Solutions and Notes to Michael Spivak's *Calculus*

A Work of All

1 Basic Properties of Numbers

1.1

Prove the following.

1.1.1

If $ax = a$ for some number $a \neq 0$, then $x = 1$.

$$\begin{aligned} ax &= a \\ x &= \frac{a}{a} \\ x &= 1 \end{aligned}$$

1.1.2

$x^2 - y^2 = (x - y)(x + y)$

$$\begin{aligned} x^2 - y^2 &= x^2 + xy - xy - y^2 \\ x^2 - y^2 &= x^2 - y^2 \end{aligned}$$

1.2

What is wrong with the following "proof"? Let $x = y$.

$$\begin{aligned} (x + y)(x - y) &= y(x - y) \\ x + y &= y \end{aligned}$$

We divide by $(x - y)$, which given $x = y$ equals 0. We cannot divide by zero.

1.3

Prove the following.

1.3.1

$\frac{a}{b} = \frac{ac}{bc}$, if $b, c \neq 0$.

$$\begin{aligned} ab^{-1} &= acb^{-1}c^{-1} \\ ab^{-1} &= ab^{-1} \end{aligned}$$

1.3.2

$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, if $b, d \neq 0$.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
$$ad + cb = ad + cb$$

1.3.3

$(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$.

$$(ab)^{-1} = a^{-1}b^{-1}$$
$$\frac{1}{ab} = \frac{1}{a}\frac{1}{b}$$
$$\frac{1}{ab} = \frac{1}{ab}$$

1.3.4

1.3.5

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}, \text{ if } b, c, d \neq 0$$
$$\frac{a}{b} \cdot (\frac{c}{d})^{-1} = \frac{ad}{bc}$$
$$ab^{-1} \cdot c^{-1} \cdot (d^{-1})^{-1} = \frac{ad}{bc}$$
$$ab^{-1} \cdot c^{-1} \cdot d = \frac{ad}{bc}$$
$$\frac{ad}{bc} = \frac{ad}{bc}$$

1.3.6

If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also determine when $\frac{a}{b} = \frac{b}{a}$.

To prove a theorem of the form A *if and only if* B, you first prove *if* A *then* B, then you prove *if* B *then* A, and that's enough to complete the proof.

if A then B:

$$
\begin{aligned}
\frac{a}{b} &= \frac{c}{d} \\
ab^{-1} &= cd^{-1} \\
ab^{-1}bd &= cd^{-1}bd \\
ad &= bc
\end{aligned}
$$

If B then A is given by the inverse direction of the above.