

# Introduction to Generating Functions for Probability

Generating functions are a transform which can be applied to any sequence of numbers. The sequence is indexed by natural numbers. Thus generating functions inherently have a discrete domain. The Wikipedia article has a fair amount of information pertaining to the various uses of generating functions, these notes are limited to some rudimentary uses in probability theory.

We begin by stating the general form for any generating function:

$$G_X(s) = \mathbb{E}(s^X) = \sum_{n=0}^{\infty} s^n \mathbb{P}(X = n) \quad (1)$$

It should be clear from this that with sufficient algebra we can produce the generating function for any known distribution. To demonstrate this we compute the generating functions for a number of common distributions below:

## 1 Generating Functions for a few Distributions

### Bernoulli

A Bernoulli random variable with  $\mathbb{P}(X = 1) = p$  and  $\mathbb{P}(X = 0) = 1 - p = q$

$$G_X(s) = \mathbb{E}(s^X) \quad (2)$$

$$= \sum_{n=0}^{\infty} s^n \mathbb{P}(X = n) \quad (3)$$

$$= s^0 \mathbb{P}(X = 0) + s^1 \mathbb{P}(X = 1) \quad (4)$$

$$= 1 \cdot q + s \cdot p \quad (5)$$

$$= sp + q \quad (6)$$

### Geometric

A geometrically distributed random variable with parameter  $q$ , such that  $\mathbb{P}(X = n) = q^n(1 - q)$  for  $n = 0, 1, 2, \dots$

$$G_X(s) = \mathbb{E}(s^X) \quad (7)$$

$$= \sum_{n=0}^{\infty} s^n \mathbb{P}(X = n) \quad (8)$$

$$= \sum_{n=0}^{\infty} s^n q^n (1 - q) \quad (9)$$

$$= (1 - q) \sum_{n=0}^{\infty} (sq)^n \quad (10)$$

$$= (1 - q) \frac{1}{1 - sq} \quad (11)$$

### Poisson

A Poisson distributed random variable with parameter  $\lambda$  recall, a Poisson random variable has a PDF

$$\mathbb{P}(X = n) = e^{-\lambda} \frac{\lambda^n}{n!}$$

$$G_X(s) = \mathbb{E}(s^X) \tag{12}$$

$$= \sum_{n=0}^{\infty} s^n \mathbb{P}(X = n) \tag{13}$$

$$= \sum_{n=0}^{\infty} s^n e^{-\lambda} \frac{\lambda^n}{n!} \tag{14}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} s^n \frac{\lambda^n}{n!} \tag{15}$$

$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda s)^n}{n!} \tag{16}$$

$$= e^{-\lambda} e^{\lambda s} \tag{17}$$

Maclaurin series

$$= e^{\lambda(s-1)} \tag{18}$$

## 2 A few Properties of Generating Functions

In this section we prove a few useful properties of generating functions.

First we prove that  $\mathbb{E}(X) = G'_X(1)$

$$G_X(s) = \sum_{n=0}^{\infty} s^n \mathbb{P}(X = n) \tag{19}$$

$$G'_X(s) = \frac{d}{ds} \sum_{n=0}^{\infty} s^n \mathbb{P}(X = n) \tag{20}$$

$$= \sum_{n=0}^{\infty} n s^{n-1} \mathbb{P}(X = n) \tag{21}$$

$$G'_X(1) = \sum_{n=0}^{\infty} n 1^{n-1} \mathbb{P}(X = n) \tag{22}$$

Assume  $X$  and  $Y$  are independent random variables, prove  $G_{X+Y} = G_X G_Y$

$$G_{X+Y}(s) = \mathbb{E}(s^{X+Y}) \tag{23}$$

$$= \mathbb{E}(s^X) \mathbb{E}(s^Y) \tag{24}$$

$$= G_X(s) G_Y(s) \tag{25}$$

Find the generating function for  $S_n = X_1 + X_2 + \dots + X_n$  where  $X_i$ , for  $i = 1, 2, 3, \dots$  are iid Bernoulli random variables.

$$G_{S_n} = \mathbb{E}(s^{S_n}) \quad (26)$$

$$= \mathbb{E}(s^{X_1+X_2+\dots+X_n}) \quad (27)$$

$$= \mathbb{E}(s^{X_1})\mathbb{E}(s^{X_2}) \dots \mathbb{E}(s^{X_n}) \quad (28)$$

$$= \mathbb{E}(s^X)^n = (ps + q)^n \text{the generating function for a Bernoulli random variable} \quad (29)$$

Prove that if  $X_1, X_2, \dots, X_N$  are iid, and  $N$  is independent of the  $X_i$  with generating  $G_N$ , then  $S_N = X_1 + X_2 + \dots + X_N$  has a generating function  $G_{S_N}(s) = G_N(G_X(s))$

$$G_{S_N}(s) = \mathbb{E}(s^{S_N}) \quad (30)$$

$$= \mathbb{E}(s^{X_1+X_2+\dots+X_N} | N = n) \quad (31)$$

$$= \sum \mathbb{E}(s^{X_1})\mathbb{E}(s^{X_2}) \dots \mathbb{E}(s^{X_N}) \cdot \mathbb{P}(N = n) \quad (32)$$

$$= \sum G_X(s)^N \mathbb{P}(N = n) \quad (33)$$

$$= G_N(G_X(s)) \quad (34)$$

### 3 Redoing HW 2 Problem 5 with Generating Functions

Suppose the number  $N$  of cars arriving during a given time period at a toll booth has a Poisson distribution with parameter  $\lambda$ . Each car has a probability  $p$  of being in a car pool. Let  $M$  be the number of car-pool cars that arrive in the given period. Show that  $M$  also has a Poisson distribution, with parameter  $p\lambda$ .

First we need to note that what we are trying to do is prove that  $G_M(s) = e^{\lambda p(s-1)}$ . Observe that what we did was show that we are trying to prove that  $M$  has a generating function of Poisson form.

$$G_M(s) = G_N(G_X(s)) \quad \text{X is Bernoulli with probability p} \quad (35)$$

$$= \sum_n G_X(s)^n \mathbb{P}(N = n) \quad (36)$$

$$= \sum_n (q + ps)^n \mathbb{P}(N = n) \quad (37)$$

$$= G_N(q + ps) \quad (38)$$

$$= e^{-\lambda + \lambda(q+ps)} \quad \text{Poisson generating function} \quad (39)$$

$$= e^{\lambda p(s-1)} \quad (40)$$