The equations of structures

Pont du Viaduc de Garabit 1884, wrought iron. Engineers: Léon Boyer, Maurice Koechlin, Builder: Gustave Alexandre Eiffel

Planck non-dimensional unit of force

Schwarzschild solution

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi$$

mass m in metres is equivalent to $m_{\mathrm{Conventional}}$ in kilogrammes
length l in metres is equivalent to $l_{\mathrm{Conventional}}$ in metres
time t in metres is equivalent to $t_{\mathrm{Conventional}}$ in seconds
dimensionless force F is equivalent to $F_{\mathrm{Conventional}}$ in newtons

$$m = \frac{G}{c^2} \times m_{\mathrm{Conventional}}$$
$$l = l_{\mathrm{Conventional}}$$
$$t = c \times t_{\mathrm{Conventional}}$$
$$F = \frac{G}{c^4} F_{\mathrm{Conventional}}$$

G = the gravitational constant = $6.674 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$
c = the speed of light = $2.998 \times 10^8 \mathrm{ms}^{-1}$

$$\frac{c^4}{G} \approx \frac{\left(2.998 \times 10^8\right)^4}{6.674 \times 10^{-11}} \approx 1.210 \times 10^{44} \mathrm{N}$$

1 non-dimensional Planck force ≈ 1.210 x 10^{44} N

Pin jointed trusses and cable networks

Equilibrium of loads and member tensions

$[\mathbf{T}]$ = column matrix of member tensions (negative element for compression)

$[\mathbf{p}]$ = column matrix of loads applied to nodes not connected to supports

$[\mathbf{A}]$ = equilibium matrix relating $[\mathbf{p}]$ to $[\mathbf{T}]$

$[\mathbf{r}]$ = column matrix of support reactions

$[\mathbf{B}]$ = equilibium matrix relating $[\mathbf{r}]$ to $[\mathbf{T}]$

$$\begin{bmatrix}\mathbf{p}\\ \mathbf{r}\end{bmatrix} = \begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}[\mathbf{T}]$$

Compatibility of displacements and member elongations

$[\mathbf{e}]$ = column matrix of member elongation increments

$[\mathbf{u}]$ = column matrix of nodal displacement increments of nodes not connected to supports

$[\mathbf{C}]$ = compatibility matrix relating $[\mathbf{u}]$ to $[\mathbf{e}]$

$[\mathbf{v}]$ = column matrix of nodal displacement increments of nodes that are connected to supports

$[\mathbf{D}]$ = compatibility matrix relating $[\mathbf{v}]$ to $[\mathbf{e}]$

$$[\mathbf{e}] = \begin{bmatrix}\mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}$$

The compatibility matrices and equilibrium matrices are the transpose of each other, $\begin{bmatrix}\mathbf{C} & \mathbf{D}\end{bmatrix} = \begin{bmatrix}\mathbf{A} \\ \mathbf{B}\end{bmatrix}^{\mathrm{T}}$

Elastic stiffness

If members are initially unstressed,

$$[\mathbf{T}] = [\mathbf{S}][\mathbf{e}]$$

$$[\mathbf{S}] = \begin{bmatrix} \dfrac{E_1 A_1}{L_1} & 0 & 0 & 0 \\ 0 & \dfrac{E_2 A_2}{L_2} & 0 & 0 \\ . & . & . & . \\ . & . & . & . \end{bmatrix}$$

E = Young's modulus, A = cross-sectional area of member, L = length of member

Elastic stiffness matrix

$\begin{bmatrix}\mathbf{p} \\ \mathbf{r}\end{bmatrix} = \begin{bmatrix}\mathbf{K_{pu}} & \mathbf{K_{pv}} \\ \mathbf{K_{ru}} & \mathbf{K_{rv}}\end{bmatrix}\begin{bmatrix}\mathbf{u} \\ \mathbf{v}\end{bmatrix}$ where the *elastic stiffness matrix*, $\begin{bmatrix}\mathbf{K_{pu}} & \mathbf{K_{pv}} \\ \mathbf{K_{ru}} & \mathbf{K_{rv}}\end{bmatrix} = \begin{bmatrix}\mathbf{A} \\ \mathbf{B}\end{bmatrix}[\mathbf{S}]\begin{bmatrix}\mathbf{C} & \mathbf{D}\end{bmatrix} = \begin{bmatrix}\mathbf{A} \\ \mathbf{B}\end{bmatrix}[\mathbf{S}]\begin{bmatrix}\mathbf{A} \\ \mathbf{B}\end{bmatrix}^{\mathrm{T}}$. For the case when $\mathbf{v} = \mathbf{0}$, $[\mathbf{p}] = [\mathbf{K_{pu}}][\mathbf{u}]$ and $[\mathbf{u}] = [\mathbf{K_{pu}}]^{-1}[\mathbf{p}]$.

Geometric stiffness matrix

If a structure already loaded or *prestressed*, geometric stiffness may be significant. This stiffness is positive for tension elements and negative for compression elements. In $\begin{bmatrix}\mathbf{p} \\ \mathbf{r}\end{bmatrix} = \begin{bmatrix}\mathbf{A} \\ \mathbf{B}\end{bmatrix}[\mathbf{T}]$ it is the *change* in $\begin{bmatrix}\mathbf{p} \\ \mathbf{r}\end{bmatrix}$ caused by the *change* in $\begin{bmatrix}\mathbf{A} \\ \mathbf{B}\end{bmatrix}$,

$$\delta\begin{bmatrix}\mathbf{p} \\ \mathbf{r}\end{bmatrix} = \delta\begin{bmatrix}\mathbf{A} \\ \mathbf{B}\end{bmatrix}[\mathbf{T}] + \begin{bmatrix}\mathbf{A} \\ \mathbf{B}\end{bmatrix}\delta[\mathbf{T}].$$

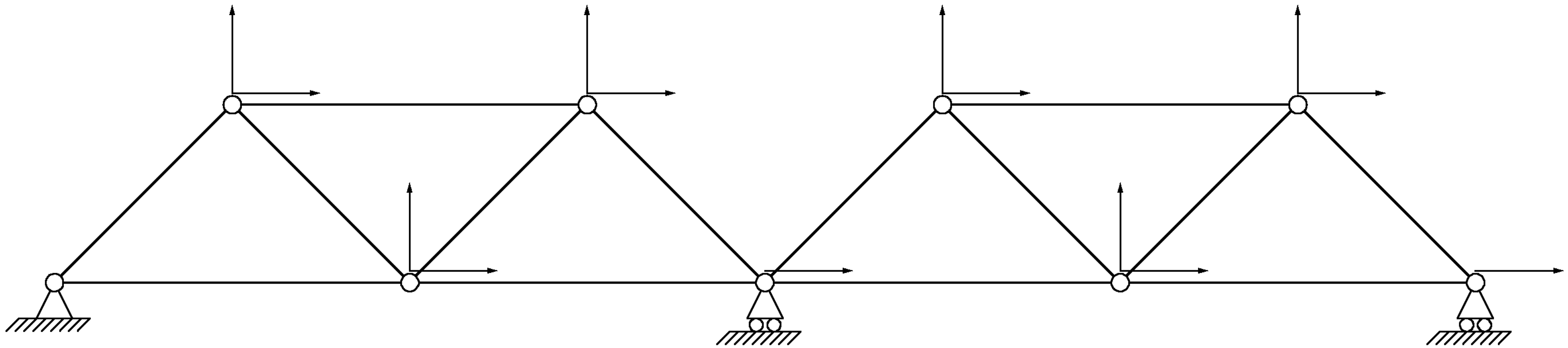
Statical determinacy

A structure is *statically determinate* if $\mathbf{A}$ in $\begin{bmatrix}\mathbf{p} \\ \mathbf{r}\end{bmatrix} = \begin{bmatrix}\mathbf{A} \\ \mathbf{B}\end{bmatrix}[\mathbf{T}]$ is square and non-singular. If this is the case, $[\mathbf{T}] = [\mathbf{A}]^{-1}[\mathbf{p}]$ and $[\mathbf{T}]$ can be determined by statics alone.

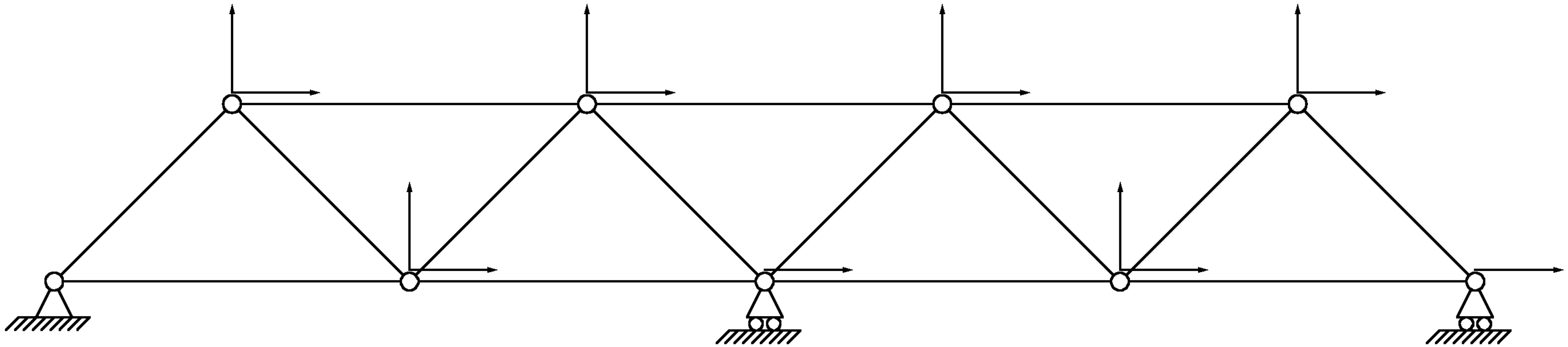
Virtual work theorem

$\begin{bmatrix}\mathbf{u} \\ \mathbf{v}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{p} \\ \mathbf{r}\end{bmatrix} = \begin{bmatrix}\mathbf{u} \\ \mathbf{v}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{A} \\ \mathbf{B}\end{bmatrix}[\mathbf{T}] = \begin{bmatrix}\mathbf{u} \\ \mathbf{v}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{C} & \mathbf{D}\end{bmatrix}^{\mathrm{T}}[\mathbf{T}]$ and thus

$$\boxed{\begin{bmatrix}\mathbf{u} \\ \mathbf{v}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{p} \\ \mathbf{r}\end{bmatrix} = [\mathbf{e}]^{\mathrm{T}}[\mathbf{T}]}$$

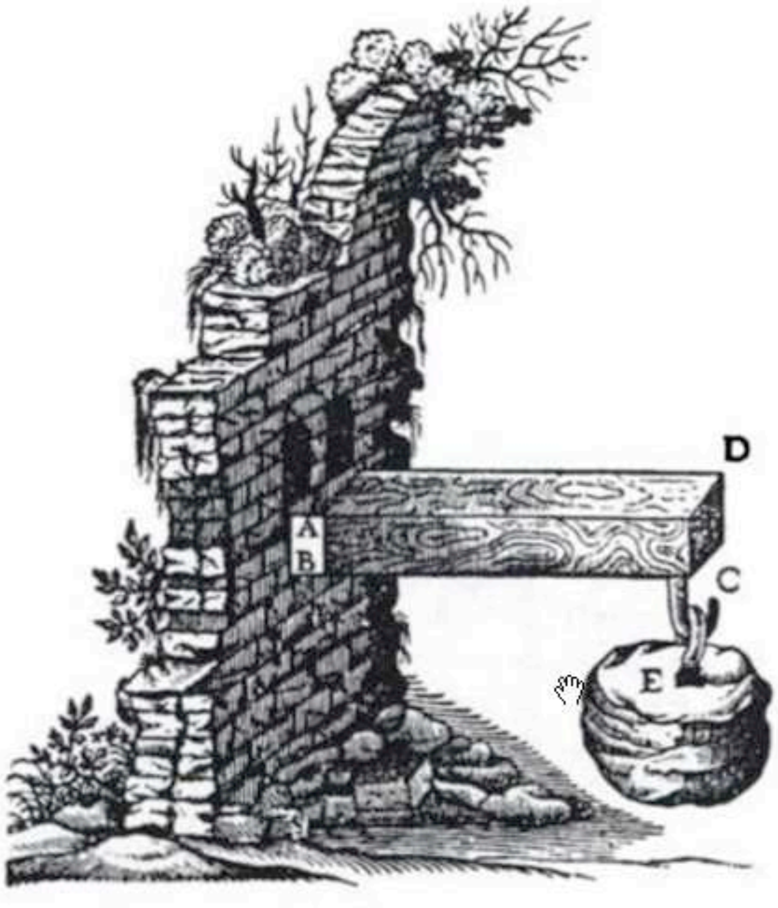


Statically determinate



Statically indeterminate

Beam



Equilibrium of loads, shear force and bending moment

p = upwards load per unit length

F = shear force

M = sagging bending moment

$$p + \frac{dF}{dx} = 0$$

$$F + \frac{dM}{dx} = 0$$

$$p = \frac{d^2M}{dx^2}$$

Compatibility of displacement and curvature

v = upwards displacement

$$\kappa = \text{curvature} = \frac{d^2v}{dx^2}\left(\frac{dv}{dx} \text{ assumed small}\right)$$

Virtual work theorem

$$\int_{\text{End 1}}^{\text{End 2}} M\kappa dx = \int_{\text{End 1}}^{\text{End 2}} M\frac{d^2v}{dx^2}dx = \left[M\frac{dv}{dx}\right]_{\text{End 1}}^{\text{End 2}} - \int_{\text{End 1}}^{\text{End 2}} \frac{dM}{dx}\frac{dv}{dx}dx$$

$$= \left[M\frac{dv}{dx}\right]_{\text{End 1}}^{\text{End 2}} - \left[\frac{dM}{dx}v\right]_{\text{End 1}}^{\text{End 2}} + \int_{\text{End 1}}^{\text{End 2}} \frac{d^2M}{dx^2}vdx$$

and thus

$$\boxed{\int_{\text{End 1}}^{\text{End 2}} M\kappa dx = \left[M\frac{dv}{dx}\right]_{\text{End 1}}^{\text{End 2}} + \left[Fv\right]_{\text{End 1}}^{\text{End 2}} + \int_{\text{End 1}}^{\text{End 2}} pvdx}$$

Three dimensional continuum

Equilibrium of loads and stresses

$\mathbf{p}$ = load per unit volume

$\boldsymbol{\sigma}$ = stress tensor

$$\mathbf{p} + \nabla \bullet \boldsymbol{\sigma} = 0$$

$$p^j + \partial_i \sigma^{ij} = 0$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$$

$$\sigma^{ij} = \sigma^{ji}$$

Compatibility of strains and displacements

$\mathbf{u}$ = displacement increment

$\boldsymbol{\gamma}$ = strain increment tensor

$$\boldsymbol{\gamma} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}\right)$$

$$\gamma_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$$

Isotropic linear elastic material

$\boldsymbol{\gamma} = \dfrac{1}{E}\left((1+\nu)\boldsymbol{\sigma} - \nu \operatorname{tr}(\boldsymbol{\sigma})\mathbf{I}\right)$, E = Young's modulus, ν = Poisson's ratio.

Virtual work

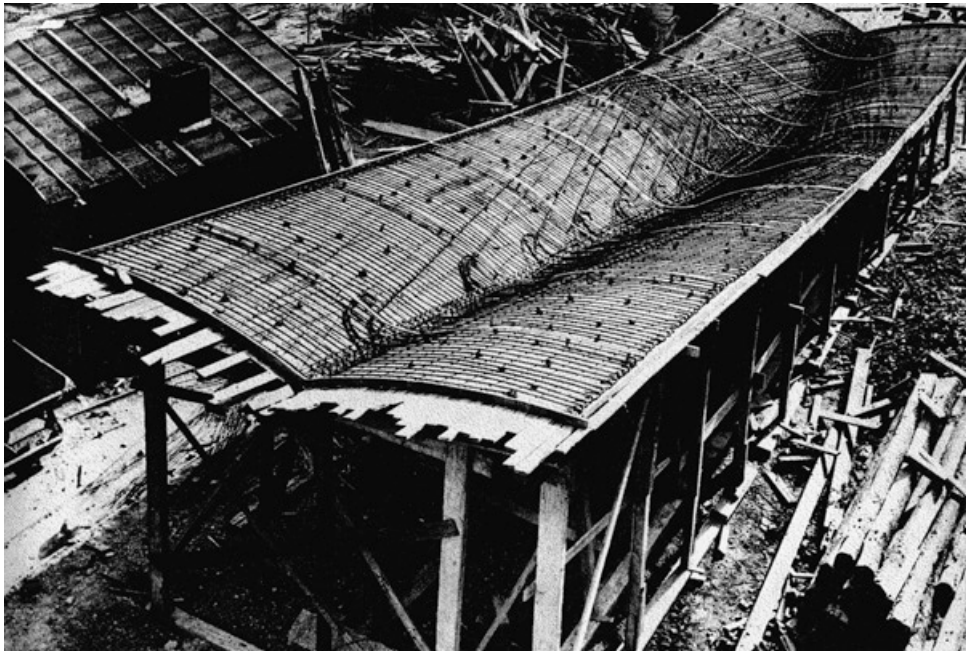
Using the divergence theorem

$$\int_{\partial V} d\mathbf{a} \bullet \boldsymbol{\sigma} \bullet \mathbf{u} + \int_V \mathbf{p} \bullet \mathbf{u}\, dV = \int_V \nabla \bullet (\boldsymbol{\sigma} \bullet \mathbf{u})\, dV + \int_V \mathbf{p} \bullet \mathbf{u}\, dV$$
$$= \int_V \partial_i\left(\sigma^{ij} u_j\right) dV + \int_V \mathbf{p} \bullet \mathbf{u}\, dV$$
$$= \int_V \left(\partial_i \sigma^{ij} u_j + \sigma^{ij} \partial_i u_j\right) dV + \int_V \mathbf{p} \bullet \mathbf{u}\, dV$$
$$= \int_V \left(\nabla \bullet \boldsymbol{\sigma} \bullet \mathbf{u} + \sigma^{ij} \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)\right) dV + \int_V \mathbf{p} \bullet \mathbf{u}\, dV$$

Writing $\boldsymbol{\sigma} : \boldsymbol{\gamma} = \operatorname{tr}\left(\boldsymbol{\sigma} \bullet \boldsymbol{\gamma}^{\mathrm{T}}\right) = \sigma^{ij}\gamma_{ij}$.

$$\boxed{\int_V \boldsymbol{\sigma} : \boldsymbol{\gamma}\, dV = \int_{\partial V} d\mathbf{a} \bullet \boldsymbol{\sigma} \bullet \mathbf{u} + \int_V \mathbf{p} \bullet \mathbf{u}\, dV}$$

Shells



Differential geometry

$$\mathbf{r} = \mathbf{r}\left(\theta^1, \theta^2\right)$$

$\mathbf{a}_\alpha = \dfrac{\partial \mathbf{r}}{\partial \theta^\alpha} = \mathbf{r}_{,\alpha}$ the covariant base vectors

$a_{\alpha\beta} = a_{\beta\alpha} = \mathbf{a}_\alpha \bullet \mathbf{a}_\beta$

are components of the metric tensor

or the coefficients of the first fundamental form

$\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta$ the contravariant base vectors

$\mathbf{a}^\alpha \bullet \mathbf{a}_\beta = \delta^\alpha_\beta$

∂_α denotes the surface covariant derivative

$$\partial_\alpha v^\beta = v^\beta{}_{,\alpha} + v^\chi \Gamma^\beta_{\alpha\chi} = \frac{\partial v^\beta}{\partial \theta^\alpha} + v^\chi \Gamma^\beta_{\alpha\chi}$$

$\Gamma^\beta_{\alpha\chi}$ are the Christoffel symbols

$\mathbf{n} = \dfrac{\mathbf{a}_1 \times \mathbf{a}_2}{\left|\mathbf{a}_1 \times \mathbf{a}_2\right|}$ = unit normal

$$\nabla \mathbf{n} = -\mathbf{b}$$

$b_{\alpha\beta} = b_{\beta\alpha} = -\mathbf{n}_{,\alpha} \bullet \mathbf{a}_\beta = \mathbf{a}_{\alpha,\beta} \bullet \mathbf{n}$

are the coefficients of the second fundamental form

The coefficients of the first and second fundamental form are not independent. They must satisfy the Gauss–Codazzi–Mainardi equations.

The antisymmetric second order surface permutation pseudotensor, $\boldsymbol{\varepsilon} = -\boldsymbol{\varepsilon}^{\mathrm{T}}$, is defined such that $\mathbf{v} = \mathbf{u} \bullet \boldsymbol{\varepsilon}$ in which $\mathbf{u}$ and $\mathbf{v}$ are orthogonal surface vectors with the same magnitude orientated clockwise from $\mathbf{u}$ to $\mathbf{v}$ looking along $\mathbf{n}$.

Bending theory of shells

Equilibrium of loads, membrane stresses and normal shea forces

$\mathbf{p}$ = load per unit area

$\boldsymbol{\sigma}$ = membrane stress tensor

$\mathbf{qn}$ = normal shear force tensor

$\mathbf{N}$ = Vector in plane of surface pointing outwards from boundary of surface

$$0 = \int_A \mathbf{p}\, dA + \int_{\partial A} (\mathbf{N} ds) \bullet (\boldsymbol{\sigma} + \mathbf{qn}) = \int_A \mathbf{p}\, dA + \int_A \nabla \bullet (\boldsymbol{\sigma} + \mathbf{qn}) dA$$

$$\mathbf{p} + \nabla \bullet (\boldsymbol{\sigma} + \mathbf{qn}) = 0$$

$$p^\alpha \mathbf{a}_\alpha + p\mathbf{n} + \left(\partial_\alpha \sigma^{\alpha\beta} - q_\alpha b^{\alpha\beta}\right)\mathbf{a}_\beta + \left(\sigma^{\alpha\beta} b_{\alpha\beta} + \partial_\alpha q^\alpha\right)\mathbf{n} = 0$$

Equilibrium of moments

$\mathbf{m} \bullet \boldsymbol{\varepsilon} = m^{\alpha\lambda} \varepsilon_{\lambda\beta} \mathbf{a}_\alpha \mathbf{a}^\beta$ = bending and torsional moment tensor

$\mathbf{c} = c_\alpha \mathbf{a}^\alpha + c\mathbf{n}$ = applied loading couple per unit area

$$
\begin{aligned}
0 &= \int_A (\mathbf{r}\times\mathbf{p}+\mathbf{c})dA + \int_{\partial A}\mathbf{r}\times\left[(\mathbf{N}ds)\bullet(\boldsymbol{\sigma}+\mathbf{qn})\right] + \int_{\partial A}\left[(\mathbf{N}ds)\bullet\mathbf{m}\bullet\boldsymbol{\varepsilon}\right] \\
&= \int_A (\mathbf{r}\times\mathbf{p}+\mathbf{c})dA - \int_{\partial A}\left[(\mathbf{N}ds)\bullet(\boldsymbol{\sigma}+\mathbf{qn})\times\mathbf{r}\right] + \int_{\partial A}\left[(\mathbf{N}ds)\bullet\mathbf{m}\bullet\boldsymbol{\varepsilon}\right] \\
&= \int_A (\mathbf{r}\times\mathbf{p}+\mathbf{c})dA - \int_A \nabla\bullet\left[(\boldsymbol{\sigma}+\mathbf{qn})\times\mathbf{r}\right]dA + \int_A \nabla\bullet\left[\mathbf{m}\bullet\boldsymbol{\varepsilon}\right]dA \\
&= \int_A \mathbf{r}\times\left[\mathbf{p}+\nabla\bullet(\boldsymbol{\sigma}+\mathbf{qn})\right]dA + \int_A \mathbf{c}dA \\
&\quad -\int_A \begin{bmatrix}\mathbf{a}^{\alpha}\bullet(\boldsymbol{\sigma}+\mathbf{qn})\times\mathbf{a}_{\alpha} + \\ \mathbf{n}\bullet(\boldsymbol{\sigma}+\mathbf{qn})\times\mathbf{n}\end{bmatrix} dA + \int_A \nabla\bullet\left[\mathbf{m}\bullet\boldsymbol{\varepsilon}\right]dA \\
&= \int_A \mathbf{c}dA - \int_A \left[\sigma^{\alpha\beta}\varepsilon_{\beta\alpha}\mathbf{n} + q^{\alpha}\varepsilon_{\alpha\beta}\mathbf{a}^{\beta}\right]dA + \int_A \nabla\bullet\left[\mathbf{m}\bullet\boldsymbol{\varepsilon}\right]dA
\end{aligned}
$$

$$
\begin{aligned}
0 &= \mathbf{c} + \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\mathbf{n} - \mathbf{q}\bullet\boldsymbol{\varepsilon} + \nabla\bullet\left[\mathbf{m}\bullet\boldsymbol{\varepsilon}\right] \\
&= \left(c_{\chi} + \left(-q^{\beta} + \partial_{\alpha}m^{\alpha\beta}\right)\varepsilon_{\beta\chi}\right)\mathbf{a}^{\chi} + \left(c + \left(\sigma^{\alpha\beta} + m^{\lambda\alpha}b_{\lambda}^{\beta}\right)\varepsilon_{\alpha\beta}\right)\mathbf{n}
\end{aligned}
$$

Compatibility of the increments of membrane strain, rotation, bending and displacement

$\mathbf{u}$ = displacement increment

$\boldsymbol{\gamma}$ = strain increment tensor

$\boldsymbol{\beta}$ = increment of unit normal

$\boldsymbol{\omega}$ = increment of rotation

$$
\begin{aligned}
\gamma_{\alpha\beta} &= \text{ increment of } a_{\alpha\beta} = \frac{1}{2}\left(\mathbf{u}_{,\alpha}\bullet\mathbf{a}_{\beta} + \mathbf{a}_{\alpha}\bullet\mathbf{u}_{,\beta}\right) \\
&= \frac{1}{2}\left(\left(u_{\lambda}\mathbf{a}^{\lambda}+u\mathbf{n}\right)_{,\alpha}\bullet\mathbf{a}_{\beta} + \mathbf{a}_{\alpha}\bullet\left(u_{\lambda}\mathbf{a}^{\lambda}+u\mathbf{n}\right)_{,\beta}\right) \\
&= \frac{1}{2}\left(\partial_{\alpha}u_{\beta} + \partial_{\beta}u_{\alpha}\right) - ub_{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\beta} &= \frac{\mathbf{u}_{,1}\times\mathbf{a}_2 + \mathbf{a}_1\times\mathbf{u}_{,2}}{\left|\mathbf{a}_1\times\mathbf{a}_2\right|} - \frac{\mathbf{a}_1\times\mathbf{a}_2\left(\mathbf{u}_{,1}\times\mathbf{a}_2 + \mathbf{a}_1\times\mathbf{u}_{,2}\right)\bullet\left(\mathbf{a}_1\times\mathbf{a}_2\right)}{\left[\left(\mathbf{a}_1\times\mathbf{a}_2\right)\bullet\left(\mathbf{a}_1\times\mathbf{a}_2\right)\right]^{\frac{3}{2}}} \\
&= \frac{\left[\left(u_{\lambda}\mathbf{a}^{\lambda}+u\mathbf{n}\right)_{,1}\times\mathbf{a}_2 + \mathbf{a}_1\times\left(u_{\lambda}\mathbf{a}^{\lambda}+u\mathbf{n}\right)_{,2}\right]\bullet\left(\mathbf{I}-\mathbf{nn}\right)}{\left|\mathbf{a}_1\times\mathbf{a}_2\right|} \\
&= \frac{\left(u_{\lambda}b_1^{\lambda}+u_{,1}\right)\mathbf{n}\times\mathbf{a}_2 + \mathbf{a}_1\times\left(u_{\lambda}b_2^{\lambda}+u_{,2}\right)\mathbf{n}}{\left|\mathbf{a}_1\times\mathbf{a}_2\right|} \\
&= -\left(u_{\lambda}b_{\alpha}^{\lambda}+u_{,\alpha}\right)\mathbf{a}^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega} &= \mathbf{n}\times\boldsymbol{\beta} + \frac{1}{2}\partial_{\alpha}u_{\beta}\varepsilon^{\alpha\beta}\mathbf{n} \\
&= -\left(u_{\lambda}b_{\alpha}^{\lambda}+u_{,\alpha}\right)\mathbf{n}\times\mathbf{a}^{\alpha} + \frac{1}{2}\partial_{\alpha}u_{\beta}\varepsilon^{\alpha\beta}\mathbf{n} \\
&= -\left(u_{\lambda}b_{\alpha}^{\lambda}+u_{,\alpha}\right)\varepsilon^{\alpha\beta}\mathbf{a}_{\beta} + \frac{1}{2}\partial_{\alpha}u_{\beta}\varepsilon^{\alpha\beta}\mathbf{n}
\end{aligned}
$$

Virtual work

$$
\begin{aligned}
&\int_{\partial A}\mathbf{N}\bullet\left[(\boldsymbol{\sigma}+\mathbf{qn})\bullet\mathbf{u} + \mathbf{m}\bullet\boldsymbol{\varepsilon}\bullet\boldsymbol{\omega}\right]ds + \int_A \mathbf{p}\bullet\mathbf{u}dA + \int_A \mathbf{c}\bullet\boldsymbol{\omega}dA \\
&= \int_{\partial A} N_{\alpha}\left(\sigma^{\alpha\beta}u_{\beta} + q^{\alpha}u + m^{\alpha\beta}\varepsilon_{\beta\chi}\omega^{\chi}\right)ds + \int_A \mathbf{p}\bullet\mathbf{u}dA + \int_A \mathbf{c}\bullet\boldsymbol{\omega}dA \\
&= \int_A \partial_{\alpha}\left(\sigma^{\alpha\beta}u_{\beta} + q^{\alpha}u + m^{\alpha\beta}\varepsilon_{\beta\chi}\omega^{\chi}\right)dA + \int_A \mathbf{p}\bullet\mathbf{u}dA + \int_A \mathbf{c}\bullet\boldsymbol{\omega}dA \\
&= \int_A \begin{pmatrix}\partial_{\alpha}\sigma^{\alpha\beta}u_{\beta} + \sigma^{\alpha\beta}\partial_{\alpha}u_{\beta} \\ +\partial_{\alpha}q^{\alpha}u + q^{\alpha}\partial_{\alpha}u + \\ \partial_{\alpha}m^{\alpha\beta}\varepsilon_{\beta\chi}\omega^{\chi} + m^{\alpha\beta}\varepsilon_{\beta\chi}\partial_{\alpha}\omega^{\chi}\end{pmatrix} dA + \int_A \mathbf{p}\bullet\mathbf{u}dA + \int_A \mathbf{c}\bullet\boldsymbol{\omega}dA
\end{aligned}
$$

$$
\begin{aligned}
&= \int_A \begin{pmatrix} \left(\partial_\alpha \sigma^{\alpha\beta} - q_\alpha b^{\alpha\beta}\right) u_\beta + q_\alpha b^{\alpha\beta} u_\beta + \sigma^{\alpha\beta} \partial_\alpha u_\beta \\ + \left(\sigma^{\alpha\beta} b_{\alpha\beta} + \partial_\alpha q^\alpha\right) u - \sigma^{\alpha\beta} b_{\alpha\beta} u + q^\alpha \partial_\alpha u - \left(c_\chi - q^\beta \varepsilon_{\beta\chi}\right) \omega^\chi + m^{\alpha\beta} \varepsilon_{\beta\chi} \partial_\alpha \omega^\chi \end{pmatrix} dA \\
&+ \int_A \mathbf{p} \bullet \mathbf{u} dA + \int_A \mathbf{c} \bullet \boldsymbol{\omega} dA \\
&= \int_A \left(q_\alpha b^{\alpha\beta} u_\beta + q^\alpha \partial_\alpha u + \sigma^{\alpha\beta} \partial_\alpha u_\beta - \sigma^{\alpha\beta} b_{\alpha\beta} u + q^\beta \varepsilon_{\beta\chi} \omega^\chi + m^{\alpha\beta} \varepsilon_{\beta\chi} \partial_\alpha \omega^\chi + c\omega\right) dA \\
&= \int_A \begin{pmatrix} q_\alpha b^{\alpha\beta} u_\beta + q^\alpha \partial_\alpha u + \sigma^{\alpha\beta} \partial_\alpha u_\beta - \sigma^{\alpha\beta} b_{\alpha\beta} u \\ - q^\beta \varepsilon_{\beta\chi} \left(u_\lambda b_\alpha^\lambda + u_{,\alpha}\right) \varepsilon^{\alpha\chi} + m^{\alpha\beta} \varepsilon_{\beta\chi} \partial_\alpha \omega^\chi + c\omega \end{pmatrix} dA \\
&= \int_A \left(q_\alpha b^{\alpha\beta} u_\beta + q^\alpha \partial_\alpha u + \sigma^{\alpha\beta} \partial_\alpha u_\beta - \sigma^{\alpha\beta} b_{\alpha\beta} u - q^\alpha \left(u_\lambda b_\alpha^\lambda + u_{,\alpha}\right) + m^{\alpha\beta} \varepsilon_{\beta\chi} \partial_\alpha \omega^\chi + c\omega\right) dA \\
&= \int_A \left(\sigma^{\alpha\beta} \left(\partial_\alpha u_\beta - b_{\alpha\beta} u\right) + m^{\alpha\beta} \varepsilon_{\beta\chi} \partial_\alpha \omega^\chi + c\omega\right) dA
\end{aligned}
$$

But

$$
\begin{aligned}
&\sigma^{\alpha\beta} \left(\partial_\alpha u_\beta - b_{\alpha\beta} u\right) + m^{\alpha\beta} \varepsilon_{\beta\chi} \partial_\alpha \omega^\chi + c\omega \\
&= \sigma^{\alpha\beta} \left(\frac{1}{2}\left(\partial_\alpha u_\beta + \partial_\alpha u_\beta\right) - b_{\alpha\beta} u\right) + \sigma^{\alpha\beta} \frac{1}{2}\left(\partial_\alpha u_\beta - \partial_\alpha u_\beta\right) + m^{\alpha\beta} \varepsilon_{\beta\chi} \partial_\alpha \omega^\chi + c\omega \\
&= \sigma^{\alpha\beta} \gamma_{\alpha\beta} + \sigma^{\alpha\beta} \varepsilon_{\alpha\beta} \omega + m^{\alpha\beta} \varepsilon_{\beta\chi} \partial_\alpha \omega^\chi + c\omega \\
&= \sigma^{\alpha\beta} \gamma_{\alpha\beta} + m^{\alpha\beta} \varepsilon_{\beta\chi} \partial_\alpha \omega^\chi - m^{\lambda\alpha} b_\lambda^\beta \varepsilon_{\alpha\beta} \omega \\
&= \sigma^{\alpha\beta} \gamma_{\alpha\beta} + m^{\alpha\beta} \varepsilon_{\beta\chi} \left(\partial_\alpha \omega^\chi - b_\alpha^\chi \omega\right) \\
&= \sigma^{\alpha\beta} \gamma_{\alpha\beta} + m^{\alpha\beta} \kappa_{\alpha\beta}
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_{\alpha\beta} &= \varepsilon_{\beta\chi} \left(\partial_\alpha \omega^\chi - b_\alpha^\chi \omega\right) \\
&= \varepsilon_{\beta\chi} \left(\partial_\alpha \left(-\left(u_\lambda b_\mu^\lambda + u_{,\mu}\right) \varepsilon^{\mu\chi}\right) - b_\alpha^\chi \frac{1}{2} \partial_\lambda u_\mu \varepsilon^{\lambda\mu}\right) \\
&= -\partial_\alpha \left(u_\lambda b_\beta^\lambda + u_{,\beta}\right) - b_\alpha^\chi \frac{1}{2}\left(\partial_\beta u_\chi - \partial_\chi u_\beta\right) \\
&= -\left(\partial_\alpha u_\lambda b_\beta^\lambda + u_\lambda \partial_\alpha b_\beta^\lambda + \partial_\alpha \partial_\beta u\right) - b_\alpha^\lambda \frac{1}{2}\left(\partial_\beta u_\lambda - \partial_\lambda u_\beta\right) \\
&= -\partial_\alpha u_\lambda b_\beta^\lambda - \partial_\beta u_\lambda b_\alpha^\lambda - u_\lambda \partial_\alpha b_\beta^\lambda - \partial_\alpha \partial_\beta u + b_\alpha^\lambda \frac{1}{2}\left(\partial_\beta u_\lambda + \partial_\lambda u_\beta\right)
\end{aligned}
$$

$$
\begin{aligned}
\chi_{\alpha\beta} &= \text{Increment of } b_{\alpha\beta} = -\boldsymbol{\beta}_{,\alpha} \bullet \mathbf{a}_\beta - \mathbf{n}_{,\alpha} \bullet \mathbf{u}_{,\beta} \\
&= \partial_\alpha \left(u_\lambda b_\beta^\lambda + u_{,\beta}\right) + b_{\alpha\lambda} \left(\partial_\beta u^\lambda - u b_\beta^\lambda\right) \\
&= \partial_\alpha u_\lambda b_\beta^\lambda + \partial_\beta u_\lambda b_\alpha^\lambda + u_\lambda \partial_\alpha b_\beta^\lambda + \partial_\alpha \partial_\beta u - u b_{\alpha\lambda} b_\beta^\lambda \\
&= \chi_{\beta\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{\alpha\beta} &= -\left(\chi_{\alpha\beta} + u b_{\alpha\lambda} b_\beta^\lambda\right) + b_\alpha^\lambda \frac{1}{2}\left(\partial_\beta u_\lambda + \partial_\lambda u_\beta\right) \\
&= -\chi_{\alpha\beta} + b_\alpha^\lambda \left(\frac{1}{2}\left(\partial_\beta u_\lambda + \partial_\lambda u_\beta\right) - u b_{\beta\lambda}\right) \\
&= -\chi_{\alpha\beta} + b_\alpha^\lambda \gamma_{\lambda\beta} \\
&= -\text{Increment of } b_{\alpha\beta} + b_\alpha^\lambda \text{ times increment of } a_{\lambda\beta} \\
&= -\text{Increment of } \left(b_\alpha^\lambda a_{\lambda\beta}\right) + b_\alpha^\lambda \text{ times increment of } a_{\lambda\beta} \\
&= -a_{\lambda\beta} \text{ times increment of } \left(b_\alpha^\lambda\right)
\end{aligned}
$$

Thus finally,

$$\int_{\partial A} \mathbf{N} \bullet \left[(\boldsymbol{\sigma} + \mathbf{qn}) \bullet \mathbf{u} + \mathbf{m} \bullet \boldsymbol{\varepsilon} \bullet \boldsymbol{\omega} \right] ds$$
$$+ \int_A \mathbf{p} \bullet \mathbf{u} dA + \int_A \mathbf{c} \bullet \boldsymbol{\omega} dA$$
$$= \int_A \boldsymbol{\sigma} : \boldsymbol{\gamma} dA + \int_A \mathbf{m} : \boldsymbol{\kappa} dA$$
$$\boldsymbol{\kappa} = -\boldsymbol{\chi} + \mathbf{b} \bullet \boldsymbol{\gamma}$$

Membrane theory of shells

Equilibrium of loads and membrane stresses

$\mathbf{p} =$ load per unit area
$\boldsymbol{\sigma} =$ membrane stress tensor
$\mathbf{N} =$ Vector in plane of surface pointing outwards from boundary of surface

$$0 = \int_A \mathbf{p} dA + \int_{\partial A} (\mathbf{N} ds) \bullet \boldsymbol{\sigma} = \int_A \mathbf{p} dA + \int_A \nabla \bullet \boldsymbol{\sigma} dA$$

$$\mathbf{p} + \nabla \bullet \boldsymbol{\sigma} = 0$$
$$p^\alpha \mathbf{a}_\alpha + p\mathbf{n} + \partial_\alpha \sigma^{\alpha\beta} \mathbf{a}_\beta + \sigma^{\alpha\beta} b_{\alpha\beta} \mathbf{n} = 0$$

If the shape and loading are known, we have 3 equations of equilibrium and 3 unknowns, σ^{11}, $\sigma^{12} = \sigma^{21}$ and σ^{22}. Thus in the membrane theory shells are statically determinate, if the shape of the shell and the boundary conditions are appropriate.

Equilibrium of moments

$$0 = \int_A \mathbf{r} \times \mathbf{p} dA + \int_{\partial A} \mathbf{r} \times \left[(\mathbf{N} ds) \bullet \boldsymbol{\sigma} \right] = \int_A \mathbf{r} \times \mathbf{p} dA - \int_{\partial A} \left[(\mathbf{N} ds) \bullet \boldsymbol{\sigma} \times \mathbf{r} \right]$$
$$= \int_A (\mathbf{r} \times \mathbf{p} + \mathbf{c}) dA - \int_A \nabla \bullet [\boldsymbol{\sigma} \times \mathbf{r}] dA$$
$$= \int_A \mathbf{r} \times [\mathbf{p} + \nabla \bullet \boldsymbol{\sigma}] dA - \int_A \left[\mathbf{a}^\alpha \bullet \boldsymbol{\sigma} \times \mathbf{a}_\alpha + \mathbf{n} \bullet \boldsymbol{\sigma} \times \mathbf{n} \right] dA$$
$$= -\int_A \sigma^{\alpha\beta} \varepsilon_{\beta\alpha} \mathbf{n} dA$$

Thus $\sigma^{\alpha\beta} = \sigma^{\beta\alpha}$.

Compatibility of the increments of membrane strain, rotation bending and displacement

$\mathbf{u} =$ displacement increment
$\boldsymbol{\gamma} =$ strain increment tensor

$$\gamma_{\alpha\beta} = \text{ increment of } a_{\alpha\beta} = \frac{1}{2}\left(\mathbf{u}_{,\alpha} \bullet \mathbf{a}_\beta + \mathbf{a}_\alpha \bullet \mathbf{u}_{,\beta} \right)$$
$$= \frac{1}{2}\left(\left(u_\lambda \mathbf{a}^\lambda + u\mathbf{n} \right)_{,\alpha} \bullet \mathbf{a}_\beta + \mathbf{a}_\alpha \bullet \left(u_\lambda \mathbf{a}^\lambda + u\mathbf{n} \right)_{,\beta} \right)$$
$$= \frac{1}{2}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha \right) - u b_{\alpha\beta}$$

A shell will try and undergo inextensional deformation (see Spivak and Lord Rayleigh), $\boldsymbol{\gamma} = \mathbf{0}$ when

$$\frac{1}{2}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha \right) - u b_{\alpha\beta} = 0.$$

Virtual work

$$\begin{aligned}
&\int_{\partial A} \mathbf{N} \bullet \boldsymbol{\sigma} \bullet \mathbf{u} ds + \int_A \mathbf{p} \bullet \mathbf{u} dA \\
&= \int_{\partial A} N_\alpha \sigma^{\alpha\beta} u_\beta ds + \int_A \mathbf{p} \bullet \mathbf{u} dA = \int_A \partial_\alpha \left(\sigma^{\alpha\beta} u_\beta \right) dA + \int_A \mathbf{p} \bullet \mathbf{u} dA \\
&= \int_A \left(\partial_\alpha \sigma^{\alpha\beta} u_\beta + \sigma^{\alpha\beta} \partial_\alpha u_\beta + \sigma^{\alpha\beta} b_{\alpha\beta} u - \sigma^{\alpha\beta} b_{\alpha\beta} u \right) dA + \int_A \mathbf{p} \bullet \mathbf{u} dA \\
&= \int_A \left(\sigma^{\alpha\beta} \partial_\alpha u_\beta - \sigma^{\alpha\beta} b_{\alpha\beta} u \right) dA = \int_A \sigma^{\alpha\beta} \gamma_{\alpha\beta} dA
\end{aligned}$$

Thus,

$$\boxed{\int_{\partial A} \mathbf{N} \bullet \boldsymbol{\sigma} \bullet \mathbf{u} ds + \int_A \mathbf{p} \bullet \mathbf{u} dA = \int_A \boldsymbol{\sigma} : \boldsymbol{\gamma} dA}$$

Finite element method

Define geometry of structure by a finite number of *shape functions* which are controlled by the position of nodes, and, possibly, an orientation associated with each node. The surface definition could by a subdivision surface or one of the shell finite elements.

The coordinates (and, if appropriate, orientations) of the nodes are the *degrees of freedom*. The forces (and moments) applied to the structure and associated with each degree of freedom are found using the virtual work equation in which the displacement is that caused by an infinitesimal change in the degree of freedom.

Chris Williams, June 2008

References

A. E. Green and W. Zerna, *Theoretical Elasticity* Oxford Universit Press, 1954.

Michael Spivak, *A Comprehensive Introduction to Differentia Geometry*, Volume 5 2nd Edition, Publish or Perish Inc., 1979

Lord Rayleigh (John William Strutt), *On Bells*, The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science (Fiftl Series), January 1890

In the case of an elastic structure, this leads to exactly the sam formulation as the minimisation of the elastic strain energy.