Exponential and Logarithmic Functions Review

1. On the same graph, sketch the two functions: $y = e^x$ and $y = \ln x$.

2. Here we will review some basic properties for exponents. Write the simplified forms of the following expressions in terms of the variables given.

   (a) $a^m \times a^n$

   (b) $\frac{b^k}{b^j}$

   (c) $(c^x)^y$

   (d) $x^k \times y^k$

   (e) $\frac{1}{a^k}$

   (f) b^0

   (g) $\sqrt[n]{x^m}$

3. Solve the following equations for x with the use of exponential identities.

   (a)
   $$487 \times 2^x = 7792$$

   (b)
   $$13^x = \frac{169 \times 7^x}{49}$$

   (c) Consider factoring out a certain number (Hint: the number is a power of 4)
   $$16^{x+1} - 4^{x+2} - 48 \cdot 4^x + 64 = 0$$

4. Rick is currently saving up for a new computer. At the start of each month he invests \$2000 into his investment account. At the end of each month, his investment returns him the principal amount (amount he put in initially) as well as an additional *interest* of 4% of his principal amount. (You can ask your parents about this but this rate is crazy good.)

   (a) Write an expression for the final amount of money (m_f) in Rick's account at the end of each month in terms of the money in his account at the start of each month (m_i)

   (b) Hence, find the amount of money Rick has at the end of the third month.

   (c) Check that your number is consistent with the number given by the following equation:
   $$m_3 = 2000((1.04)^3 + (1.04)^2 + (1.04))$$

   (d) Suppose that the computer costs \$10000, and that Rick wants to purchase it in 3 months, how much must he put in per month in order to do so?

   (e) Suppose that the computer costs \$10000, as before, but Rick puts down a principal amount of \$8000 at the start of the first month, how much must the monthly interest rate be for Rick to be able to afford the computer by the end of the fourth month? In other words, what is the percentage of the interest he receives in each month?

5. Alfred, Elvis and Mavis are 3 people living in the same town. One day, they went on an adventure but they have accidentally contracted a disease on the way back. Upon transmission, 2 people in the town will be infected, but the original patient will be cured. Assume that the disease cannot re-infect people who are currently sick.

   (a) At the end of the first day, the disease transmits itself from Alfred, Elvis and Mavis. How many people are now infected?

   (b) At the end of the second day, the disease transmit itself again from all of its patients. How many people are now infected?

   (c) It was found that the disease will transmit itself from all patients at the end of each day. Hence, write an expression for the number of people infected – n_i, at the end of the ith day.

   (d) The town has a population of 1536 people. How many days would it take for everyone to be infected?

   (e) On the 7th day of the outbreak, a doctor Iris managed to find a cure for the disease. At least how many people must she cure on the 7th day to control the disease? Hint: How many people must be cured for the net amount of infected people n_i to be lower at the end of the 7th day as compared to the end of the 6th day?

   (f) Instead of finding the cure on the 7th day, if Iris had found the cure on the 3rd day, at least how many people must she cure by the end of the 3rd day in order to control the disease?

6. Write down the simplified form for the following expressions using logarithm properties.

   (a) $\log a + \log b$

   (b) $\log a - \log b$

   (c) $\log c^k$

   (d) $\log_x x$

   (e) $\log_y y^m$

   (f) $\log_k x$; Write your answer using the ln function.

7. In the following expressions, express y in terms of x. One example has been done for you. All letters except x and y represent constants.
   Example: $x = \ln y \rightarrow$ Answer: $y = e^x$

   (a) $\ln y = mx + c$

   (b) $\ln y = m \ln x + c$

   (c) $\frac{\ln y}{\ln x} = kx$

   (d) $\ln y = \ln a + \ln x - bx$

   (e) $e^{xy} = \ln x$

8. Solve the following equations involving logarithms and their properties. You may express your answers in terms of e.

   (a) $\ln\left(\sqrt{\frac{x+1}{x-1}}\right) = 1$

   (b) $\log_2 (x-3) + \log_2 (x+1) + 1 - \log_2 (3x-2) = 0$

   (c) $\log_5 x = \log_{25} x + 2$

9. The population of a city in the tth year is modeled as follows:

$$P(t) = 125000(1 + r)^t$$

   (a) What is the city's initial population?
   (b) If $r = 0.05$, when will the city's population exceed 200000?
   (c) The city's industries are dying and people are now leaving the city, if instead $r = -0.15$, when will the town's population be halved?

10. The scale for severity of earthquakes, the Richter scale is defined as follows:

$$M = \log_{10} \frac{I}{I_0}$$

where I is the intensity of the earthquake and I_0 is a constant to be determined.

   (a) Two earthquakes have magnitude difference $M_1 - M_2 = 3$, how much more energy was released in earthquake 1 compared to earthquake 2? In other words, calculate I_1/I_2.
   (b) It was found that a magnitude 7 earthquake had an intensity of $I = 1000\text{Wm}^{-2}$. Calculate I_0 From this information

11. A student is investigating on the radioactive decay of a material. He logged the following information.

Time(s)	Particles remaining
0	8000
3	6682
6	5581
9	4662
12	3894
15	3252

   (a) The student argues that the number of particles remaining is inversely proportional to the time elapsed. Is he correct? (Hint: Find if the number of particles remaining can be represented by the function $N(t) = \frac{k}{t}$, where k is a constant.)
   (b) Plot a graph of $\ln N$ against t, where N represents the number of particles remaining.
   (c) What does the graph look like, from there on, show that the decay of a radioactive substance can be represented by

$$N = N_0 e^{-kt}$$

   Also, determine the values of N_0 and k for this experiment.
   (d) How long would it take for 98% of the particles to be decayed away?

12. The amount of Internet users in a village over m months ever since the broadband company started providing services can be represented by the following equation

$$U(m) = \frac{1420}{1 + 10e^{-0.4m}}$$

   (a) How many people used the Internet when it was just introduced to the village?
   (b) After a very long time (i.e. $m \to \inf$) How many people in the village will use the internet?
   (c) Assuming that there are 1476 residents in the village, how many months would it take for half of them to become Internet users?

13. The gravitational force is dependent on the distance between the astronomer and the planet. An astromer records the data for Earth to study how the force drops off as distance increases. He obtains the following data:

Distance (km)	Gravitational Force Strength (N Kg^{-1})
6400	10.0
9600	4.44
12600	2.58
16000	1.60
20000	1.02

(a) Plot a graph of $\log g$ against $\log r$, where g is the gravitational force strength and r is the distance from Earth.

(b) Hence show that g and r follow the inverse-square law, which is

$$g = \frac{k}{r^2}$$

Determine the value of the constant k.

(c) From this formula, what happens to the gravitational force for small values of r?

14. (Harder) Find the ratio $\frac{AB}{BC}$ in terms of a and b in the following figure.

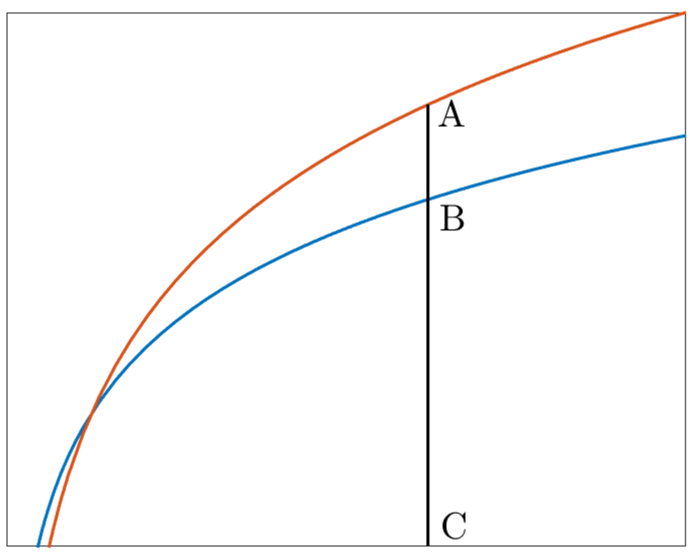

Figure 1: Orange line: $y = \log_a x$; Blue line: $y = \log_b x$