Conformal Transformations in General Relativity

Berat Cacu

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1 Introduction

In mathematics, a conformal transformation is a transformation that keeps the structure of space invariant. In general, it is rescaling of local distances and hence keeps angles unchanged.

In general relativity, distance is defined by the the spacetime metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

So, the conformal transformation of the spacetime has the form

$$d\tilde{s}^2 = \Omega^2(x^\mu) ds^2$$

where $d\tilde{s}^2$ is the distance in the conformal frame and $\Omega^2(x^\mu)$ is a nonvanishing conformal factor which is a function of spacetime coordinates.

One important thing to note is that these are not coordinate transformations; they rescale the distances locally. Hence, under a conformal mapping, the space-time structure does not change. So, in general we have :

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \tag{1}$$

and also the inverse matric also transforms trivially as:

$$\tilde{g}^{\mu\nu} = \Omega^{-2}(x) g^{\mu\nu} \tag{2}$$

We say that $\tilde{g}_{\mu\nu}$ is conformal to $g_{\mu\nu}$ if there exist a non-vanishing conformal factor $\Omega(x)$. One of the most important properties of conformal transformations is that they preserve angles between coordinate axes. Hence, the structure of the spacetime is preserved. An immediate result is that the type of physical spacetime is also preserved in a conformal mapping. Suppose you have a null vector A^μ such that

$$g_{\mu\nu} A^\mu A^\nu = 0$$

then,

$$\tilde{g}_{\mu\nu} A^\mu A^\nu = \Omega^2 g_{\mu\nu} A^\mu A^\nu = 0$$

Hence, the same result applies for spacelike and timelike vectors. In general, if a 4-vector is null, spacelike or timelike in ordinary frame with metric $g_{\mu\nu}$, then it is also null, spacelike or timelike accordingly in a conformal frame with metric $\tilde{g}_{\mu\nu}$ conformal to $g_{\mu\nu}$.

Now, define a null path such that a path $x^\mu(\lambda)$ is null if

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

Then one immediately concludes that:

$$\tilde{g}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \Omega^2 g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

So we can also say that conformal transformations leave light cones invariant.[1] It is also an immediate result of the angle-preserving property of conformal transformations, since light cones are 45° paths in Minkowski diagrams.

After understanding basics of a conformal transformation, we can look how the tools related and derived from metric changes under a conformal transformation. First of all, let look at the connection. Christoffel connections are calculated as:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}) \tag{3}$$

Let the metric transform as

$$\tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} \tag{4}$$

and hence

$$\tilde{g}^{\mu\nu} = \Omega^{-2}(x)g^{\mu\nu} \tag{5}$$

We will now compute the transformed connection:

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2}\tilde{g}^{\mu\sigma}(\partial_\alpha \tilde{g}_{\beta\sigma} + \partial_\beta \tilde{g}_{\alpha\sigma} - \partial_\sigma \tilde{g}_{\alpha\beta}) \tag{6}$$

By chain rule:

$$\partial_\alpha(\Omega^2 g_{\beta\sigma}) = 2\Omega\partial_\alpha\Omega g_{\beta\sigma} + \Omega^2\partial_\alpha g_{\beta\sigma} \tag{7}$$

and same for other terms:

$$\partial_\beta(\Omega^2 g_{\alpha\sigma}) = 2\Omega\partial_\beta\Omega g_{\alpha\sigma} + \Omega^2\partial_\beta g_{\alpha\sigma} \tag{8}$$

$$\partial_\sigma(\Omega^2 g_{\alpha\beta}) = 2\Omega\partial_\sigma\Omega g_{\alpha\beta} + \Omega^2\partial_\sigma g_{\alpha\beta} \tag{9}$$

Substitute all into the expression for $\tilde{\Gamma}^\mu_{\alpha\beta}$:

[1] Sean Carroll, *Spacetime and Geometry*, Cambridge University Press, 2004, Appendix G.

$$\tilde{\Gamma}^{\mu}_{\alpha\beta} = \frac{1}{2}\Omega^{-2}g^{\mu\sigma}[2\Omega(\partial_\alpha\Omega)g_{\beta\sigma} + \Omega^2\partial_\alpha g_{\beta\sigma} + 2\Omega(\partial_\beta\Omega)g_{\alpha\sigma} + \Omega^2\partial_\beta g_{\alpha\sigma} - 2\Omega(\partial_\sigma\Omega)g_{\alpha\beta} - \Omega^2\partial_\sigma g_{\alpha\beta}] \tag{10}$$

Then we simplify the expression to isolate the transformed connection:

$$\tilde{\Gamma}^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta} + \delta^{\mu}_{\alpha}\partial_\beta(\ln\Omega) + \delta^{\mu}_{\beta}\partial_\alpha(\ln\Omega) - g_{\alpha\beta}g^{\mu\sigma}\partial_\sigma(\ln\Omega) \tag{11}$$

and we define the tensor

$$C^{\mu}_{\alpha\beta} = \tilde{\Gamma}^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\alpha\beta} \tag{12}$$

Now this expression is a tensor since it is the difference of two connections. Indeed this will be very helpful when we try to undedrstand covariant derivatives in conformal frames. Now, lets look at the covariant derivative in conformal frame:

For covariant derivative with metric $g_{\mu\nu}$ we have:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\sigma}V^\sigma \tag{13}$$

and corresponding covariant differentiation in conformal frame with conformal connection reads :

$$\tilde{\nabla}_\mu V^\nu = \partial_\mu V^\nu + \tilde{\Gamma}^{\nu}_{\mu\sigma}V^\sigma \tag{14}$$

Now,by using the tensor that we defined in 12 we have:

$$\tilde{\Gamma}^{\nu}_{\mu\sigma} = \Gamma^{\nu}_{\mu\sigma} + C^{\nu}_{\mu\sigma} \tag{15}$$

and when we plug in we have:

$$\boxed{\tilde{\nabla}_\mu V^\nu = \nabla_\mu V^\nu + C^{\nu}_{\mu\sigma}V^\sigma} \tag{16}$$

This relation plays an important role when we try to look at the geodesic equation or Riemann curvature. Also, we can note that, by the same consideration we have:

$$\boxed{\tilde{\nabla}_\mu \omega_\nu = \nabla_\mu \omega_\nu - C^{\sigma}_{\mu\nu}\omega_\sigma} \tag{17}$$

We can now look at the geodesic equation for a conformal frame. Assume X^μ is a geodesic for $g_{\mu\nu}$ parametrized by λ such that

$$U^\beta\nabla_\beta U^\mu = 0, \quad U^\mu = \frac{dx^\mu}{d\lambda} \tag{18}$$

Then for a conformal frame we have:

$$U^\beta\tilde{\nabla}_\beta U^\mu = U^\beta(\nabla_\beta U^\mu + C^{\mu}_{\beta\nu}U^\nu) = U^\beta\nabla_\beta U^\mu + U^\beta C^{\mu}_{\beta\nu}U^\nu \tag{19}$$

Observe that the first term in the right handside vanishes and recall that we have:

$$C^{\mu}_{\beta\nu} = 2\delta^{\mu}_{(\beta}\partial_{\nu)}\ln\Omega - g_{\beta\nu}g^{\mu\sigma}\partial_\sigma\ln\Omega \tag{20}$$

Then we have :

$$\Rightarrow U^\beta \tilde{\nabla}_\beta U^\mu = U^\beta C^\mu_{\beta\nu} U^\nu \tag{21}$$

$$= 2U^\mu U^\rho \partial_\rho \ln \Omega - g_{\beta\nu} g^{\mu\rho} U^\beta U^\nu \partial_\rho \ln \Omega \tag{22}$$

Thus, in general X^μ fails to be a geodesic with $\tilde{g}_{\mu\nu}$. But, in special cases it indeed does not. Consider a null geodesic with $g_{\mu\nu}U^\mu U^\nu = 0$. Then second term in 22 is zero and X^μ is also a geodesic for $\tilde{g}_{\mu\nu}$ with an affine parameter $\alpha = 2U^\rho \partial_\rho \ln \Omega$.[2]

Thus null geodesics are conformally invariant, i.e., null geodesics for $\tilde{g}_{\mu\nu}$ coincide with those with $g_{\mu\nu}$ with an affine parameter $\tilde{\lambda}$ for $\tilde{g}_{\mu\nu}$-geodesics related to the affine parameter λ for $g_{\mu\nu}$-geodesics by:[3]

$$\frac{d\tilde{\lambda}}{d\lambda} = c\Omega^2, \quad c \in \mathbb{R} \tag{23}$$

2 Riemann Curvature and Weyl Tensor

Much of the foundational work on conformal transformations originates from the contributions of Hermann Weyl. He made profound advances in both mathematics and physics, including general relativity. After Einstein published his theory of general relativity, Weyl became deeply interested in its geometric structure and began studying Einstein's field equations.

Weyl sought to understand the part of spacetime curvature that is independent of the energy-momentum content. Since the Ricci tensor $R_{\mu\nu}$ vanishes in vacuum, he focused on the components of the Riemann curvature tensor that remain even in the absence of matter.

To isolate this purely gravitational, trace-free portion of the curvature, he introduced what is now known as the **Weyl tensor** $C^\rho{}_{\sigma\mu\nu}$. It is defined as the trace-free part of the Riemann tensor and encapsulates the tidal and conformal properties of spacetime curvature. In n-dimensional spacetime, it is given by:[4]

$$R^\rho{}_{\sigma\mu\nu} = C^\rho{}_{\sigma\mu\nu} + \frac{2}{n-2}\left(\delta^\rho_{[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]}{}^\rho\right) - \frac{2}{(n-1)(n-2)} R\, \delta^\rho_{[\mu} g_{\nu]\sigma}$$

This decomposition shows how the Riemann tensor splits into the Weyl tensor and terms involving the Ricci tensor and scalar curvature. In particular, the Weyl tensor vanishes identically in conformally flat spacetimes. Now one can observe that every contraction of Weyl tensor vanishes identically:

Start by contracting the first and third indices:

$$C^\mu{}_{\sigma\mu\nu} = R_{\sigma\nu} - \frac{2}{n-2}\left(\delta^\mu_{[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]}{}^\mu\right) + \frac{2}{(n-1)(n-2)} R \delta^\mu_{[\mu} g_{\nu]\sigma} \tag{24}$$

[2]Look at the Appendix A

[3]Robert M. Wald, General Relativity, University of Chicago Press, 1984, Appendix D

[4]Sean Carroll, Spacetime and Geometry, Cambridge University Press, 2004

Evaluate each term:

$$\delta^\mu_{[\mu} R_{\nu]\sigma} = \frac{1}{2}(\delta^\mu_\mu R_{\nu\sigma} - \delta^\mu_\nu R_{\mu\sigma}) = \frac{1}{2}(nR_{\nu\sigma} - R_{\nu\sigma}) = \frac{1}{2}(n-1)R_{\nu\sigma} \tag{25}$$

$$g_{\sigma[\mu} R_{\nu]}{}^\mu = \frac{1}{2}(g_{\sigma\mu} R_\nu{}^\mu - g_{\sigma\nu} R_\mu{}^\mu) = \frac{1}{2}(R_{\nu\sigma} - g_{\sigma\nu} R) \tag{26}$$

$$\delta^\mu_{[\mu} g_{\nu]\sigma} = \frac{1}{2}(\delta^\mu_\mu g_{\nu\sigma} - \delta^\mu_\nu g_{\mu\sigma}) = \frac{1}{2}(ng_{\nu\sigma} - g_{\nu\sigma}) = \frac{1}{2}(n-1)g_{\nu\sigma} \tag{27}$$

Now plug into the full contraction:

$$C^\mu{}_{\sigma\mu\nu} = R_{\sigma\nu} - \frac{2}{n-2}\left(\frac{1}{2}(n-1)R_{\nu\sigma} - \frac{1}{2}(R_{\nu\sigma} - g_{\nu\sigma}R)\right) + \frac{2}{(n-1)(n-2)} R \cdot \frac{1}{2}(n-1)g_{\nu\sigma} \tag{28}$$

$$= R_{\sigma\nu} - \frac{1}{n-2}\left((n-1)R_{\nu\sigma} - R_{\nu\sigma} + g_{\nu\sigma}R\right) + \frac{R}{n-2} g_{\nu\sigma} \tag{29}$$

$$= R_{\sigma\nu} - \frac{1}{n-2}\left((n-2)R_{\nu\sigma} + g_{\nu\sigma}R\right) + \frac{R}{n-2} g_{\nu\sigma} \tag{30}$$

$$= R_{\sigma\nu} - R_{\nu\sigma} = 0 \tag{31}$$

Since the Ricci tensor is symmetric, $R_{\sigma\nu} = R_{\nu\sigma}$, the contraction vanishes.
Therefore:

$$\boxed{C^\mu{}_{\sigma\mu\nu} = 0} \tag{32}$$

By the same process and computations one can also shows that all contractions vanish identically.

Weyl's insight thus links conformal geometry with the local curvature of spacetime, playing a central role in general relativity, gravitational radiation, and gauge theories of gravity. Indeed Ricci scalar does not contain all the information related to curvature of spacetime. In n dimensions for $n \geq 2$ Riemann tensor $R^\mu_{\nu\sigma\rho}$ has $\frac{n^2(n^2-1)}{12}$ components due to its symmetries and Ricci curvature has $\frac{n(n+1)}{2}$ components.[5]. In dimension 4, Riemann tensor has 20 independent components while Ricci tensor contains the information of only 10 of them. Hence in 4 dimensions, generally Weyl tensor does not vanishes and contains information related to trace-free part of the Riemann curvature tensor. One can easily observe that in dimension three Riemann and Ricci tensor has same number of independent components and thus Weyl tensor vanishes because all the information about curvature is contained in Ricci curvature tensor.

The Weyl tensor is also called as conformal tensor because in a conformal transformation Weyl tensor is invariant. We now try to show this by the technique we have developed.First, lets us look at how Riemann tensor changes under a conformal transformation.
Indeed, ıf we use the techniques we derived, we could get a formula for Riemann tensor in conformal frame $\tilde{R}^\mu_{\nu\rho\sigma}$ by using the definition of Riemann tensor in terms of connection. But if we use the formula for Riemann tensor by using Lee brackets and formula for covariant

[5]Sean Carroll, Spacetime and Geometry, Cambridge University Press, 2004

derivative in conformal frame, it is straightforward.

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]V^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu}V^\sigma \tag{33}$$

Now expand the commutator using $\tilde{\nabla}_\mu = \nabla_\mu + C_\mu$:

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]V^\rho = [\nabla_\mu + C_\mu, \nabla_\nu + C_\nu]V^\rho \tag{34}$$

This becomes:

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]V^\rho = [\nabla_\mu, \nabla_\nu]V^\rho + (\nabla_\mu C^\rho_{\nu\sigma} - \nabla_\nu C^\rho_{\mu\sigma})V^\sigma + C^\rho_{\mu\lambda}C^\lambda_{\nu\sigma}V^\sigma - C^\rho_{\nu\lambda}C^\lambda_{\mu\sigma}V^\sigma \tag{35}$$

Therefore, the Riemann tensor transforms as:

$$\boxed{\tilde{R}^\rho{}_{\sigma\mu\nu} = R^\rho{}_{\sigma\mu\nu} + \nabla_\mu C^\rho_{\nu\sigma} - \nabla_\nu C^\rho_{\mu\sigma} + C^\rho_{\mu\lambda}C^\lambda_{\nu\sigma} - C^\rho_{\nu\lambda}C^\lambda_{\mu\sigma}} \tag{36}$$

This derivation keeps the transformation law manifestly covariant and reflects the purely geometric content of curvature under conformal rescaling.

Explicit Computation of Ricci Tensor and Scalar under Conformal Transformation

Let the metric rescale as:

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu} \tag{37}$$

The correction tensor is defined by:

$$C^\rho_{\mu\nu} = \delta^\rho_\mu \partial_\nu \ln\Omega + \delta^\rho_\nu \partial_\mu \ln\Omega - g_{\mu\nu} g^{\rho\lambda} \partial_\lambda \ln\Omega \tag{38}$$

Compute its contraction:

$$C^\mu_{\mu\sigma} = \delta^\mu_\mu \partial_\sigma \ln\Omega + \delta^\mu_\sigma \partial_\mu \ln\Omega - g_{\mu\sigma} g^{\mu\lambda} \partial_\lambda \ln\Omega \tag{39}$$

$$= n\partial_\sigma \ln\Omega + \partial_\sigma \ln\Omega - \partial_\sigma \ln\Omega = n\partial_\sigma \ln\Omega \tag{40}$$

The Ricci tensor transforms as:

$$\tilde{R}_{\nu\sigma} = R_{\nu\sigma} + \nabla_\mu C^\mu_{\nu\sigma} - \nabla_\nu C^\mu_{\mu\sigma} + C^\mu_{\mu\lambda}C^\lambda_{\nu\sigma} - C^\mu_{\nu\lambda}C^\lambda_{\mu\sigma} \tag{41}$$

Since $C^\mu_{\mu\sigma} = n\partial_\sigma \ln\Omega$,

$$\nabla_\nu C^\mu_{\mu\sigma} = n\nabla_\nu \nabla_\sigma \ln\Omega \tag{42}$$

Now compute:

$$C^\mu_{\mu\lambda}C^\lambda_{\nu\sigma} = n\partial_\lambda \ln\Omega \cdot \left(\delta^\lambda_\nu \partial_\sigma \ln\Omega + \delta^\lambda_\sigma \partial_\nu \ln\Omega - g_{\nu\sigma} g^{\lambda\rho} \partial_\rho \ln\Omega\right) \tag{43}$$

$$= n\left(\partial_\nu \ln\Omega \partial_\sigma \ln\Omega + \partial_\sigma \ln\Omega \partial_\nu \ln\Omega - g_{\nu\sigma}(\nabla \ln\Omega)^2\right) \tag{44}$$

Also:

$$C^{\mu}_{\nu\lambda}C^{\lambda}_{\mu\sigma} = \partial_\mu \ln\Omega \cdot (\delta^\mu_\nu \partial_\sigma \ln\Omega + \delta^\mu_\sigma \partial_\nu \ln\Omega - g_{\nu\sigma} g^{\mu\rho}\partial_\rho \ln\Omega) \tag{45}$$

Combine all terms:

$$\tilde{R}_{\nu\sigma} = R_{\nu\sigma} - (n-2)\nabla_\nu \nabla_\sigma \ln\Omega - g_{\nu\sigma}\nabla^2 \ln\Omega \tag{46}$$
$$+ (n-2)\nabla_\nu \ln\Omega \nabla_\sigma \ln\Omega - (n-2) g_{\nu\sigma}(\nabla \ln\Omega)^2 \tag{47}$$

The scalar curvature is then:

$$\tilde{R} = \Omega^{-2} g^{\nu\sigma}\tilde{R}_{\nu\sigma} \tag{48}$$

$$\tilde{R} = \Omega^{-2}\left[R - 2(n-1)\nabla^2 \ln\Omega - (n-1)(n-2)(\nabla \ln\Omega)^2\right] \tag{49}$$

$$\boxed{\tilde{R}_{\nu\sigma} = R_{\nu\sigma} - (n-2)\nabla_\nu \nabla_\sigma \ln\Omega - g_{\nu\sigma}\nabla^2 \ln\Omega + (n-2)\nabla_\nu \ln\Omega \nabla_\sigma \ln\Omega - (n-2) g_{\nu\sigma}(\nabla \ln\Omega)^2} \tag{50}$$

$$\boxed{\tilde{R} = \Omega^{-2}\left[R - 2(n-1)\nabla^2 \ln\Omega - (n-1)(n-2)(\nabla \ln\Omega)^2\right]} \tag{51}$$

3 Conformal Invariance of Weyl Tensor

The most important of property of the Weyl tensor is that it is invariant under conformal transformations. Under given transformation of metric $\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$ we showed how connections, geodesic equations, covariant derivative, Riemann tensor, Ricci tensor and finally Ricci scalar transform. With these fundamental transformations and the formula for Weyl tensor:

$$R^\rho{}_{\sigma\mu\nu} = C^\rho{}_{\sigma\mu\nu} + \frac{2}{n-2}\left(\delta^\rho_{[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]}{}^\rho\right) - \frac{2}{(n-1)(n-2)} R\, \delta^\rho_{[\mu} g_{\nu]\sigma} \tag{52}$$

one can also shows that Weyl tensor is invariant under such transformations. This is indeed a very long and messy computation since it contains square terms of $C^\rho_{\mu\lambda}$ and you should insert all the transformations related to Riemann and Ricci tensors given in 36 and 50 and with transformation of scalar in 51. But after carefully inserted transformations and by a serious brute force, one can shows that Weyl tensor remains invariant under conformal transformations[6] and we have:

$$\tilde{C}^\rho_{\sigma\mu\nu} = C^\rho_{\sigma\mu\nu} \tag{53}$$

One should be careful about the fully covariant form of the Weyl tensor since it is not trivially invariant. Indeed, one can raise index by the metric to conclude that:

$$\tilde{C}_{\rho\sigma\mu\nu} = \Omega^2(x) C_{\rho\sigma\mu\nu} \tag{54}$$

[6] Sean Carroll, Spacetime and Geometry, Cambridge University Press, 2004,3.7

4 Weyl Tensor in 3-Dimensions

In three dimensions, Riemann tensor has 6 independent components and this is identical with the number of independent components of Ricci tensor in 3 dimensions. Hence all the information related to Riemann tensor is contained in Ricci tensor and scalar, so Weyl tensor identically vanishes. Now we will try to show this. Note that by equation 52, Weyl tensor in dimension 3 reduces such that:

$$R^{\rho}{}_{\sigma\mu\nu} = C^{\rho}{}_{\sigma\mu\nu} + \frac{2}{n-2}\left(\delta^{\rho}_{[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]}{}^{\rho}\right) - \frac{2}{(n-1)(n-2)}R\,\delta^{\rho}_{[\mu}g_{\nu]\sigma} \tag{55}$$

For $n = 3$ case we have:

$$C^{\rho}_{\sigma\mu\nu} = R^{\rho}_{\sigma\mu\nu} - 2\left(\delta^{\rho}_{[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R^{\rho}_{\nu]}\right) + R\delta^{\rho}_{[\mu}g_{\nu]\sigma} \tag{56}$$

In $n = 3$, the Riemann tensor has 6 independent components, which is exactly the number of independent components of the Ricci tensor. Hence, the Riemann tensor can be derived from the Ricci tensor and Ricci scalar.

Consider the ansatz:

$$R^{\rho}{}_{\sigma\mu\nu} = a\left(\delta^{\rho}_{[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R^{\rho}{}_{\nu]}\right) + bR\delta^{\rho}_{[\mu}g_{\nu]\sigma} \tag{57}$$

By contracting $\rho = \mu$, we obtain:

$$R_{\nu\sigma} = a\left(\delta^{\rho}_{[\rho}R_{\nu]\sigma} - g_{\sigma[\rho}R^{\rho}{}_{\nu]}\right) + bR\delta^{\rho}_{[\rho}g_{\nu]\sigma} \tag{58}$$

$$= \frac{a}{2}\left[(n-2)R_{\nu\sigma} + g_{\nu\sigma}R\right] + \frac{bR}{2}(n+1)g_{\nu\sigma} \tag{59}$$

Matching coefficients:

$$1 = \frac{a}{2}(n-2) \Rightarrow a = \frac{2}{n-2} \tag{60}$$

$$\frac{a}{2} + \frac{b}{2}(n+1) = 0 \Rightarrow \frac{2}{n-2} + b(n-1) = 0 \tag{61}$$

Solving this gives for $n = 3$:

$$a = 2, \quad b = -1 \tag{62}$$

Thus the Riemann tensor in 3D is:

$$\boxed{R^{\rho}{}_{\sigma\mu\nu} = 2\left(\delta^{\rho}_{[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R^{\rho}{}_{\nu]}\right) - R\delta^{\rho}_{[\mu}g_{\nu]\sigma}} \tag{63}$$

Note that this is exactly the term in 56 and hence Weyl tensor vanishes identically for n=3 case.

5 Conformal Flatness

A manifold with the metric $g_{\mu\nu}$ is called conformally flat if the metric is conformal to Minkovski metric, i.e.

$$g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu} \tag{64}$$

where $\eta_{\mu\nu}$ is Minkovski metric. A manifold is conformally flat if and only if Weyl tensor vanishes. Indeed if the manifold conformally flat and Weyl tensor vanishes in flat manifolds, one can conclude that in both manifolds Weyl tensor vanishes since it is invariant under conformal transformations.

It is well known that every 2-dimensional pseudo-Riemannian manifold is conformally flat[7]. Indeed, ın 2 dimension the curvature is fully determined by curvature scalar and there is only one independent component. Hence the conformal factor can be found by some partial differential equation contains curvature scalar.
For dimension 3, we showed that Weyl tensor vanishes, so for the conformal flatness we use *Cotton tensor*.

6 The Cotton Tensor in Three Dimensions

In three-dimensional differential geometry, the Weyl tensor vanishes identically. To describe the conformal structure of a 3D Riemannian manifold, one introduces the **Cotton tensor**. In a general n-dimensional manifold, the Cotton tensor is defined by:

$$C_{\mu\nu\rho} := \nabla_\rho R_{\mu\nu} - \nabla_\nu R_{\mu\rho} - \frac{1}{2(n-1)}\left(\nabla_\rho R\, g_{\mu\nu} - \nabla_\nu R\, g_{\mu\rho}\right) \tag{65}$$

In three dimensions ($n = 3$), this simplifies to:

$$\boxed{C_{\mu\nu\rho} = \nabla_\rho R_{\mu\nu} - \nabla_\nu R_{\mu\rho} - \frac{1}{4}\left(\nabla_\rho R\, g_{\mu\nu} - \nabla_\nu R\, g_{\mu\rho}\right)} \tag{66}$$

- **Antisymmetry:** $C_{\mu\nu\rho} = -C_{\mu\rho\nu}$
- **Trace-free:** $g^{\nu\rho}C_{\mu\nu\rho} = 0$
- **Conformal Flatness Criterion in 3D:** $C_{\mu\nu\rho} = 0$ if and only if the manifold is conformally flat.
- **Relation to Weyl tensor in $n > 3$:** For $n \geq 4$, the divergence of the Weyl tensor gives:

$$\nabla^\lambda C_{\lambda\mu\nu\rho} \propto (n-3)C_{\mu\nu\rho} \tag{67}$$

Thus, in 3D, the Cotton tensor captures all conformal information.

[7] Emergence of the Cotton Tensor for Describing Gravity,Junpei Harada,2021

7 Weyl Tensor and Geodesic Deviation

The geodesic deviation equation describes how the separation vector ξ^μ between neighboring geodesics changes due to spacetime curvature:

$$\frac{D^2\xi^\mu}{D\tau^2} = R^\mu{}_{\nu\rho\sigma} u^\nu \xi^\rho u^\sigma \tag{68}$$

Here, u^μ is the tangent vector to the geodesic, and $R^\mu{}_{\nu\rho\sigma}$ is the Riemann tensor.

In dimensions $n \geq 4$, the Riemann tensor can be decomposed as:

$$R^\mu{}_{\nu\rho\sigma} = C^\mu{}_{\nu\rho\sigma} + (\text{Ricci terms}) \tag{69}$$

Substituting this into the geodesic deviation equation gives:

$$\frac{D^2\xi^\mu}{D\tau^2} = \left[C^\mu{}_{\nu\rho\sigma} + (\text{trace terms})\right] u^\nu \xi^\rho u^\sigma \tag{70}$$

- **Free tidal forces:** The Weyl tensor encodes the curvature that is independent of matter; it represents the "free gravitational field."
- **Vacuum spacetimes:** In vacuum ($R_{\mu\nu} = 0$), the Riemann tensor reduces to the Weyl tensor:
$$\frac{D^2\xi^\mu}{D\tau^2} = C^\mu{}_{\nu\rho\sigma} u^\nu \xi^\rho u^\sigma \tag{71}$$
- **Gravitational waves:** In the wave zone, the spacetime is approximately vacuum, and the Weyl tensor carries the tidal effects of gravitational radiation.
- **Geometric meaning:** The Weyl tensor is conformally invariant and represents how geodesics deviate due to spacetime's intrinsic conformal curvature.

In general relativity, the Weyl tensor plays a central role in describing the part of spacetime curvature that leads to observable geodesic deviation in the absence of matter.

A Appendix A

In his book on general relativity, Wald states that any path parametrized with λ that satisfies equation:

$$T^a \nabla_a T^b = \alpha T^b \tag{72}$$

for some scalar function $\alpha(\lambda)$, can be reparametrized by a new parameter $\tilde{\lambda}$ such that the path satisfies the geodesic equation. Now we give a proof to this and in our example $\alpha{=}2U^\rho \partial_\rho \ln \Omega$ We are given that a curve with tangent vector

$$T^a = \frac{dx^a}{d\lambda} \tag{73}$$

satisfies the non-affinely parametrized geodesic equation:

$$T^a \nabla_a T^b = \alpha T^b \tag{74}$$

for some scalar function $\alpha(\lambda)$.

We aim to find a new parameter $\tilde{\lambda}$ and corresponding tangent vector

$$\hat{T}^a = \frac{dx^a}{d\tilde{\lambda}} \tag{75}$$

such that it satisfies the affinely parametrized geodesic equation:

$$\hat{T}^a \nabla_a \hat{T}^b = 0 \tag{76}$$

We assume a reparametrization:

$$\frac{d\tilde{\lambda}}{d\lambda} = f(\lambda) > 0 \quad \Rightarrow \quad \hat{T}^a = \frac{1}{f} T^a \tag{77}$$

Now compute:

$$\hat{T}^a \nabla_a \hat{T}^b = \frac{1}{f} T^a \nabla_a \left(\frac{1}{f} T^b \right) = \frac{1}{f} T^a \left(-\frac{f'}{f^2} T^b + \frac{1}{f} \nabla_a T^b \right) = -\frac{f'}{f^3} T^a T^b + \frac{1}{f^2} T^a \nabla_a T^b \tag{78}$$

Using the assumption $T^a \nabla_a T^b = \alpha T^b$, we substitute:

$$\hat{T}^a \nabla_a \hat{T}^b = \left(-\frac{f'}{f^3} T^a + \frac{\alpha}{f^2} \right) T^b \tag{79}$$

To have an affinely parametrized geodesic (i.e., the left-hand side vanishing), we require:

$$\left(-\frac{f'}{f^3} T^a + \frac{\alpha}{f^2} \right) = 0 \quad \Rightarrow \quad \frac{f'}{f} = \alpha \tag{80}$$

This is a first-order linear differential equation. It has the solution:

$$f(\lambda) = \exp\left(\int \alpha(\lambda)\, d\lambda \right) \tag{81}$$

Therefore, defining:

$$\frac{d\tilde{\lambda}}{d\lambda} = f(\lambda) = \exp\left(\int \alpha(\lambda) d\lambda \right) \tag{82}$$

we obtain a reparametrization $\tilde{\lambda}$ such that $\hat{T}^a = \frac{dx^a}{d\tilde{\lambda}}$ satisfies the affine geodesic equation.

Conclusion: Any geodesic curve satisfying $T^a \nabla_a T^b = \alpha T^b$ can be reparametrized so that it satisfies $\hat{T}^a \nabla_a \hat{T}^b = 0$. Thus, Wald's proposition is proven.