and thus

$$\pi_i = \pi_0 r_i \tag{8.22}$$

where

$$r_i = \Pi_{k=1}^{i} \frac{u_{k-1}}{d_k} \tag{8.23}$$

where $r_i = 0$ for $i > m$ if the chain has no states past m.

Then since the π_i must sum to 1, we have that

$$\pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty} r_i} \tag{8.24}$$

and the other π_i are then found via (8.22).

Note that the chain might be finite, i.e. have $u_i = 0$ for some i. In that case it is still a birth/death chain, and the formulas above for π still apply.

8.7 Cell Communications Model

Let's consider a more modern example of this sort, involving cellular phone systems. (This is an extension of the example treated in K.S. Trivedi, *Probability and Statistics, with Reliability and Computer Science Applications* (second edition), Wiley, 2002, Sec. 8.2.3.2, which is in turn is based on two papers in the *IEEE Transactions on Vehicular Technology.*)

We consider one particular cell in the system. Mobile phone users drift in and out of the cell as they move around the city. A call can either be a **new call**, i.e. a call which someone has just dialed, or a **handoff call**, i.e. a call which had already been in progress in a neighboring cell but now has moved to this cell.

Each call in a cell needs a **channel**.[2] There are n channels available in the cell. We wish to give handoff calls priority over new calls.[3] This is accomplished as follows.

The system always reserves g channels for handoff calls. When a request for a new call (i.e. a non-handoff call) arrives, the system looks at X_t, the current number of calls in the cell. If that

[2]This could be a certain frequency or a certain time slot position.

[3]We would rather give the caller of a new call a polite rejection message, e.g. "No lines available at this time, than suddenly terminate an existing conversation.

number is less than n-g, so that there are more than g idle channels available, the new call is accepted; otherwise it is rejected.

We assume that new calls originate from within the cells according to a Poisson process with rate λ_1, while handoff calls drift in from neighboring cells at rate λ_2. Meanwhile, call durations are exponential with rate μ_1, while the time that a call remains within the cell is exponential with rate μ_2.

8.7.1 Stationary Distribution

We again have a birth/death process, though a bit more complicated than our earlier ones. Let $\lambda = \lambda_1 + \lambda_2$ and $\mu = \mu_1 + \mu_2$. Then here is a sample balance equation, focused on transitions into (left-hand side in the equation) and out of (right-hand side) state 1:

$$\pi_0 \lambda + \pi_2 2\mu = \pi_1(\lambda + \mu) \tag{8.25}$$

Here's why: How can we enter state 1? Well, we could do so from state 0, where there are no calls; this occurs if we get a new call (rate λ_1) or a handoff call (rate λ_2. In state 2, we enter state 1 if one of the two calls ends (rate μ_1) or one of the two calls leaves the cell (rate μ_2). The same kind of reasoning shows that we leave state 1 at rate $\lambda + \mu$.

As another example, here is the equation for state n-g:

$$\pi_{n-g}[\lambda_2 + (n-g)\mu] = \pi_{n-g+1} \cdot (n-g+1)\mu + \pi_{n-g-1}\lambda \tag{8.26}$$

Note the term λ_2 in (8.26), rather than λ as in (8.25).

Using our birth/death formula for the π_i, we find that

$$\pi_k = \begin{cases} \pi_0 \frac{A^k}{k!}, & \text{k} \leq \text{n-g} \\ \pi_0 \frac{A^{n-g}}{k!} A_1^{k-(n-g)}, & \text{k} \geq \text{n-g} \end{cases} \tag{8.27}$$

where $A = \lambda/\mu$, $A_1 = \lambda_2/\mu$ and

$$\pi_0 = \left[\sum_{k=0}^{n-g-1} \frac{A^k}{k!} + \sum_{k=n-g}^{n} \frac{A^{n-g}}{k!} A_1^{k-(n-g)} \right]^{-1} \tag{8.28}$$

8.7.2 Going Beyond Finding the π

One can calculate a number of interesting quantities from the π_i:

- The probability of a handoff call being rejected is π_n.
- The probability of a new call being dropped is

$$\sum_{k=n-g}^{n} \pi_k \tag{8.29}$$

- Since the per-channel utilization in state i is i/n, the overall long-run per-channel utilization is

$$\sum_{i=0}^{n} \pi_i \frac{i}{n} \tag{8.30}$$

- The long-run proportion of accepted calls which are handoff calls is the rate at which handoff calls are accepted, divided by the rate at which calls are accepted:

$$\frac{\lambda_2 \sum_{i=0}^{n-1} \pi_i}{\lambda_1 \sum_{i=0}^{n-g-1} \pi_i + \lambda_2 \sum_{i=0}^{n-1} \pi_i} \tag{8.31}$$