SINGULARITY ANALYSIS OF GENERATING FUNCTIONS

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Abstract. While generating functions are often studied combinatorially as formal power series, we can also treat them as complex analytic functions and study how properties of the functions govern the asymptotics of the underlying combinatorial sequence. This expository paper introduces methods for finding asymptotics of generating function coefficients, progressing from elementary techniques to more advanced approaches. We begin with basic methods applicable to rational functions, develop techniques for meromorphic functions, and culminate with the general method of singularity analysis. Throughout, we illustrate these methods with examples from combinatorics. This paper assumes background knowledge in generating functions and complex analysis.

1. Introduction

Generating functions bridge between discrete combinatorial structures and continuous analytic functions. While often studied combinatorially as formal objects, treating them as functions allows us to exploit their analytic properties to gain insights into the asymptotic behavior of combinatorial sequences. The key idea is that the analytic properties of a generating function near its singularities govern the asymptotic growth of its coefficients.

In this section, we establish some definitions and find asymptotics for coefficients of some simple generating functions.

Definition 1.1. A *singularity* of a function f is a point where the function ceases to be defined or analytic.

Definition 1.2. Two functions $f(x)$ and $g(x)$ are said to be asymptotically equivalent, denoted $f(x) \sim g(x)$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

We will see that the location of a function's singularities determines the exponential growth of its coefficients, and the type of singularity determines the subexponential factor.

Theorem 1.3 (Pringsheim's Theorem [5])**.** *If $f(z)$ can be represented by the series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with each a_n being nonnegative and has a radius of convergence of R, then the point $z = R$ is a singularity of $f(z)$.

Proof. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

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for nonnegative coefficients a_n and some radius of convergence R. First, note that we can assume without loss of generality that $R = 1$ since all other cases can be reduced to this by multiplying each of the coefficients by a factor. Now suppose for the sake of contradiction that $z = 1$ is not a singularity and that $f(z)$ is analytic in some disc of radius $r < 1$ centered at $z = 1$. We will show that the power series of f converges in a disc with radius greater than 1, which is a contradiction.

First, we choose some point z_0 on the real axis close to $z = 1$, specifically such that $1 - \frac{r}{2} < z_0 < 1$. The series expansion of f about z_0 is given by

$$f(z) = \sum_{m=0}^{\infty} b_m (z - z_0)^m.$$

We can now find b_n in terms of a_i's. Let $y = z - z_0$. Then,

$$f(z) = f(y + z_0) = \sum_{n=0}^{\infty} b_n y^n = \sum_{n=0}^{\infty} a_n (y + z_0)^n.$$

Expanding with the binomial theorem,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (y + z_0)^n \\ &= (a_0 + a_1 z_0 + a_2 z_0^2 + \cdots) + y(a_1 + 2a_2 z_0 + \cdots) + y^2(a_2 + 3a_3 z_0 + \cdots) + \cdots . \end{aligned}$$

In general, the coefficient of y^m is both b_m and

$$\sum_{k=0}^{\infty} a_{m+k} \binom{m+k}{m} z_0^k.$$

Therefore,

$$b_m = \sum_{n=m}^{\infty} \binom{n}{m} a_n z_0^{n-m},$$

which implies $b_m \geq 0$ for all m. Now, note that $\sum_{m=0}^{\infty} b_m (z - z_0)^m$ converges at $2 - z_0$ since $2 - z_0 < 1 + \frac{r}{2} < z_0 + r$. Therefore,

$$\begin{aligned} f(2 - z_0) &= \sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} \binom{n}{m} a_n z_0^{n-m} \right) (2 - 2z_0)^m \\ &= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} z_0^{n-m} (2 - 2z_0)^m \\ &= \sum_{n=0}^{\infty} a_n ((z_0) + (2 - 2z_0))^n \\ &= \sum_{n=0}^{\infty} a_n (2 - z_0)^n, \end{aligned}$$

where we have convergence at each step. Therefore, for some $2 - z_0 > 1$, we have $a_n = o((2 - z_0)^{-n})$, thus the radius of convergence is strictly greater than 1, which is a contradiction. ■

This theorem will be very important to us in this section because combinatorial generating functions are all represented with power series of nonnegative coefficients. It ensures that such functions always have a real, positive singularity on the boundary of their disc of convergence.

Definition 1.4. Singularities of functions analytic at 0 with magnitude equal to the radius of convergence of the function are called *dominant singularities.*

We can now use Pringsheim's Theorem to find dominant singularities of some combinatorial classes.

Definition 1.5. We say that a sequence $\{a_n\}$ is of *exponential order* K^n if and only if

$$\limsup |a_n|^{1/n} = K.$$

A sequence is *subexponential* if it is of exponential order 1^n.

For any sequence $\{a_n\}$, we can write $a_n = K^n\theta(n)$ for some K and some subexponential factor $\theta(n)$. If $\{a_n\}$ is of exponential order K^n, then for any $\varepsilon > 0$, we have $|a_n| > (K-\varepsilon)^n$ for infinitely many values of n and $|a_n| < (K+\varepsilon)^n$ except for only finitely many values of n. Now, recall the exponential growth formula.

Theorem 1.6 (Exponential Growth Formula). *If $f(z)$ is analytic at the origin and R is the distance of the nearest singularity of f from the origin,*

$$R = \sup\{r \mid f \textit{ is analytic on } |z| < r\},$$

then the coefficients $f_n = [z^n]f(z)$ are of exponential order $\left(\frac{1}{R}\right)^n$. Additionally, if all coefficients are nonnegative, we can define

$$R = \sup\{r \mid f \textit{ is analytic for all } 0 \leq z < r\}.$$

Proof. Let R' be the radius of convergence of $f(z)$ at $z = 0$. We know that $R \geq R'$ since it is analytic for all $|z| < r$. However, we also know that there must be a singularity on the boundary of the disc of convergence, so $R \leq R'$. Therefore, $R = R'$. Furthermore, for functions with nonnegative coefficients such as combinatorial generating functions, $z = R'$ is a singularity of $f(z)$ by Pringsheim's Theorem, so we need only use the less strict formulation of

$$R = \sup\{r \mid f \text{ is analytic for all } 0 \leq z < r\}.$$

Now that we know $R = R'$, by the definition of radius of convergence, for all $\varepsilon > 0$, $\lim_{n\to\infty} f_n(R-\varepsilon)^n = 0$. Therefore, for all n sufficiently large, $|f_n|(R-\varepsilon)^n < 1$, and $|f_n|^{1/n} < \frac{1}{R-\varepsilon}$ except for a finite number of n. Similarly, $|f_n|(R+\varepsilon)^n$ diverges to infinity, so $|f_n|^{1/n} > (R+\varepsilon)^{-1}$ infinitely often.

■

We can now use this formula to find the asymptotic behavior of some example generating functions.

First, we start with the Motzkin numbers [1]. Let $\mathcal{M}$ be the class of unary-binary trees, meaning that each node can have 0, 1, or 2 children. We will find a generating function for $\mathcal{M}$ and then find asymptotics for the coefficients.

First, we set $\Omega = \{0, 1, 2\}$ to get the specification,

$$\mathcal{M} \cong \mathcal{Z} \times \left(\mathcal{E} + \mathcal{M} + \mathcal{M}^2\right).$$

Solving the quadratic equation $M(z) = z + zM(z) + zM(z)^2$, taking the negative root of the quadratic after checking some coefficients, we get

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z} = z + z^2 + 2z^3 + 4z^4 + 9z^5 + \cdots .$$

Now, we can factor the expression under the square root to get

$$M(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z},$$

so $z = \frac{1}{3}$ is the closest singularity to the origin, where M must have a branch point. Thus, $M(z)$ is of exponential order 3^n.

2. Rational and Meromorphic Functions

We have seen how the location of singularities determines the exponential growth rate, but now we show how the type of singularity determines the corresponding subexponential factor $\theta(n)$.

First, consider rational functions, which are functions of the form $\frac{P(x)}{Q(x)}$, for polynomials P and Q.

Theorem 2.1. *Suppose $f(z)$ is a rational function that is analytic at zero with poles at $\alpha_1, \alpha_2, \ldots, \alpha_m$. Then, there exists polynomials $P_1, P_2, \ldots, P_m$, such that P_j has degree equal to the order of the pole of f at α_j minus one, so that for all sufficiently large n,*

$$[z^n]f(z) = \sum_{j=1}^{m} \frac{P_j(n)}{\alpha_j^n}.$$

Proof. Let $f(z)$ be a rational function of the form $f(z) = \frac{A(z)}{B(z)}$ for polynomials A and B. It must have a partial fraction decomposition of

$$f(z) = Q(z) + \sum_{j=1}^{n} \sum_{r=1}^{m_j} \frac{c_{j,r}}{(z - \alpha_j)^r}$$

where $Q(z)$ has degree $(\deg A - \deg B)$ and m_j is the multiplicity of the pole at α_j. Now, with the binomial theorem, we can find that the contribution to the z^k term in $(z - \alpha_j)^{-r}$ is

$$c_{j,r}\binom{-r}{k} z^k (-\alpha_j)^{-r-k} = \frac{(-1)^r c_{j,r}}{\alpha_j^{r+k}} z^k \binom{k+r-1}{k} = z^k \frac{(-1)^r c_{j,r}}{\alpha_j^{r+k}} \binom{k+r-1}{r-1}.$$

This coefficient is a polynomial of degree $r - 1$ in n, thus we have proven the theorem. ■

Furthermore, we have shown that each pole α_j of degree r contributes

$$\sum_{k=1}^{r} \frac{(-1)^k c_{j,k}}{\alpha_j^{n+k}} \binom{n+k-1}{k-1}$$

to the coefficient of z^n in f. This extends to the more general case of meromorphic functions too. To see this, we first look at the Laurent expansion of a meromorphic function. If a function $f(z)$ is meromorphic on some closed disk $|z| \leq R$ and α_j is a pole of f of order r, then in some punctured disk centered at α_j, we have the expansion

$$f(z) = \sum_{k=1}^{r} \frac{c_{j,k}}{(z-\alpha_j)^k} + \sum_{k=0}^{\infty} d_{j,k}(z-\alpha_j)^k.$$

Definition 2.2. In the expansion above, we denote

$$\sum_{k=1}^{r} \frac{c_{j,k}}{(z-\alpha_j)^k}$$

to be the *principal part* of the expansion of f about singularity α_j.

We have seen before that the principal part of f about α_j can be expressed as

Lemma 2.3. *The principal part of f about α_j can be expressed as*

$$\sum_{n=0}^{\infty} \left(z^n \sum_{k=1}^{r} \frac{(-1)^k c_{j,k}}{\alpha_j^{n+k}} \binom{n+k-1}{k-1} \right)$$

for some constants $c_{j,k}$.

Proof. Just as in the proof of Theorem 2.1, we can use the Binomial Theorem to find that the contribution of

$$\sum_{k=1}^{r} \frac{c_{j,k}}{(z-\alpha_j)^k}$$

to the coefficient of z^n is

$$\sum_{k=1}^{r} c_{j,k} \binom{-k}{n} (-\alpha_j)^{-n-k} = \sum_{k=1}^{r} c_{j,k} (-1)^k \alpha_k^{-n-k} \binom{n+k-1}{n-1}.$$

Summing these up for all n, we get the desired expression. ■

We can use these principal parts to modify f to make it analytic. In particular, if we subtract the principal part of f at α_i from f, we get a new function that is analytic at f. If we subtract the principal parts of all singularities in some disk, we make a new function that is analytic on that entire disk. With this idea, we can greatly extend Theorem 2.1.

Theorem 2.4. *Suppose $f(z)$ is meromorphic in some region containing the origin with poles at $\alpha_1, \ldots, \alpha_m$ such that $R = |\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_m|$. Let $\alpha_1, \ldots, \alpha_s$ be all the poles of smallest magnitude (that is, of magnitude R) and let α_{s+1} have magnitude R'. Then for any $R' > \varepsilon > 0$, we have*

$$[z^n]f(z) = [z^n] \left(\sum_{j=1}^{s} \sum_{k=1}^{r} \frac{c_{j,k}}{(z-\alpha_j)^k} \right) + O\left(\frac{1}{(|R'|-\varepsilon)^n} \right).$$

Proof. Notice that each

$$\sum_{k=1}^{r} \frac{c_{j,k}}{(z-\alpha_j)^k}$$

is the principal part of the singularity at α_j so

$$f(z) - \sum_{k=1}^{r} \frac{c_{j,k}}{(z-\alpha_j)^k}$$

is analytic at α_j. Let the sum of all the principal parts be

$$g(z) = \sum_{j=1}^{s} \sum_{k=1}^{r} \frac{c_{j,k}}{(z-\alpha_j)^k}.$$

If we subtract the principal parts of each singularity of magnitude R, we get a function that is analytic at all the singularities of f, and so analytic on all of $|z| \le R$, which additionally means it is analytic on $|z| < R'$, the next smallest singularity of f. Finally, we finish by either invoking Theorem 1.6 on $f(z) - g(z)$ or by bounding the Cauchy Integral Formula with the ML inequality as

$$\Big|[z^n]f(z) - g(z)\Big| = \left|\frac{1}{2\pi i}\right| \left|\oint_{|z|=r} \frac{f(z)-g(z)}{z^n} \cdot \frac{dz}{z}\right| \le \frac{1}{2\pi} \frac{f(a)-g(a)}{r^{n+1}} 2\pi r = O\left(\frac{1}{r^n}\right)$$

for some a and for any r such that $f(z) - g(z)$ is analytic on $|z| \le r$. Since $f(z) - g(z)$ is analytic on all $|z| < R'$, we can use $r = R' - \varepsilon'$ for any $R' > \varepsilon' > 0$ and the bound holds, which completes the proof.

■

We can now look at some examples and apply this theorem to generating functions. First, consider the Fibonacci numbers given by $F(z) = \frac{z}{1-z-z^2}$. We will find an approximation for the n^{th} Fibonacci number. First, notice that the poles of F are at $\alpha_1 = \frac{-1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{-1-\sqrt{5}}{2}$, with the former being the dominant singularity. Notice that

$$\lim_{z\to\alpha_1} \frac{-z}{(z-\alpha_1)(z-\alpha_2)}(z-\alpha_1) = \frac{-\alpha_1}{\alpha_1 - \alpha_2} = -\frac{\alpha_1}{\sqrt{5}},$$

so the principal part of α_1 is

$$\frac{-\alpha_1}{\sqrt{5}}(z-\alpha_1)^{-1}.$$

Therefore, by Lemma 2.3, the principal part contributes

$$(-1)\frac{-\alpha_1}{\sqrt{5}}(\alpha_1^{-(n+1)}) = \frac{1}{\sqrt{5}\alpha_1^n}$$

to the coefficient of z^n, hence

$$f_n = \frac{1}{\alpha_1^n\sqrt{5}} + O\left(\frac{1}{(\alpha_2-\varepsilon)^n}\right) = \frac{\phi^n}{\sqrt{5}} + O\left(\frac{1}{\phi^n}\right),$$

which agrees with Binet's formula for the n^{th} Fibonacci number.

Next, we consider the ordered Bell numbers. Suppose we wish to rank n objects, where some objects we like better than others, but objects are also allowed to be equally desirable. Let b_n denote the number of ways we can rank them. For instance, $b_2 = 3$ because we can prefer object 1 over object 2, or we can prefer object 2 over object 1, or we can like them equally. The coefficients b_n are the ordered Bell numbers, and we now find asymptotics for this combinatorial class.

We know this is a sequence of set partitions where the order of objects in a group doesn't matter. Thus there is the specification

$$\mathcal{B} = \textsc{Seq}(\textsc{Set}_{\ge 1}(\mathcal{Z})).$$

Therefore, the exponential generating function is

$$B(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n = \frac{1}{1-(e^z-1)} = \frac{1}{2-e^z} = 1 + z + \frac{3}{2}z^2 + \frac{13}{6}z^3 + \frac{75}{24}z^4 + \cdots .$$

Now, the singularities of this function occur when $e^z = 2$, or at $z = \log(2) + 2\pi k i$, for $k \in \mathbb{Z}$. There is only one singularity of lowest magnitude $\log(2)$, and we can calculate its principal part. Since

$$\lim_{z\to\log(2)} \frac{z-\log(2)}{(2-e^z)} = \lim_{z\to\log(2)} \frac{1}{-e^z} = -\frac{1}{2},$$

the principal part is $\frac{-1/2}{z-\log(2)}$ and from our earlier expansion of the contribution of a principal part of a pole to coefficients, we see that this principal part contributes

$$\frac{1}{2(\log(2))^{n+1}}$$

to the coefficient of z^n. Furthermore, R' in this case is $\sqrt{(\log(2))^2 + (2\pi)^2} \approx 6.32$. By Theorem 2.4, we have

$$b_n = n!\left(\frac{1}{2(\log(2))^{n+1}} + O((.16)^n)\right).$$

3. Singularity Analysis

We now turn to analyzing asymptotics for functions whose singularities are of more exotic forms. There are many methods of doing this such as with Tauberian theorems and Darboux's methods. However, we will look at the method of singularity analysis introduced by Flajolet and Odlyzko [3]. We will consider only functions with a single dominant singularity since functions with multiple dominant singularities can be decomposed into this case and then added back together.

Definition 3.1. Define a domain to be a Δ-domain at ζ if it can be written as

$$\{z \mid |z| < R,\ z \neq \zeta,\ |\arg(z-\zeta)| > \phi\}$$

for some $R > |\zeta|$ and $0 < \phi < \frac{\pi}{2}$ and call a function Δ-analytic if it is analytic in some Δ-domain.

These domains are defined in this way so that we can use countour integration on a Hankel-type path to prove the following transfer theorems.

Theorem 3.2. *Let $\alpha, \beta \in \mathbb{R}$ and let $f(z)$ be a Δ-analytic function. Then,*

$$f(z) = O\left((1-z)^{-\alpha}\right) \implies [z^n]f(z) = O(n^{\alpha-1})$$

and

$$f(z) = o\left((1-z)^{-\alpha}\right) \implies [z^n]f(z) = o(n^{\alpha-1}).$$

Proof. We use the Cauchy integral formula on the domain $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ where

$$\begin{aligned}
\Gamma_1 &= \{z \mid |z-1| = \frac{1}{n}, \, |\arg(z-1)| \geq \theta\} \\
\Gamma_2 &= \{z \mid |z-1| \geq \frac{1}{n}, \, |\arg(z-1)| = \theta, \, |z| \leq r\} \\
\Gamma_3 &= \{z \mid |\arg(z-1)| \geq \theta, \, |z| = r\} \\
\Gamma_4 &= \{z \mid |z-1| \geq \frac{1}{n}, \, |\arg(z-1)| = -\theta, |z| \leq r\}
\end{aligned}$$

and $1 < r < R$, $\phi < \theta < \frac{\pi}{2}$. Note that from the definition of a Δ-domain, $f(z)$ is analytic on and inside this entire contour. Therefore,

$$\begin{aligned}
|[z^n]f(z)| &= \frac{1}{2\pi i}\left(\int_{\Gamma_1} \frac{f(z)}{z^{n+1}}\,dz + \int_{\Gamma_2} \frac{f(z)}{z^{n+1}}\,dz + \int_{\Gamma_3} \frac{f(z)}{z^{n+1}}\,dz + \int_{\Gamma_4} \frac{f(z)}{z^{n+1}}\,dz\right) \\
&\leq \frac{1}{2\pi}\sum_{j=1}^{4}\int_{\Gamma_j} \frac{|f(z)|}{|z|^{n+1}}\,|dz|.
\end{aligned}$$

Now, we evaluate the integral for each of the contour components as $n \to \infty$. On Γ_1, the greatest value of $|f(z)|$ is $c \cdot \frac{1}{n}$ for some c, so the ML-inequality gives

$$\int_{\Gamma_1} \frac{|f(z)|}{|z|^{n+1}}\,|dz| \leq \frac{\left(\frac{c}{n}\right)^{\alpha}}{\left(1-\frac{1}{n}\right)^{n+1}}\left(\frac{2\pi}{n}\right) = O\left(n^{-\alpha-1}\cdot\left(\frac{n}{n-1}\right)^{n+1}\right).$$

Since for $n > 2$,

$$\left(\frac{n}{n-1}\right)^{n+1} = \left(1+\frac{1}{n-1}\right)^{n-1}\left(1+\frac{1}{n-1}\right)^{2} \leq e\left(1+\frac{1}{n-1}\right)^{2}$$

is a decreasing function and $\left(\frac{2}{2-1}\right)^{2+1} = 8$, for $n > 2$,

$$\left(\frac{n}{n-1}\right)^{n+1} \leq 8$$

and so

$$\int_{\Gamma_1} \frac{|f(z)|}{|z|^{n+1}}\,|dz| = O\left(n^{-\alpha-1}\cdot\left(\frac{n}{n-1}\right)^{n+1}\right) = O(n^{-\alpha-1}).$$

Now, we can evaluate the integral over Γ_2 and Γ_4. Due to symmetry, the same bound will work for both contours. Setting $z = 1 + \frac{te^{i\theta}}{n}$, we get

$$\int_{\Gamma_2} \frac{|f(z)|}{|z|^{n+1}}\,|dz| = \int_1^{\infty} O\left(\left(\frac{t}{n}\right)^{-\alpha}\right)\left|1+\frac{te^{i\theta}}{n}\right|^{-n-1}\,dt.$$

However,

$$\left|1+\frac{te^{i\theta}}{n}\right| \geq 1+\frac{t\cos(\theta)}{n},$$

so

$$\int_{\Gamma_2} \frac{|f(z)|}{|z|^{n+1}}\,|dz| = O(n^{\alpha-1}\int_1^{\infty} t^{-\alpha}\left(1+\frac{t\cos(\theta)}{n}\right)^{-n}\,dt).$$

This integral is bounded above by some constant for any given α since for $0 < \theta < \frac{\pi}{2}$, as $n \to \infty$,

$$\int_1^\infty t^{-\alpha} \left(1 + \frac{t\cos(\theta)}{n}\right)^{-n} dt \to \int_1^\infty t^{-\alpha} e^{-t\cos\theta}\, dt$$

converges. Therefore,

$$\left|1 + \frac{te^{i\theta}}{n}\right| = O(n^{\alpha-1}).$$

Finally, consider Γ_3. Since $f(z)$ is bounded on the domain, we have

$$\int_{\Gamma_3} \frac{|f(z)|}{|z|^{n+1}} |dz| = O(r^{-n-1} \cdot (2\pi r)) = O(r^{-n}),$$

which becomes exponentially small. Putting these 4 contours together, we get that $[z^n]f(z) = O(n^{\alpha-1})$.

For the second part of the theorem, use the same contour of $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$. We are given that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|z-1| < \delta \implies |f(z)| < \varepsilon|1-z|^{-\alpha}$ for all z. We need to prove that for this $\varepsilon > 0$, there also exists a n_0 such that

$$n \geq n_0 \implies [z^n]f(z) < \varepsilon K n^{\alpha-1}$$

for some fixed K.

Over Γ_1, we can take some $n_0 > \frac{1}{\delta}$. That way, $|f(z)| < \varepsilon|1-z|^{-\alpha}$ over Γ_1. Therefore, similar to the proof of the first part of the theorem, for $n > n_0$,

$$\int_{\Gamma_1} \frac{|f(z)|}{|z|^{n+1}}|dz| < 8\varepsilon n^{\alpha-1} = o(n^{\alpha-1}).$$

The integrals over Γ_2 and Γ_4 are similar to that of the first part except it is helpful to break up the integral,

$$\int_1^\infty \left|f\left(1 + \frac{te^{i\theta}}{n}\right)\right| \left|1 + \frac{t\cos(\theta)}{n}\right|^{-n} dt$$

into one from 1 to $\log^2(n)$ and from $\log^2(n)$ to ∞ and taking different bounds for each. We get that the integral is actually $o(n^{\alpha-3}$. The integral over Γ_3 is entirely analogous to that in the first part of the theorem. Each contour's contribution is $\frac{1}{2\pi}o(n^{\alpha-1})$, so we have $[z^n]f(z) = o(n^{\alpha-1})$ and we are done. ■

It is also possible to extend these transfer theorems to more classes of functions. For example, functions of the form

$$(1-z)^\alpha \left(\log\frac{1}{1-z}\right)^\beta \left(\log\log\frac{1}{1-z}\right)^\gamma$$

and large and slow varying functions are also discussed in [3].

Corollary 3.3. *Let $f(z)$ be a Δ-analytic function. Then, if $f(z) \sim (1-z)^{-\alpha}$ for $\alpha \notin \{0\} \cup \mathbb{Z}^-$ and $z \to 1$ is in Δ, we have*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}.$$

Proof. We know that $f(z) = (1-z)^{-\alpha} + o((1-z)^{-\alpha})$. First, from the Taylor series expansion, we know that

$$[z^n](1-z)^{-\alpha} = \binom{n+\alpha-1}{n} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}.$$

Now, asymptotically by Stirling's formula, we can expand this as

$$[z^n](1-z)^{-\alpha} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \frac{\alpha(\alpha-1)}{2n} + \frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24n^2} + O\left(\frac{1}{n^3}\right)\right).$$

Applying Theorem 3.2 to the expansion $f(z) = (1-z)^{-\alpha} + o((1-z)^{-\alpha})$, we immediately get the result. ■

4. Advanced examples

Now we revisit some earlier examples and apply singularity analysis to obtain more precise asymptotic results.

First, let's look again at the Motzkin numbers and use the method of singularity analysis to get sharper bounds on the coefficients of the generating function than with the elementary techniques of the first section. Recall that the Motzkin numbers satisfy

$$M(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z}.$$

This has a dominant singularity at $z = \frac{1}{3}$ and a secondary singularity at $z = -1$. Taking a branch cut along the real axis, we know that $M(z)$ is Δ-analytic. We can rewrite $M(z)$ as

$$M(z) = -\frac{1}{2} + \frac{1}{2z} - \frac{\sqrt{1+z}}{2z}(1-3z)^{1/2}.$$

The series expansion as $z \to \frac{1}{3}$ of $\frac{\sqrt{1+z}}{2z}$ is just the Taylor series,

$$\begin{aligned} M(z) &= 1 - \left(\sqrt{3} + \frac{7}{8}\sqrt{3}(1-3z) + O((1-3z)^2)\right)(1-3z)^{1/2} \\ &= 1 - \sqrt{3}(1-3z)^{1/2} - \frac{7}{8}\sqrt{3}(1-3z)^{3/2} + O((1-3z)^{5/2}). \end{aligned}$$

From Corollary 3.3, we can write the asymptotic expansion of $[z^n](1-3z)^{1/2}$ as

$$[z^n](1-3z)^{1/2} \sim \frac{n^{-(1/2)-1}}{\Gamma(-1/2)}3^n = \frac{3^n}{\sqrt{4\pi n^3}}.$$

In this case, we will need another term of the expansion, so using the expansion we got with Stirling's formula, we have

$$\begin{aligned} [z^n](1-3z)^{1/2} &= \frac{n^{-(1/2)-1}}{\Gamma(-1/2)}3^n\left(1 + \frac{(-1/2)(-3/2)}{2n} + O\left(\frac{1}{n^2}\right)\right) \\ &= \frac{3^n}{\sqrt{4\pi n^3}}\left(1 + \frac{3}{8n} + O\left(\frac{1}{n^2}\right)\right) \end{aligned}$$

Therefore, $[z^n]M(z)$ has a general asymptotic expansion of

$$[z^n]M(z) = \frac{3^n}{\sqrt{4\pi n^3}} \cdot \sqrt{3}\left(1 + \frac{3}{8n} + O\left(\frac{1}{n^2}\right)\right) - \frac{3^n n^{-5/2}}{4\sqrt{\pi}/3} \cdot \frac{7}{8}\sqrt{3}\left(1 + O\left(\frac{1}{n}\right)\right) + O((1-3z)^{5/2})$$
$$= \sqrt{\frac{3}{4\pi n^3}} 3^n \left(1 - \frac{15}{16}n + O\left(\frac{1}{n^2}\right)\right).$$

This is a much better bound than the one of 3^n found without singularity analysis in the introduction and can easily be improved arbitrarily much by using more terms of the expansion of $\frac{\sqrt{1+z}}{2z}$ and $[z^n](1-3z)^{1/2}$.

Our last example is permutations with cycles of odd length. This combinatorial class $\mathcal{P}$ has specification

$$\mathcal{P} = \textsc{Set}(\textsc{Cyc}_{\text{odd}}(\mathcal{Z})),$$

so we have the exponential generating function

$$P(z) = e^{\log(\frac{1+z}{1-z})/2} = \sqrt{\frac{1+z}{1-z}}.$$

This has two dominant singularities at $z = 1$ and $z = -1$. While we have not considered two dominant singularities in this paper yet, it is easy to show that singularity analysis still works if $P(z)$ is analytic on a domain that is the intersection of two rotated Δ-domains each about one of the singularities. A proof of this fact can be found in [4]. This function, $P(z)$, clearly does satisfy the condition, so asymptotics for $[z^n]P(z)$ can then be found by summing the results of normal singularity analysis for each of the dominant singularities. First, as $z \to 1$, we have

$$P(z) = \frac{1}{\sqrt{1-z}}\left(2^{1/2} - 2^{-3/2}(1-z) - 2^{-9/2}(1-z)^2 + O((1-z)^3)\right)$$

and as $z \to -1$,

$$P(z) = \sqrt{1+z}\left(2^{-1/2} + 2^{-5/2}(1+z) + O((1+z)^2)\right).$$

By singularity analysis, $z = 1$ contributes to $[z^n]P(z)$ a total of

$$\frac{2^{1/2}}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + \frac{1}{128n^2}\right) + \frac{2^{-3/2}}{\sqrt{\pi n^3}}\left(\frac{1}{2} + \frac{3}{16n}\right) - \frac{2^{-9/2}}{\sqrt{\pi n^5}}\left(\frac{3}{4}\right) + O(n^{-7/2})$$

and $z = -1$ contributes

$$-\frac{(-1)^n 2^{-1/2}}{\sqrt{\pi n^3}}\left(\frac{1}{2} + \frac{3}{16n}\right) + \frac{2^{-5/2}}{\sqrt{\pi n^5}}\left(\frac{3}{4}\right) + O(n^{-7/2}).$$

Summing these, we have

$$[z^n]P(z) = \frac{2^{1/2}}{\sqrt{\pi n}} - \frac{(-1)^n 2^{-3/2}}{\sqrt{\pi n^3}} - \frac{9 \cdot 2^{-13/2}}{\sqrt{\pi n^5}} + O(n^{-7/2}).$$

This result provides a highly accurate approximation of the number of permutations with odd-length cycles, again showing the effectiveness of singularity analysis in obtaining precise asymptotic estimates.

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