Entropy Tensor of Spacetime

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Abstract

In this paper, we propose a covariant formalism for the entropy of spacetime by introducing a rank-2 entropy tensor $S_{\mu\nu}$, defined as the energy-momentum tensor $T_{\mu\nu}$ divided by a scalar temperature field T, in analogy with the classical definition of entropy. This construction provides a natural extension of thermodynamic relations to curved spacetime and enables a local, covariant description of entropy flow. We argue that this framework is compatible with the ideas put forth by Verlinde and Jacobson, particularly in the context of emergent gravity and local horizon thermodynamics. The entropy tensor formalism also aligns with general relativity through its compatibility with the Einstein field equations. Within this framework, we recover several well-known results in gravitational thermodynamics, including the Bekenstein-Hawking area law and the Clausius relation at local Rindler horizons. Finally, we show that the divergence of the entropy tensor provides a natural covariant generalization of the entropic force as a four-vector, linking spacetime dynamics to thermodynamic gradients. This suggests that the entropy tensor may serve as a foundational tool for exploring the thermodynamic structure of spacetime in both classical and emergent gravity scenarios.

Keywords: entropy tensor, spacetime thermodynamics, gravitational entropy

1. Introduction and Overview on Entropic Gravity

General theory of relativity is one of the most noble theories in the history of science. This theory treats spacetime as a 4 dimensional manifold M with a metric $g_{\mu\nu}$ and explains gravity as a manifestation of the curvature of the spacetime. Thus, Einstein field equations are tensorial equations that connect matter-energy with the curvature of spacetime via the formula:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1}$$

After Einstein's discoveries, the theory of relativity and its applications to other areas of physics became one of the most actively studied subjects. Richard C. Tolman was among the first physicists to explore the implications of relativity in other domains, particularly in thermodynamics. He investigated how the fundamental equations and concepts of classical thermodynamics should be modified in a relativistic context [1, 2]. One of his key results was that, in contrast to classical thermodynamics, a system in thermal equilibrium in a gravitational field exhibits a temperature gradient. He formulated this phenomenon,now known as Tolman's Law,as:

$$T(x^\mu)\sqrt{-g_{00}} = constant. \tag{2}$$

Max Planck was also among the early physicists who recognized the foundational significance of Einstein's theory of relativity and sought to integrate it with classical physics, particularly thermodynamics. Although Planck himself made important contributions to formulating thermodynamic principles compatible with special relativity, the debate over how to reconcile heat, temperature, and entropy with Lorentz invariance remained unsettled.

A particularly notable episode in this discourse occurred in the early 1950s, when Max von Laue presented arguments concerning the transformation behaviour of heat in relativistic systems. His views reignited Einstein's interest in the subject, prompting a series of letters between Einstein and von Laue. In these letters, Einstein expressed dissatisfaction with the standard formulations of relativistic thermodynamics, specifically disputing the proposed transformation laws for heat and temperature. He sought a more conceptually coherent framework consistent with the principle of relativity.

This intellectual exchange has been carefully reconstructed and critically analysed by Chuang Liu in a detailed historical study [3]. Liu not only contextualizes Einstein's late-stage concerns, but also sheds light on the broader confusion and ambiguity that plagued early attempts to build a consistent relativistic theory of thermodynamics. This "strange episode," as Liu calls it, reflects both the depth of the problem and the philosophical weight Einstein placed on the unification of mechanics and thermodynamics within the relativistic domain.

In the most contemporary studies on the thermodynamic interpretation of gravity, Ted Jacobson and Erik Verlinde have made foundational contributions. Jacobson famously argued that by considering *local Rindler horizons* and applying the Clausius relation $\delta Q = T\,dS$ to spacetime, one can derive the Einstein field equations as an equation of state [4]. His approach suggests that gravity is not a fundamental interaction, but rather an emergent thermodynamic phenomenon — a macroscopic consequence of underlying microscopic degrees of freedom.

Building on this conceptual foundation, Verlinde introduced a complementary perspective in which gravity arises as an entropic force [5]. By invoking the *holographic principle* and *Bekenstein's entropy bounds*, Verlinde derived Newton's law of gravitation from statistical arguments about the distribution of entropy on holographic screens. Both Jacobson's and Verlinde's frameworks emphasize the need to reinterpret gravitational dynamics as arising from thermodynamic and information-theoretic principles.

For such an interpretation to have full physical meaning, it becomes essential to construct proper relativistic analogues of classical thermodynamic quantities — such as *entropy*, *heat*, and *temperature* — within the language of four-vectors and tensors. In this regard, Pandey proposed a formalism that introduces *four-entropy* and *four-heat* vectors [6]. While his model extends the thermodynamic structure into a covariant framework, Pandey maintains the assumption that *temperature is a Lorentz-invariant scalar*, justifying this choice by stating:

> “By treating temperature as a Lorentz-invariant scalar, the transformation $T' = T$ avoids the inconsistencies seen with other transformations, ensuring that no artificial temperature gradients or unphysical entropy changes occur in the moving frame.” [6]

This scalar treatment of temperature attempts to preserve thermodynamic consistency across reference frames, though it remains a point of philosophical and physical debate in relativistic thermodynamics.

In this paper, we propose a covariant formalism for the entropy of spacetime as a rank-2 tensor constructed from the energy-momentum tensor and temperature scalar:

$$S_{\mu\nu} = \frac{T_{\mu\nu}}{T}. \tag{3}$$

We argue that this theory can be used as an initiative in the studies of covariant entropy and entropic gravity.

2. Entropy Tensor

We propose a covariant tensor formalism for the entropy of spacetime. Building on the foundational ideas of Bekenstein and Hawking, and further developed by Verlinde and Jacobson, we argue that a fully relativistic theory of thermodynamics requires a covariant formulation of entropy. Such a framework would allow quantities like entropy flux and entropic force (associated with the divergence of the entropy flux) to be treated in a mathematically rigorous and coordinate-independent way.

To this end, we propose a rank-2 tensor for entropy that is structurally compatible with the energy-momentum tensor $T_{\mu\nu}$ and a scalar temperature field T. The entropy tensor is defined as

$$S_{\mu\nu} = \frac{T_{\mu\nu}}{T}. \tag{4}$$

This definition captures the intuition that entropy density is energy density divided by temperature, generalized to the relativistic, tensorial setting. A natural consequence of this definition is that, when contracted with the four-velocity u^ν of an observer, the entropy tensor yields the entropy four-flux

$$s^\mu = S^{\mu\nu} u_\nu. \tag{5}$$

This vector s^μ represents the flow of entropy through spacetime from the perspective of a co-moving observer. Its covariant divergence $\nabla_\mu s^\mu$ can be interpreted as the local entropy production rate, and plays a central role in the covariant formulation of the second law of thermodynamics.

A direct consequence of the entropy tensor formalism, compatible with the Bekenstein area law, arises when we treat the temperature scalar as in classical thermodynamics and assume the holographic principle.

Let N be the number of bits associated with a region of spacetime — interpreted as the number of fundamental degrees of freedom. In classical thermodynamics, temperature in equilibrium is defined as the energy per degree of freedom. Assuming $T^{00} = \rho$ as the local energy density, the total energy in a region of area A is $E = \rho A$. Then, the temperature per bit becomes

$$T = \frac{E}{N} = \frac{\rho A}{N}.$$

Using the entropy tensor definition $S^{00} = \frac{T^{00}}{T}$, we find

$$S^{00} = \frac{\rho}{\frac{\rho A}{N}} = \frac{N}{A}. \tag{6}$$

Assuming that the holographic principle gives $N = \frac{A}{4G}$, we obtain a constant entropy density:

$$S^{00} = \frac{1}{4G}$$

Integrating this over the surface of the region yields the following.

$$S = \oiint_{\mathcal{H}} S^{00}\, dA = \oiint_{\mathcal{H}} \frac{1}{4G}\, dA = \frac{A}{4G} \tag{7}$$

which reproduces the Bekenstein-Hawking entropy formula. This shows that the entropy tensor formalism naturally encodes the area law under classical thermodynamic and holographic assumptions.

3. Covariant Conservation of Entropy in FLRW Spacetime

In classical thermodynamics, when the system is in an adiabatic process, the entropy of the system is conserved. The proper version of this law is the covariant conservation of entropy tensor. In general, we shall state this as:

$$\nabla_\mu S^{\mu\nu} = 0. \tag{8}$$

Consider the FLRW metric:

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2).$$

The nonzero Christoffel symbols are

$$\Gamma^0_{ij} = -a\dot{a}\delta_{ij}, \quad \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j.$$

The energy-momentum tensor for a perfect fluid is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \quad , \quad S^{\mu\nu} = \frac{1}{T}T^{\mu\nu}.$$

The covariant divergence of entropy tensor gives

$$\nabla_\mu S^{\mu\nu} = \nabla_\mu \left(\frac{(\rho + p)}{T} u^\mu u^\nu \right) + \nabla_\mu \left(\frac{p}{T} g^{\mu\nu} \right).$$

Now consider $\nabla_\mu S^{\mu 0}$. For co-moving fluid it gives

$$T^{00} = \rho, \quad T^{0i} = 0, \quad T^{ij} = pg^{ij}.$$

Then

$$S^{00} = \frac{\rho}{T}, \quad S^{ij} = \frac{p}{T} g^{ij}$$

From the covariant derivative,

$$\nabla_\mu S^{\mu\nu} = \partial_\mu S^{\mu\nu} + \Gamma^\mu_{\mu\lambda} S^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} S^{\mu\lambda}.$$

In particular,

$$\nabla_\mu S^{\mu 0} = \partial_t S^{00} + 3\frac{\dot{a}}{a} S^{00} = 0.$$

Substituting $S^{00} = \frac{\rho}{T}$

$$\partial_t \left(\frac{\rho}{T}\right) + 3\frac{\dot{a}}{a}\left(\frac{\rho}{T}\right) = 0, \frac{d}{dt}\ln\left(\frac{\rho}{T}\right) + 3\frac{d}{dt}\ln(a) = 0$$

So we have $\frac{\rho}{T}a^3$=constant.

It is well known in cosmology that, for radiation dominated universe $\rho \propto a^{-4}$. From the previous result from covariant conservation of entropy tensor we have $\frac{\rho}{T}a^3$=constant and hence $T \propto a^{-1}$ which is in agreement with adiabatic expansion of radiation dominated stage.

4. Local Rindler Horizons

In his seminal paper [4], Jacobson derives Einstein's field equations as an equation of state by invoking the thermodynamics of local Rindler horizons. There is, however, a conceptual distinction between Jacobson's approach and the entropy tensor formalism developed in this work.

Jacobson's method interprets the flow of heat across a local causal horizon as a geometrically motivated variation of energy, using the boost-energy current of matter fields. In this view, the heat flux plays the role of a horizon operator. By contrast, our entropy tensor formalism establishes a *local* relationship between the matter energy content and the entropy of spacetime, without presupposing a horizon structure.

Despite this difference in motivation, the entropy tensor formalism recovers the Clausius relation in the setting of local Rindler horizons, provided one carefully defines the entropy flux across the horizon.

Jacobson defines the heat flux across the horizon as [4]:

$$\delta Q = \int_H T_{\mu\nu}\chi^\mu d\Sigma^\nu, \tag{9}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter, χ^μ is the approximate local boost Killing vector that generates the Rindler horizon, and $d\Sigma^\nu$ is the surface element on the horizon.

From the entropy tensor formalism, we postulate the relation:

$$T_{\mu\nu} = TS_{\mu\nu},$$

which allows us to rewrite the heat flux in terms of the entropy tensor:

$$\begin{aligned}\delta Q &= \int_H T_{\mu\nu}\chi^\mu d\Sigma^\nu = \int_H TS_{\mu\nu}\chi^\mu d\Sigma^\nu \\ &= T\int_H S_{\mu\nu}\chi^\mu d\Sigma^\nu. \end{aligned}\tag{10}$$

Two remarks are in order:

- First, the temperature is assumed to be constant over the horizon. This assumption is consistent with both the derivation and with Tolman's law, which states that $T\sqrt{-g_{00}} = \text{const}$. Since the norm of the Killing vector is constant along the horizon generator, so too must be the temperature. This means that local Rindler horizons are somehow constant temperature curves of the spacetime.

- Second, we define the entropy flux across the horizon, following the spirit of Jacobson's approach, as:
$$dS = \int_H S_{\mu\nu}\chi^\mu d\Sigma^\nu. \tag{11}$$

Under these definitions, the Clausius relation follows immediately:
$$\delta Q = TdS. \tag{12}$$

This result demonstrates that the covariant entropy tensor formalism is not only consistent with the relativistic thermodynamic behaviour of horizons, but also naturally reproduces well-known relations—such as the Clausius relation—when appropriate geometric definitions are applied. In particular, by identifying entropy flux with the contraction $S_{\mu\nu}\chi^\mu d\Sigma^\nu$ and assuming isothermal conditions along the horizon, the formalism yields established results of horizon thermodynamics in a manifestly covariant framework.

5. Entropy Tensor and Curvature

The covariant definition of entropy in terms of the energy-momentum tensor naturally leads to a relation between entropy and spacetime curvature, as governed by the Einstein field equations. From the definition of the entropy tensor:
$$S_{\mu\nu} = \frac{T_{\mu\nu}}{T},$$
and the Einstein equation (1), we obtain:
$$S_{\mu\nu} = \frac{1}{\kappa T}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right), \tag{13}$$
which expresses a direct link between the entropy content of spacetime and its curvature.

This result follows immediately from the entropy tensor formalism and the Einstein equations. For example, in vacuum ($T_{\mu\nu} = 0$), the entropy tensor vanishes identically:
$$S_{\mu\nu} = 0,$$
reflecting the absence of matter-energy content. Of course, this does not include quantum effects, such as vacuum fluctuations or quantum fields, which may contribute to entropy in the semi-classical regime. The inclusion of such quantum corrections could enrich this curvature-entropy relation and is a direction for further development.

Taking the trace of the expression 13 yields a scalar relation:
$$R = 4\Lambda - \kappa TS, \tag{14}$$
where $S = S^\mu{}_\mu$ is the trace of the entropy tensor. This equation suggests that the scalar curvature R is determined by the total entropy density S, the local temperature T, and the cosmological constant Λ. It provides a scalar constraint that could serve as a condition for local thermodynamic equilibrium or be interpreted as a curvature-based entropy balance law.

6. Entropic Force

Pursuing the philosophy behind the construction of the entropy tensor, we now define an entropic force in curved spacetime. In classical thermodynamics, the entropic force is given by:

$$F = T\nabla S, \tag{15}$$

where S is the entropy and T is the temperature.

With the introduction of a rank-2 entropy tensor, this concept generalizes naturally. We propose the covariant form:

$$F^\nu = T\nabla_\mu S^{\mu\nu}, \tag{16}$$

which can be understood as the relativistic analogue of equation (15). This definition reinforces the necessity of representing entropy as a rank-2 tensor in order to preserve covariance under general coordinate transformations.

To explore the behavior of this force, we again consider the spatially flat FLRW metric:

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$

and assume a perfect fluid energy-momentum tensor. The corresponding entropy tensor is given by:

$$S^{\mu\nu} = \frac{1}{T}\left[(\rho + p)u^\mu u^\nu + pg^{\mu\nu}\right],$$

where $u^\mu = (1, 0, 0, 0)$ is the co-moving four-velocity, and ρ and p are energy density and pressure, respectively.

We now compute the time component of the entropic force:

$$F^0 = T\nabla_\mu S^{\mu 0} = \dot{\rho} + 3H\rho - \rho\frac{\dot{T}}{T}, \tag{17}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Using the energy conservation equation for the FLRW universe,

$$\dot{\rho} + 3H(\rho + p) = 0,$$

we obtain the entropic force as:

$$\boxed{F^0 = -3Hp - \rho\frac{\dot{T}}{T}}. \tag{18}$$

The first term, proportional to the pressure and Hubble parameter, may be interpreted as a cosmological friction force. The second term arises from a temporal temperature gradient and reflects entropy production or non-equilibrium effects. This expression provides a dynamical thermodynamic correction to inertial motion in an expanding universe.

7. Conclusion and Discussion

In this work, we introduced a rank-2 entropy tensor for the entropy of spacetime. This tensor, defined as the energy-momentum tensor divided by a scalar temperature field, provides a local and geometric extension of thermodynamic relations to curved spacetime.

We demonstrated that this formalism is conceptually consistent with the approaches of Verlinde and Jacobson in the context of entropic gravity. Furthermore, we applied the entropy tensor to both FLRW cosmology and local Rindler horizons, showing that it recovers known thermodynamic results such as the Clausius relation and the Bekenstein-Hawking area law under appropriate assumptions. We expect that Tolman law arises from the covariant conservation of entropy tensor.

Despite its consistency with many classical results, the current construction does not incorporate quantum corrections. In particular, near black hole horizons or in vacuum spacetimes where the classical energy-momentum tensor vanishes, the entropy tensor also vanishes, failing to capture quantum contributions such as those arising from Hawking radiation or vacuum fluctuations. Addressing this limitation is an important direction for future work.

Since the entropy tensor is defined in terms of the energy-momentum tensor, it naturally relates to the curvature of spacetime through the Einstein field equations. While the precise physical interpretation of this curvature-entropy connection remains to be fully understood, we hope that the formalism presented here may offer a foundation for further studies exploring the thermodynamic structure of spacetime, particularly in semi-classical or quantum gravitational regimes.

References

[1] R. C. Tolman and P. Ehrenfest, "Temperature equilibrium in a static gravitational field", Physical Review **36**, 1791–1798 (1930).

[2] R. C. Tolman, *Relativity, thermodynamics and cosmology* (Oxford University Press, 1934).

[3] C. Liu, "Einstein and relativistic thermodynamics in 1952: a historical and critical study of a strange episode in the history of modern physics", The British Journal for the History of Science **25**, 185–206 (1992).

[4] T. Jacobson, "Thermodynamics of spacetime: the einstein equation of state", Physical Review Letters **75**, 1260–1263 (1995).

[5] E. Verlinde, "On the origin of gravity and the laws of newton", Journal of High Energy Physics **2011**, 1–27 (2011).

[6] B. Pandey, "Resolving inconsistencies in relativistic thermodynamics with four-entropy and four-heat current", version v3, arXiv preprint arXiv:2410.10930 (2024).