not hold.

Task 1.3

For each of the situations below, fill in the blanks to describe when the situation holds. Fill in with "some", "all" or "no".

a) $\models \exists x \in \mathbb{Z}. \forall y \in \mathbb{Z}. p$ if for ____ states σ, it is true that $\sigma[x \mapsto \alpha_1][y \mapsto \alpha_2] \models p$ for ____ $\alpha_1 \in \mathbb{Z}$ and ____ $\alpha_2 \in \mathbb{Z}$.

Ans. $\models \exists x \in \mathbb{Z}. \forall y \in \mathbb{Z}. p$ if for <u>all</u> states σ, it is true that $\sigma[x \mapsto \alpha_1][y \mapsto \alpha_2] \models p$ for <u>some</u> $\alpha_1 \in \mathbb{Z}$ and <u>all</u> $\alpha_2 \in \mathbb{Z}$.

This statement asserts that there exists at least one integer value for x such that for all integer values of y, for some integer values α_1 and α_2 if we substitute x with α_1 and y with α_2 in some state σ, the predicate p is true.

b) $\models \neg(\forall x \in \mathbb{Z}. \exists y \in \mathbb{Z}. q)$ if for ____ states σ, it is true that $\sigma[x \mapsto \alpha_1][y \mapsto \alpha_2] \models q$ for ____ $\alpha_1 \in Z$ and ____ $\alpha_2 \in Z$.

Ans. $\models \neg(\forall x \in \mathbb{Z}. \exists y \in \mathbb{Z}. q)$ if for <u>some</u> states σ, it is true that $\sigma[x \mapsto \alpha_1][y \mapsto \alpha_2] \models q$ for <u>some</u> $\alpha_1 \in Z$ and <u>all</u> $\alpha_2 \in Z$.

Since this statement is a negated statement, it means we are trying to find cases where the ($\forall x \in \mathbb{Z}.\exists y \in \mathbb{Z}.q$) is false so that the negated statement is True and holds. As per my understanding, it will not be true for all states so it should be "some" states. The negation of the universal quantifier ($\forall x \in \mathbb{Z}$) in the statement means that there exists at least one x in $\mathbb{Z}$ for which the negated statement ($\exists y \in \mathbb{Z}.q$) is true. Similarly, the negation of the existential quantifier ($\exists y \in \mathbb{Z}.q$) means that q is true for all values of y in $\mathbb{Z}$.

2. IMP Syntax and Semantics

Task 2.1

Evaluate each of the following expressions with the state.

$$\sigma = \{x = 5, y = 2, z = 1, w = T, v = F, a = [8; 2; 5]\}$$

a) $\sigma(x * y)$

Ans. $\sigma(x * y) = \sigma(x) * \sigma(y) = 5 * 2 = 10$.

b) $\sigma(\text{if } x > y \text{ then } x - z \text{ else } y - z)$

Ans. To solve $\sigma(\text{if } x > y \text{ then } x - z \text{ else } y - z)$, we first solve the conditional $\sigma(x > y)$.

$\sigma(x > y) = \sigma(x) > \sigma(y) = 5 > 2 = T$ (True)

As the conditional value is True, if part of the if else will be solved.

$\sigma(x - z) = \sigma(x) - \sigma(z) = 5 - 1 = 4$.

c) $\sigma(a[z] + x)$

Ans. $\sigma(a[z] + x) = \sigma(a[z]) + \sigma(x) = \sigma(a[\sigma(z)]) + \sigma(x) = \sigma(a[1]) + \sigma(x) = 2 + 5 = 7$.

d) $\sigma(w \vee v)$

Ans. $\sigma(w \vee v) = \sigma(w) \vee \sigma(v) = T \vee F = T$.

e) $\sigma(a[size(a) - z])$

Ans.
$\sigma(a[size(a) - z])$
$= \sigma(a[\sigma(size(a) - z)])$

= σ(a[σ(size(a)) – σ(z)])
= σ(a[|σ(a)| – σ(z)])
= σ(a[3 – 1])
= σ(a[2])
= 5.

Task 2.2

Write a program (statement) in the syntax of IMP that “truncates” an array a to length x, setting any elements after that to 0.

- You can assume that both a and x are in the state at the beginning of the program.
- If S is your program, and ⟨S, σ⟩ $\rightarrow^*$ ⟨skip, σ′⟩, then for all 0 ≤ i < x, we should have σ′(a[i]) = σ(a[i]) and for all x ≤ i < |σ(a)|, we should have σ′(a[i]) = 0.
- As an example, if σ = {a = [1; 2; 3; 4; 5], x = 3}, then σ′(a) = [1; 2; 3; 0; 0].
- If x ≥ |σ(a)| (i.e., x is bigger than the length of a), then the array is not changed.
- If x ≤ 0, then σ′(a) should have all 0s.
- It’s OK if your program changes the value of x.
- Your program should never access an out-of-bounds array element for any proper state (a state that contains values for both x and a).

Ans.

```
i := 0̅;
while (i < σ(size(a)))
do
    if (i < x) then
      i := i + 1̅;
      skip
    else
      a[i] := 0̅;
      i := i + 1̅
    fi
od
```

I applied a very simple logic by traversing the array. I counted the number of indices I have seen till the value of x. If I cross the value x, I start truncating the value of the array to 0, index by index.

Since, I am learning IMP language, I tried this first in Python and converted it to IMP using the rules taught in class.

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```python
a = [1,2,3,4,5]
x = 3

i=0
while i<len(a):
    if i<x:
        i = i+1
        continue
    else:
        a[i] = 0
        i = i+1

print(a)
```

```
[1, 2, 3, 0, 0]
```

```python
a = [1,2,3,4,5]
x = 6

i=0
while i<len(a):
    if i<x:
        i = i+1
        continue
    else:
        a[i] = 0
        i = i+1

print(a)
```

```
[1, 2, 3, 4, 5]
```

```python
a = [1,2,3,4,5]
x = -1

i=0
while i<len(a):
    if i<x:
        i = i+1
        continue
    else:
        a[i] = 0
        i = i+1

print(a)
```

```
[0, 0, 0, 0, 0]
```

I tried three values of x to check if my program works for all the cases given in the question.

Task 2.3

Consider the program S = while x > y do x := y od, in the state σ = {x = 3, y = 2}.

a) Evaluate the program using the small-step operational semantics we saw in class, i.e., give a series of configurations such that $\langle S, \sigma\rangle \rightarrow \langle S', \sigma'\rangle \rightarrow^{*} \langle skip, \sigma''\rangle$.

Ans.
Given, S = while x > y do x := y od
σ = {x = 3, y = 2}

$\langle S, \{x = 3, y = 2\}\rangle$	(Using While rule)
$\rightarrow \langle$if x > y then x := y; S else skip fi, $\{x = 3, y = 2\}\rangle$	(Using if else rule)
$\rightarrow \langle x := y; S, \{x = 3, y = 2\}\rangle$	(Using x:=e rule)
$\rightarrow \langle skip; S, \{x = 3, y = 2\}[x \mapsto y]\rangle$	(Using skip & statement rule)
$\rightarrow \langle S, \{x = 3, y = 2\}[x \mapsto 2]\rangle$	(State Update)
$\rightarrow \langle S, \{x = 2, y = 2\}\rangle$	(Using While rule)
$\rightarrow \langle$if x > y then x := y; S else skip fi, $\{x = 2, y = 2\}\rangle$	(Using if else rule)
$\rightarrow \langle skip, \{x = 2, y = 2\}\rangle$	(Final Evaluated State)

b) What is M (S, σ)? Explain how you know (potentially by showing your calculation, but other explanations are also acceptable).

Ans.
By Definition of Big step semantics,

$$M(S, \sigma) = \{\sigma'\} \Leftrightarrow \langle S, \sigma\rangle \rightarrow^{*} \langle skip, \sigma'\rangle$$

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Given, S = while x > y do x := y od
σ = {x = 3, y = 2}

Evaluating by Big step semantics:
Let Σ_k be the set of states we might have after running the loop k times:
$\Sigma_0 = \{\sigma\}$
$\Sigma_{k+1} = \cup_{\sigma \in \Sigma_k} M(S, \sigma)$

Applying the definition to our problem,
Taking while x > y do x := y od in the state {x = 3, y = 2}

$\Sigma_0 = \{\{ x = 3, y = 2\}\}$
$\Sigma_1 = \cup_{\sigma \in \Sigma_0} M(x := y, \sigma) = M(x := y, \{x = 3, y = 2\}) = M(x := 2, \{x = 3, y = 2\}) = \{\{x = 2, y = 2\}\}$

Here, we have reached a Σ_k, such that for all $\sigma \in \Sigma_k$, the conditional expression (x > y) is false, so the loop stops.

We get, M(S, σ) = {σ'} = {{x = 2, y = 2}}.

3. One more wrap-up question

Task 3.1

How long (approximately) did you spend on this homework, in total hours of actual working time?

Ans. Totally I spent 4.5 hours on this assignment.