Solutions and Notes to Michael Spivak's *Calculus*

1 Basic Properties of Numbers

1.1

Prove the following.

1.1.1

If $ax = a$ for some number $a \neq 0$, then $x = 1$.

$$
\begin{aligned}
ax &= a \\
x &= \frac{a}{a} \\
x &= 1
\end{aligned}
$$

1.1.2

$x^2 - y^2 = (x - y)(x + y)$

$$
\begin{aligned}
x^2 - y^2 &= x^2 + xy - xy - y^2 \\
x^2 - y^2 &= x^2 - y^2
\end{aligned}
$$

1.2

What is wrong with the following "proof"? Let $x = y$.

$$
\begin{aligned}
(x + y)(x - y) &= y(x - y) \\
x + y &= y
\end{aligned}
$$

We divide by $(x - y)$, which given $x = y$ equals 0. We cannot divide by zero.

1.3

Prove the following.

1.3.1

$\frac{a}{b} = \frac{ac}{bc}$, if $b, c \neq 0$.

$$
\begin{aligned}
ab^{-1} &= acb^{-1}c^{-1} \\
ab^{-1} &= ab^{-1}
\end{aligned}
$$

1.3.2

$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, if $b, d \neq 0$.

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{ad+bc}{bd} \\ ad + cb &= ad + cb \end{aligned}$$

1.3.3

$(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$.

$$\begin{aligned} (ab)^{-1} &= a^{-1}b^{-1} \\ \frac{1}{ab} &= \frac{1}{a}\frac{1}{b} \\ \frac{1}{ab} &= \frac{1}{ab} \end{aligned}$$

1.3.4

1.3.5

$$\begin{aligned} \frac{\frac{a}{b}}{\frac{c}{d}} &= \frac{ad}{bc}, \text{ if } b, c, d \neq 0 \\ \frac{a}{b} \cdot (\frac{c}{d})^{-1} &= \frac{ad}{bc} \\ ab^{-1} \cdot c^{-1} \cdot (d^{-1})^{-1} &= \frac{ad}{bc} \\ ab^{-1} \cdot c^{-1} \cdot d &= \frac{ad}{bc} \\ \frac{ad}{bc} &= \frac{ad}{bc} \end{aligned}$$

1.3.6

If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also determine when $\frac{a}{b} = \frac{b}{a}$.

To prove a theorem of the form A *if and only if* B, you first prove *if* A *then* B, then you prove *if* B *then* A, and that's enough to complete the proof.

if A then B:

$$\begin{aligned}\frac{a}{b} &= \frac{c}{d}\\ ab^{-1} &= cd^{-1}\\ ab^{-1}bd &= cd^{-1}bd\\ ad &= bc\end{aligned}$$

If B then A is given by the inverse direction of the above.

1.4

Find all numbers x for which:

1.4.1

$$\begin{aligned}4 - x &< 3 - 2x\\ x &< -1\\ x &\in \{(-1, \infty)\}\end{aligned}$$

1.4.2

$$\begin{aligned}5 - x^2 &< 8\\ -3 &< x^2\end{aligned}$$

Every real number squared is positive, hence $\in \{\mathbb{R}\}$

1.4.3

$$\begin{aligned}5 - x^2 &< -2\\ -x^2 &< -7\\ x^2 &> 7\\ x &> \sqrt{7}\end{aligned}$$

$x \in \{x > \sqrt{7}\}$

1.4.4

$$(x-1)(x-3) > 0$$
$$\text{then } x \neq 1 \text{ and } x \neq 3$$

$x \in \{x > 3, x < 1\}$

1.4.5

$x^2 - 2x + 2$ draws a parabola which is greater zero for all $x \in \{\mathbb{R}\}$.