The restricted Three Body Problem and orbital mechanics

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Introduction

Maintaining stable orbits around celestial bodies has been the foundation of space travel since man first escaped the prison that is our planet over 60 years ago aboard Vostok I [1]. Without this very basic capability, our capacity to navigate our own cosmic backyard, let alone the great beyond, is handicapped. The restricted three-body problem does unfortunately limit our ability to compute these orbits analytically and thus forces us to use numerical methods to solve the differential equations of motion instead. This project aims to use these methods to simulate the orbit of a spacecraft at the Earth-Moon L_2 point.

Background

To solve the differential equations that govern the motion of three or more gravitational bodies, we must use numerical methods. In this work, simulations were produced using two approximations, the first of which was the Taylor expansion:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + a\mathbf{x}'_n + \frac{a^2}{2}\mathbf{x}''_n + O(a^3)$$

$$\mathbf{x}'_{n+1} = \mathbf{x}'_n + a\mathbf{x}''_n + O(a^2)$$

The second method used was the fourth order Runge-Kutta formula:

$$\mathbf{x}_{n+1}^{(n)} = \mathbf{x}_n^{(n)} + \frac{a}{6}\left(\mathbf{x}_n^{(n+1)} + 2\mathbf{z}_1^{(n+1)} + 2\mathbf{z}_2^{(n+1)} + \mathbf{z}_3^{(n+1)}\right) + O(a^5)$$

In the Runge-Kutta (RK4) method, the same formula can be used to compute both the velocity and the position of the spacecraft using only an expression for its acceleration. **The RK4 computation is therefore more streamlined and theoretically simpler to implement in Python. It is also more accurate, leaving each computation with an error on the order of a^5.** This is significantly better than the Taylor integrator, although this disparity can easily be mitigated by including higher order terms of the Taylor expansion. [2]

The step size a is directly related to the accuracy of the simulation and must be adjusted to provide a sufficient level of accuracy without compromising the efficiency of the simulation. After repeated testing, a step size of $a = 10$ seconds provided a sufficient level of accuracy while maintaining an acceptable computation efficiency.

The motion of the Earth and the Moon were modelled with **simple harmonic motion**, with each travelling in a circle around the barycentre of the two body system, approximately 4700km from the centre of the Earth. This formed the origin of the *x-y* coordinate system on which the simulations were run. Further development of this work will involve a similar coordinate basis centred upon the barycentre of whichever gravitational system is locally gravitationally dominant.

Lagrange points

Lagrange points occur in space when the gravitational forces of two or more large bodies balance, causing any object placed there to **remain stationary with respect to the two bodies**. However, all Lagrange point orbits are **unstable in nature** and therefore require constant corrections and careful control of initial conditions to ensure that they do not decay [3]. The Earth Moon L2 point's distance from the system' barycentre was calculated using

$$r_{L2} = \left(d \times \left(\frac{m_{moon}}{m_{earth}}\right)^{\frac{1}{3}}\right) - \frac{d m_{moon}}{m_{earth} + m_{moon}}$$

Where d is the Earth-Moon separation.

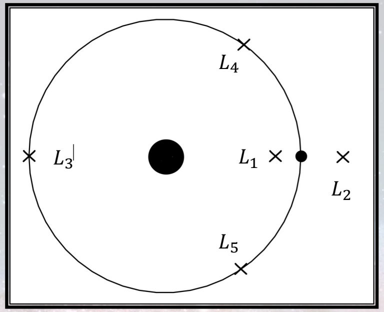

Fig. 1: Lagrangian point positions in a two-body system.

Results and comparison

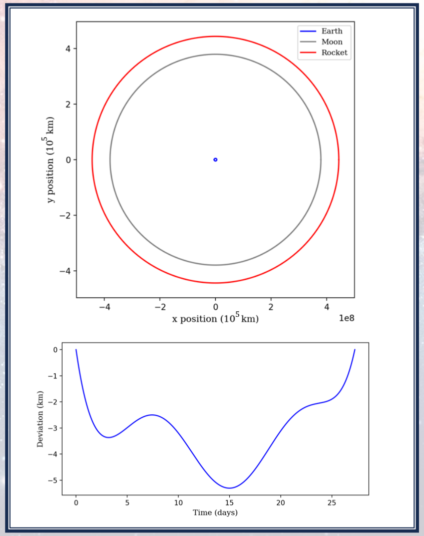

Fig. 2: Positions of the Earth, Moon and spacecraft as calculated using Taylor expansion after 1 orbit. The deviation of the rocket from a perfect circle is also shown.

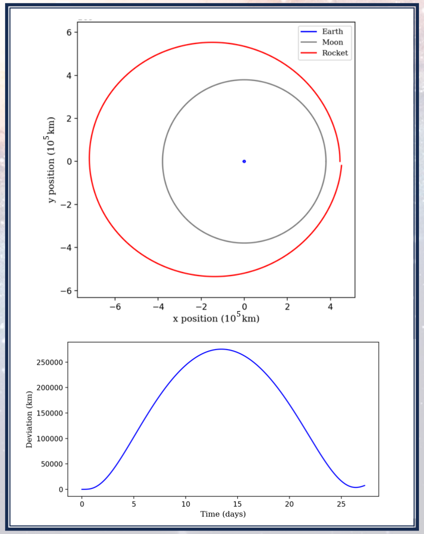

Fig. 3: Positions of the Earth, Moon and spacecraft as calculated using the RK4 method after 1 orbit. Another deviation plot is shown with a much greater magnitude.

Conclusions

The implementation of the Taylor and RK4 numerical approximations clearly highlighted the differences that exist between the two, particularly regarding their implementation in a practical, adaptable orbital simulation.

The Taylor series approximation was **significantly more stable** throughout the computation and has proved efficient with a time step of $a = 10$s. The initial radius of the spacecraft was set to be exactly $\boldsymbol{r_i} = \mathbf{1.0067879641561020 55}\,\boldsymbol{r_{L2}}$ and the initial velocity of the spacecraft was 1184.74ms^{-1}, calculated to match the Moon's angular velocity. The maximum deviation from a perfect circle modelled at r_i was approximately $-$ 5km over one full orbit, where a negative value indicates the spacecraft's orbit was too tight. **The final radius of the spacecraft $\boldsymbol{r_f}$ was $\mathbf{1.12}\boldsymbol{cm}$ closer to the Earth than the initial radius.**

Interestingly, **attempts at further optimisation beyond this were futile due to the resolution of the simulation**. The difference between the r_i and r_f snapped either to -1.12cm or 4.36cm or some multiple of the difference between them, returning an approximate resolution of 5.24cm. To improve the accuracy of the Taylor integrator, higher order expansions will be used in future iterations of this milestone programme.

The implementation of the RK4 approximation was **less intuitive, less stable and the improved accuracy was overall not worth the concessions**. As before a time step of $a = 10$s was used, resulting in the position plots seen in figure 3. The launch radius was $\boldsymbol{r_i} = \mathbf{1.0078}\boldsymbol{r_{L2}}$ and the maximum deviation was approximately 270000km, several orders of magnitude greater than the Taylor integrator. Despite returning to a point within 2% or approximately 6000km of its original radius r_i, the numerical accuracy of the simulation is inadequate, and the path traced is not physically correct. Presently, issues exist with the implementation of the RK4 routine in the simulation code. To rectify this, **future development will be carried out during this project to make full use of SciPy's integration routines as opposed to the RK4 integrator**. This will allow for more robust and consistent integration methods that leave less room for error and issues with implementation.

In conclusion, an **optimal initial launch point of 0.73% greater than $\boldsymbol{r_{L2}}$ was calculated** by taking the mean of the optimised launch points from each integrator.

[1] Solin Burgess, Rex Hall (June 2, 2010). *"The first Soviet cosmonaut team: their lives, legacy, and historical impact"*. Praxis. p. 356. ISBN 978-0-387-84823-5.

[2] Kutta, Wilhelm (1901). *"Beitrag zur näherungsweisen Integration totaler Differentialgleichungen." Zeitschrift für Mathematik und Physik"*, 46: 435–453.

[3] Crawford, I. and Joy, K. H. (2014). *"Lunar exploration: opening a window into the history and evolution of the inner solar system."* Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 372(2024), 20130315.