University of St Andrews

Final year project

Equilibria of Self-Gravitating Fluids

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I certify that this project report has been written by me, is a record of work carried out by me, and is essentially different from work undertaken for any other purpose or assessment.

Bruce Muller, April 28, 2011

Abstract

Using the equations of motion a relation is derived that may be used to find possible configurations to the problem of a uniformly rotating homogeneous body that satisfy equilibrium. The equilibrium is a force balance between gravitational, centripetal and pressure forces. Configurations that have an ellipsoidal figure are sought and the gravitational potential at an internal point is determined and consequently a relation for the gravitational potential energy is derived. Using these relations two possible equilibrium configurations are found:

- The Maclaurin spheroids
- The Jacobi ellipsoids

where the Maclaurin spheroids constitute a sequence of oblate spheroids that rotate about their polar axis with uniquely determined angular velocities for each member, and the Jacobi ellipsoids constitute a sequence of ellipsoids that rotate about their polar axis with a uniquely defined relationship between their principal semi-axis ratios and corresponding angular velocities.

Investigation of these two sequences reveals the property that the Jacobi ellipsoids bifurcate from the Maclaurin spheroids at a specific point along the sequence which uniquely binds and relates these two sequences of ellipsoidal equilibrium configurations.

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Chapter 1

Introduction

Consider a rotating distribution of mass that is subject to its own gravitational field and pressure gradients. Since this configuration is rotating we also have a centripetal effect that acts to repel matter away from the axis of rotation, where as gravity will have the opposite effect. It is the combination of gravity, rotation and pressure gradients that constitutes the force balance which we shall call equilibrium. This state of affairs gives rise to the the theory of the conditions necessary for a distribution of matter to sustain its shape or form in time as defined by its boundary, usually under a uniform rotation, and we will call such a distribution an equilibrium configuration. Using this theory we can begin to model astronomical masses such as gas giants, galaxies and other astrophysical phenomenon.

The problem of the equilibrium for a uniformly rotating homogeneous mass is the primary concern of this report. The aim is to provide a survey of the key aspects of this problem and its solution. The method outlined here is primarily based on the work by Chandrasekhar in his book *Ellipsoidal Figures of Equilibrium* [5].

1.1 Conditions for equilibrium

Before we proceed it is important to highlight the following approximations that will be made throughout the rest of this report. We have that:

- The configurations rotate uniformly.
- The configurations are homogeneous.
- The configurations are self-gravitating.
- The configurations are an ideal fluid.

Homogeneity indicates that we have uniformly distributed mass or, in other words, a constant density. The term self-gravitating implies that the configurations in question are of sufficient mass to take their own gravitational field into account for the equilibrium. A justification of the the above assumptions is that the resulting dynamics are greatly simplified.

In order for a homogeneous configuration of uniformly rotating matter to be in a state of equilibrium there must be a force balance. The forces that are in balance are:

- Gravity due to the mass of the configuration.
- Centripetal force due to the rotation.
- Pressure gradients present within the configuration.

These three forces define the equilibrium and are the only forces that we will take into account within our analysis.

You can imagine the centripetal effect as the same that you would achieve when slinging a ball on a string around your body. As soon as you let go of the string the ball will proceed to move off at a tangent to its original state of equilibrium. For the ball, the outward rotation is balanced by the tension in the string but in our model we have that the rotation is balanced by gravity. Though it is important to note that we also have pressure gradients present within the rotating body that contribute to the force balance.

1.2 Overview

In Chapter 2 we will state fundamental definitions which will be used and we will also see how we can derive the relevant virial equations which can be used to directly analyse the problem of possible equilibria for a uniformly rotating homogeneous body.

The key to finding possible equilibrium configurations is determining the gravitational potential of the distribution and this is the objective of Chapter 3. Here we will introduce homoeoidal shells and seek to construct ellipsoidal figures to find whether or not there exists ellipsoidal equilibria.

In Chapter 4 fundamental integrals to the problem are sought to be expressed in terms of the standard incomplete elliptic integral functions of the two kinds, which enabled the defining equations for the forthcoming sequences of equilibrium configurations to be expressed in terms of these well understood functions. It also enabled the illustrations in Chapter 5 to be computed.

Chapter 5 will outline the possible equilibrium figures and we shall see that there exist ellipsoidal equilibria, the shape of which is defined by its principal axis. These equilibria come in the form of two unique sequences:

- The Maclaurin spheroids.
- The Jacobi ellipsoids.

and leading on from this we will look at the bifurcation properties of these sequences.

The aims of this report are:

- To explore possible equilibrium configurations for a uniformly rotating homogeneous mass.
- To survey theory leading to these configurations.
- To review the bifurcation properties of these configurations

This project is intended to be read by interested mathematicians at or above undergraduate Honours level. Knowledge of basic mathematical notation is assumed. Index notation will be used and a basic knowledge of this is useful.

Chapter 2

The Virial Equations

2.1 The virial method

The virial method involves taking the moments of the relevant hydrodynamical equations and results in the virial equations of various orders. The hydrodynamical equations in question will be the momentum equations, otherwise known as the equations of motion, in which the gravitational field of the distribution of matter is taken into account and ultimately these equations need to be applied to a rotating frame of reference.

How does one take the moments of these equations?

The moments of these equations involve simply multiplying the momentum equations, sequentially, by $1, x_j, x_j x_k, x_j x_k x_l$ etc. and integrating over the volume V which the fluid occupies instantaneously. We will first apply this method to the momentum equations in a non-rotating frame of reference and then we will apply the method to the momentum equations relevant under rotating conditions.

Why are the virial equations useful? The purpose of deriving the virial equations, in particular the virial equations of the second order, is so that we may find solutions to the problem of the equilibrium of a uniformly rotating homogeneous fluid in an entirely systematic way. The second-order virial equations derived in this chapter are important in describing the force-balance that exists for rotating bodies that are modeled as a fluid. Hence these equations will be central to the derivation of possible equilibrium configurations derived in Chapter 5.

Before we proceed we will need to define quantities best expressed in terms of tensors which will be required later on.

2.2 Definitions: The moments of density, pressure, velocity and gravitational potential

The moments of density, pressure, velocity and gravitational potential help to describe the distribution of these quantities. The following definitions are essential for deriving and describing the virial equations.[1] Furthermore it should be noted that when using index notation the convention is that we sum over repeated indices.

The moment of inertia tensors

Consider a homogeneous distribution of matter (of mass M) in a volume V. We have

[1]For more detailed definitions and information see Chandrasekhar [5, p. 15-18]

$$M = \int_V \rho \; d\mathbf{x} \tag{2.1}$$

where ρ is the density of the distribution, assumed constant throughout. The moment of inertia is defined as

$$I = \int_V \rho |\mathbf{x}|^2 d\mathbf{x} \tag{2.2}$$

where $\mathbf{x}$ is a point in space.

The moment of inertia tensor of rank-one is given by

$$I_i = \int_V \rho x_i \, d\mathbf{x} \equiv 0 \tag{2.3}$$

and notice that this has been set equivalent to zero by choice; this fixes the origin as the centre of mass for any configuration we consider.

The moment of inertia tensor of rank-two is defined as

$$I_{ij} = \int_V \rho x_i x_j \, d\mathbf{x} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}, i = 1,2,3, j = 1,2,3 \tag{2.4}$$

and this will give further information about the distribution of matter; it is a more informative quantity than I or I_i. The moment of inertia tensor I_{ij} is a 3×3 matrix in three-dimensional space. The tensor I_{ij} is symmetric in its indices and its trace is equivalent to the scalar moment of inertia:

$$I_{ii} = I \tag{2.5}$$

where we sum over repeated indices. Note that the trace of a matrix is the sum of its diagonal elements.

The distribution of pressure

Within the treatment of this project, only the scalar pressure will be required:

$$\Pi = \int_V p \; d\mathbf{x} \tag{2.6}$$

which helps to describe the prevalent distribution of pressure p and we will assume the pressure p is isotropic. We will not need to consider the moments of higher rank (i.e. $\Pi_i, \Pi_{ij}, etc...$).

The distribution of velocity

The total kinetic energy of the motions in the system is given by

$$K = \frac{1}{2} \int_V \rho |\mathbf{u}|^2 \; d\mathbf{x} \tag{2.7}$$

where $|\mathbf{u}|^2 = u_1^2 + u_2^2 + u_3^2$ is the square of the velocity of a fluid element at a point $\mathbf{x}$.

However a more enlightening quantity is the rank-two kinetic-energy tensor

$$K_{ij} = \frac{1}{2}\int\limits_V \rho u_i u_j \ d\mathbf{x} \tag{2.8}$$

and is also symmetric in its indices, i.e.

$$K_{ij} = K_{ji}$$

and also note that $K_{ii} = K$.

Gravitational potential and potential energy

We will also need to consider a gravitational potential due to a distribution of matter of density ρ, given by[2]

$$U(\mathbf{x}) = -G\int\limits_V \frac{\rho}{|\mathbf{x}' - \mathbf{x}|} \ d\mathbf{x}' \tag{2.9}$$

where G is the constant of gravitation. This is a solution of Poisson's equation

$$\nabla^2 U = 4\pi G\rho \tag{2.10}$$

at an internal point of the distribution for constant density ρ, which indicates that the potential is of the form

$$U = Ax_1^2 + Bx_2^2 + Cx_3^2 + D$$

where A,B,C and D are constants and x_i is a point within the body. The determination of the potential is explored in Chapter 3. The gravitational potential energy, directly associated with the gravitational potential, is given by

$$E = -\frac{1}{2}\int\limits_V \rho U \ \mathrm{d}\mathbf{x} \tag{2.11}$$

As before we will require tensor generalisations of the scalar quantities U and E denoted by U_{ij} and E_{ij} respectively. Like the moment of inertia tensor I_{ij} and the kinetic-energy tensor K_{ij}, we have

$$U_{ii} = U \quad \text{and} \quad E_{ii} = E$$

and the generalisations required are given by

$$U_{ij}(\mathbf{x}) = G\int\limits_V \rho\frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{x} - \mathbf{x}'|^3} \ d\mathbf{x}' \tag{2.12}$$

$$E_{ij} = -\frac{1}{2}\int\limits_V \rho U_{ij} \ d\mathbf{x} \tag{2.13}$$

A form which will be useful follows from the definition of U_{ij}:

[2]See Binney [6, p. 30] for more detail

$$
\begin{aligned}
E_{ij} &= -\frac{1}{2}\int_V \rho U_{ij}\ d\mathbf{x} \\
&= -\frac{1}{2}G\int_V\int_V \rho^2\frac{(x_i-x_i')(x_j-x_j')}{|\mathbf{x}-\mathbf{x}'|^3}\ d\mathbf{x}d\mathbf{x}' \\
&= -G\int_V\int_V \rho^2\frac{x_i(x_j-x_j')}{|\mathbf{x}-\mathbf{x}'|^3}\ d\mathbf{x}d\mathbf{x}' \\
&= G\int_V \rho x_i\frac{\partial}{\partial x_j}\int_V \frac{\rho}{|\mathbf{x}-\mathbf{x}'|}\ d\mathbf{x}'d\mathbf{x} \\
&= \int_V \rho x_i\frac{\partial U}{\partial x_j}\ d\mathbf{x}
\end{aligned} \tag{2.14}
$$

2.3 The virial equations in a non-rotating frame of reference

Now that we have stated the general tensors required we shall consider deriving the virial equations. Consider a non-rotating fluid described by its homogeneous density and an isotropic pressure $p(\mathbf{x},t)$. Remember that we assume throughout this report that the bodies under consideration are massive enough to take their own gravitational field into account and this is what we mean by self-gravitating. For our purposes we will seek configurations satisfying a force balance between gravity, inertial forces and pressure gradients. Under these circumstances the motion of the fluid is governed by the well known hydrodynamical equations of motion:

$$
\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho\frac{\partial U}{\partial x_i} \quad , \quad i = 1,2,3 \tag{2.15}
$$

where

$$
\frac{DH}{Dt} = \frac{\partial H}{\partial t} + u_j\frac{\partial H}{\partial x_j} \tag{2.16}
$$

is the material or convective derivative and remember that we sum over repeated indices. It describes the variation in time of a quantity H associated with a fluid element as we follow the element with its motion in the fluid.

As previously stated we may derive the virial equations of the various orders by multiplying the equations of motion (2.15) by $1, x_i, x_ix_j, x_ix_jx_k, etc.$, and integrating over the entire volume V instantaneously occupied by the fluid. *For our purposes we shall only need to derive the virial equations of second order.*

2.3.1 The virial equations of second order

Multiplying equation (2.15) by x_j and then integrating over the volume V obtains

$$
\int_V \rho x_j\frac{Du_i}{Dt}\ d\mathbf{x} = \int_V x_j\left(-\frac{\partial p}{\partial x_i} + \rho\frac{\partial U}{\partial x_i}\right)\ d\mathbf{x} \tag{2.17}
$$

$$
= \int_V -x_j\frac{\partial p}{\partial x_i}\ d\mathbf{x} + \int_V \rho x_j\frac{\partial U}{\partial x_i}\ d\mathbf{x} \tag{2.18}
$$

Before proceeding we will require the following property:

PROPERTY:

$$\int_V \rho(\mathbf{x},t)\frac{DA}{Dt}\ d\mathbf{x} = \frac{D}{Dt}\int_V \rho(\mathbf{x},t)A(\mathbf{x},t)\ d\mathbf{x} \tag{2.19}$$

where $A(\mathbf{x},t)$ is any characteristic of a fluid element. See Chandrasekhar [5, p. 21] for an explanation of this property.

Firstly we can reduce the form of the term on the left-hand side of (2.17) by using (2.19) and the definition of the kinetic-energy tensor K_{ij} given by (2.8), thus

$$\begin{aligned}\int_V \rho x_j \frac{Du_i}{Dt}\ d\mathbf{x} &= \int_V \rho\left(\frac{D}{Dt}(u_i x_j) - u_i u_j\right)\ d\mathbf{x} \\ &= \frac{D}{Dt}\int_V \rho u_i x_j\ d\mathbf{x} - \int_V \rho u_i u_j\ d\mathbf{x} = \frac{D}{Dt}\int_V \rho u_i x_j\ d\mathbf{x} - 2K_{ij}\end{aligned} \tag{2.20}$$

Secondly we can reduce the the first term of (2.18) using integration by parts as follows

$$-\int_V x_j \frac{\partial p}{\partial x_i}\ d\mathbf{x} = -\int_S p x_j\ d\mathbf{S} + \int_V p\frac{\partial x_j}{\partial x_i}\ d\mathbf{x} \tag{2.21}$$

where S is a surface bounding the volume V. Note that $d\mathbf{S} = n_i dS$ is the outward unit surface normal to S. The first term on the right-hand side of (2.21) vanishes on the boundary of V due to the fact that p is isotropic by assumption (i.e symmetric in the coordinate axis). Also δ_{ij} is defined as

$$\delta_{ij} \equiv \frac{\partial x_j}{\partial x_i} \quad = \begin{cases} 1 & ,i = j \\ 0 & ,i \neq j \end{cases}$$

and therefore we have

$$-\int_V x_j \frac{\partial p}{\partial x_i}\ d\mathbf{x} = \delta_{ij}\int_V p\ d\mathbf{x} = \delta_{ij}\Pi \tag{2.22}$$

where we have used the definition of Π in (2.6).

Thirdly we can reduce the second term of (2.18) by using equation (2.14), thus

$$\int_V \rho x_j \frac{\partial V}{\partial x_i}\ d\mathbf{x} = E_{ij} \tag{2.23}$$

Finally combining these results we obtain the equation (Chandrasekhar [5, p. 22])

$$\frac{D}{Dt}\int_V \rho u_i x_j\ d\mathbf{x} = 2K_{ij} + E_{ij} + \delta_{ij}\Pi \tag{2.24}$$

which represents a set of nine equations due to the fact that tensors of rank-two in three-dimensional space have a total of nine elements.

For this system to be in a stationary state, equation (2.24) becomes

$$2K_{ij} + E_{ij} = -\delta_{ij}\Pi \tag{2.25}$$

Note that since all the tensors in this expression are symmetric in their indices i and j, we have a total of six *unique* integral relations which *must* be satisfied for conditions to be, and remain to be, stationary. We will now extend this derivation into a rotating frame of reference so that we may investigate the problem of a rotating configuration.

2.4 The virial equations in a rotating frame of reference

Consider a homogeneous distribution of mass rotating with a uniform angular velocity $\mathbf{\Omega}$. In order to analyse problems of the equilibrium of such rotating configurations we will need to refer the equations of motion (2.15) to a frame of reference rotating with angular velocity $\mathbf{\Omega}$.

2.4.1 The equations of motion in a rotating frame of reference

Given a Cartesian position vector $\underline{\mathbf{x}} = x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}}$ we have

$$\frac{d\underline{\mathbf{x}}}{dt} = \frac{dx}{dt}\underline{\mathbf{i}} + \frac{dy}{dt}\underline{\mathbf{j}} + \frac{dz}{dt}\underline{\mathbf{k}} + x\frac{d\underline{\mathbf{i}}}{dt} + y\frac{d\underline{\mathbf{j}}}{dt} + z\frac{d\underline{\mathbf{k}}}{dt} \tag{2.26}$$

If we denote the absolute reference frame by the Cartesian coordinates ($\underline{\mathbf{I}}$,$\underline{\mathbf{J}}$,$\underline{\mathbf{K}}$) and the rotating reference frame by the Cartesian coordinates ($\underline{\mathbf{i}}$,$\underline{\mathbf{j}}$,$\underline{\mathbf{k}}$), it can be shown that for a solid and uniform rotation Ω we have

$$\underline{i} = \cos(\Omega t)\underline{I} + \sin(\Omega t)\underline{J}$$
$$\underline{j} = -\sin(\Omega t)\underline{I} + \cos(\Omega t)\underline{J}$$

and hence we may find that

$$\frac{d\underline{i}}{dt} = \Omega\underline{k} \times \underline{i}\ ,$$
$$\frac{d\underline{j}}{dt} = \Omega\underline{k} \times \underline{j}\ ,$$
$$\frac{d\underline{k}}{dt} = \Omega\underline{k} \times \underline{k}$$

and thus (2.26) becomes

$$\left(\frac{d\underline{\mathbf{x}}}{dt}\right)_a = \left(\frac{d\underline{\mathbf{x}}}{dt}\right)_r + \underline{\Omega} \times \underline{\mathbf{x}} \tag{2.27}$$

where $\underline{\Omega} = \Omega\underline{k}$ i.e. a solid rotation about the z-axis. Using index notation this becomes

$$(u_i)_a = (u_i)_r + \varepsilon_{ilm}\Omega_l x_m \tag{2.28}$$

where subscript a denotes calculation relative to the absolute frame and subscript r denotes calculation relative to the rotating frame. We define ε_{ilm} as follows

$$\varepsilon_{ilm} \equiv \begin{cases} 1 & \text{if i,j,k are an even permutation of 1,2,3} \\ -1 & \text{if i,j,k are an odd permutation of 1,2,3} \\ 0 & \text{otherwise} \end{cases}$$

We will denote the absolute velocity by u_i and the relative velocity (i.e the velocity calculated in the rotating frame) by v_i. Where there is no subscript we calculate with respect to the rotating frame.

$$\left(\frac{d\underline{\mathbf{x}}}{dt}\right)_a = (u_i)_a \equiv u_i \quad - \text{Absolute velocity} \tag{2.29}$$

$$\left(\frac{d\underline{\mathbf{x}}}{dt}\right)_r = (u_i)_r \equiv v_i \quad - \text{Relative velocity} \tag{2.30}$$

If we differentiate (2.28) with respect to time in the absolute frame of reference we obtain

$$\begin{aligned}
\left(\frac{d}{dt}\right)_a u_i &= \left(\frac{Du_i}{Dt}\right)_a \\
&= \left(\frac{d}{dt}\right)_a \left[\frac{dx_i}{dt} + \varepsilon_{ilm}\Omega_l x_m\right] \\
&= \frac{d}{dt}\left(\frac{dx_i}{dt}\right)_a + \varepsilon_{ilm}\Omega_l \left(\frac{dx_m}{dt}\right)_a \\
&= \frac{d}{dt}\left[\frac{dx_i}{dt} + \varepsilon_{ilm}\Omega_l x_m\right] + \varepsilon_{ilm}\Omega_l \left[\frac{dx_m}{dt} + \varepsilon_{mnq}\Omega_n x_q\right] && (2.31) \\
&= \frac{d^2x_i}{dt^2} + 2\varepsilon_{ilm}\Omega_l \frac{dx_m}{dt} + \varepsilon_{ilm}\Omega_l \varepsilon_{mnq}\Omega_n x_q \\
&= \frac{dv_i}{dt} + 2\varepsilon_{ilm}\Omega_l v_m + \varepsilon_{ilm}\Omega_l \varepsilon_{mnq}\Omega_n x_q && (2.32)
\end{aligned}$$

Note that in (2.31) we have to introduce n and q so that there is no conflict with l and m. Also note that in the above calculation, where subscript a is not used we are calculating in the rotating reference frame. In equation (2.32) it should be noted that

$$\frac{dv_i}{dt} = \frac{Dv_i}{Dt} \quad - \text{ This is the relative acceleration}$$

$$2\varepsilon_{ilm}\Omega_l v_m \quad - \text{ This is the Coriolis acceleration}$$

$$\varepsilon_{ilm}\Omega_l \varepsilon_{mnq}\Omega_n x_q \quad - \text{ This is the centripetal acceleration}$$

It will be convenient to express the centripetal acceleration term $\varepsilon_{ilm}\Omega_l \varepsilon_{mnq}\Omega_n x_q$ in an alternative form, using the vector identity

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$$

written in index notation as

$$\varepsilon_{ilm} a_l \varepsilon_{mnq} b_n c_q = (a_k c_k) b_i - (a_k b_k) c_i \tag{2.33}$$

where we sum over repeated index k, this allows us to write

$$\varepsilon_{ilm}\Omega_l \varepsilon_{mnq}\Omega_n x_q = \Omega_i(\Omega_k x_k) - \Omega^2 x_i \tag{2.34}$$

where $\Omega^2 = \Omega_k\Omega_k = \Omega_1^2 + \Omega_2^2 + \Omega_3^2$. We then from (2.32) have

$$\left(\frac{Du_i}{Dt}\right)_a = \frac{Dv_i}{Dt} + 2\varepsilon_{ilm}\Omega_l v_m + \Omega_i(\Omega_k x_k) - \Omega^2 x_i \tag{2.35}$$

Finally if we substitute this expression for the absolute acceleration into our equations of motion (2.15) given again below

$$\rho\left(\frac{Du_i}{Dt}\right)_a = -\frac{\partial p}{\partial x_i} + \rho\frac{\partial U}{\partial x_i}$$

to produce the equations of motion, $i = 1, 2, 3$, for a rotating frame of reference in the form (Chandrasekhar [5, p. 25])

$$\rho\frac{Dv_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho\frac{\partial U}{\partial x_i} + \rho\Omega^2 x_i - \rho\Omega_k x_k \Omega_i + 2\rho\varepsilon_{ilm} v_l \Omega_m \tag{2.36}$$

where we should note that

$$2\rho\varepsilon_{ilm}\Omega_l v_m = -2\rho\varepsilon_{ilm} v_l \Omega_m$$

which represents the change of sign that occurs when switching the elements in a cross-product in index notation.

2.4.2 The virial equations of second order

To derive the virial equations of second order in the rotating frame, we use the same method as that for deriving the analogous equations in the non-rotating frame. First multiply equation (2.36) by x_j and then integrate over the entire volume V that the fluid instantaneously occupies. Carrying this out with use of equation (2.24) from the non-rotating derivation and equation (2.4) for I_{ij} we have

$$\frac{D}{Dt}\int_V \rho v_i x_j \; d\mathbf{x} = 2K_{ij} + E_{ij} + \delta_{ij}\Pi \tag{2.37}$$

$$+\,\Omega^2\int_V \rho x_i x_j \; d\mathbf{x} - \Omega_i\Omega_k\int_V \rho x_k x_j \; d\mathbf{x} + 2\varepsilon_{ilm}\Omega_m\int_V \rho v_l x_j \; d\mathbf{x} \tag{2.38}$$

$$= 2K_{ij} + E_{ij} + \delta_{ij}\Pi + \Omega^2 I_{ij} - \Omega_i\Omega_k I_{kj} + 2\varepsilon_{ilm}\Omega_m\int_V \rho v_l x_j \; d\mathbf{x} \tag{2.39}$$

For a steady state we have no dependency on time and this equation becomes

$$2K_{ij} + E_{ij} + \delta_{ij}\Pi + \Omega^2 I_{ij} - \Omega_i\Omega_k I_{kj} + 2\varepsilon_{ilm}\Omega_m\int_V \rho v_l x_j \; d\mathbf{x} = 0 \tag{2.40}$$

For an equilibrium configuration we want to consider the case where there are no relative motions v_i, that is motions relative to the rotating frame. Note that in deriving equation (2.28) we have chosen rotation to be along $\underline{\mathbf{k}}$. Under these conditions the first and last terms on the left-hand side of equation (2.40) vanish and the equation becomes

$$\begin{aligned} E_{ij} + \Omega^2 I_{ij} - \Omega_i\Omega_k I_{kj} &= E_{ij} + \Omega^2 I_{ij} - (\Omega_i\Omega_1 I_{1j} + \Omega_i\Omega_2 I_{2j} + \Omega_i\Omega_3 I_{3j}) \\ &= E_{ij} + \Omega^2 I_{ij} - \Omega_i\Omega_3 I_{3j} \\ &= E_{ij} + \Omega^2(I_{ij} - \delta_{i3} I_{3j}) = -\delta_{ij}\Pi \end{aligned} \tag{2.41}$$

where $\Omega_1 = \Omega_2 = 0$ on the first line due to the choice or orientation of coordinate axis. On the third line above we take Ω^2 out as a common factor, where $\Omega^2 \equiv \Omega_1^2 + \Omega_2^2 + \Omega_3^2 = \Omega_3^2$, and because only Ω_3 is non-zero in the vector $\mathbf{\Omega}$ we can replace $\Omega_i\Omega_3$ with $\Omega^2\delta_{i3}$ since $\delta_{13} = \delta_{23} = 0$ and $\delta_{33} = 1$.

Expression (2.41) is a tensor equation of rank-two tensors that represents a total of nine equations ($i = 1, 2, 3,\ j = 1, 2, 3$). We may write these out as (Chandrasekhar [5, p. 25])

$$\begin{aligned}
E_{11} + \Omega^2 I_{11} &= -\Pi \\
E_{22} + \Omega^2 I_{22} &= -\Pi \\
E_{33} &= -\Pi \\
E_{12} + \Omega^2 I_{12} &= 0 \\
E_{21} + \Omega^2 I_{21} &= 0 \\
E_{13} + \Omega^2 I_{13} &= 0 \\
E_{23} + \Omega^2 I_{23} &= 0 \\
E_{31} &= 0 \\
E_{32} &= 0
\end{aligned}$$

These are the second order virial equations that we seek. They are central on our journey to find possible equilibria to the problem of a uniformly rotating body. They define the essential relationship between the characteristic moment of inertia and gravitational potential energy tensors of the equilibrium configurations. For equilibrium to be satisfied we must seek solutions to these equations as they are derived from the governing equations motion in a rotating frame of reference under static conditions.

In order to find such solutions we must first determine the gravitational potential energy tensor of the equilibrium configurations in terms of their moment of inertia tensor and this is the subject of the next chapter.

Chapter 3

Ellipsoidal Potential

3.1 Introduction

Astronomical bodies, while existing in many different shapes and sizes, commonly take an ellipsoidal form. Indeed one may suspect that our own planet is perfectly spherical when in fact it is closer to spheroidal, being approximately an oblate spheroid. An oblate spheroid is a particular case of an ellipsoid, where the equatorial axes are equal and of greater length than the polar axis. To be more precise, one can form an oblate spheroid from the revolution of an ellipse about its minor axis. Figure 3.1a illustrates an oblate spheroid and we will see in Chapter 5 that this specific spheroid with its specific semi-axes lengths is an equilibrium configuration given a particular angular velocity. Figure 3.1b illustrates an ellipsoid with three unequal axes.

The main assertion of this chapter will be to seek an *ellipsoidal* potential and consequently from this point we will seek ellipsoidal equilibrium configurations. Before we may find possible ellipsoidal equilibrium figures we will need to determine their gravitational potential energy, which is directly related to the gravitational potential of a body. This is done in two steps:

1. Determine the gravitational potential of an ellipsoid at an internal point.

2. Determine the gravitational potential of an ellipsoid at an external point.

The first of these is more readily obtained than the second but we must find that the gravitational potential matches at the ellipsoidal boundary of the body. The focus of this chapter will be to provide an overview of the determination of the gravitational potential for a homogeneous ellipsoidal configuration. In this chapter we have chosen to use Chandrasekhar's [5, p, 38-45] method for the internal potential and that of Kellog [3, p, 190-194] for the external potential.

3.2 Preliminaries

Definition 1. *An ellipsoid*[1] *is a three dimensional closed quadratic surface defined by, in Cartesian coordinates* x_i,

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \tag{3.1}$$

where $a_i, i = 1, 2, 3,$ *denote the principal axes of the ellipsoid.*

[1]See http://mathworld.wolfram.com/Ellipsoid.html (accessed 28th April 2010)

Note that the precise shape of the ellipsoid is defined by the semi-axes a_i $(i = 1, 2, 3)$. From this point we will define a_1 and a_2 to be the equatorial radii and a_3 to be the polar radius.

In subsequent sections we will endeavour to determine the gravitational potential at an internal and external point of a homogeneous ellipsoid and in order to achieve this we will first calculate the potential interior of a homoeoid.

Definition 2. *A homoeoid is a shell comprised of concentric and similar outer and inner bounding ellipsoids. The outer bounding ellipsoidal surface is given by*

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$$

and the inner bounding ellipsoidal surface is given by

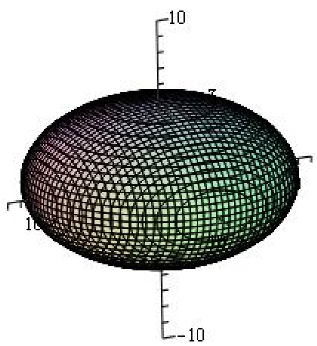
$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = m^2$$

where $0 \leq m \leq 1$.

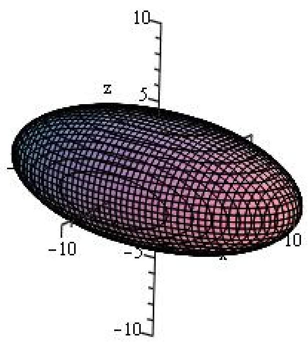
Figure 3.2 illustrates examples of a homoeoid. On the basis of homoeoidal shells we will construct the potential at a point interior to a solid ellipsoid and finally we will find the potential at a point exterior to a solid ellipsoid.

The final result of this chapter will be an expression of the potential energy tensor E_{ij} in terms of the principal axes of the ellipsoid a_i and the moment of inertia tensor I_{ij}. Once this gravitational potential energy has been found we can then proceed to use it within the virial equations we have already derived. We will be able to formulate possible equilibrium configurations under the assumption that their figure is ellipsoidal.

It is worth noting that since the potential is a solution to Poisson's equation, the general form of the potential will be quadratic, if the density is constant.



(a) Oblate Spheroid: $a_1 = 10$, $a_2 = 10$, $a_3 = 5$



(b) Ellipsoid: $a_1 = 10$, $a_2 = 5$, $a_3 = 4$

Figure 3.1: Example of an oblate spheroid and an ellipsoid

3.3 Potential interior of a homoeoid

Before we can find the potential of a homogeneous ellipsoid at an interior point we will find the gravitational potential at an interior point of a homoeoid. Consider the following theorem.

Lemma 1. *At any internal point of a homogeneous homoeoid the potential is constant.*

For proof of Lemma 1 see Chandrasekhar [5, p. 39]. We will now state a theorem for the constant potential inside a homoeoid.

Theorem 1. *The constant potential inside a homoeoid with semi-axis a_i and ma_i, with $m < 1$ (where* m *is real) of the outer and inner bounding ellipsoids respectively is given by*

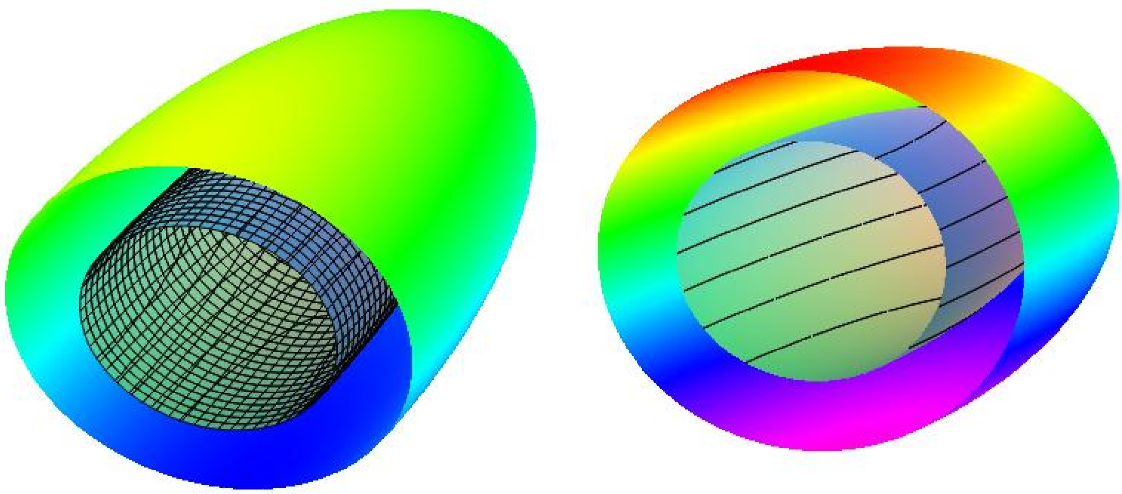
$$U = \frac{1}{2}G\rho(1 - m^2) \int_S r^2 \, d\omega \tag{3.2}$$

where $\boldsymbol{r}$ is the radius vector drawn from the centre of the homoeoid to a point on the surface S of the outer bounding ellipsoid. ($d\omega$ is the angle of an elementary cone, see proof)

Proof. We know from Lemma 1 that the potential at any internal point of a homoeoid is a constant. Consider an elementary cone of angle $d\omega$ with its vertex at the centre, the contribution of the matter contained within the cone and the homoeoid is given by

$$dU = G \int_{r_1}^{r_2} \rho r \, d\omega dr = \frac{1}{2}G\rho(r_2^2 - r_1^2) \, d\omega \tag{3.3}$$

where r_1 and r_2 are the inner and outer radii of the bounding ellipsoids respectively. We have that $r_1 = mr_2$ and thus integrating (3.3) over all the angles $d\omega$ gives the required result. □



(a) (b)

Figure 3.2: Examples of a homoeoid. This is a cross section through its centre for illustrative purposes.

3.4 Auxiliary results

Before we can find the gravitational potential at an interior point of an ellipsoid we need to prove a series of lemmas.

Lemma 2. *Given a homoeoid with semi-axis a_i and ma_i, with $m < 1$, of the outer and inner bounding ellipsoids respectively*

$$\int_S r^2 \, d\omega = 2\pi a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta} \tag{3.4}$$

where $\Delta^2 = (a_1^2 + u)(a_2^2 + u)(a_3^2 + u)$, S is the surface of the outer bounding ellipsoid and $\boldsymbol{r}$ is the radius vector from the centre to a point on S.

Proof. Consider an ellipsoid with semi-axis a_i $(i = 1, 2, 3)$ whose surface is represented by S,

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \tag{3.5}$$

In spherical polar coordinates we have

$$x_1 = r \sin\theta \cos\phi, \;\; x_2 = r \sin\theta \sin\phi, \;\; x_3 = r\cos\theta$$

where ϕ is the longitude relative to the x_1, x_2-axis and θ is the co-latitude relative to the x_3 -axis.

Substituting these values into (3.5) gives us the equation for the ellipsoidal surface in these coordinates

$$\sin^2\theta \left(\frac{\cos^2\phi}{a_1^2} + \frac{\sin^2\phi}{a_2^2} \right) + \frac{\cos^2\theta}{a_3^2} = \frac{1}{r^2} \tag{3.6}$$

from which we may calculate our required integral. Placing our ellipsoid at the origin in our coordinate system, we see that the ellipsoid is symmetrical about all three axes and therefore we only require to integrate over one of the eight quadrants for the surface integral. Thus we have on using equation (3.6)

$$\int_S r^2 \, d\omega = 8 \int_0^{\pi/2}\int_0^{\pi/2} \frac{\sin\theta \; d\theta d\phi}{\sin^2\theta \left(\frac{\cos^2\phi}{a_1^2} + \frac{\sin^2\phi}{a_2^2} \right) + \frac{\cos^2\theta}{a_3^2}} \tag{3.7}$$

where $d\omega = \sin\theta d\theta d\phi$ is an elemental surface area on the ellipsoidal surface associated with the elementary half-cone we met in proving Theorem 1. Now, using the substitution

$$t = \tan\phi \quad \Rightarrow \quad dt = \sec^2 \; d\phi$$

we may transform the integral (3.7) as follows

$$\int_S r^2 \, d\omega = 8 \int_0^{\pi/2} \sin\theta d\theta \int_0^\infty \frac{\cos^2\phi \; dt}{\sin^2\theta \left(\frac{\cos^2\phi}{a_1^2} + \frac{\sin^2\phi}{a_2^2} \right) + \frac{\cos^2\theta}{a_3^2}}$$

and dividing the top and bottom of this integral by $\cos^2\phi$ and noting that

$$\frac{1}{\cos^2\phi} = 1 + \tan^2\phi = 1 + t^2$$

we obtain

$$\int_S r^2 \; d\omega = 8 \int_0^{\pi/2} \sin\theta d\theta \int_0^\infty \frac{dt}{\frac{\sin^2\theta}{a_1^2} + \frac{\cos^2\theta}{a_3^2} + (\frac{\sin^2\theta}{a_2^2} + \frac{\cos^2\theta}{a_3^2})t^2} \tag{3.8}$$

We may now use a well-known (Routh [2, p. 99]) standard integral

$$\int_0^\infty \frac{dt}{A + Bt^2} = \left[\frac{1}{\sqrt{AB}} \tan^{-1}(\sqrt{\frac{B}{A}}t) \right]_0^\infty = \frac{1}{\sqrt{AB}} \frac{\pi}{2}$$

to evaluate the integral on the right hand side of (3.8) to give

$$\int_S r^2 \; d\omega = 4\pi \int_0^{\pi/2} \frac{\sin\theta d\theta}{\sqrt{\frac{\sin^2\theta}{a_1^2} + \frac{\cos^2\theta}{a_3^2}} \sqrt{\frac{\sin^2\theta}{a_2^2} + \frac{\cos^2\theta}{a_3^2}}} \tag{3.9}$$

Multiplying the top and bottom of the integral in (3.9) by $a_1 a_2 a_3^2 \sec^2\theta$ gives

$$\int_S r^2 \; d\omega = 4\pi a_1 a_2 a_3^2 \int_0^{\pi/2} \frac{\sec^2\theta \sin\theta d\theta}{\sqrt{a_1^2 + a_3^2 \tan^2\theta} \sqrt{a_2^2 + a_3^2 \tan^2\theta}} \tag{3.10}$$

Now use the following substitution

$$u = a_3^2 \tan^2\theta \quad \Rightarrow \quad du = 2a_3^2 \sin\theta \sec^3\theta d\theta$$

and (3.10) becomes

$$\begin{aligned} \int_S r^2 \; d\omega &= 4\pi a_1 a_2 a_3^2 \int_0^\infty \frac{du}{(\sqrt{a_1^2 + u}\sqrt{a_2^2 + u}) 2a_3^2 \sec\theta} \\ &= 2\pi a_1 a_2 \int_0^\infty \frac{du}{(\sqrt{a_1^2 + u}\sqrt{a_2^2 + u}) \sec\theta} \end{aligned} \tag{3.11}$$

where

$$\sec\theta = \frac{a_3}{a_3}(1 + \tan^2\theta)^{\frac{1}{2}} = \frac{(a_3^2 + a_3^2 \tan^2\theta)^{\frac{1}{2}}}{a_3} = \frac{\sqrt{a_3^2 + u}}{a_3}$$

and hence equation (3.11) becomes

$$\int_S r^2 \; d\omega = 2\pi a_1 a_2 a_3 \int_0^\infty \frac{du}{\sqrt{a_1^2 + u}\sqrt{a_2^2 + u}\sqrt{a_3^2 + u}} = 2\pi a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta}$$

as required. Note this becomes

$$\int_S r^2 \; d\omega = 2\pi I \tag{3.12}$$

where I is defined as follows. $\square$

Definition 3.

$$I = a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta} \tag{3.13}$$

$$A_i = a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta(a_i^2 + u)} \tag{3.14}$$

where a_i $(i = 1, 2, 3)$ *are the principal axes of an ellipsoid and* $\Delta^2 = (a_1^2 + u)(a_2^2 + u)(a_3^2 + u)$.

Lemma 3.

$$A_i = \frac{I}{a_i^2} - \frac{1}{a_i}\frac{\partial I}{\partial a_i} \tag{3.15}$$

where I and A_i are as defined (3.13) and (3.14) respectively.

Proof. We will show this is true for $i = 1$. It will be more convenient to rearrange this expression and show that

$$\frac{\partial I}{\partial a_1} = \frac{I}{a_1} - a_1 A_1$$

We must differentiate I with respect to a_1 and use the definitions of Δ, I and A_1

$$\begin{aligned}
\frac{\partial I}{\partial a_1} &= a_2 a_3 \int_0^\infty \frac{du}{\Delta} + a_1 a_2 a_3 \int_0^\infty \frac{\partial}{\partial a_1}\frac{1}{\Delta}\, du \\
&= a_2 a_3 \int_0^\infty \frac{du}{\Delta} - a_1^2 a_2 a_3 \int_0^\infty \frac{(a_2^2 + u)(a_3^2 + u)}{\Delta^3}\, du \\
&= a_2 a_3 \int_0^\infty \frac{du}{\Delta} - a_1^2 a_2 a_3 \int_0^\infty \frac{du}{\Delta(a_1^2 + u)} \\
&= \frac{I}{a_1} - a_1 A_1
\end{aligned}$$

as required. This also true for $i = 2, 3$ by symmetry.

□

Lemma 4.

$$\int_S r^4 l_i^2 \; d\omega = \pi a_i^3 \frac{\partial I}{\partial a_i} \tag{3.16}$$

where l_i $(i = 1, 2, 3)$ *are the direction cosines of the radius vector* $\boldsymbol{r}$.

Proof. Given (3.4) we have

$$\int_S r^2 \; d\omega = 2\pi I \tag{3.17}$$

and differentiating (3.17) with respect to a_i to obtain

$$\int_S r \frac{\partial r}{\partial a_i} \; d\omega = \pi \frac{\partial I}{\partial a_i} \tag{3.18}$$

Now differentiating (3.6) written in the form (summing over repeated indices)

$$\frac{1}{r^2} = \frac{l_i^2}{a_i^2} \tag{3.19}$$

with respect to a_i generates

$$\frac{1}{r^3}\frac{\partial r}{\partial a_i} = \frac{l_i^2}{a_i^3}$$

where in using index notation we sum over repeated indices and substituting into (3.18) obtains the lemma. □

Lemma 5.

$$\sum_{i=1}^{3} A_i = 2 \tag{3.20}$$

Proof. Given that $\Delta^2 = (a_1^2+u)(a_2^2+u)(a_3^2+u)$ we have

$$\frac{d}{du}\Delta^2 = 2\Delta\frac{d\Delta}{du} = (a_2^2+u)(a_3^2+u)+(a_1^2+u)(a_3^2+u)+(a_1^2+u)(a_2^2+u)$$

and dividing by Δ^2 gives

$$2\frac{1}{\Delta}\frac{d\Delta}{du} = \sum_{i=1}^{3}\frac{1}{(a_i^2+u)} \tag{3.21}$$

Hence on using result (3.21) and the definition of A_i given by (3.14), we have

$$\begin{aligned}
\sum_{i=1}^{3} A_i &= a_1a_2a_3\int_0^\infty \left(\sum_{i=1}^{3}\frac{1}{(a_i^2+u)}\right)\frac{du}{\Delta} \\
&= 2a_1a_2a_3\int_0^\infty \frac{1}{\Delta^2}\frac{d\Delta}{du}\,du \\
&= 2a_1a_2a_3\int_{\Delta=a_1a_2a_3}^{\infty}\frac{1}{\Delta^2}\,d\Delta \\
&= 2a_1a_2a_3\left[\frac{1}{\Delta}\right]_{\infty}^{a_1a_2a_3} = 2
\end{aligned}$$

as required.

□

Lemma 6.

$$\sum_{i=1}^{3} a_i^2 A_i = I \tag{3.22}$$

Proof. It can be shown (Chandrasekhar [5, p. 41]) that

$$\int_S r^2 l_i^2\,d\omega = 2\pi a_i^2 A_i \tag{3.23}$$

where l_i are the direction cosines of the radius vector $\underline{\mathbf{r}}$. Summing equation (3.23) over i and using the result (3.12) gives

$$2\pi \sum_{i=1}^{3} a_i^2 A_i = \int_S \sum_{i=1}^{3} l_i^2 r^2 \ d\omega = \int_S r^2 \ d\omega = 2\pi I$$

as required, where we use the fact that $l_1^2 + l_2^2 + l_3^2 = 1$. □

3.5 Potential of ellipsoid at an internal point

Using the lemmas just stated, we now have the capability to find the potential at an internal point x_i of a solid homogeneous ellipsoid, i.e. an ellipsoid composed of uniformly distributed matter.

Theorem 2. *The potential of a solid homogeneous ellipsoid at an internal point x_i is given by*

$$U = \pi G\rho \left(I - \sum_{i=1}^{3} A_i x_i^2 \right) \tag{3.24}$$

where the ellipsoid has semi-axes denoted by a_i. G *is the constant of gravitation and ρ is the constant density of the homogeneous ellipsoid.*

Proof. (Chandrasekhar [5, p. 44])

Consider an elementary cone of angle $d\omega$ with its vertex at a point within a homogeneous ellipsoid. Where the elementary cone is composed of two cones joined at the vertex which we will call "half-cones". We may construct the potential U at such a point by considering the differential quantity dU of the gravitational potential due to the matter contained within both the ellipsoid and the elementary cone considered.

On the basis of homoeoidal shells and by Theorem 1 (where $m \to 0$) we may find the elemental potential dU contributed by such an elementary cone to be given by

$$dU = \frac{1}{2} G\rho (R_1^2 + R_2^2) \ d\omega \tag{3.25}$$

where R_1 is the straight line distance from the vertex of the elementary cone and the intersection of one half-cone axis with the boundary of the ellipsoid, and R_2 is the corresponding distance for the other half-cone of the elementary cone.

Now integrating the above quantity (3.25) for dU over all elementary cones (or angles $d\omega$) and accounting for the fact that each elementary cone is counted twice we have over the surface S of the ellipsoid

$$U = \frac{1}{4} G\rho \int_S (R_1^2 + R_2^2) \ d\omega \tag{3.26}$$

Let the vertex of the elementary cone considered be at the point denoted by the vector x_i ($x_i = \underline{\mathbf{x}} = (x_1, x_2, x_3)$). We require to know the values of the two points of intersection of the elementary cone-axis with the ellipsoidal boundary. Given that $\underline{\mathbf{r}}$ is the radius vector from the centre of the ellipsoid and taken parallel to any elementary cone-axis considered, we may write these points of intersection as

$$x_i + l_i R_1 \quad \text{and} \quad x_i + l_i R_2 \tag{3.27}$$

where l_i is the direction cosine of $\underline{\mathbf{r}}$. Therefore we have R_1 and R_2 equal the roots of the equation

$$\sum_{i=1}^{3} \left(\frac{x_i + l_i R}{a_i} \right)^2 = 1 \tag{3.28}$$

since the points (3.27) lie on the ellipsoidal surface. Expanding equation (3.28) gives

$$R^2 \sum_{i=1}^{3} \frac{l_i^2}{a_i^2} + 2R \sum_{i=1}^{3} \frac{x_i l_i}{a_i^2} + \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} - 1 = 0 \tag{3.29}$$

Note that from the proof of Lemma 4 we have equation (3.19) written as

$$\sum_{i=1}^{3} \frac{l_i^2}{a_i^2} = \frac{1}{r^2} \tag{3.30}$$

thus equation (3.29) becomes

$$\frac{R^2}{r^2} + 2R \sum_{i=1}^{3} \frac{x_i l_i}{a_i^2} + \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} - 1 = 0 \tag{3.31}$$

which clearly has the form of a quadratic in R and hence we may deduce the value of R_1 and R_2, or more usefully $R_1^2 + R_2^2$. Taking

$$a \equiv \frac{1}{r^2} \ , \ b \equiv 2 \sum_{i=1}^{3} \frac{x_i l_i}{a_i^2} \ , \ c \equiv \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} - 1 \tag{3.32}$$

we may then use the well known formula for quadratic equations

$$R_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.33}$$

and taking the square of both R_1 and R_2 and adding the result produces

$$R_1^2 + R_2^2 = \frac{b^2}{a^2} - \frac{2c}{a} \tag{3.34}$$

Now substituting in the values for a,b and c gives us the equation

$$R_1^2 + R_2^2 = 4r^4 \left(\sum_{i=1}^{3} \frac{x_i l_i}{a_i^2} \right)^2 + 2r^2 \left(1 - \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} \right) \tag{3.35}$$

Substituting equation (3.35) into equation (3.26) provides the potential

$$U = \frac{1}{2} G\rho \int_S \left[2r^4 \sum_{i=1}^{3} \frac{x_i^2 l_i^2}{a_i^4} + r^2 \left(1 - \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} \right) \right] \, d\omega \tag{3.36}$$

$$= G\rho \sum_{i=1}^{3} \frac{x_i^2}{a_i^4} \int_S l_i^2 r^4 \, d\omega + \frac{G\rho}{2} \left(1 - \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} \right) \int_S r^2 \, d\omega \tag{3.37}$$

and we may use Lemma 4 and Lemma 2 to ascertain the values of the first and second integrals respectively in (3.37) in an alternative form, thus

$$U = \pi G\rho \sum_{i=1}^{3} x_i^2 \frac{1}{a_i}\frac{\partial I}{\partial a_i} + \pi G\rho I \left(1 - \sum_{i=1}^{3}\frac{x_i^2}{a_i^2}\right) \tag{3.38}$$

$$= \pi G\rho \left[\sum_{i=1}^{3} x_i^2 \left(\frac{1}{a_i}\frac{\partial I}{\partial a_i} - \frac{I}{a_i^2}\right) + I\right] \tag{3.39}$$

Finally we may use Lemma 3 to express (3.39) as required

$$U = \pi G\rho \left[I - \sum_{i=1}^{3} x_i^2 \left(\frac{I}{a_i^2} - \frac{1}{a_i}\frac{\partial I}{\partial a_i}\right)\right] \tag{3.40}$$

$$= \pi G\rho \left[I - \sum_{i=1}^{3} x_i^2 A_i\right] \tag{3.41}$$

□

3.6 External potential

It is more complicated to derive an expression for the gravitational potential at an exterior point of a homogeneous ellipsoid. This section will provide a brief outline of the route one may take to derive such an expression.[2]

Consider the divergence theorem

$$\iiint\limits_{V} \nabla \cdot \underline{x}\; dV = \iint\limits_{S} \underline{x} \cdot \underline{n}\; dS = \iint\limits_{S} r\cos\theta\; dS \tag{3.42}$$

where V is the volume of an ellipsoid (Kellog [3, p. 39] Exercise 3), S is its bounding surface, $\underline{\mathbf{x}}$ is the position vector (x_1, x_2, x_3), $\underline{\mathbf{n}}$ is the outward directed normal to the surface S and θ is the angle between the vector $\underline{\mathbf{n}}$ and $\underline{\mathbf{x}}$ and clearly $\nabla \cdot \underline{\mathbf{x}} = 3$.

Imagine a homogeneous ellipsoid and consider an elementary cone with vertex at the ellipsoidal centre cutting out an element ΔS from the ellipsoidal surface, then the the corresponding volume element enclosed by both the ellipsoid and the elementary cone is, using (3.42), given by

$$\Delta V = \frac{1}{3}\iint\limits_{\Delta S} r\cos\theta\; dS = \frac{1}{3}\iint\limits_{\Delta S} p\; dS \tag{3.43}$$

where p is the perpendicular distance from the centre of the ellipsoid to the point of integration on the ellipsoidal surface.

Let us consider a basic-ellipsoid from which we may generate a family of similar ellipsoids through multiplication of the dimensions by a scaling factor of h. We can see that the above volume integral can be expressed for such a member of a family of similar ellipsoids, with a scale factor of h_1 by simply multiplying the quantity by h_1^3, so

$$\Delta V = \frac{h_1^3}{3}\iint\limits_{\Delta S_0} p_0\; dS_0 \tag{3.44}$$

[2]For a detailed account and derivation of properties used see Kellog [3, p. 178-194]

where the subscript denotes integration taking place over the basic-ellipsoid, over the surface ΔS_0 of the basic-ellipsoid. Now consider a homoeoidal shell, that is the volume enclosed between two similar and concentric ellipsoids with the inner ellipsoid having scaling factor h_1 and the outer ellipsoid having scaling factor h_2. The volume cut out by the elementary cone considered in the above argument is, on using (3.44), given by

$$\Delta V = \frac{h_2^3 - h_1^3}{3} \iint_{\Delta S_0} p_0 \, dS_0 = \overline{h}^2 \Delta h \overline{p}_0 \Delta S_0 \tag{3.45}$$

where quantities with an over-line denote mean values. We observe that $p = \overline{h} p_0$ and $\Delta S = \overline{h}^2 \Delta S_0$ on the ellipsoid $h = \overline{h}$ and thus (3.45) becomes

$$\Delta V = \frac{\overline{p} \Delta S \Delta h}{\overline{h}} \tag{3.46}$$

The external potential of a homogeneous homoeoidal shell may be shown by Newton's Law of gravitation to be given by

$$U = \lim_{\Delta V \to 0} \sum_{\Delta V} \frac{\rho G \Delta V}{r} = \rho G \lim_{\Delta V \to 0} \sum_{\Delta V} \frac{\overline{p} \Delta S \Delta h}{\overline{h} r} = \rho G \int_{h_1}^{h_2} \frac{1}{h} \iint_S \frac{p \, dS}{r} \, dh \tag{3.47}$$

Consider the specific case of the solid ellipsoid by setting $h_1 = 0$ and $h_2 = 1$, and with scale-factor h we will thus have semi-axes ha_1, ha_2, ha_3. We may express the inner integral in (3.47) as (Kellog [3, p. 189])

$$\iint_S \frac{p \, dS}{r} = \frac{D}{2} \int_\lambda^\infty \frac{ds}{\sqrt{\varphi(h,s)}} \tag{3.48}$$

where

$$\varphi(h,s) = (a_1^2 h^2 + s)(a_2^2 h^2 + s)(a_3^2 h^2 + s)$$

and $\lambda(h)$ is the greatest root of the equation

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i h^2 + \lambda} = 1 \tag{3.49}$$

which is commonly known as the ellipsoidal coordinate using an ellipsoidal coordinate system. Here D is the spread of mass over the ellipsoid and it may also be shown (Kellog [3, p. 191]) that such a distribution of mass is given by $4\pi a_1 a_2 a_3 h^3$.

Thus (3.48) becomes

$$\frac{D}{2} \int_\lambda^\infty \frac{ds}{\sqrt{\varphi(h,s)}} = 2\pi a_1 a_2 a_3 h^3 \int_\lambda^\infty \frac{ds}{\sqrt{\varphi(h,s)}} \tag{3.50}$$

and hence (3.47) gives

$$U = 2\pi\rho G \; a_1 a_2 a_3 \int_0^1 h^2 \int_\lambda^\infty \frac{ds}{\sqrt{\varphi(h,s)}} \, dh \tag{3.51}$$

Using the following substitution

$$s = h^2 t \quad \Rightarrow \quad ds = h^2 dt$$

and noting that φ becomes $\varphi(t) = h^6(a_1^2 + t)(a_2^2 + t)(a_3^2 + t)$, gives

$$U = 2\pi\rho G \; a_1 a_2 a_3 \int_0^1 h \int_w^\infty \frac{dt}{\sqrt{\varphi(t)}} \; dh \tag{3.52}$$

where $w = \lambda/h^2$. Using integration by parts gives us

$$\int_0^1 h \int_w^\infty \frac{dt}{\sqrt{\varphi(t)}} \; dh = \left[\frac{h^2}{2} \int_w^\infty \frac{dt}{\sqrt{\varphi(t)}} \right]_0^1 + \frac{1}{2} \int_0^1 \frac{h^2}{\sqrt{\varphi(w)}} \frac{dw}{dh} \; dh \tag{3.53}$$

where we have used Leibnitz's rule of integration:

$$\frac{d}{dh} \int_{a(h)}^{b(h)} f(h,t) \; dt = \frac{db}{dh} f(h,b) - \frac{da}{dh} f(h,a) + \int_{a(h)}^{b(h)} \frac{\partial}{\partial h} f(h,t) \; dt$$

Since $\lambda = wh^2$ we may, upon using equation (3.49), write

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i^2 + w} = h^2 \tag{3.54}$$

Therefore equation (3.53) may use w as the variable of integration for the integral on the right hand side of (3.53). Thus with the knowledge that, as h tends to zero, w tends to infinity, and as h tends to one, w tends to λ, we see that

$$U = \pi G \rho a_1 a_2 a_3 \left[\int_\lambda^\infty \frac{dt}{\sqrt{\varphi(t)}} - \int_\lambda^\infty \left\{ \sum_{i=1}^{3} \frac{x_i^2}{a_i^2 + w} \right\} \frac{dw}{\sqrt{\varphi(w)}} \right] \tag{3.55}$$

where we have used (3.54) for h^2. That is the gravitational potential at an external point of a solid homogeneous ellipsoid may be written as

$$U = \pi G \rho a_1 a_2 a_3 \int_\lambda^\infty \left(1 - \sum_{i=1}^{3} \frac{x_i^2}{a_i^2 + u} \right) \frac{du}{\Delta} \tag{3.56}$$

where $\Delta^2 = (a_1^2 + u)(a_2^2 + u)(a_3^2 + u)$ and remembering that λ is the ellipsoidal coordinate of the point under consideration and is equal to the positive root of the equation

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i^2 + \lambda} \tag{3.57}$$

The gradient of the gravitational potential is equal to a gravitational force, thus a discontinuity in the potential across the boundary of the homogeneous body would imply an infinite force under which the configuration would not be able to maintain its bounding figure. At the ellipsoidal boundary we have that $\lambda = 0$ and thus equation (3.56) for the external potential is in agreement with equation (3.24) for the internal potential, and hence allows that the gravitational potential is continuous across the ellipsoidal boundary as required.

3.7 Gravitational potential energy

Now that since we have the expression for the gravitational potential at an interior point we may proceed to find the corresponding expression for the gravitational potential energy tensor at an interior point.

Theorem 3. *For a homogeneous ellipsoid, the gravitational potential energy tensor E_{ij} is given by*

$$E_{ij} = -2\pi G\rho A_i I_{ij} \tag{3.58}$$

where I_{ij} is the moment of inertia tensor given by (2.4) *and A_{ij} is as defined in* (3.14)*.*

Proof. Firstly, see that from (2.14) we have

$$E_{ij} = \int\limits_V \rho x_i \frac{\partial U}{\partial x_j}\, dV \tag{3.59}$$

for the potential energy in terms of the potential. Secondly, differentiating the interior potential (3.24) with respect to space gives

$$\frac{\partial U}{\partial x_j} = \pi G\rho a_1 a_2 a_3 \int\limits_0^\infty \frac{-2x_j}{a_j^2 + u}\, du \tag{3.60}$$

Substituting (3.60) into (3.59) obtains

$$E_{ij} = -2\pi G\rho a_1 a_2 a_3 \int\limits_0^\infty \frac{du}{\Delta(a_i^2 + u)} \int\limits_V \rho x_i x_j\, dV = -2\pi G\rho A_i I_{ij}$$

□

Equation (3.58) provides a relationship between the tensors E_{ij} and I_{ij} for an ellipsoidal configuration and will be crucial in deriving the defining equations of sequences of equilibrium figures in Chapter 5. However before moving to this stage it is worth exploring how the integrals A_i may be expressed in terms of well understood functions.

Chapter 4

The Standard Incomplete Elliptic Integrals

4.1 Introduction

In the previous chapter an overview of the derivation of the relevant potential energy tensor for a homogeneous ellipsoid was provided. It is now possible to illustrate ellipsoidal configurations that satisfy equilibrium but before doing so it will be convenient to express the integrals A_i in terms of the well-known standard incomplete elliptic integrals of the two kinds.[1] They are most useful in this form because we may then express the defining equations of two sequences of equilibria in terms of these well-known functions. However we will also be able to express A_i in terms of the eccentricity for oblate spheroidal figures. Eccentricity is measure of how flattened a configuration is and we will see that this is a more informative quantity. We also want to find the integrals A_i in this form as it will enable straight-forward numerical computation of the defining equilibria equations given in Chapter 5 and thus we may produce illustrative figures of some possible equilibria. (see figures 5.5, 5.4, 5.3 and 5.2)

This is the main motivation for this chapter, which is devoted to the expression of the integrals A_i in terms of the standard incomplete elliptic integrals of the two kinds (See Chandrasekhar [5, p. 43]), which are defined by

$$F(\theta,\phi) = \int_0^{\phi} \frac{d\varphi}{(1-\sin^2\theta\sin^2\varphi)^{\frac{1}{2}}} \tag{4.1}$$

$$E(\theta,\phi) = \int_0^{\phi} (1-\sin^2\theta\sin^2\varphi)^{\frac{1}{2}}\ d\varphi \tag{4.2}$$

with the definitions

$$\sin^2\theta = \frac{a_1^2-a_2^2}{a_1^2-a_3^2} \quad \text{and} \quad \cos\phi = \frac{a_3}{a_1} \tag{4.3}$$

First of all we want to determine A_1 in terms of the above elliptic integrals F and E and this is done by using a series of substitutions.

[1]See Lyttleton [4, p. 41]

4.2 Determination of A_1 in terms of the elliptic integrals

Theorem 4.

$$A_1 = \frac{2a_2a_3}{a_1^2 \sin^3\phi \sin^2\theta}\left[F(\theta,\phi) - E(\theta,\phi)\right] \tag{4.4}$$

where a_i $(i = 1,2,3)$ are the principal semi-axes of the ellipsoid. Also, θ, ϕ, F and E are as defined above.

Proof. Start with the definition of A_1(see (3.14)), we have

$$A_1 = a_1a_2a_3 \int_0^\infty \frac{du}{(a_1^2+u)^{\frac{3}{2}}(a_2^2+u)^{\frac{1}{2}}(a_3^2+u)^{\frac{1}{2}}} \tag{4.5}$$

remembering that $\Delta^2 = (a_1^2+u)(a_2^2+u)(a_3^2+u)$. Firstly make the following transformation of variables

$$t = a_1^2 + u \quad \Rightarrow \quad dt = du$$

where the limits of integration transform as

$$\begin{aligned} u = 0 \quad &\Rightarrow \quad t = a_1^2 \\ u = \infty \quad &\Rightarrow \quad t = \infty \end{aligned}$$

and we have

$$\begin{aligned} A_1 &= a_1a_2a_3 \int_{a_1^2}^\infty \frac{dt}{t^{\frac{3}{2}}(a_2^2 - a_1^2 + t)^{\frac{1}{2}}(a_3^2 - a_1^2 + t)^{\frac{1}{2}}} \\ &= a_1a_2a_3 \int_{a_1^2}^\infty \frac{dt}{t^{\frac{5}{2}}(\frac{a_2^2-a_1^2}{t}+1)^{\frac{1}{2}}(\frac{a_3^2-a_1^2}{t}+1)^{\frac{1}{2}}} \end{aligned} \tag{4.6}$$

Secondly make the further transformation of variables

$$w = \frac{1}{t} \quad \Rightarrow \quad dw = -\frac{1}{t^2}\, dt$$

where the limits of integration transform as

$$\begin{aligned} t = a_1^2 \quad &\Rightarrow \quad w = \frac{1}{a_1^2} \\ t = \infty \quad &\Rightarrow \quad w = 0 \end{aligned}$$

so therefore (4.6) becomes

$$\begin{aligned} A_1 &= -a_1a_2a_3 \int_{\frac{1}{a_1^2}}^0 \frac{w^{\frac{1}{2}}\, dw}{((a_2^2-a_1^2)w+1)^{\frac{1}{2}}((a_3^2-a_1^2)w+1)^{\frac{1}{2}}} \\ &= a_1a_2a_3 \int_0^{\frac{1}{a_1^2}} \frac{w^{\frac{1}{2}}\, dw}{(1-(a_1^2-a_2^2)w)^{\frac{1}{2}}(1-(a_1^2-a_3^2)w)^{\frac{1}{2}}} \end{aligned}$$

Using the definition of $\sin^2\theta$ in (4.3) we can eliminate $(a_1^2 - a_2^2)$ in the denominator of the integral, so

$$A_1 = a_1 a_2 a_3 \int_0^{\frac{1}{a_1^2}} \frac{w^{\frac{1}{2}}\ dw}{(1 - \sin^2\theta(a_1^2 - a_3^2)w)^{\frac{1}{2}}(1 - (a_1^2 - a_3^2)w)^{\frac{1}{2}}}$$

Finally, use the substitution

$$\sin^2\varphi = (a_1^2 - a_3^2)w \quad \Rightarrow \quad 2\sin\varphi\cos\varphi d\varphi = (a_1^2 - a_3^2)dw$$

where the limits of integration transform as

$$\begin{aligned} w &= 0 \quad \Rightarrow \quad \varphi = 0 \\ w &= \frac{1}{a_1^2} \quad \Rightarrow \quad \varphi = \sin^{-1}\left(\frac{a_1^2 - a_3^2}{a_1^2}\right)^{\frac{1}{2}} \end{aligned}$$

Note that the upper limit becomes, on using the definition of ϕ in (4.3),

$$\begin{aligned} \varphi = \sin^{-1}\left(\frac{a_1^2 - a_3^2}{a_1^2}\right)^{\frac{1}{2}} \ \text{ or } \ \sin^2\varphi = \frac{a_1^2 - a_3^2}{a_1^2} \\ \Rightarrow 1 - \cos^2\varphi = 1 - \frac{a_3^2}{a_1^2} \Rightarrow \cos\varphi = \frac{a_3}{a_1} \Rightarrow \varphi = \cos^{-1}\frac{a_3}{a_1} = \phi \end{aligned} \tag{4.7}$$

and therefore A_1 now becomes

$$\begin{aligned} A_1 &= a_1 a_2 a_3 \int_0^{\phi} \frac{\frac{2\sin\varphi\cos\varphi}{a_1^2 - a_3^2}\left(\frac{\sin^2\varphi}{a_1^2 - a_3^2}\right)^{\frac{1}{2}}}{(1 - \sin^2\theta\sin^2\varphi)^{\frac{1}{2}}(1 - \sin^2\varphi)^{\frac{1}{2}}}\ d\varphi \\ &= \frac{2a_1 a_2 a_3}{(a_1^2 - a_3^2)^{\frac{3}{2}}} \int_0^{\phi} \frac{\sin^2\varphi\ d\varphi}{(1 - \sin^2\theta\sin^2\varphi)^{\frac{1}{2}}} \end{aligned} \tag{4.8}$$

where we have used $\cos\varphi = (1 - \sin^2\varphi)^{\frac{1}{2}}$. Now note that from the definition of ϕ in (4.3) we have

$$\sin^3\phi = \frac{(a_1^2 - a_3^2)^{\frac{3}{2}}}{a_1^3} \tag{4.9}$$

Using (4.9) and multiplying (4.8) by $\frac{\sin^2\theta}{\sin^2\theta}$ produces

$$
\begin{aligned}
A_1 &= \frac{2a_2a_3}{a_1^2\sin^3\phi\sin^2\theta}\int_0^{\phi}\frac{\sin^2\theta\sin^2\varphi\;d\varphi}{(1-\sin^2\theta\sin^2\varphi)^{\frac{1}{2}}} \\
&= \frac{2a_2a_3}{a_1^2\sin^3\phi\sin^2\theta}\int_0^{\phi}\frac{1-(1-\sin^2\theta\sin^2\varphi)\;d\varphi}{(1-\sin^2\theta\sin^2\varphi)^{\frac{1}{2}}} \\
&= \frac{2a_2a_3}{a_1^2\sin^3\phi\sin^2\theta}\left[\int_0^{\phi}\frac{d\varphi}{(1-\sin^2\theta\sin^2\varphi)^{\frac{1}{2}}}-\int_0^{\phi}(1-\sin^2\theta\sin^2\varphi)^{\frac{1}{2}}\;d\varphi\right] \\
&= \frac{2a_2a_3}{a_1^2\sin^3\phi\sin^2\theta}\left[F(\theta,\phi)-E(\theta,\phi)\right]
\end{aligned}
$$

as required and note that we have used the definitions (4.1) and (4.2) in the above derivation. □

4.3 Determination of I in terms of elliptic integrals

Now that A_1 has been determined, we will want to express A_2 and A_3 in terms of elliptic integrals also. The best way to do this is to use Lemma 4 in the previous chapter in conjunction with Lemma 5 but before this can be done we first need to determine I in terms of the elliptic integral F.

Theorem 5.

$$
I = \frac{2a_1a_2a_3}{(a_1^2-a_3^2)^{\frac{1}{2}}}F(\theta,\phi) \tag{4.10}
$$

where a_i $(i = 1,2,3)$ are the principal semi-axes of the ellipsoid. Also, θ, ϕ, F and E are as defined.

Proof. Proceeding in a similar fashion as before we start from the definition of I (see (3.13))

$$
I = a_1a_2a_3\int_0^{\infty}\frac{du}{(a_1^2+u)^{\frac{1}{2}}(a_2^2+u)^{\frac{1}{2}}(a_3^2+u)^{\frac{1}{2}}}
$$

Introducing the transformation of variables

$$
t = a_1^2 + u \quad \Rightarrow \quad dt = du
$$

we thus have

$$
\begin{aligned}
I &= a_1a_2a_3\int_{a_1^2}^{\infty}\frac{dt}{t^{\frac{1}{2}}(a_2^2-a_1^2+t)^{\frac{1}{2}}(a_3^2-a_1^2+t)^{\frac{1}{2}}} \\
&= a_1a_2a_3\int_{a_1^2}^{\infty}\frac{dt}{t^{\frac{3}{2}}(\frac{a_2^2-a_1^2}{t}+1)^{\frac{1}{2}}(\frac{a_3^2-a_1^2}{t}+1)^{\frac{1}{2}}}
\end{aligned}
$$

and making a further transformation

$$w = \frac{1}{t} \quad \Rightarrow \quad dw = -\frac{1}{t^2}dt$$

to provide us with

$$\begin{aligned}
I &= -a_1a_2a_3 \int_{\frac{1}{a_1^2}}^{0} \frac{dw}{w^{\frac{1}{2}}((a_2^2 - a_1^2)w + 1)^{\frac{1}{2}}((a_3^2 - a_1^2)w + 1)^{\frac{1}{2}}} \\
&= a_1a_2a_3 \int_{0}^{\frac{1}{a_1^2}} \frac{dw}{w^{\frac{1}{2}}(1 - (a_1^2 - a_2^2)w)^{\frac{1}{2}}(1 - (a_1^2 - a_3^2)w)^{\frac{1}{2}}} \\
&= a_1a_2a_3 \int_{0}^{\frac{1}{a_1^2}} \frac{dw}{w^{\frac{1}{2}}(1 - \sin^2\theta(a_1^2 - a_3^2)w)^{\frac{1}{2}}(1 - (a_1^2 - a_3^2)w)^{\frac{1}{2}}}
\end{aligned}$$

where we have used the definition of $\sin^2\theta$ in (4.3).

Finally, use the transformation of variables

$$\sin^2\varphi = (a_1^2 - a_3^2)w \quad \Rightarrow \quad 2\sin\varphi\cos\varphi d\varphi = (a_1^2 - a_3^2)dw$$

where we use the same argument as in the determination of A_1 (see (4.7)), therefore

$$\begin{aligned}
I &= a_1a_2a_3 \int_{0}^{\phi} \frac{\frac{2\sin\varphi\cos\varphi d\varphi}{(a_1^2 - a_3^2)}}{\left(\frac{\sin^2\varphi}{a_1^2 - a_3^2}\right)^{\frac{1}{2}}(1 - \sin^2\theta\sin^2\varphi)^{\frac{1}{2}}(1 - \sin^2\varphi)^{\frac{1}{2}}} \\
&= \frac{2a_1a_2a_3}{(a_1^2 - a_3^2)^{\frac{1}{2}}} \int_{0}^{\phi} \frac{d\varphi}{(1 - \sin^2\theta\sin^2\varphi)^{\frac{1}{2}}} \\
&= \frac{2a_1a_2a_3}{(a_1^2 - a_3^2)^{\frac{1}{2}}} F(\theta, \phi)
\end{aligned}$$

as required and note that we have used the definition (4.1).

□

4.4 Determination of A_3 in terms of the elliptic integrals

Theorem 6.

$$A_3 = \frac{2a_2a_3}{a_1^2\sin^3\phi\cos^2\theta}\left[\frac{a_2}{a_3}\sin\phi - E(\theta, \phi)\right] \tag{4.11}$$

where a_i $(i = 1, 2, 3)$ are the principal semi-axes of the ellipsoid. Also, θ, ϕ, F *and* E *are as defined.*

Proof. Lemmas 5 and 6 as given in the previous chapter are

$$A_1 + A_2 + A_3 = 2 \tag{4.12}$$

$$a_1^2A_1 + a_2^2A_2 + a_3^2A_3 = I \tag{4.13}$$

respectively. Using (4.12) to eliminate A_2 from (4.13) obtains

$$a_1^2 A_1 + a_2^2(2 - A_1 - A_3) + a_3^2 A_3 = I$$

Rearranging the above equation for A_3 gives

$$A_3 = \frac{1}{a_3^2 - a_2^2}(I + A_1(a_2^2 - a_1^2) - 2a_2^2) \tag{4.14}$$

where we have already obtained the required expressions for A_1 and I. Substituting the expression for A_1 given by (4.4) into (4.14) and we have

$$A_3 = \frac{1}{a_3^2 - a_2^2}(I - 2a_2^2 + \alpha[F(\theta,\phi) - E(\theta,\phi)])$$

where we have defined

$$\alpha \equiv \frac{2a_2 a_3}{a_1^2 \sin^3\phi \sin^2\theta}(a_2^2 - a_1^2)$$

and we now have

$$\begin{aligned} A_3 &= \frac{1}{a_3^2 - a_2^2}(I - 2a_2^2 + \alpha F(\theta,\phi) - \alpha E(\theta,\phi)) \\ &= \underbrace{\frac{\alpha}{a_3^2 - a_2^2}}_{1}(\underbrace{\frac{I}{\alpha} - \frac{2a_2^2}{\alpha} + F(\theta,\phi)}_{2} - E(\theta,\phi)) \end{aligned} \tag{4.15}$$

and we want to tidy up parts 1 and 2 of the above equation as marked. Firstly note that from (4.3) we have

$$\sin^2\theta = \frac{a_1^2 - a_2^2}{a_1^2 - a_3^2} \quad \Rightarrow \quad \cos^2\theta = \frac{a_2^2 - a_3^2}{a_1^2 - a_3^2}$$

and therefore part one of equation (4.15) becomes

$$\begin{aligned} \frac{\alpha}{a_3^2 - a_2^2} &= \frac{2a_2 a_3}{a_1^2 \sin^3\phi \sin^2\theta}\frac{(a_2^2 - a_1^2)}{(a_3^2 - a_2^2)} \\ &= \frac{2a_2 a_3}{a_1^2 \sin^3\phi}\frac{(a_1^2 - a_3^2)}{(a_3^2 - a_2^2)}\frac{(a_2^2 - a_1^2)}{(a_1^2 - a_2^2)} = \frac{2a_2 a_3}{a_1^2 \sin^3\phi \cos^2\theta} \end{aligned}$$

Remembering the definition of ϕ in (4.3) we have

$$\cos\phi = \frac{a_3}{a_1} \quad \Rightarrow \quad \sin^3\phi = \frac{(a_1^2 - a_3^2)^{\frac{3}{2}}}{a_1^3}$$

and also note that using the expression for I in (4.10) gives

$$\begin{aligned} \frac{I}{\alpha} &= \frac{a_1^2 \sin^3\phi \sin^2\theta}{2a_2 a_3 (a_2^2 - a_1^2)}\frac{2a_1 a_2 a_3}{(a_1^2 - a_3^2)^{\frac{1}{2}}}F(\theta,\phi) \\ &= \frac{2a_1^3 a_2 a_3}{2a_2 a_3 (a_2^2 - a_1^2)(a_1^2 - a_3^2)^{\frac{1}{2}}}\frac{(a_1^2 - a_3^2)^{\frac{3}{2}}}{a_1^3}\frac{(a_1^2 - a_2^2)}{(a_1^2 - a_3^2)}F(\theta,\phi) = -F(\theta,\phi) \end{aligned}$$

Part two of equation (4.15) becomes

$$\begin{aligned}
\frac{I}{\alpha} - \frac{2a_2^2}{\alpha} + F(\theta,\phi) &= -F(\theta,\phi) - \frac{2a_2^2}{\alpha} + F(\theta,\phi) = -\frac{2a_2^2}{\alpha} \\
&= -2a_2^2 \frac{a_1^2 \sin^3\phi \sin^2\theta}{2a_2 a_3 (a_2^2 - a_1^2)} \\
&= -2a_2^2 \frac{a_1^2}{2a_2 a_3 (a_2^2 - a_1^2)} \frac{(a_1^2 - a_3^2)^{\frac{3}{2}}}{a_1^3} \frac{(a_1^2 - a_2^2)}{(a_1^2 - a_3^2)} \\
&= \frac{a_2}{a_3} \frac{(a_1^2 - a_3^2)^{\frac{1}{2}}}{a_1} = \frac{a_2}{a_3} \sin\phi
\end{aligned}$$

and therefore equation (4.15) becomes

$$A_3 = \frac{2a_2 a_3}{a_1^2 \sin^3\phi \cos^2\theta} \left[\frac{a_2}{a_3} \sin\phi - E(\theta,\phi) \right]$$

as required.

□

4.5 Determination of A_2 in terms of the elliptic integrals

Theorem 7.

$$A_2 = \frac{2a_2 a_3}{a_1^2 \sin^3\phi \sin^2\theta \cos^2\theta} [E(\theta,\phi) - \cos^2\theta F(\theta,\phi) - \frac{a_3}{a_2} \sin^2\theta \sin\phi] \tag{4.16}$$

where a_i $(i = 1, 2, 3)$ *are the principal semi-axes of the ellipsoid. Also,* θ*,* ϕ*,* F *and* E *are as defined.*

Proof. Using (4.4), (4.11) and (4.12) gives

$$\begin{aligned}
A_2 &= 2 - A_1 - A_3 \\
&= 2 - \frac{\beta}{\sin^2\theta} [F(\theta,\phi) - E(\theta,\phi)] - \frac{\beta}{\cos^2\theta} \left[\frac{a_2}{a_3} \sin\phi - E(\theta,\phi) \right] \\
&= \frac{\beta}{\sin^2\theta \cos^2\theta} \left[\frac{2\sin^2\theta \cos^2\theta}{\beta} - \cos^2\theta (F - E) - \sin^2\theta \left(\frac{a_2}{a_3} \sin\phi - E \right) \right]
\end{aligned}$$

where

$$\beta = \frac{2a_2 a_3}{a_1^2 \sin^3\phi}$$

and simplifying the above equation remembering to use the definitions (4.3) gives us

$$A_2 = \frac{2a_2 a_3}{a_1^2 \sin^3\phi \sin^2\theta \cos^2\theta} [E(\theta,\phi) - \cos^2\theta F(\theta,\phi) - \frac{a_3}{a_2} \sin^2\theta \sin\phi]$$

as required. □

4.5.1 Summary of A_i in terms of elliptic integrals

The above theorems show the determination of the integral functions A_i $(i = 1, 2, 3)$ in terms of the incomplete elliptic integrals of the two kinds $F(\theta, \phi)$ and $E(\theta, \phi)$. To summarise

$$A_1 = \frac{2a_2a_3}{a_1^2 \sin^3\phi \sin^2\theta}\left[F(\theta,\phi) - E(\theta,\phi)\right] \tag{4.17}$$

$$A_2 = \frac{2a_2a_3}{a_1^2 \sin^3\phi \sin^2\theta \cos^2\theta}[E(\theta,\phi) - \cos^2\theta F(\theta,\phi) - \frac{a_3}{a_2}\sin^2\theta \sin\phi] \tag{4.18}$$

$$A_3 = \frac{2a_2a_3}{a_1^2 \sin^3\phi \cos^2\theta}\left[\frac{a_2}{a_3}\sin\phi - E(\theta,\phi)\right] \tag{4.19}$$

4.6 Spheroidal A_i in terms of eccentricity

It will be required to express A_i when $a_1 = a_2 > a_3$. Because we set $a_1 = a_2$ the ellipsoid in question is a spheroid which is symmetric around the a_3 axis and therefore we have $A_1 = A_2$. Using Taylor expansions on F and E, we see that using (4.4) A_1 becomes

$$A_1 = \frac{2a_2a_3}{a_1^2\sin^3\phi\sin^2\theta}\left[\int_0^\phi (1 + \frac{1}{2}\sin^2\theta\sin^2\varphi + (\text{Higher powers of } \sin^2\theta))\ d\varphi \right.$$
$$\left. - \int_0^\phi (1 - \frac{1}{2}\sin^2\theta\sin^2\varphi - (\text{Higher powers of } \sin^2\theta))\ d\varphi\right]$$

but from (4.3), we see that $\sin^2\theta = 0$ as $a_1 = a_2$. Therefore the higher powers with $\sin^2\theta$ vanish and the first term with $\sin^2\theta$ cancels with the $\sin^2\theta$ outside the integral, we have that

$$\begin{aligned}
A_1 &= \frac{2a_2a_3}{a_1^2\sin^3\phi\sin^2\theta}\left[\int_0^\phi (1 + \frac{1}{2}\sin^2\theta\sin^2\varphi)\ d\varphi - \int_0^\phi (1 - \frac{1}{2}\sin^2\theta\sin^2\varphi)\ d\varphi\right] \\
&= \frac{2a_3}{a_1\sin^3\phi}\int_0^\phi \sin^2\varphi\ d\varphi \\
&= \frac{2a_3}{a_1\sin^3\phi}\int_0^\phi \frac{1}{2}(1 - \cos 2\varphi)d\varphi \\
&= \frac{a_3}{a_1\sin^3\phi}(\phi - \frac{1}{2}\sin 2\phi) \qquad (4.20)
\end{aligned}$$

We want to express A_1 in terms of the eccentricity e of an oblate spheroid, e is defined as

$$e = (1 - \frac{a_3^2}{a_1^2})^{\frac{1}{2}}$$

where physically eccentricity is a measure of how flat the defining ellipse is. In the case of the oblate spheroids the defining ellipse is a cross-section through the polar-axis a_3. We may visualise the eccentricity by looking at Figure 5.1 in Chapter 5 where the flatter spheroids have a higher eccentricity.

Note that on using $\cos\phi = a_3/a_1$

$$e = (1 - \frac{a_3^2}{a_1^2})^{\frac{1}{2}} = (1 - \cos^2\phi)^{\frac{1}{2}} \Rightarrow \cos\phi = (1 - e^2)^{\frac{1}{2}}$$

and therefore $a_3/a_1 = (1-e^2)^{\frac{1}{2}}$. Also, on using the trigonometric identity $\sin^2\phi + \cos^2\phi = 1$ we have

$$\cos\phi = \frac{a_3}{a_1} \Rightarrow \sin\phi = (1 - \frac{a_3^2}{a_1^2})^{\frac{1}{2}} = e$$

Putting this together and using $\sin 2\phi = 2\sin\phi\cos\phi$, we have for (4.20)

$$\begin{aligned}
A_1 &= \frac{(1-e^2)^{\frac{1}{2}}}{e^3}(\phi - \sin\phi\cos\phi) \\
&= \frac{(1-e^2)^{\frac{1}{2}}}{e^3}(\sin^{-1} e - e(1-e^2)^{\frac{1}{2}}) \\
&= \frac{(1-e^2)^{\frac{1}{2}}}{e^3}\sin^{-1} e - \frac{1-e^2}{e^2}
\end{aligned}$$

Doing the same for A_3 in (4.11) and noting that $\cos\theta = 1,\ \sin\theta = 0$ and again using the Taylor expansion for E produces

$$\begin{aligned}
A_3 &= \frac{2a_2a_3}{a_1^2\sin^3\phi\cos^2\theta}\left[\frac{a_2}{a_3}\sin\phi - E(\theta,\phi)\right] \\
&= \frac{2(1-e^2)^{\frac{1}{2}}}{e^3}\left[\frac{e}{(1-e^2)^{\frac{1}{2}}} - \int_0^\phi (1 - \frac{1}{2}\sin^2\theta\sin^2\varphi) - (\text{Higher powers of } \sin^2\theta)\right] \\
&= \frac{2(1-e^2)^{\frac{1}{2}}}{e^3}\left[\frac{e}{(1-e^2)^{\frac{1}{2}}} - \int_0^\phi 1\right] \\
&= \frac{2(1-e^2)^{\frac{1}{2}}}{e^3}\left[\frac{e}{(1-e^2)^{\frac{1}{2}}} - \sin^{-1} e\right] \\
&= \frac{2}{e^2} - \frac{2(1-e^2)^{\frac{1}{2}}}{e^3}\sin^{-1} e
\end{aligned}$$

In summary we have

$$A_1 = A_2 = \frac{(1-e^2)^{\frac{1}{2}}}{e^3}\sin^{-1} e - \frac{1-e^2}{e^2} \tag{4.21}$$

$$A_3 = \frac{2}{e^2} - \frac{2(1-e^2)^{\frac{1}{2}}}{e^3}\sin^{-1} e \tag{4.22}$$

4.7 Further definitions

It will be convenient to state the following definitions which will be used in the forthcoming chapter.

Definition 4.

$$A_{ij} = a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta(a_i^2 + u)(a_j^2 + u)} \tag{4.23}$$

$$B_{ij} = a_1 a_2 a_3 \int_0^\infty \frac{u \; du}{\Delta(a_i^2 + u)(a_j^2 + u)} \tag{4.24}$$

Relations (Chandrasekhar [5, p. 54]) which follow from the above definitions and the definition of A_i are

$$A_i - A_j = (a_j^2 - a_i^2) A_{ij} \tag{4.25}$$

$$a_i^2 A_i - a_j^2 A_j = (a_i^2 - a_j^2) B_{ij} \tag{4.26}$$

$$B_{ij} = A_j - a_i^2 A_{ij} \tag{4.27}$$

Chapter 5

Equilibria

5.1 Introduction

In Chapter 2 we derived equations from the equations of motion, the second order virial equations (2.41), that must be satisfied for equilibrium (i.e. our force balance) to hold and for our configurations to maintain their rotating figure. In our journey to find possible equilibrium configurations, the relevant virial equations are central not only because they are derived from the equations of motion themselves but also because it determines the unique relationship between fundamental characteristics of the rotating figure's composite fluid as a whole for equilibrium to be satisfied. These fundamental characteristics are the moment of inertia and gravitational potential energy tensors I_{ij} and E_{ij} respectively.

In Chapter 3, under the assumption that the equilibrium figures we are seeking are ellipsoidal and homogeneous, we found the gravitational potential of the ellipsoid at an internal point and thus determined a formula for the gravitational potential energy of an ellipsoid in terms of its moment of inertia and principal semi-axes given by (3.58).

Remarkably, we now have all we need to determine some possible sequences of equilibrium configurations and the focus of this chapter is on two such sequences - *The Maclaurin spheroids* and the *Jacobi ellipsoids*.

These sequences are uniquely defined by their governing equations that are derived in this chapter. We will conclude with a significant property that links these two sequences of equilibria, which we call bifurcation.

5.2 Preliminaries

Before proceeding with the main argument we will need to express the values of the diagonal components of the moment of inertia tensor I_{ij} in terms of the the principal axes a_1,a_2 and a_3.

5.2.1 Finding I_{ii}

For an ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$$

and in spherical polar coordinates (r, θ, ϕ) we have

$$x_1 = a_1 r \sin\theta\cos\phi$$
$$x_2 = a_2 r \sin\theta\sin\phi$$
$$x_3 = a_3 r \cos\theta$$

We also have,

$$dx_1 dx_2 dx_3 = a_1 a_2 a_3 r^2 \sin\theta dr d\theta d\phi$$

From the definition of the moment of inertia tensor I_{ij}, we see that in particular

$$I_{11} = \rho \int\limits_V x_1^2 \; dx_1 dx_2 dx_3$$

where

$$x_1^2 = a_1^2 r^2 \sin^2\theta\cos^2\phi$$

Putting this together with use of the trigonometric identities

$$\cos 2\theta = 1 - 2\sin^2\theta \text{ and } \cos 2\phi = 2\cos^2\phi - 1$$

obtains

$$\begin{aligned}
I_{11} &= \rho a_1^3 a_2 a_3 \int\limits_{r=0}^{1}\int\limits_{\theta=0}^{\pi}\int\limits_{\phi=0}^{2\pi} r^4 \sin^3\theta\cos^2\phi \; dr d\theta d\phi \\
&= \rho a_1^3 a_2 a_3 \frac{1}{4}\int\limits_{r=0}^{1}\int\limits_{\theta=0}^{\pi}\int\limits_{\phi=0}^{2\pi} r^4 \sin\theta(1-\cos 2\theta)(\cos 2\phi + 1) \; dr d\theta d\phi \\
&= \rho a_1^3 a_2 a_3 \frac{1}{4}\frac{1}{5}\int\limits_{\theta=0}^{\pi} \sin\theta(1-\cos 2\theta) \; d\theta \int\limits_{\phi=0}^{2\pi} (\cos 2\phi + 1) \; d\phi \\
&= \rho a_1^3 a_2 a_3 \frac{1}{4}\frac{1}{5} 2\pi \int\limits_{\theta=0}^{\pi} \sin\theta - \sin\theta\cos 2\theta \; d\theta \\
&= \rho a_1^3 a_2 a_3 \frac{1}{4}\frac{1}{5} 2\pi \left(2 - \int\limits_{\theta=0}^{\pi} \sin\theta\cos 2\theta \; d\theta \right) \qquad (5.1)
\end{aligned}$$

Using integration by parts we have

$$\begin{aligned}
\int\limits_0^{\pi} \sin\theta\cos 2\theta \; d\theta &= [-\cos\theta\cos 2\theta]_0^{\pi} - \int\limits_0^{\pi} 2\cos\theta\sin 2\theta \; d\theta \\
&= 2 - 2\left[[\sin\theta\sin 2\theta]_0^{\pi} - \int\limits_0^{\pi} 2\sin\theta\cos 2\theta \; d\theta \right] \\
&= 2 + 4\int\limits_0^{\pi} \sin\theta\cos 2\theta \; d\theta
\end{aligned}$$

and thus this gives us

$$\int_0^\pi \sin\theta \cos 2\theta \; d\theta = -\frac{2}{3}$$

Returning to (5.1) we get

$$I_{11} = \rho a_1^3 a_2 a_3 \frac{4\pi}{15}$$

where I_{22} and I_{33} are found in a similar way. In summary we have

$$I_{11} = \rho a_1^3 a_2 a_3 \frac{4\pi}{15} \tag{5.2}$$

$$I_{22} = \rho a_1 a_2^3 a_3 \frac{4\pi}{15} \tag{5.3}$$

$$I_{33} = \rho a_1 a_2 a_3^3 \frac{4\pi}{15} \tag{5.4}$$

5.2.2 Virial equations revisited

Next we want to use the second order virial equations derived in Chapter 2 as it must be satisfied for a uniformly rotating homogeneous configuration to be in equilibrium. That is, under a state of equilibrium we have (Chandrasekhar [5, p. 25])

$$E_{ij} + \Omega^2 (I_{ij} - \delta_{i3} I_{3j}) = -\delta_{ij} \Pi$$

where we have chosen the rotation to be about the x_3-axis.

Let us write out the different components of the above virial equation. First the diagonal components for $i = j = 1, i = j = 2$ and $i = j = 3$ give

$$E_{11} + \Omega^2 I_{11} = E_{22} + \Omega^2 I_{22} = E_{33} = -\Pi \tag{5.5}$$

and the non-diagonal components give

$$E_{12} + \Omega^2 I_{12} = 0 \tag{5.6}$$

$$E_{21} + \Omega^2 I_{21} = 0 \tag{5.7}$$

$$E_{13} + \Omega^2 I_{13} = 0 \tag{5.8}$$

$$E_{31} = 0 \tag{5.9}$$

$$E_{23} + \Omega^2 I_{23} = 0 \tag{5.10}$$

$$E_{32} = 0 \tag{5.11}$$

The tensors E_{ij} and I_{ij} are symmetric and thus we have from (5.9) and (5.11)

$$E_{13} = E_{23} = 0$$

and upon using (5.8) and (5.10) we have also

$$I_{13} = I_{23} = 0$$

Therefore our tensors E_{ij} and I_{ij} look like

$$E_{ij} = \begin{pmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & E_{33} \end{pmatrix} \text{ and } I_{ij} = \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

So far, we can see that E_{12} and I_{12} are not zero but are related by (5.6). However we may choose to orientate the coordinate system about the x_3-axis to make I_{12} (and hence I_{21}) vanish. Using (5.6) E_{12} also vanishes (and hence E_{21}), thus our tensors E_{ij} and I_{ij} are diagonal and we see that

$$I_{12} = I_{21} = 0 \quad \textit{and} \quad E_{12} = E_{21} = 0 \tag{5.12}$$

5.3 Maclaurin spheroids

First let us state the defining assumption for the Maclaurin Spheroids, that is

The body is symmetric about its axis of rotation.

or more specifically we will assume that we have an oblate spheroid rotating about its polar axis. Note that for an oblate spheroid the equatorial axes are equal and the polar axis is less than the equatorial, that is we have $a_1 = a_2$ and $a_3 < a_1$ where Figure 5.1 illustrates oblate spheroids. The family of spheroids is a particular subset of the family of ellipsoids.

Since we have assumed the body is symmetric about its axis of rotation we may take

$$E_{11} = E_{22} \text{ and } I_{11} = I_{22} \tag{5.13}$$

Therefore from (5.5) we only require that

$$E_{11} + \Omega^2 I_{11} = E_{33} \tag{5.14}$$

We should note however that, without assuming an axis of symmetry, we do not require that (5.13) holds for equilibrium to be satisfied but we may still arrange that (5.12) is true.

Stating the relationship (3.58), we have for the gravitational potential energy of a homogeneous ellipsoid

$$\frac{E_{ij}}{\pi G \rho} = -2A_i I_{ij}$$

where A_i is as defined in (3.14).

Using the above expression in equation (5.14) to eliminate the potential energy tensor produces

$$-2A_1 I_{11} + \Omega^2 I_{11} = -2A_3 I_{33}$$

where we take Ω^2 to be measured in units of $\pi G \rho$. Rearranging for Ω^2 yields

$$\Omega^2 = 2\left(A_1 - \frac{I_{33}}{I_{11}} A_3\right)$$

and substituting in (5.2) and (5.4) into the above equation yields

$$\Omega^2 = 2\left(A_1 - \frac{a_3^2}{a_1^2} A_3\right) \tag{5.15}$$

This equation defines the Maclaurin sequence of spheroids but we may go further and substitute our values for A_1 and A_3 in terms of the eccentricity found in Chapter 4, (4.21) and (4.22), into equation (5.15) to find that

$$\Omega^2 = \frac{2(1-e^2)^{\frac{1}{2}}}{e^3}(3-2e^2)\sin^{-1} e - \frac{6}{e^2}(1-e^2) \tag{5.16}$$

where we have

$$e = \left(1 - \frac{a_3^2}{a_1^2}\right)^{\frac{1}{2}}$$

the eccentricity of a member of the Maclaurin sequence. Recall that the eccentricity is a measure of the *flatness* of a figure, as $e \to 1$ the figure becomes increasingly flattened at the poles, this is equivalent to $a_3/a_1 \to 0$.

Equation (5.16) is known as Maclaurin's formula, which was first derived by Colin Maclaurin in 1742. It describes a unique sequence of homogeneous oblate spheroids that rotate uniformly about their polar-axis and that satisfy our force balance under the given formulation. Each member of the sequence is prescribed a unique angular velocity which is the defining feature of the Maclaurin Spheroids.

Figures 5.2 and 5.3 illustrate[1] exactly how the angular velocity varies across the Maclaurin sequence of spheroids accorfing to the defining equations. For Figure 5.2 We see that the angular velocity increases to a maximum of about 0.45 at an eccentricity of around 0.93 and then decreases to zero as the figures become increasingly flattened. It is worth stressing that there are two possible Maclaurin Spheroids for any one angular velocity which is not above the maximum angular velocity of $\Omega = 0.67$. One is more more flattened than the other and this property was first noticed by Thomas Simpson in 1743(Chandrasekhar [5, p. 4]). Hence we have an upper limit for the rate of rotation for a configuration to be in equilibrium under our model.

We may see the usefulness of this result in astronomical measurements. Given measurements of a figure's rotation rate, one may predict the shape of the figure to be one of two possibilities under this model. On the other hand if you were to measure the figure's shape, you could predict how fast the figure is rotating, assuming it to be a member of the Maclaurin sequence. Of course this would only be appropriate if the assumption of homogeneity and the ideal fluid approximation are suitable for the body under consideration.

Figure 5.1 illustrates the Maclaurin spheroids at six points along the sequence starting from a spheroid departing only slightly from a sphere in Figure 5.1a to the spheroid with the highest rate of rotation in 5.1d to the highly flattened spheroids in Figures 5.1e and 5.1f.

5.3.1 The angular momentum of the Maclaurin spheroids

At one extreme the sequence of Maclaurin spheroids become highly flattened "discs" with a low angular velocity and at the other extreme we have a figure departing only slightly from a sphere with a low angular velocity. While the angular velocity reaches a maximum, it is worth noting that the angular momentum steadily increases with eccentricity as may be seen in Figure 5.4, so for each value of the angular momentum there is only one possible equilibrium configuration. The angular momentum L of the rotating body is given by

$$L = \frac{\Omega\sqrt{3}}{5(1-e^2)^{\frac{1}{3}}}$$

[1]See the appendix for the Maple code used to generate these figures

where L is measured in units of $(GM^3\bar{a})^{\frac{1}{2}}$ and $\bar{a} = (a_1^2 a_3)^{\frac{1}{3}}$, M is the mass of the spheroid and $\bar{a}$ is the radius of a sphere of mass M (Chandrasekhar [5, p. 78]).

In Figure 5.5 we can see how the angular momentum varies with the a_3/a_1 ratio for the Maclaurin spheroids. As this ratio decreases the figures becomes increasingly flattened and we can see how the angular momentum tends to infinity as the figures tend to highly flattened "discs" - known as *Maclaurin discs* (Binney [6]).

5.4 Jacobi ellipsoids

In the preceding section we illustrated a particular solution to the problem of a uniformly rotating fluid - the sequence of Maclaurin spheroids. While the Maclaurin sequence was derived under the assumption of rotational symmetry, it is worth asking the question - "Are there possible solutions of ellipsoidal equilibrium configurations to the problem of a uniformly rotating body with three unequal axes?" It was assumed that the Maclaurin spheroids were the only equilibria for the problem of a uniformly rotating homogeneous mass for almost a century after Maclaurin's discovery. It was not until 1834 that Jacobi realised that rotational symmetry, although sufficient for, was not necessary for equilibrium to be satisfied. Jacobi said:

> *One would make a grave mistake if one supposed that the spheroids of revolution are the only admissible figures of equilibrium ... in fact a simple consideration shows that ellipsoids with three unequal axes can very well be figures of equilibrium; and that one can assume an ellipse of arbitrary shape for the equatorial section and determine the third axis (which is also the least of the three axes) and the angular velocity of rotation such that the ellipsoid is a figure of equilibrium.*(Chandrasekhar [5, p. 5])

Such figures of equilibrium are known as *The Jacobi ellipsoids* of which is the focus of this section and accordingly we will not assume symmetry about the axis of rotation in the following derivation (Chandrasekhar [5, p. 101]) of the defining equations.

Again we will take the second order virial tensor equation

$$E_{ij} + \Omega^2(I_{ij} - \delta_{i3}I_{3j}) = -\delta_{ij}\Pi$$

and we need only consider the diagonal components given by (5.5) as we may still arrange that (5.12) is true, so we have

$$E_{11} + \Omega^2 I_{11} = E_{22} + \Omega^2 I_{22} = E_{33}$$

which must be satisfied for equilibrium to hold. Substituting in the expression for the potential energy tensor (3.58) into the above expression obtains

$$\Omega^2 I_{11} - 2A_1 I_{11} = \Omega^2 I_{22} - 2A_2 I_{22} = -2A_3 I_{33} \tag{5.17}$$

Now substituting the values for I_{11}, I_{22} and I_{33} given by (5.2), (5.3) and (5.4) into (5.17) provides

$$\Omega^2 a_1^2 - 2A_1 a_1^2 = \Omega^2 a_2^2 - 2A_2 a_2^2 = -2A_3 a_3^2$$

Let us now add $2A_{12}a_1^2 a_2^2$ to the above relation to give

$$a_1^2(\Omega^2 - 2(A_1 - a_2^2 A_{12})) = a_2^2(\Omega^2 - 2(A_2 - a_1^2 A_{12})) = 2(A_{12}a_1^2 a_2^2 - A_3 a_3^2)$$

where A_{12} is as defined by (4.23), and using the definition of B_{12} given by (4.24) produces

$$a_1^2(\Omega^2 - 2B_{12}) = a_2^2(\Omega^2 - 2B_{12}) = 2(A_{12}a_1^2a_2^2 - A_3a_3^2) \tag{5.18}$$

We seek solutions for $a_1 \neq a_2$ and for this condition to be satisfied with the above equalities (5.18) we *must* have that

$$\Omega^2 = 2B_{12} \quad \textit{and} \tag{5.19}$$

$$A_{12}a_1^2a_2^2 = A_3a_3^2 \tag{5.20}$$

These are the defining equations for the sequence of ellipsoidal equilibrium configurations known as the Jacobi ellipsoids. Equation (5.19) determines the rate of rotation for each member of the Jacobi sequence of ellipsoids.

Equation (5.20) is as Chandrasekhar best puts it:

> *a geometric restriction on the ellipsoid: it determines a unique relation between the ratios of the axes, a_2/a_1 and a_3/a_1, in order that equilibrium may at all be possible; it is therefore the distinguishing feature of the Jacobi figures.*
> (Chandrasekhar [5, p. 102])

Figure 5.6 illustrates this unique relationship between the ratios a_2/a_1 and a_3/a_1 for Jacobi ellipsoids. Given the value of a_2/a_1 we may determine a_3/a_1 by (5.20) to be a member of the Jacobi ellipsoids, and each member is then further defined by a unique prescribed angular velocity given by equation (5.19). Hence applying (5.19) onto the line given by Figure 5.6 produces the red line in Figure 5.3 for the Jacobi ellipsoids.

Figures 5.3 and 5.2 illustrate[2] the variation of Ω^2 across the Jacobi sequence of ellipsoidal equilibrium configurations and we can clearly see how the rotation rate steadily decreases along the sequence as the figures become increasingly elongated or "cigar-shaped", which is illustrated in Figure 5.7 at six different points along the sequence. Note that in Figure 5.2 we use the eccentricity of the elliptical cross-section through the a_1 and a_3 semi-axes but in Figure 5.3 we illustrate a more complete picture.

5.4.1 The angular momentum of the Jacobi ellipsoids

It is possible to illustrate the angular momentum for the Jacobi sequence as shown in Figures 5.5 and 5.4. The angular momentum for the Jacobi sequence is given by the relation

$$L = \frac{\sqrt{3}}{10}\frac{a_1^2 + a_2^2}{\overline{a}^2}\Omega$$

where $\overline{a} = \sqrt[3]{a_1a_2a_3}$ and L is measured in units of $(GM^3\overline{a})^{\frac{1}{2}}$ where M is the mass of a sphere of radius $\overline{a}$ (Chandrasekhar [5, p. 103]). Looking at figure 5.5 and 5.4 we observe how the angular momentum tends to infinity as the figures become increasingly flattened and elongated.

[2] These illustrations were generated numerically using Maple 13, by fixing values of a_2/a_1, using (5.20) to determine the corresponding values of a_3/a_1 (illustrated in figure 5.6) and using (5.19) to find the corresponding values of angular velocity.

5.5 Bifurcation

A significant highlight in this investigation is that of bifurcation between the two sequences of Maclaurin and Jacobi. It is known that the Jacobi sequence of ellipsoids bifurcates from the Maclaurin sequence of spheroids at a certain point and this can be seen in Figures 5.5, 5.4, 5.3 and 5.2. Bifurcation occurs when $a_1 = a_2$ for the Jacobi sequence and equations (5.19) and (5.20) become

$$a_1^4 A_{11} = a_3^2 A_3 \tag{5.21}$$

$$\text{and} \quad \Omega^2 = 2B_{11} \tag{5.22}$$

We may show (Chandrasekhar [5, p. 102]) that this is also a member of the Maclaurin spheroids. For the Maclaurin spheroids, from (5.15), we have

$$\Omega^2 = 2\left(A_1 - \frac{a_3^2}{a_1^2}A_3\right) = 2(1 - \frac{a_3^2}{a_1^2})(A_3 - a_1^2 A_{13}) = 2(\frac{a_1^2 - a_3^2}{a_1^2})B_{13}$$

where we used the definition (4.23) and relations (4.25) and (4.27). This together with equation (5.22) gives us

$$\begin{aligned}
& & (a_1^2 - a_3^2)B_{13} &= a_1^2 B_{11} \\
&\Leftrightarrow & (a_1^2 - a_3^2)(A_1 - a_3^2 A_{13}) &= a_1^2(A_1 - a_1^2 A_{11}) \\
&\Leftrightarrow & a_1^4 A_{11} &= a_3^2(a_1^2 - a_3^2)A_{13} + a_3^2 A_1 \\
& & &= a_3^2((a_1^2 - a_3^2)A_{13} + A_1) = a_3^2 A_3
\end{aligned}$$

which is identical to equation (5.21). Therefore fascinatingly the Jacobi ellipsoid defined by (5.21) and (5.22) is also a member of the Maclaurin spheroids and we see that the Jacobi sequence bifurcates from the Maclaurin sequence from the point where $\Omega^2 = 2B_{11}$, as can be seen in the given illustrations.

From Figures 5.2 and 5.3 we can see that, as we move past the point of bifurcation, the Jacobi sequence immediately loses its angular velocity whilst the Maclaurin sequence continues to a maximum. The spheroid of bifurcation is illustrated in Figure 5.1c and Figure 5.7a.

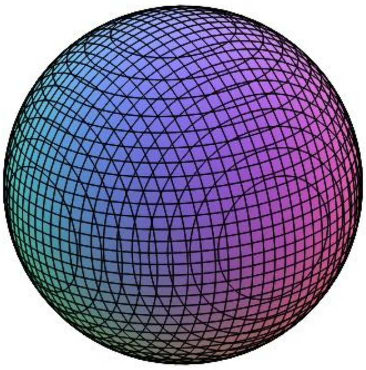
Figures 5.5 and 5.4 again contrast the Maclaurin spheroids with the Jacobi ellipsoids, we can see the point of bifurcation and that with increasing angular momentum both sequences tend to infinite angular momentum after bifurcation as they become infinitely flattened as seen in Figures 5.1 and 5.7.

In conclusion we find that by numerical calculation the Jacobi sequence bifurcates from the Maclaurin sequence at

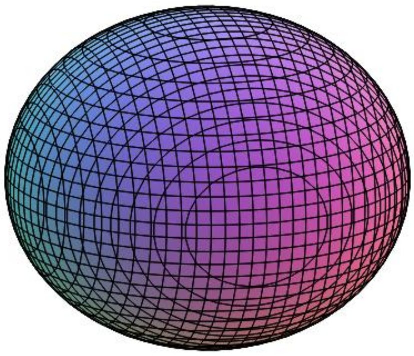
$$e = 0.813 \quad and \quad \frac{\Omega^2}{\pi G \rho} = 0.374$$

where e is the eccentricity of the a_1, a_3 section of the figures. Equivalently we have the point of bifurcation is

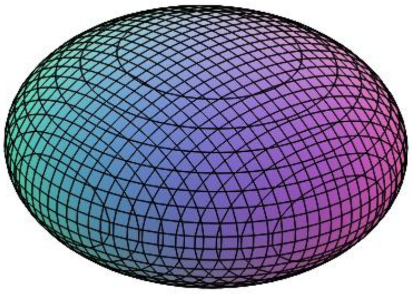
$$\frac{a_2}{a_1} = 1 \quad and \quad \frac{a_3}{a_1} = 0.582 \quad and \quad \frac{\Omega^2}{\pi G \rho} = 0.374$$



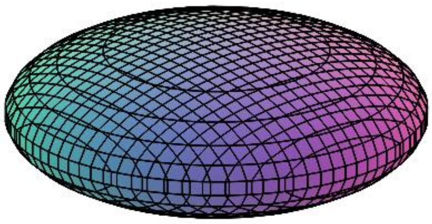
(a) $\Omega = 0.00534$



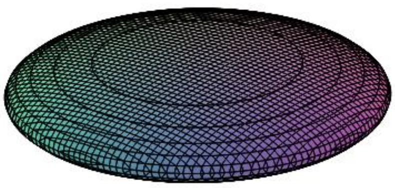
(b) $\Omega = 0.4082$



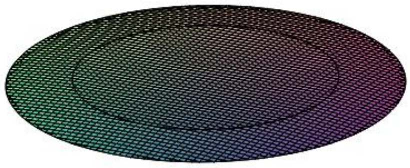
(c) Bifurcation spheroid $\Omega = 0.61174$



(d) Maximum $\Omega = 0.67032$ spheroid



(e) $\Omega = 0.61483$



(f) $\Omega = 0.35412$

Figure 5.1: Maclaurin Spheroid Sequence

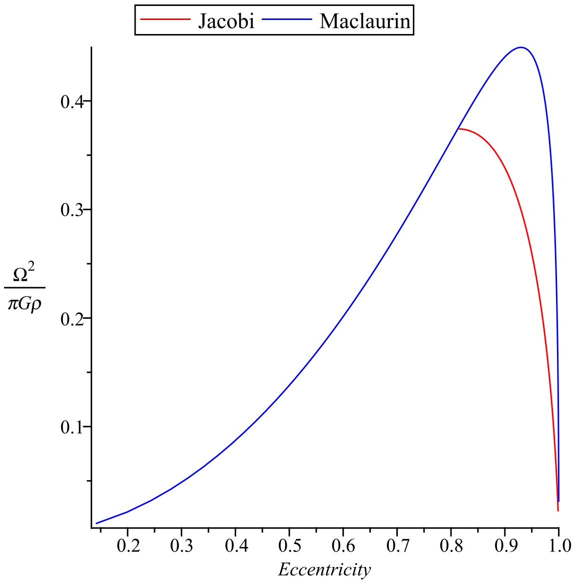


Figure 5.2: Variation of angular velocity squared (in units of $\pi G\rho$) with eccentricity of the configurations for both the Maclaurin and Jacobi sequences. Where the eccentricity is that of the (a_3, a_1) section. The point where the Jacobi ellipsoids bifurcate from the Maclaurin spheroids can clearly be seen.

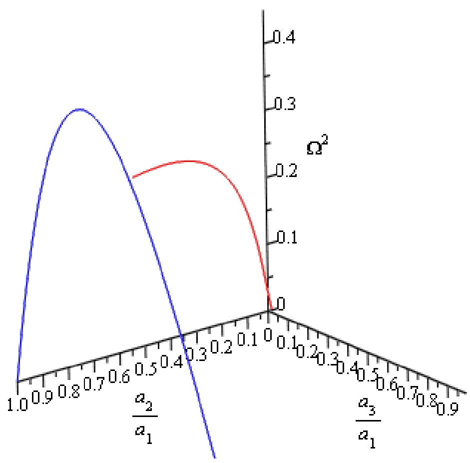


Figure 5.3: Variation of angular velocity squared (in units of $\pi G\rho$) with axes-ratios. Blue depicts the Maclaurin sequence, red depicts the Jacobi sequence. The point at which the Jacobi sequence bifurcates from the Maclaurin sequence is clearly seen.

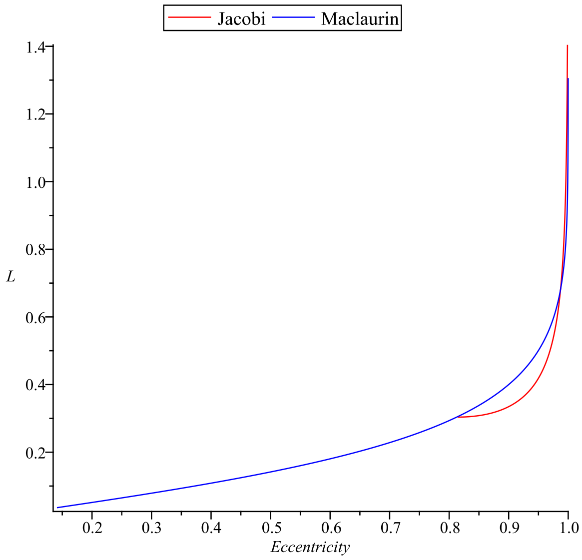


Figure 5.4: Variation of angular momentum (L) with eccentricity of configuration for both the Maclaurin and Jacobi sequences. Where the eccentricity is that of the (a_3, a_1) section. The point where the Jacobi ellipsoids bifurcate from the Maclaurin spheroids can clearly be seen.

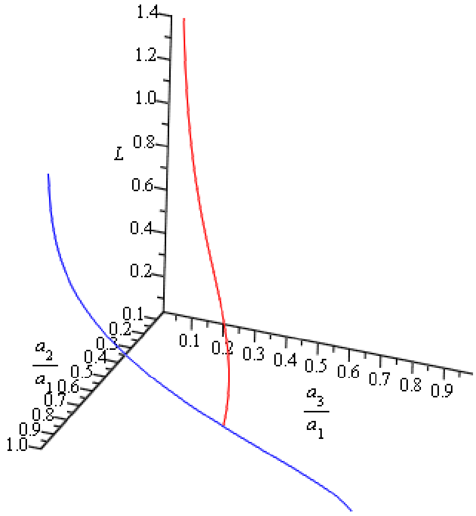


Figure 5.5: Variation of angular momentum (L) with axes-ratios. Blue depicts the Maclaurin sequence, red depicts the Jacobi sequence. The point of bifurcation is clearly seen. Both sequences tend to infinity as

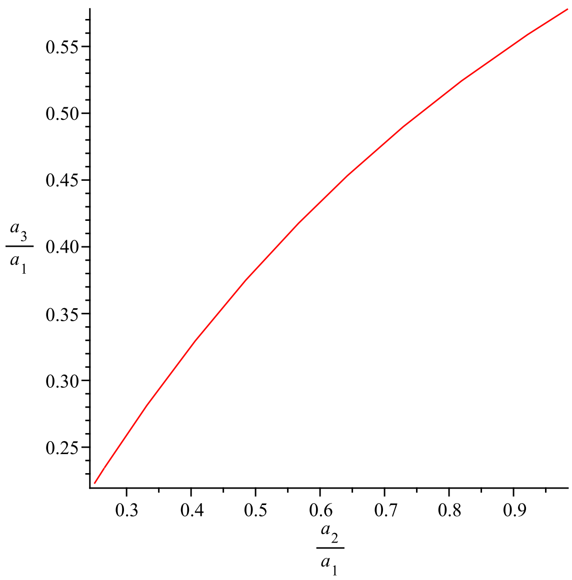
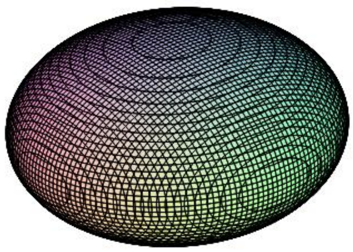
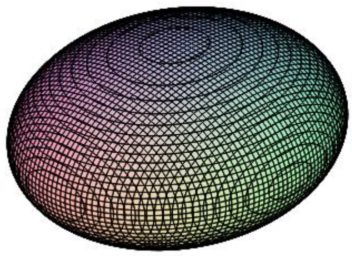


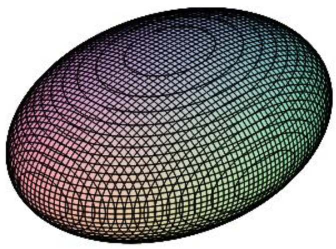
Figure 5.6: Relationship between the ratios of the axes $\frac{a_3}{a_1}$ and $\frac{a_2}{a_1}$ for the Jacobi ellipsoids given by (5.20). Applying equation (5.19) to this relationship provides the illustration of the angular velocity for the Jacobi ellipsoids given in figure 5.3.



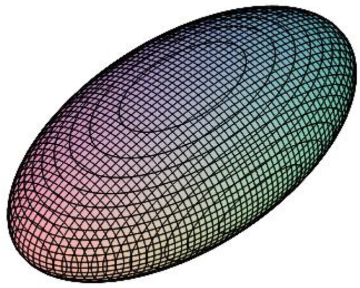
(a) Bifurcation spheroid $\Omega = 0.61174$



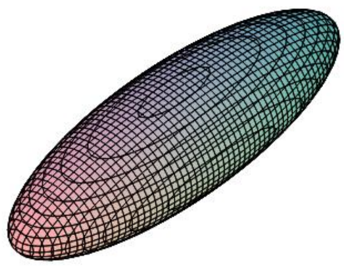
(b) $\Omega = 0.610$



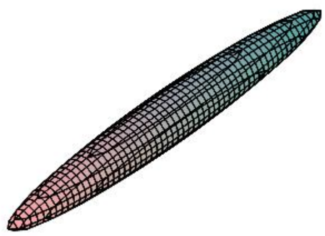
(c) $\Omega = 0.599$



(d) $\Omega = 0.562$



(e) $\Omega = 0.477$



(f) $\Omega = 0.280$

Figure 5.7: Jacobi Ellipsoid sequence.5.7a is equivalent to 5.1c

Chapter 6

Concluding Remarks

We have explored possible ellipsoidal equilibrium configurations to the problem of a uniformly rotating homogeneous body and found two possibilities:

- The Maclaurin spheroids.
- The Jacobi ellipsoids.

where the Maclaurin spheroids constitute a sequence of oblate spheroids that rotate about their polar axes. This sequence of spheroids is uniquely defined by a specific angular velocity for each specific member of the sequence as given by Maclaurin's formula (5.16). We observe how the angular velocity varies across this sequence of spheroids in Figures 5.3 and 5.2, clearly the angular velocity of the sequence increases to a maximum and then decreases to zero as the sequence becomes increasingly flattened at the poles. This sequence of spheroids is also evident from Figure 5.1 where we may clearly observe this flattening along the sequence.

The Jacobi ellipsoids constitute a sequence of ellipsoids with three unequal semi-axes that also rotate about their polar axes. This sequence of ellipsoids is uniquely defined by a particular relationship between the ratios of their semi-axes (how a_2/a_1 is related to a_3/a_1) as given by equation (5.20) and is observed in Figure 5.6. Each member of the Jacobi sequence is further defined by its angular velocity as given by equation (5.19). We observe how the angular velocity varies across this sequence of ellipsoids in Figures 5.3 and 5.2, clearly the angular velocity of the sequence decreases from a maximum as the sequence becomes increasingly elongated at the equator and flattened at the poles. This sequence of ellipsoids is also evident from Figure 5.7 where we may clearly observe this elongation and flattening as we proceed along the sequence.

Interestingly the Maclaurin spheroids and the Jacobi ellipsoids are uniquely related to each other as we observe that the Jacobi sequence bifurcates from the Maclaurin sequence. This occurs for a specific angular rotation and axis ratio a_3/a_1 when $a_1 = a_2$ as given by

$$\frac{a_2}{a_1} = 1 \quad and \quad \frac{a_3}{a_1} = 0.582 \quad and \quad \frac{\Omega^2}{\pi G \rho} = 0.374$$

and evidently the first member of the Jacobi sequence is a spheroid which also belongs to the Maclaurin sequence. We observe this bifurcation in Figures 5.5, 5.4, 5.3 and 5.2.

It is natural to ask the question of whether there are other possible equilibrium configurations. The answer is that there are other possible ellipsoidal equilibrium configurations if we take into account internal motions of a *uniform vorticity* in addition to uniform rotation. There exists *Dedekind ellipsoids* which maintain ellipsoidal equilibrium purely from internal motions and these are a further subset of equilibria known as *Riemann ellipsoids* (Chandrasekhar [5]). These equilibria and the formulation under the addition of uniform vorticity are beyond the scope of this report and could be the object of further study.

It would also be worth investigating the stability of possible equilibrium configurations and it would be beneficial to know what equilibria maintain their form under small disturbances or departures from their flow and to see whether this theory is in agreement with astronomical observations. The books by Lyttleton [4] and Chandrasekhar[5] stated in the bibliography provide detailed accounts related to stability analysis of such equilibria.

It is important to note that astronomical bodies do not generally have a uniform distribution of mass which we have assumed throughout this report. We have also made the assumption that astronomical bodies can be approximated as a fluid and whilst many rotating bodies are fluidic in nature, such as Saturn and Jupiter, we find that it is not appropriate to make this approximation for a large number of astronomical configurations. Therefore the theory outlined here should only be taken as a first approximation or starting point for an investigation into the possible figures of a rotating distribution of mass.

Bibliography

[1] I. Todhunter, *A History of The Mathamatical Theories of Attraction and The Figure of The Earth*. Macmillan and Company, 1873. [Republished by Dover Publications, Inc. in 1962]

[2] E.J. Routh, *A Treatise on Analytical Statics*. Cambridge University Press, 2nd Volume, 1892, reprinted 1922.

[3] O.D. Kellog, *Foundations of Potential Theory*. Berlin Verlag Von Julius Springer, 1929.

[4] R.A. Lyttleton, *The Stability of Rotating Liquid Masses*. Cambridge University Press, 1953.

[5] S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium*. Yale University Press, 1st Edition, 1969.

[6] J. Binney and S. Tremaine, *Galactic Dynamics*. Princeton University Press, 1st Edition, 1987.

Appendix A

Maple Code

What follows is the code used within Maple 13 to generate the illustrations contained in Chapter 5.

The following code was used to define the incomplete elliptic integrals of the two kinds E and F as well as the integrals A_i in terms of E and F as determined in Chapter 4.

```
E1:=EllipticE(sin(arccos(alpha3)),sqrt((1-alpha2^2)/(1-alpha3^2))):

F1:=EllipticF(sin(arccos(alpha3)),sqrt((1-alpha2^2)/(1-alpha3^2))):

A1:=(2*alpha2*alpha3*(1-alpha3^2))/((sin(arccos(alpha3)))^3*(1-alpha2^2))*(F1-E1):

A2:=(2*alpha2*alpha3*(1-alpha3^2)^2)/((1-alpha2^2)*(alpha2^2-alpha3^2)
    *(sin(arccos(alpha3)))^3)*(E1-F1*(alpha2^2-alpha3^2)/(1-alpha3^2)
    -sin(arccos(alpha3))*(1-alpha2^2)*alpha3/(alpha2*(1-alpha3^2))):

A3:=(2*alpha2*alpha3*(1-alpha3^2)*(alpha2/alpha3*sin(arccos(alpha3))-E1))
    /((sin(arccos(alpha3)))^3*(alpha2^2-alpha3^2)):
```

We may verify that these are correct by checking that

$$A_1 + A_2 + A_3 = 2$$

using `simplify(A1+A2+A3)` we see that this is indeed the case. Furthermore, the same integrals above were required to be defined in a similar way:

```
E2:=(alpha2,alpha3)->EllipticE(sin(arccos(alpha3)),
    sqrt((1-alpha2^2)/(1-alpha3^2))):

F2:=(alpha2,alpha3)->EllipticF(sin(arccos(alpha3)),
    sqrt((1-alpha2^2)/(1-alpha3^2))):

A12:=(alpha2,alpha3)->(2*alpha2*alpha3*(1-alpha3^2))
    /((sin(arccos(alpha3)))^3*(1-alpha2^2))*(F2(alpha2,alpha3)
        -E2(alpha2,alpha3)):

A22:=(alpha2,alpha3)->(2*alpha2*alpha3*(1-alpha3^2)^2)/((1-alpha2^2)
    *(alpha2^2-alpha3^2)*(sin(arccos(alpha3)))^3)*(E2(alpha2,alpha3)
    -F2(alpha2,alpha3)*(alpha2^2-alpha3^2)/(1-alpha3^2)
    -sin(arccos(alpha3))*(1-alpha2^2)*alpha3/(alpha2*(1-alpha3^2))):
```

```
A32:=(alpha2,alpha3)->(2*alpha2*alpha3*(1-alpha3^2)
    *(alpha2/alpha3*sin(arccos(alpha3))-E2(alpha2,alpha3)))
    /((sin(arccos(alpha3)))^3*(alpha2^2-alpha3^2)):
```

and then the following function was defined

```
OM := (alpha2, alpha3) ->
    (2*A12(alpha2, alpha3)-2*alpha2^2*A22(alpha2, alpha3))/(1-alpha2^2);
```

and used in the following "do" loop

```
AL3 := []; AL2 := []; OMEG := []; i := 1; L := []; ECC1 := [];
for alpha2 from 0.99 to 0.5e-1 by -0.01 do
 AL2 := [op(AL2), alpha2];
 AL3 := [op(AL3), fsolve(sqrt(alpha2^2*(A1-A2)/((alpha2^2-1)*A3))
         = alpha3, alpha3)];
 OMEG := [op(OMEG), OM(AL2[i], AL3[i])];
 L := [op(L), (1/10)*sqrt(3)*(1+AL2[i]^2)*abs(sqrt(OMEG[i]))
         /(AL2[i]*AL3[i])^(2/3)];
 ECC1 := [op(ECC1), abs(sqrt(1-AL3[i]^2))]; i := i+1
end do:
```

This provided all the necessary data to plot the Jacobi sequence in figures 5.3, 5.2, 5.5 and 5.4, which were plotted using the `pointplot3d` Maple procedure.

The following code was used to numerically construct the required data for illustrating the Maclaurin sequence in figures 5.3, 5.2, 5.5 and 5.4.

```
X := []; Y := []; Z := []; i := 1;LM := []; ECC := [];
M1:=(alpha3)->alpha3*arcsin(sqrt(1-alpha3^2))/
     (1-alpha3^2)^(3/2)-alpha3^2/(1-alpha3^2):
M3:=(alpha3)->2/(1-alpha3^2)-2*alpha3
     *arcsin(sqrt(1-alpha3^2))/(1-alpha3^2)^(3/2)
OZ := (alpha3) -> 2*M1(alpha3)-2*alpha3^2*M3(alpha3):
for alpha3 from 0.99 to 0.05 by -0.01 do
 X := [op(X), 1];Y := [op(Y), alpha3]; Z := [op(Z), OZ(Y[i])];
 LM := [op(LM), (1/5)*sqrt(3)*(1/alpha3)^(2/3)*abs(sqrt(Z[i]))];
 ECC := [op(ECC), abs(sqrt(1-Y[i]^2))]; i := i+1
end do
```

The following code was used to plot figure 5.6

```
implicitplot(alpha2^2*(A2-A1)/(1-alpha2^2) = alpha3^2*A3,alpha2 = 0 .. 1,
             alpha3 = 0 .. 1,numpoints = 1000)
```

Figures 5.1 and 5.7 were generated using the `implicplot3d` procedure.