Linear Regression: Introduction & Estimation

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Equation of a Line

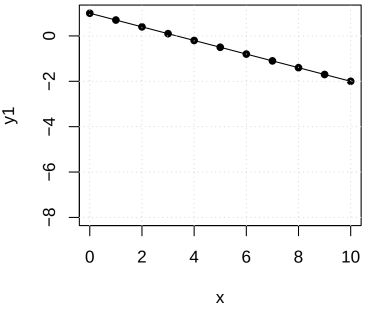
The equation of a straight line is

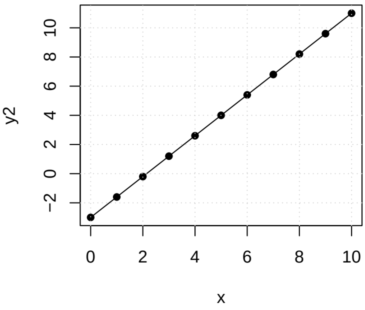
$$\mathrm{y} = \mathrm{bx} + \mathrm{a}$$

where b is the slope of the line and a is the intercept.

Linear Relationships

Below are examples of perfect linear relationships.





Interpretation of slope

For any two points (x_1, y_1) and (x_2, y_2),

$$b = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope is the exact rate of change in y for every unit increase in x.

Interpretation of intercept

When $x = 0$,

$$y = b(0) + a = a$$

The intercept is the exact value of y when $x = 0$.

Example: Housing Prices

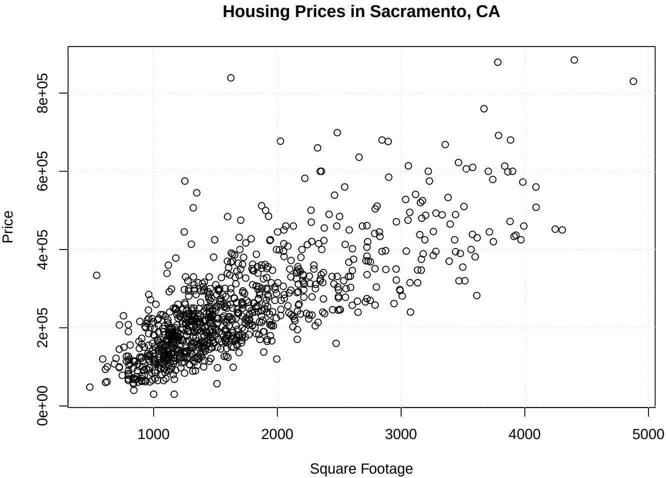
- How much should you pay for a house?
- Factors that influence price:
  - Location
  - Year built
  - Amenities
  - Square footage
- To determine a price, we might model price as a function of square footage:

  $$\text{Price} = \text{f}(\text{Square Footage}) + \varepsilon$$

  f is called the regression function.

Motivating Example: Housing Prices

Scatter plots help determine the functional relationship between two variables.



Simple Linear Regression

The simplest model is where f is a linear function:

$$\mathrm{f(x)} = \beta_0 + \beta_1 \mathrm{x}.$$

The model becomes

$$\mathrm{Y_i} = \beta_0 + \beta_1 \mathrm{X_i} + \varepsilon_{\mathrm{i}}.$$

$$\mathrm{Price_i} = \beta_0 + \beta_1 \mathrm{Square\ Footage_i} + \varepsilon_{\mathrm{i}}.$$

$\mathrm{Y_i}$ is called the dependent variable or the response and $\mathrm{X_i}$ is called the independent variable or predictor.

Parameters of SLR

Under SLR, the model is

$$\mathrm{Y_i} = \beta_0 + \beta_1 \mathrm{X_i} + \varepsilon_\mathrm{i}.$$

The parameters of the model are the intercept β_0 and the slope β_1. These are unknown and must be estimated using data.

The Error Term

Under SLR, the model is

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i.$$

The error term ε_i accounts for the fact that

- not all the points lie exactly on the regression line and
- Y cannot be perfectly predicted from X alone

The Error Term

Under SLR, the model is

$$\mathrm{Y_i} = \beta_0 + \beta_1 \mathrm{X_i} + \varepsilon_\mathrm{i}.$$

We assume the error term ε_i satisfies

- $\varepsilon_\mathrm{i} \sim \mathcal{N}(0, \sigma^2)$
- the error terms are all independent of each other (mutually independent)

Regression Models the Conditional Expectation

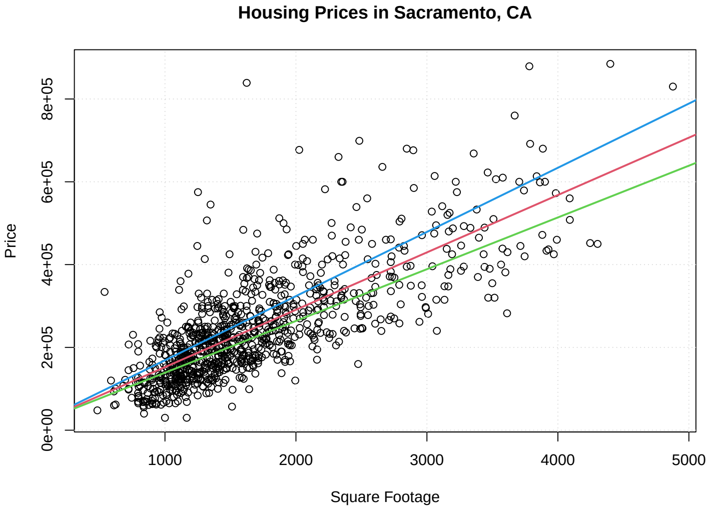
In SLR, $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$ so that $\mathbb{E}(\varepsilon_i) = 0$. Treating X_i as a given constant, we have

$$\begin{aligned}\mathbb{E}[Y_i|X_i] &= \mathbb{E}(\beta_0 + \beta_1 X_i + \varepsilon_i) \\ &= \beta_0 + \beta_1 X_i + \mathbb{E}(\varepsilon_i) \\ &= \beta_0 + \beta_1 X_i\end{aligned}$$

$\mathbb{E}[Y_i|X_i]$ is the conditional expectation of Y_i given X_i - it depends on the value X_i.

Line of Best Fit

Many different lines “fit” the data, which is the best?



Residuals

Given some β_0 and β_1, the predicted value of $\mathrm{Y_i}$ at $\mathrm{X_i}$ is

$$\hat{\mathrm{Y}}_\mathrm{i} = \beta_0 + \beta_1 \mathrm{X_i}$$

The ith residual is

$$\hat{\varepsilon}_\mathrm{i} = \mathrm{Y_i} - \hat{\mathrm{Y}}_\mathrm{i} = \mathrm{Y_i} - \beta_0 - \beta_1 \mathrm{X_i}$$

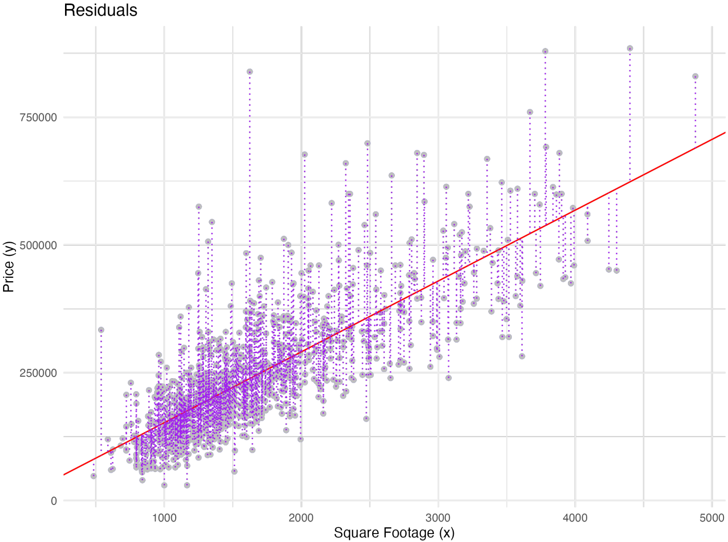
Least Squares Criterion

We choose the line with intercept β_0 and slope β_1 that minimizes the sum of squared residuals,

$$\mathrm{RSS} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \hat{\varepsilon}_{\mathrm{i}}^2 = \sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{Y}_{\mathrm{i}} - \beta_0 - \beta_1 \mathrm{X}_{\mathrm{i}})^2$$

The values of β_0 and β_1 that minimize RSS are called the least squares estimators.

Residuals of the Housing Data



Least Squares Estimators

Use principles of calculus to find the minimizers of the residual sum of squares,

$$\mathrm{RSS} = \sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{Y_i} - \beta_0 - \beta_1 \mathrm{X_i})^2.$$

Least Squares Estimators

Define

$$\mathrm{SXY} = \sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{X_i} - \bar{\mathrm{X}})(\mathrm{Y_i} - \bar{\mathrm{Y}}) \quad \text{and} \quad \mathrm{SXX} = \sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{X_i} - \bar{\mathrm{X}})^2.$$

The least squares estimators are

$$\hat{\beta}_1 = \frac{\mathrm{SXY}}{\mathrm{SXX}} \quad \text{and} \quad \hat{\beta}_0 = \bar{\mathrm{Y}} - \hat{\beta}_1 \bar{\mathrm{X}}.$$

Covariance

The quantity

$$\mathrm{SXY} = \sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{X_i} - \bar{\mathrm{X}})(\mathrm{Y_i} - \bar{\mathrm{Y}})$$

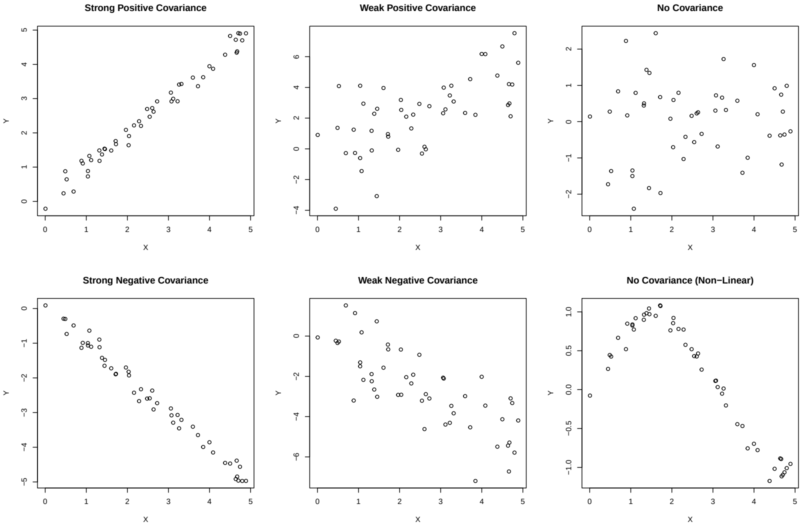
is an estimator of the covariance between X and Y.

Covariance

The covariance $\mathrm{Cov}(X, Y)$ between X and Y

- quantifies the strength of the linear relationship between X and Y,
- can be positive or negative, and
- could be near zero if the relationship is non-linear.

Covariance



Strong Positive Covariance
Weak Positive Covariance
No Covariance
Strong Negative Covariance
Weak Negative Covariance
No Covariance (Non−Linear)
X
Y

Correlation

Standardizing the covariance gives the correlation ρ:

$$\rho = \frac{\mathrm{Cov}(\mathrm{X}, \mathrm{Y})}{\sigma_{\mathrm{X}} \sigma_{\mathrm{Y}}}$$

It has the same interpretation but is restricted to lie between -1 and 1. Correlation of 1 is a perfect positive linear relationship.

Interpretation of Slope Parameter

Recall

$$\mathbb{E}[\mathrm{Y_i}|\mathrm{X_i}] = \beta_0 + \beta_1 \mathrm{X_i}$$

- The slope is the change in the expected value of Y for every unit increase in X.
- The intercept is the expected value of Y when X = 0.

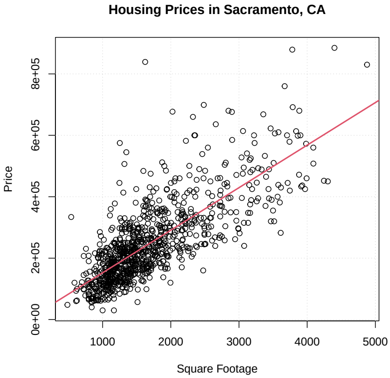
Example: Housing Prices

For the housing data,

$$\hat{\beta}_0 = 13859.393$$

and

$$\hat{\beta}_1 = 138.546$$



No Intercept Model

Sometimes it is appropriate to fit the SLR model with no intercept:

$$Y_i = \beta_1 X_i + \varepsilon_i$$

Is this appropriate for the housing data?

Linear Model in R

A linear model can be fit using

```
> model <- lm(price ~ sqft)
> model <- lm(price ~ sqft - 1) # no intercept
> summary(model)
```

Linear Model in R

```
> summary(model)

Call:
lm(formula = price ~ sqft, data = Sacramento)

Residuals:
    Min      1Q  Median      3Q     Max 
-231889  -54717  -11822   38993  600141 
```

$\hat{\beta}_0$

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 13859.393   6948.714   1.995   0.0464 *  
sqft          138.546      3.796  36.495   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

$\hat{\beta}_1$

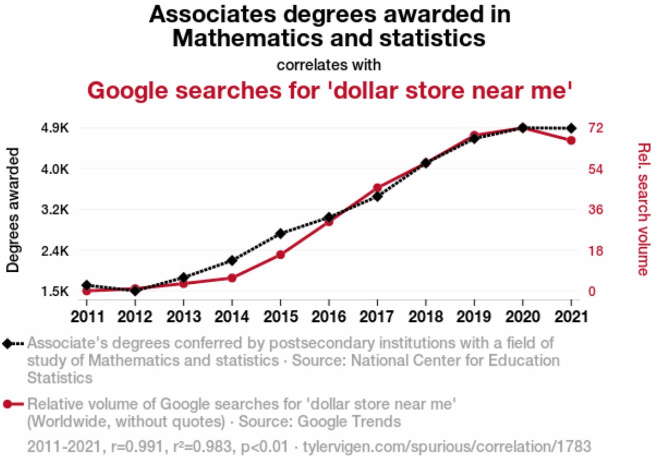
```
Residual standard error: 84130 on 930 degrees of freedom
Multiple R-squared:  0.5888,	Adjusted R-squared:  0.5884 
F-statistic:  1332 on 1 and 930 DF,  p-value: < 2.2e-16
```

Association and Causation

A strong relationship between two variables does not always imply a causal relationship.

A strong association between two variables is often due to lurking variables that we are not aware of.

Association and Causation



https://www.tylervigen.com/spurious-correlations

Association and Causation

- The best evidence for causal relationships comes from properly designed randomized experiments.
- Observational studies can show a strong association, but it is not appropriate to conclude causation.

Does Smoking Cause Lung Cancer?

- Unethical to investigate this with an experiment.
- Observational studies have demonstrated an association between lung cancer and smoking.
- Evidence has been collected from many studies.
- It is plausible that smoking causes cancer, but the conclusion is not as strong as evidence from a randomized experiment.