# Lab Assignment 1 Computational Finance

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**Lab assignment**

**15 Februari 2023**

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# Introduction

In this assignment we have explored different methods to simulate stock prices and to calculate option values. Binomial tree and Euler approximation are used to simulate stock price process. Black-Scholes model is used to provide the analytical solution for option pricing as a comparison to the option price calculated by backward iterating from the payoff at maturity. We have also explored delta hedging for European options where some meaningful result is derived after comparing the PnL of the delta hedging replicating portfolio.

# Methods

In order to complete this assignment and write a report, we have applied several mathematical and computational methods and techniques.

## Part I

For the theoretical problems of part I, we have applied basic calculus (i.e. limits and continuity) as well as arbitrage strategies. An arbitrage strategy is an investment strategy such that the investor has a probability of 1 of not losing money, and a positive probability of making a positive amount of money. Usually we will consider strategies in which an investor makes money *almost surely*, which means that they have a strategy which makes money with probability 1.

## Part II

First, we write and execute a binomial tree program through Python to evaluate European call option prices and examine the differences between the first approximate method and the analytical Black Scholes model with relation to the estimation of the values and some parameters such as the one used for hedging the portfolio of the underlying asset. Then, we analyze how the approach for increasing the number of steps in the binomial tree converges and, with this, how computationally complex this technique could be. Lastly, we code the American both put and call options again in Python and test the outcomes after performing experiments with various volatility settings like before.

## Part III

For the first two problems of this part, we apply multivariable calculus (i.e. partial derivatives) as well as the fair value of $c_t$ as was derived in the given appendix. We apply the put-call parity which we have developed in part I. As for the simulation, we first simulate the stock price process using the Euler method, then we calculate the theoretical delta for each day. Then we choose to hedge daily or weekly. We then constructed the replicating portfolio of a short position on a call option as buying delta amount of stock and keeping the rest in the bank to gain interest. At each timestep, you consider the time-value of the previous balance, then rebalance your portfolio with the newly calculated delta, then recalulate the balance accordingly. This is later validated against the option value along the timeline. The PnL of the strategy can then be calculated at time maturity (as it is a European option, you can only have realized PnL at the end).

# Results

## Part I

In this part we will consider theoretical problems, having to do with arbitrage and financial contracts.

### 1

First, consider an interest factor $r > 0$ over a period of length 1 (for an arbitrary unit). In each period, there are $n \in \mathbb{N}$ moments of payout (here $0 \notin \mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). Then for each $n$, if one invests $C > 0$ units of money into the bank, the worth of the investment at time 1 equals

$$C \cdot \underbrace{\left(1 + r \cdot \frac{1}{n}\right) \dots \left(1 + r \cdot \frac{1}{n}\right)}_{n \text{ times}} = C\left(1 + \frac{r}{n}\right)^n.$$

Therefore, if we keep our $C$ invested up until time $\Delta t > 0$ we have an amount

$$A_{n,\Delta t}(C) = C \cdot \underbrace{\left(1 + r \cdot \frac{1}{n}\right) \dots \left(1 + r \cdot \frac{1}{n}\right)}_{n\Delta t \text{ times}} = C\left(1 + \frac{r}{n}\right)^{n\Delta t}.$$

Now observe that going from (small) intercompound periods, which translates to large $n$, from continuous compounding is equivalent to taking the limit of $A_{n,\Delta t}$ as $n$ goes to positive infinity. Note that we can write

$$A_{n,\Delta t}(C) = \left[\left(1 + \frac{r}{n}\right)^n\right]^{\Delta t},$$

and that $y \mapsto y^{\Delta t}$, $y \mapsto C \cdot y$ are continuous functions on $\{y > 0\}$. Therefore we can interchange function and limit to find

$$A_{\Delta t} \coloneqq \lim_{n\to\infty} A_{n,\Delta t} = C\left[\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n\right]^{\Delta t} = C\left[e^r\right]^{\Delta t} = Ce^{r\Delta t},$$

which is the quantity we were after.

### 2

Next, consider a coupon bond with a principal value of 50K EURO along with quarterly payments of 300 EURO. The maturity is $T = 2$, with units in years. The fair value $V_0$ of this contract is such that, including discounting of money, we have $V_0$ equalling the value of the principal value along with the eight quarterly payments, all discounted to time zero. There is a risk-free constant compounding interest of 1.5%. Therefore, the one-year discounting equals $e^{-0.0015}$ and the quarterly discounting is $e^{-0.0015/4}$. Thus we determine the fair price to be

$$\begin{aligned} V_0 &= 50.000 \cdot e^{-0.0015\cdot 2} + 300 \cdot e^{-0.0015/4} + 300 \cdot e^{(-0.0015/4)\cdot 2} + \dots + 300 \cdot e^{(-0.0015/4)\cdot 8} \\ &= 50.000 \cdot e^{-0.003} + 300 \sum_{k=1}^{8} e^{-0.0015k/4} \\ &\approx 50.882, 20. \end{aligned}$$

### 3

Third, we consider a *forward* contract with strike price $K > 0$ and maturity $T > 0$. We specify the strike price to be equal to the forward price $F$. At time zero, the fair price to enter this contract is denoted by $F_0$ We will show using arbitrage strategies that the equality

$$F_0 = S_0 \exp\{rT\}.$$

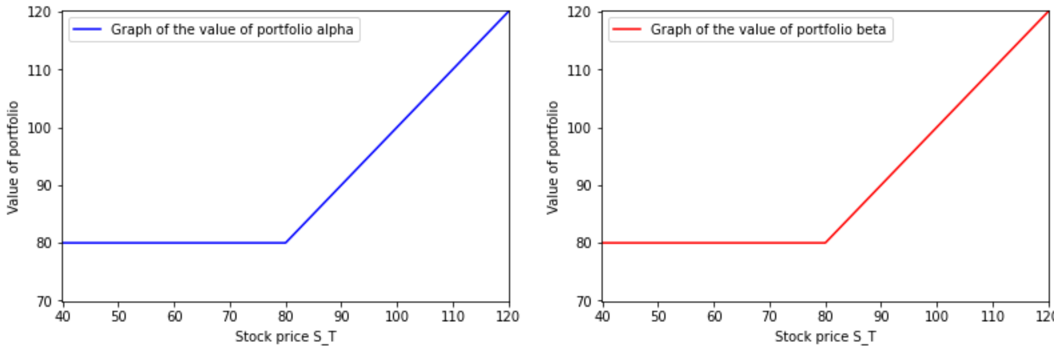


Figure 1: Graphs depicting the value of portfolio's $\alpha_T$ (blue) and $\beta_T$ (red) as a function of $S_T$ for a strike price $K = 80$. Here we consider values of $S_T$ between 40 and 120. As one can see, the payoffs are equal.

must hold, where $S_0$ is the market value of the stock $S_t$ at time zero and $r$ is the risk-free interest rate.
First, assume that we have $F_0 > S_0 \exp\{rT\}$. We construct an arbitrage strategy for the *short* position holder (i.e. the person who ends up selling the stock at maturity). The short position starts with borrowing $S_0$ from the bank, with which they buy a stock. At maturity, the short receives $K$ from the buyer, which equals $F_0$ by assumption. Then they pay back $S_0 \exp\{rT\}$ to the bank, which is due to the risk-free interest rate we saw in the first problem. This leaves the short position with $F - S_0 \exp\{rT\}$, which is positive by assumption. This is a strategy in which the short makes money with probability one, and is therefore an arbitrage strategy.

Reversely, assume $F_0 < S_0 \exp\{rT\}$. We construct an arbitrage strategy for the long position. Without loss of generality, assume that the long already has one stock to their name. We may do this since if there exists a person with a stock, then this is indeed an arbitrage strategy for them. Since there must exist a person with a stock, we may assume as such. The long position sells their stock for $S_0$ and invest this into the risk-free market. At maturity, this will be worth $S_0 \exp\{rT\}$. Now they will pay $F_0$ to purchase a stock. In the end, the long starts with a stock, and ends with a stock *in addition to* having $S_0 \exp\{rT\} - F_0 > 0$ in money. This is thus an arbitrage strategy for the long position.

**4**

We now consider two different portfolio's: Portfolio $\alpha_t$ has a European call option for a stock $S_t, t \in [0, T]$ combined with an investment of $K \exp\{-r(T-t)\}$ in the risk-free market at time $t$. We denote the value of the European call at time $t \in [0, T]$ by $C_t$. The second portfolio, call it $\beta$, is given by a European put option with value process $P_t$ and one share of the stock $S_t$.

The value at maturity of $C_t$ (i.e. $C_T$) is equal to $\max\{S_T - K, 0\}$, by an arbitrage argument. At time $T$, the money invested will be $K \exp\{-rT\} \exp\{rT\} = K$, such that $\alpha_T = \max\{S_T - K, 0\} + K = \max\{S_T, K\}$. On the other hand, portfolio $\beta_T$ at time $T$ is worth $\max\{K - S_T, 0\}$ from the put option, together with $S_T$ from the stock value. Therefore $\beta_T = \max\{K - S_T, 0\} + S_T = \max\{K, S_T\}$. Note $\alpha_T = \beta_T$. In Figure 1 we see a plot of the value of portfolio $\alpha_T$ (which is equal to the value of portfolio $\beta_T$ as a function of the stock price $S_T$ for given strike price $K$.

**5**

Consider the put-call parity

$$c_t + K\exp\{-r(T-t)\} = p_t + S_t, \tag{1}$$

where $t \in [0, T]$. In the previous exercise we saw that this parity holds for $t = T$. We now show, via arbitrage arguments, that it must hold for all other values of $t$ as well. To this end, assume that there exists $t^* \in [0, T)$ such that

$$c_{t^*} + K\exp\{-r(T-t^*)\} > p_{t^*} + S_{t^*}.$$

Recall the notation $\alpha_t$ and $\beta_t$ for the portfolio's consisting of the left-hand side (LHS) and right-hand side (RHS) of Equation (1). The person holding portfolio $\alpha_{t^*}$ will now sell their portfolio and purchase $\beta_{t^*}$. The amount of money left (by the assumed inequality this quantity is positive), call it $M > 0$, will be invested in the risk-free market. At maturity this person will have $\max\{S_T, K\} + M\exp\{r(T-t^*)\}$. If this person would not have traded their portfolio $\alpha_{t^*}$ they would have only ended up with $\max\{S_T, K\}$. Analogously, if we have

$$c_{t^*} + K\exp\{-r(T-t^*)\} < p_{t^*} + S_{t^*}$$

the person holding portfolio $\beta_{t^*}$ will sell their portfolio, purchase portfolio $\alpha_{t^*}$ and put the remainder in the bank, which implies that they end up with $\max\{S_T, K\} + N\exp\{r(T-t^*)\}$ for some $N > 0$. This is again an arbitrage strategy.

Since $t^* \in [0, T)$ was arbitrary, and by question 4 we already have $c_T + K = p_T + S_T$ it follows that for all $t \in [0, T]$

$$c_t + K\exp\{-r(T-t)\} = p_t + S_t,$$

and we are done.

## Part II

**1**

Having a European call option on a non-dividend-paying stock with a maturity time of one year and strike price of €99 means that the buyer has the right and not the obligation to exercise the underlying security (stock) with this price at the expiration of it. Also, assuming that the interest rate is 6%, today's stock price is €50, and the volatility, the way that up and down movements of the stock price in the money market are measured, is 20%. In order to compute the value of the given option, we will construct the binomial tree method in Python with 50 steps during one year. For each step, the up and down factors are computed, and we compute the risk free rate accordingly. Afterwards, we calculate backwards given the payoff at maturity the option value at each time-step using backward iteration. As a result, under the no-arbitrage principle the fair value of option is 11.877161172520175.

**2**

By comparing the analytical Black-Scholes option value with the one obtained from the binomial tree method one could easily see that there is no such a big difference. The value of the option using the Black-Scholes formula is 11.5442, almost the same as before without changing the given volatility. Furthermore, as the volatility decreases to zero the absolute error between the exact Black-Scholes solution and the approximate one from the binomial tree becomes smaller and smaller. In Figure 2 we see how the difference between the theoretical Black-Scholes option value and the binomial tree option value behaves in relation to the volatility. We have drawn graphs for different values of $n$, where $n$ of course denotes the amount of steps in the tree model. Clearly as $\sigma \downarrow 0$ we have a the stock price converging to a constant, hence the value $\max S - K, 0$ is constant. In both the BT and BS model, this implies a constant value of $c_{BT}$

and $c_{BS}$. Surprisingly, the same holds if we take volatility to infinity as can be seen in the second panel of Figure 2. Note that this convergence happens independent of the value of $n$. In the case of $\sigma \downarrow 0$ we find

$$\lim_{\sigma \downarrow 0} u = \lim_{\sigma \downarrow 0} \exp\{\sigma\sqrt{\Delta t}\} = 1,$$

hence for all $n \in \mathbb{N}$ we find using question (3) from this Part

$$\begin{aligned}
\lim_{\sigma \downarrow 0} c_{BT} &= \lim_{\sigma \downarrow 0} \exp\{-rT\} \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} \cdot \max\{S_0 u^k d^{n-k} - K, 0\} \\
&= \exp\{-rT\} \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} \cdot \max\{S_0 - K, 0\} && (u, d \to 1) \\
&= \exp\{-rT\} \max\{S_0 - K, 0\} \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} \\
&= \exp\{-rT\} \max\{S_0 - K, 0\}, && \text{(Newton's Binomial)}
\end{aligned}$$

and similarly

$$\lim_{\sigma \downarrow 0} d_1 = \lim_{\sigma \downarrow 0} \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

which goes to either positive or negative infinity depending on the sign of $\log(S_0/K)$ since the denominator will go to zero while the numerator goes to something that is zero if and only if $\log(S_0/K) = -rT$. As a result, since in this assignment we have $\log(S_0/K) + rT = \log(100/99) + 0.06 > 0$ it follows

$$\lim_{\sigma \downarrow 0} c_{BS} = \lim_{\sigma \downarrow 0} S_0 \mathcal{N}(d_1) - K \exp\{-rT\} \mathcal{N}(d_2) = S_0 - K \exp\{-rT\}.$$

Hence under our values of $S, K$ and $r$ we indeed find $c_{BT} - c_{BS} \to 0$ as the volatility vanishes.

If we instead consider the limit as $\sigma \to \infty$, for the Black-Scholes model we find

$$\lim_{\sigma \to \infty} d_1 = \lim_{\sigma \to \infty} \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \lim_{\sigma \to \infty} \frac{1}{2}\sigma\sqrt{T} = \infty,$$

and similarly

$$\lim_{\sigma \to \infty} d_2 = \lim_{\sigma \to \infty} -\frac{1}{2}\sigma\sqrt{T} = -\infty.$$

Thus

$$\lim_{\sigma \to \infty} c_{BS} = \lim_{\sigma \to \infty} S_0 \mathcal{N}(d_1) - K \exp\{-rT\} \mathcal{N}(d_2) = S_0.$$

Consider now the binomial tree model for $\sigma \to \infty$. We have $q = (\exp\{\sigma\sqrt{\Delta t}\} - d)/(u - d)$ which has the property that it goes to zero 'equally' as fast as $d$ when we take $\sigma \to \infty$, only there is a factor $\exp\{r\Delta t\}$ to consider for each power in the $q$-factor. In particular $q^n$ will yield an $\exp\{rT\}$ term. Hence as in addition $(1 - q) \to 1$ we find

$$\begin{aligned}
\lim_{\sigma \to \infty} c_{BT}(n) &= \lim_{\sigma \to \infty} \exp\{-rT\} \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} \max\{S_0 u^k d^{n-k} - K, 0\} \\
&= \exp\{-rT\} \sum_{k=0}^{n} \binom{n}{k} \max\{\lim_{\sigma \to \infty} \exp\{rk\Delta t\} S_0 u^k d^k d^{n-k} - K, 0\} \\
&= \exp\{-rT\} \sum_{k=0}^{n} \binom{n}{k} \max\{\lim_{\sigma \to \infty} \exp\{rk\Delta t\} S_0 d^{n-k} - K d^k, 0\}. && (ud = 1)
\end{aligned}$$

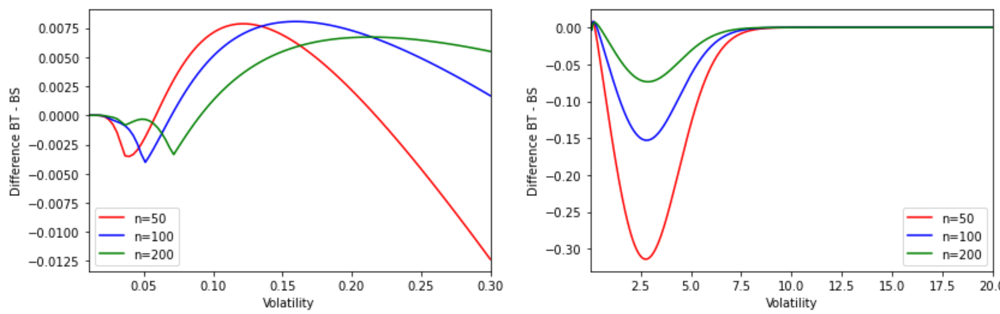


Figure 2: Graphs depicting the difference in European call option fair price between binomial tree and Black-Scholes model, $c_{BT} - c_{BS}$, as a function of volatility $\sigma$, where we take $n = 50$ (red), $n = 100$ (blue) and $n = 200$ (green).

Clearly $d^{n-k} \to 0$ if and only if $k < n$, so the only term remaining will be the $k = n$ term, at which point $d^{n-k} = 1$ and $Kd^k \to 0$, hence

$$\exp\{-rT\} \sum_{k=0}^{n} \binom{n}{k} \max\{\lim_{\sigma\to\infty} \exp\{rk\Delta t\} S_0 d^{n-k} - Kd^k, 0\} = \exp\{-rT\} \binom{n}{n} \exp\{rT\} S_0 = S_0.$$

We thus find that as $\sigma \to \infty$ we indeed find $c_{BT} - c_{BS} \to 0$.

**3**

We consider the convergence of the $n$-period binomial tree model to the Black-Scholes model. We shall see that the fair price of the European call option at time 0 in the $n$-step model will converge to the Black-Scholes theoretical value as $n \to \infty$. Note that

$$\mathbb{P}(S_n = S_0 u^k d^{n-k}) = \binom{n}{k} q^k (1-q)^{n-k},$$

where $q$ is the risk-neutral move-up probability. Hence taking maturity $T = 1$ yields

$$\begin{aligned} c_0(n) &= \exp\{-rT\} \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} \cdot \max\{S_0 u^k d^{n-k} - K, 0\} \\ &= \exp\{-rT\} \sum_{k=\gamma}^{n} \binom{n}{k} q^k (1-q)^{n-k} \cdot (S_0 u^k d^{n-k} - K), \end{aligned}$$

where $\gamma = \min\{i \in \{1, \ldots, n\} \mid i \geq \delta\}$ such that $\delta$ is the solution to $K = S_0 u^\delta d^{n-\delta}$ or equivalently

$$\delta = \frac{\log(K/S_0) - \log(d^n)}{\log(u/d)},$$

where $\delta$ is very similar to $-d_1$ with $d_1$ from the Black-Scholes model.

Now we implement the choices $u = 1/d$ and $u = \exp\{\sigma\sqrt{\Delta t}\}$, where by construction $n = (\Delta t)^{-1}$. We now find with $\hat{q} = qu \exp\{-r/n\}$

$$c_0(n) = S_0 \sum_{k=\gamma}^{n} \binom{n}{k} (\hat{q})^k (1-\hat{q})^{n-k} - K \exp\{-rT\} \sum_{k=\gamma}^{n} \binom{n}{k} q^k (1-q)^{n-k},$$

as we have to absorb the $u^k d^{n-k}$ terms in the first sum. Now as we take $n$ to infinity, note that we have in each term the CDF of the sum of i.i.d. Bernoulli variables distributed with

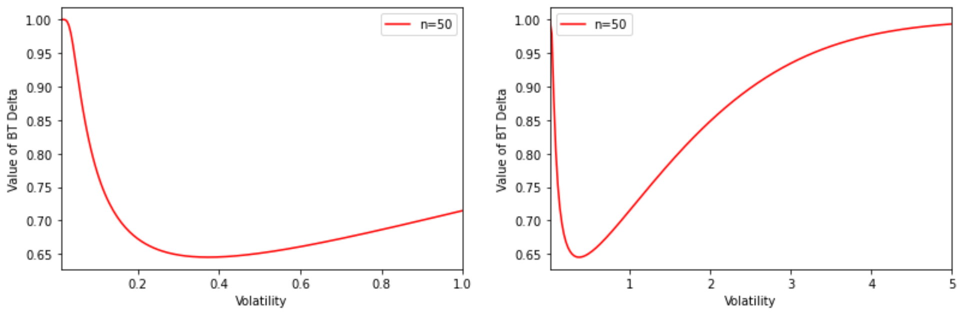


Figure 3: The behaviour of the binomial tree Delta as a function of volatility, as seen for $n = 50$ steps.

parameters $\hat{q}$ and $q$ respectively. Since they have finite second moment, we apply the central limit theorem to find

$$\sum_{k=\gamma}^{n} \binom{n}{k} (\hat{q})^k (1-\hat{q})^{n-k} \to 1 - \mathcal{N}(-d_1) = \mathcal{N}(d_1),$$

and

$$\sum_{k=\gamma}^{n} \binom{n}{k} q^k (1-q)^{n-k} \to 1 - \mathcal{N}(-d_2) = \mathcal{N}(d_2),$$

where we derive the equalities $\mathcal{N}(x) = 1 - \mathcal{N}(-x)$ in Part III. Thus we find that our binomial tree model converges to the Black-Scholes model as $n$ goes to infinity, since now

$$c_0(n) \to c_0 = S_0 \mathcal{N}(d_1) - K \exp\{-rT\} \mathcal{N}(d_2),$$

where clearly $T = 1$.

In terms of computational complexity, we have to implement a double for-loop that has a reach depending on $N$. Therefore, we have a computational complexity of $\mathcal{O}(N^2)$. Note that this is significantly better than if we were to implement a tree inversion algorithm which has complexity $\mathcal{O}(2^N)$.

**4**

The parameter $\Delta$ at time $t = 0$ reflects the number of shares the agent must purchase at the initial time in order to establish a hedge and having a European call option the latter means for the holder either to remain stable or to sell the option in the money-market. Furthermore, it depicts in parallel the exposure in a particular direction of an option position where this refers to reducing the risk associated with an investment strategy in the trading world.
We begin by considering the Delta value of the Binomial tree hedge as a function of the volatility $\sigma$. In Figure 3 we see the Delta parameter for different volatility levels. Note that, as the volatility either becomes very small of very large, the value of $\Delta$ seems to converge to 1. We explore this limiting behaviour while comparing the binomial tree Delta to the analytical Black-Scholes value, as well as give an economic interpretation to this event. The choice of $n$ does not matter when concerning Figure 3, as this only impacts the $\sqrt{\Delta t}$ term in the $\exp\{\pm\sigma\sqrt{\Delta t}\}$ element in the calculation of $\Delta$, which becomes negligible one we take $\sigma$ to either 0 or $\infty$.

We now investigate the difference in the Delta parameter for the binomial tree model, which we shall denote $\Delta_{BT,n}(\sigma)$ as a function of the number of steps $n$ as well as the volatility $\sigma$, and the Delta parameter for the Black-Scholes model. The latter one we denote by $\Delta_0(\sigma)$, again a function of $\sigma$.

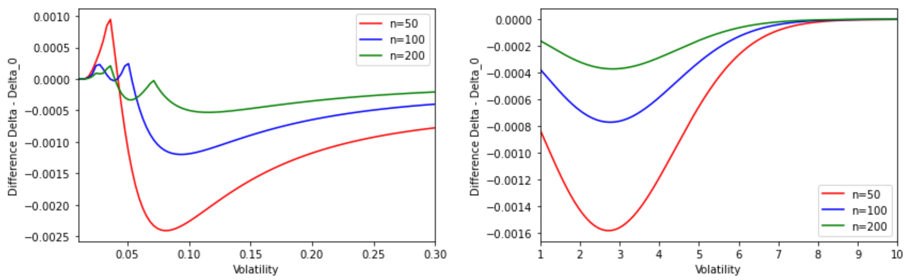


Figure 4: The behaviour of the difference between the binomial tree delta and the Black-Scholes delta as a function of the volatility $\sigma$. In both panels 1 and 2 we have repeated this proces for $n = 50$ (red), $n = 100$ (blue) and $n = 200$ (green).

Figure 4 shows us the difference between said $\Delta_{BT,n}$ and $\Delta_0$ for different volatilities. In the first panel we consider 'smaller' values of $\sigma$, which are considered more normal. The second panel shows the behaviour of the delta's when we consider a very high volatility. Note that as either $\sigma \downarrow 0$ and $\sigma \to \infty$ we seem to have $\Delta_{BT,n} - \Delta_0 \to 0$, regardless of the number of steps $n$. We can easily interpret the first limit from an economic stance, as volatility going to zero means that the underlying asset (the stock $S_t$) will remain constant. This implies that the contract price (ignoring discountation) will remain constant. Hence, in either model we should converge to a constant delta hedge. In this case we see both $\Delta_{BT,n} \to 1$ and $\Delta_0 \to 1$. We will see that this is partly because, in our assignment, we have $S_0 = 100 > 99 = K$.

Let us consider a mathematical approach to the $\sigma \downarrow 0$ limit. We have $u, d \to 1$ as $\sigma \downarrow 0$ independent of the $\sqrt{\Delta t}$ term in the exponentials for $u$ and $d$. Hence

$$\begin{aligned}
\lim_{\sigma \downarrow 0} \Delta_{BT,n} = \lim_{\sigma \downarrow 0} \frac{f_u - f_d}{S_0(u-d)} &= \lim_{\sigma \downarrow 0} \frac{\max\{S_0 u - K, 0\} - \max\{S_0 d - K, 0\}}{S_0(u-d)} \\
&= \lim_{\sigma \downarrow 0} \frac{(S_0 u - K) - (S_0 d - K)}{S_0(u-d)} && (S_0 u, S_0 d \to S_0 > K) \\
&= \lim_{\sigma \downarrow 0} \frac{S_0}{S_0} = 1.
\end{aligned}$$

Secondly, by an exercise in Part III we have $\Delta_0 = \mathcal{N}(d_1)$ where

$$d_1 = \frac{\log(S_0/K) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}.$$

As we have by assumption $S_0 > K$ $d_1$ will limit to something that is positive divided by something going to zero from above, which implies $\lim_{\sigma \downarrow 0} d_1 = \infty$. Hence

$$\Delta_0 \to \mathcal{N}(\infty) = 1,$$

and $\Delta_0, \Delta_{BT,n}$ will indeed converge to the same thing.

The intuition for growing volatility is less obvious. The mathematics, however, work in a similar way. Observe that also $\lim_{\sigma \to \infty} d_1 = \infty$, hence analogous to before $\lim_{\sigma \to \infty} \Delta_0 = 1$. For the binomial tree model, we have (slightly handwavy)

$$\begin{aligned}
\lim_{\sigma \to \infty} \Delta_{BT,n} = \lim_{\sigma \to \infty} \frac{f_u - f_d}{S_0(u-d)} &= \lim_{\sigma \to \infty} \frac{\max\{S_0 u - K, 0\} - \max\{S_0 d - K, 0\}}{S_0(u-d)} \\
&= \lim_{\sigma \to \infty} \frac{S_0 u - K - 0}{S_0(u-d)} && (S_0 d \downarrow 0 \text{ as } d \downarrow 0) \\
&= \frac{\lim_{\sigma \to \infty} (S_0 u - K)}{\lim_{\sigma \to \infty} S_0(u-d)} = 1,
\end{aligned}$$

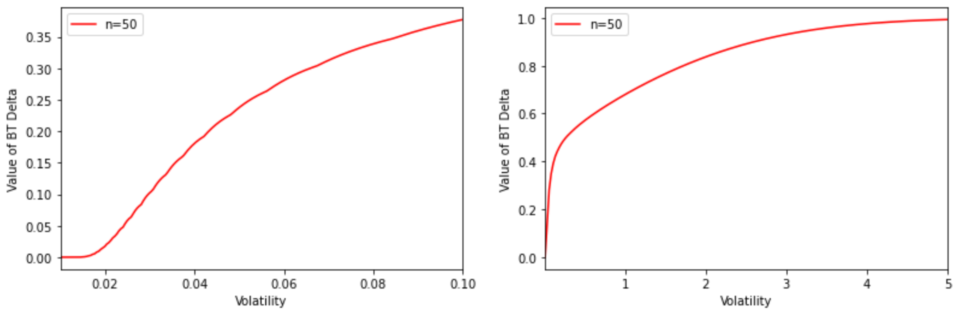


Figure 5: Value of the binomial tree Delta for different values of $\sigma$, for the choice $n = 50$. Again, the choice of $n$ is arbitrary by previous reasoning.

as clearly the $-K$ will not contribute in the limit, neither will the $d$ term. So indeed the delta's from the Black-Scholes model and the binomial tree model with converge to one. We stress that this is independent of the number of steps $n$. We can argue that, were $K < S_0$ the case, the value of Delta would drop to zero. This is because the $d_2$ term will approach $-\infty$ instead of $\infty$, meaning that $\Delta_0 \to S_0\mathcal{N}(-\infty) = 0$. We can see this result in the first panel of Figure 5, in which we have replaced the original strike price $K = 90$ with $\hat{K} = 110$. This much higher value is to account for the $rT$-term in $d_1$, such that the value of the numerator will be negative as $\sigma \downarrow 0$. We show the Delta value for the binomial tree model, which behaves in a similar fashion as the Delta for the Black-Scholes model. This is because the $\max\{S_0u - K, 0\}$ and $\max\{S_0d - K, 0\}$ terms in the derivation of $\lim_{\sigma\downarrow 0} \Delta_{BT,n}$ both *equal* zero for some small $\sigma < \varepsilon$, whereas the denominator only *approaches* zero. Again, the difference between $\Delta_0$ and $\Delta_{BT,n}$ converges to zero if either $\sigma \downarrow 0$ and $\sigma \to \infty$.

This argument does not hold if $\sigma \to \infty$, as the $\log(S_0/K)$-term in $d_1$ will disappear as $\sigma \to \infty$. Additonally, this is equivalently demonstrated in Figure 5, in the second panel.

An economic explanation is that, as the volatility of our asset increases, so could the potential payoff as stock prices could surge. On the other hand, if the stock drops, it drops hard. The seller of a call option with this underlying stock must therefore protect themselves fully against the implications of the stock value soaring off to infinity or dropping to zero by buying exactly this stock. This aligns with a choice of $\Delta_0 = 1$, and the given logic applied for the binomial tree as well.

**5**

American options have a flexible exercise period that extends through the maturity time. The decision to exercise or hold either a call or put American option at any time $t$ is solely based on the time value $t$ and the value of the underlying stock $S(t)$. One chooses that specific time during the given period of the contract with the aim of optimizing the value of the option.

The option has both time and intrinsic value. The option's intrinsic value which is a measure of an option's profitability based on the strike price and the market price of the underlying stock is always higher than 0. On the other hand, the money has also a time value i.e. the projected volatility of the underlying asset and the remaining time before the option expires. Therefore it is preferable to postpone paying the strike price by executing the option as late as feasible.

In case of an American call option the best/optimal policy for the owner is to hold the option until the maturity time and then exercise, that is $t = T = 1$. Furthermore, the value of it coincides the value of the European call option.

Before start proving it, recall that the value of the American option at time $t \leq T$, if $P(s)$

is the payoff when the stock price is given by $s$ is as follows

$$V_t = \sup_{\tau} \{P(S_\tau) e^{-r(\tau - t)} | \mathcal{F}_t\} \tag{2}$$

for every $\tau \in [t, T]$, given the filtration $(\mathcal{F}_t)$, $0 \leq t \leq T$ of the Wiener process $W_t$.

The call option has payoff $(s - K)_+$ where s and K is the stock and strike price, respectively. Note that this function is convex. Also, the times in which the owner is about to exercise the option must be stopping times w.r.t $(\mathcal{F}_t)$.

Hence, let $\tau \leq T$ be an arbitrary stopping time. The value at time zero, if the American call option were exercised at the time $\tau$ would be

$$\mathbb{E}(S_\tau - K)_+ e^{r\tau} \leq \mathbb{E}(S_\tau e^{-r\tau} - K e^{-rT})_+ \tag{3}$$

since $\tau \leq T$. From (2) one could easily understand that the discounted price process $(S_t e^{-rt})_{t \geq 0}$ is a martingale. Now, using the Jensen's inequality for the convex payoff function for time $T$ we have that

$$\mathbb{E}(S_\tau - K)_+ e^{r\tau} \leq \mathbb{E}(S_\tau e^{-rT} - K e^{-rT})_+ \tag{4}$$

As a result, the call option value at time zero to the holder intending to exercise it at time $\tau$ would not be greater than that of the European call with strike $K$ and expiration $T$. And again, by (2) it follows that the American call value is equal to European's call. Thus, the optimal policy for the holder would be to exercise the option at the expiration day. Running the code in Python,we can see that the outcome for the American put value is 11.877161172520175, the same as in European call option.

Regarding to an American put option, the best policy is not necessarily applies at maturity time. The concept in that case is to hold the option until the exercise be a rational benefit for the holder. The payoff of this contract is $\max(K - S)_+$. Let's consider for ease $t = 0$ (the same goes for any other time by using the discounting part). Take the following time as an exercise point of the American put option

$$\tau = \min\{t \geq 0 : S_t \leq K - K e^{r(t-T)}\} \tag{5}$$

Assume that $\tau \leq T$, then the option will be exercised if $S_\tau \leq K(1 - e^{r(t-T)})$ which implies that the payoff is at least $K e^{r(t-T)}$. In a matter of investing in a risk free asset, the value at the maturity time will be $K$ which is greater than any possible payoff of the European put option with strike price $K$.

Now, if we take $\tau \geq T$ then the value of the American put option will be nothing more than the European one. So, we conclude that in every case the following statement

$$V_E(t, S_t) \leq V_A(t, S_t) \tag{6}$$

holds true, that is, the payoff of the American put is greater than that of the European put. As we have derived in earlier parts of this assignment, if one takes very small volatility ($\sigma \downarrow 0$), the inequality in Equation (6) will become an equality as the stock price process converges to a constant. This derivation is completely analogous to the ones done in earlier questions of this part.

A direct implication of Equation (6) is that the put-call parity which we shall derive in Part III for the Black-Scholes model cannot hold for American options. If we assume that the put-call parity indeed holds for such options, then we would come to the conclusion that, since American and European calls are of equal value, the American and European put options would also be of equal value since we apply the put-call parity in both cases. As this is not the case (demonstrated above), the put-call parity cannot hold for American options. An intuition as to why this is the case, we have considered the put-call parity before in an environment where it was a *critical* assumption that we could only exercise at maturity as we used an arbitrage technique in Part I to show the validity of this parity.

# Part III

In this part, we consider the Black-Scholes model for the price of an underlying asset $S_t$ and contracts associated to this asset.

### 1

We consider the fair value of a European call option with underlying stock value $S_t$. The appendix tells us that the fair value of such a contract at time $t \in [0, T)$ is given by

$$c_t = S_t \mathcal{N}(d_1) - K \exp\{r\tau\} \mathcal{N}(d_2),$$

with

$$d_1 = \frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \ d_2 = d_1 - \sigma\sqrt{\tau},$$

where $\tau = T - t$. We define $\Delta_t := \frac{\partial C_t}{\partial S_t}$ and show that $\Delta_t = S_t$ for all $t \in (0, T)$. Note that $\partial_s d_1 = \partial_s d_2 = (S_t \sigma \sqrt{\tau})^{-1}$ and that $\mathcal{N}(d_1) = \mathcal{N}(d_1(S_t))$ and $\mathcal{N}(d_2) = \mathcal{N}(d_2(S_t))$ are of the form

$$f(x) = \int_{-\infty}^{h(x)} g(t) dt,$$

with $\lim_{x\to\infty} f(x) = 1$ as well as

$$f'(x) = g(h(x)) \cdot h'(x).$$

Hence we have

$$\begin{aligned}
\Delta_t &= \frac{\partial}{\partial S_t} \left[ S_t \mathcal{N}(d_1) - K e^{-r\tau} \mathcal{N}(d_2) \right] \\
&= \mathcal{N}(d_1) + S_t \partial_s \mathcal{N}(d_1) - e^{-r\tau} K \partial_s \mathcal{N}(d_2) \\
&= \mathcal{N}(d_1) + \frac{S_t}{\sqrt{2\pi}} \frac{\partial}{\partial S_t} \left[ \int_{\infty}^{d_1} e^{-\frac{1}{2}x^2} \, dx \right] - \frac{K e^{-r\tau}}{\sqrt{2\pi}} \frac{\partial}{\partial S_t} \left[ \int_{\infty}^{d_2} e^{-\frac{1}{2}x^2} \, dx \right] \\
&= \mathcal{N}(d_1) + \frac{S_t}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{\partial d_1}{\partial S_t} - \frac{K e^{-r\tau}}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{\partial d_2}{\partial S_t} \\
&= \mathcal{N}(d_1) + (\sqrt{2\pi} S_t \sigma \sqrt{\tau})^{-1} \left( S_t e^{-\frac{1}{2}d_1^2} - K e^{-r\tau} e^{-\frac{1}{2}(d_1^2 - 2\sigma\sqrt{\tau} + \sigma^2\tau)} \right) \\
&= \mathcal{N}(d_1) + (\sqrt{2\pi} S_t \sigma \sqrt{\tau})^{-1} e^{-\frac{1}{2}d_1^2} \left( S_t - K e^{\sigma\sqrt{\tau}d_1 - (r + \frac{1}{2}\sigma^2)\tau} \right).
\end{aligned}$$

Now noting that by definition $\sigma\sqrt{\tau}d_1 - (r + \frac{1}{2}\sigma^2)\tau = \log(S_t/K)$, we find

$$\begin{aligned}
\Delta_t &= \mathcal{N}(d_1) + (\sqrt{2\pi} S_t \sigma \sqrt{\tau})^{-1} e^{-\frac{1}{2}d_1^2} \left( S_t - K e^{\log(S_t/K)} \right) \\
&= \mathcal{N}(d_1) + (\sqrt{2\pi} S_t \sigma \sqrt{\tau})^{-1} e^{-\frac{1}{2}d_1^2} (S_t - S_t) \\
&= \mathcal{N}(d_1),
\end{aligned}$$

hence $\Delta_t = \mathcal{N}(d_1)$.

### 2

We use the information gathered, together with the put-call parity shown in Part I, to show the equality

$$p_t = e^{-r\tau} K \mathcal{N}(-d_2) - S_t \mathcal{N}(-d_1). \tag{7}$$

Recall that the put-call parity gave us

$$c_t + Ke^{-r\tau} = p_t + S_t,$$

hence we have

$$\begin{aligned} p_t = c_t + Ke^{-r\tau} - S_t &= S_t\mathcal{N}(d_1) - Ke^{-r\tau}\mathcal{N}(d_2) + Ke^{-r\tau} - S_t \\ &= Ke^{-r\tau}(1 - \mathcal{N}(d_2)) - S_t(1 - \mathcal{N}(d_1)). \end{aligned}$$

If we can show that $1 - \mathcal{N}(\alpha) = \mathcal{N}(-\alpha)$ for all $\alpha \in \mathbb{R}$, we are done since then

$$p_t = Ke^{-r\tau}\mathcal{N}(-d_2) - S_t\mathcal{N}(-d_1)$$

holds. Let thus $\alpha \in \mathbb{R}$ and write $f(x)$ for the standard normal density, then

$$\begin{aligned} 1 - \mathcal{N}(\alpha) = \int_{-\infty}^{\infty} f(x)\,dx - \int_{-\infty}^{\alpha} f(x)\,dx &= \int_{\alpha}^{\infty} f(x)\,dx \\ &= \int_{-\infty}^{-\alpha} f(x)\,dx \qquad \text{(Symmetry of } f \text{ around 0)} \\ &= \mathcal{N}(-\alpha). \end{aligned}$$

Thus Equation (7) indeed holds for all $t$.

**3**

We now turn to simulation. We consider a short position in a European call option. The maturity is $T = 1$, there is a yearly 6% interest rate (continuously compounded) and initial stock price $S_0 = 100$. We assume a *true* volatility of $\sigma_0 = 0.2$.
First of all, we would like to examine the difference between hedging daily and monthly. We can clearly see from the delta graph in figure 6 that they follow the same trend. As a matter of fact every week the delta value is the same as they are both computed from the theoretical value of Black-Scholes (i.e. the increments in delta are independent of previous delta values). However, we can see that with more data point available in the daily hedging, you construct a more accurate hedging simulation with less replication error.
We have also examined that delta converges either to 0 or to 1 at maturity. Intuitively this make sense, take an example where the call option is out of the money (precisely this simulation in the graph). You would not have to buy any stock to deliver at maturity, hence with a zero delta at maturity. We have mathematically derived this in Part II of the 'results' section of the report. We then move on to experiment with the situation where we have a disagreement between the implied volatility and the realized volatility . As can be seen from the graph, the replicating portfolio value with the implied volatility agrees with the theoretical option value from Black-Scholes.

As shown in previous exercises, the price of an option increases if the volatility increases. Therefore if we sell an option and delta hedge it where realized volatility is higher than expected, we expect to lose money on average. Same idea where realized volatility is lower then we expect to make money. This is illustrated with the PnL which is denoted as follows:

$$PnL = P_T - (S_T - K)^+$$

where $P_T$ is the replicating portfolio value at maturity. This is validated with 45 simulations around the central point where volatility is 20%.

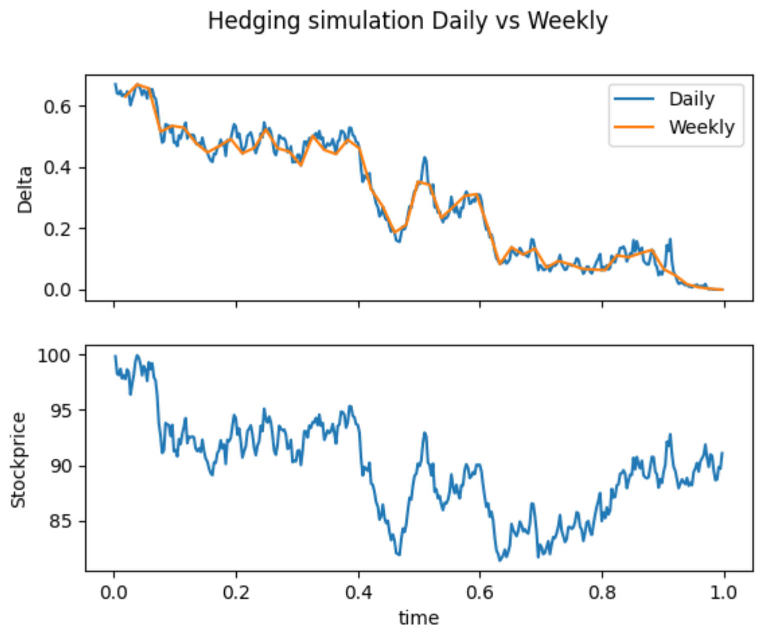


Figure 6: Hedging simulation

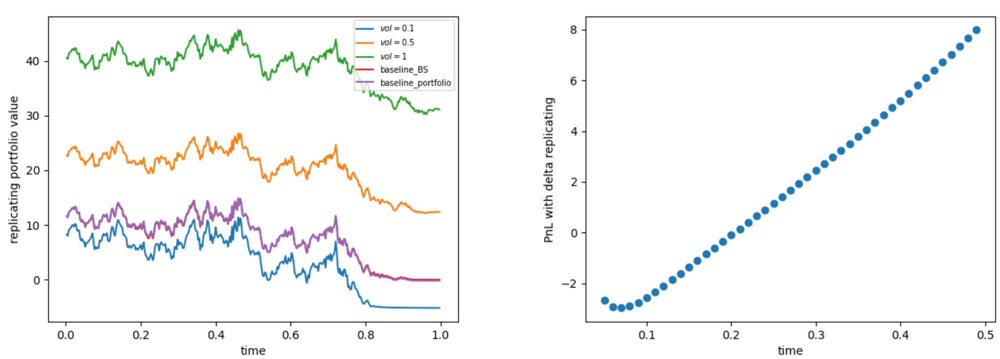


Figure 7: Replicating portfolio with different volatilities (left) and PnL plot for different volatilities (right)

## Conclusion and discussion

We conclude this report by stating a few observations and remarks as a result of our work.

Firstly, we have found out why a market without arbitrage is wishable. In Part I we discovered what strategies could be applied if there were arbitrage opportunities, making it easy for fast and observant traders to earn free money. This is clearly not desirable as a fair and competitive market is more balanced and more probable to survive.

In addition, we have found that the binomial tree model and Black-Scholes model are closely related, in the sense that as the binomial tree model considers more and more steps it resembles the Black-Scholes model more closely. This was made rigorous in Part II. Interestingly, if one considers very high or very low volatility, the binomial tree and Black-Scholes models seem to become indistinguishable, regardless of the step complexity of the binomial tree model. We have given economic interpretations as well as mathematical derivations for these occurrences. Furthermore, we have seen that American option value differs from European option value if the option is a put option. For call options, as was derived in Part II, the best course of action is to exercise one's call option at maturity, whereas for put options this is not the case. There is therefore, besides the obvious differences in definition, a different trading strategy involved when dealing with either European or American options. As was seen, this difference disappears if the volatility disappears. We concluded that the put-call parity does not hold for American options, as American put option were always worth at least the same as European put options.

We found that the Black-Scholes model also has an analytical value for the fair price of a European put option, where we used the put-call parity. We conclude that the Black-Scholes model can be used to analytically price European options, and can be used to approximate the value of American options. In addition, the delta parameter behaves exactly as you would think it behaves: it changes depending on your position being in-the-money or out-of-the-money. Finally, we conclude that if the implied and realized volatility disagree, the replicating portfolio based on the implied volatility agrees with the Black-Scholes' theoretical value.