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THE KINETICS OF ELECTRON COOLING OF BEAMS IN HEAVY PARTICLE STORAGE DEVICES

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ABSTRACT

A study is made of the kinetic features of the method of electron cooling of beams in heavy particle storage devices[1]. In the first part of this work, Landau's integral of collisions[2] for Coulomb interaction is used to obtain a kinetic equation for a beam of particles passing periodically through an accompanying stream of electrons. Apart from collisions with electrons, account is taken of scattering on coherent fluctuations (of a non-thermodynamic origin) of the space charge of the electron beam, and on atoms of residual gas. The final kinetic equation is the Fokker-Planck equation in action variables. In the second part, an investigation is made on this basis into the effect, on the kinetic process, of deviations in the electron stream from the state of thermodynamic equilibrium (in the accompanying system). A qualitative investigation is made into the dependence of the attenuation speed and of the stabilized values of angular and energy spread on the velocity distribution of electrons and on the spatial inhomogeneity of the electron beam in a radial direction. An evaluation is made of the permissible level of coherent noise in the electron stream. A solution is obtained for the kinetic equation in the region of small angles ($\theta_p < \theta_e$). An important theorem is established concerning the sum of decrements of the oscillations. An investigation is also made into the nature of the solution in the region of large amplitudes.

INTRODUCTION

Synchrotron radiation, which is used in electron and positron storage rings to obtain dense beams, is, as we know, practically absent in the case of heavy particles. G.I. Budker proposed a method known as the "electron cooling" of beams in heavy particle storage devices[1], based on the transfer of the thermal energy of the beam to an accompanying electron stream whose temperature is much lower. As a first approximation, the kinetic process can be described as the usual relaxation of a two-component plasma[1,2,3]. If the electron stream is constantly renewed, then at a full equalization of the temperatures the angular and relative energy spread of the protons (anti-protons) decreases $\sqrt{M/m}$ times in relation to the electron spread. The "relaxation" time, with an electron density $n = 10^8\ \mathrm{cm}^{-3}$, a proton energy $W = 1$ GeV, an initial angular spread $\theta_p \lesssim \theta_e = 3 \times 10^{-3}$ and a coefficient for filling the orbit with the electron beam $\eta = 0.1$, according to an estimate for Maxwellian distributions of free particles, is ~ 100 sec.

In reality, the kinetics of electron cooling may prove much more complex as a focused beam of protons (antiprotons) moving in the vicinity of a closed orbit cannot be considered as a gas of free particles. On the other hand, the electron beam cannot strictly be considered as a thermostat, since possible (and sometimes inevitable) deviations from the state of thermodynamic equilibrium in the electron current will not be attenuated (unlike what occurs in a closed system) and these may have an influence on the kinetic process.

A kinetic equation which enables proper account to be taken of the effect of the above peculiarities on the kinetics of electron cooling has been obtained in the first part of the present work. In Section 1, Landau's integral of collisions[2] and its relativistic generalization were used as a basis to obtain the collision term for paired collisions of protons with electrons of the beam and atoms of residual gas in action-phase variables. In Section 2 deduction is made of the coherent part of the collision term, which describes particle diffusion on fluctuations of a non-thermodynamic origin in an electron stream. Section 3 provides substantiation for the averaging of equation coefficients according to the phases, and then describes the kinetic process by means of a Fokker-Planck equation in a space of action variables.

In the second part of this work, a study was made, on this basis, of the effect of "non-ideal features" of the electron stream on the attenuation rate of momentum spread and its stabilized width. In Section 1 a qualitative study is made of the part played by deviations in electron distribution in the accompanying system from the Maxwellian. It is shown that Coulomb collisions may lead to instability in the oscillations at individual degrees of freedom if the energy of orderly motion of the electrons exceeds the thermal energy. As a result of the coupling of radial and longitudinal motion in accelerators, the kinetic process is quite sensitive to spatial inhomogeneity of the beam of electrons. In Section 2, evaluations were made of the permissible values of the gradients of the average velocity, density and temperature of the electrons above which the proton oscillations become unstable. In Section 3 an evaluation is made of the permissible level of coherent noise in the electron current and an examination made of other questions. The solution of the kinetic equation in the range of the small angles $\theta_p < \theta_e$ (θ_e is the electron angular spread) is obtained in Section 4. In addition, an important theorem is established concerning the positiveness of the sum of decrements of the oscillations, and its non-dependence on the gradients of electron distribution in the phase space $(\vec{p}, \vec{r})$. Finally, an investigation is made in Section 5 into the nature of the solution in the range $\theta_p > \theta_e$ and the dependence of the attenutation rate on the cross-section of the electron beam.

PART I. THE KINETIC EQUATION

1. THE COLLISION TERM OF PAIRED INTERACTION

The kinetic equation for the distribution function of interacting particles in an external field has the following general form

$$\frac{\partial}{\partial t} f(\vec{p},\vec{r},t) + [\mathcal{H};f] = -\frac{\partial j_\alpha}{\partial p_\alpha} \tag{I.1.1}$$

where $\mathcal{H}$ is the Hamiltonian of a particle in the external field, $\vec{j}$ is the vector of the particle current in the phase space, resulting from the interaction (in Cartesian coordinates $\vec{p}$, $\vec{r}$, the current vector has only momentum components). If a transformation is made, in the equation, from the usual momenta and coordinates to canonical integrals of motion in the external field (for example, action-phase) the Poisson bracket drops out of the equation:

$$\frac{\partial}{\partial t} f(C,t) = -\frac{\partial}{\partial C_i} j_i \; . \tag{I.1.2}$$

The components j_i are linked with j_α by the usual rule of vector transformation:

$$j_i = \frac{\partial C_i}{\partial p_\alpha} j_\alpha \tag{I.1.3}$$

The kinetic equation for a beam of protons can be obtained conveniently in the accompanying system, where the interaction of protons and electrons is Coulombian (the relative velocities are non-relativistic). Since the distribution function is a relativistic invariant[4], there is no need to include a special designation for it in this system.

The right-hand member (I.1.1) can be represented in the form[5,6]

$$div\,\vec{j} = [e\varphi_c\,;f] + div\,\vec{j}^{st} \tag{I.1.4}$$

where ϕ_c is the potential of the self-consistent field of the system of charges,

$\vec{j}^{st}$ is the collision current proper.

A contribution to $\vec{j}^{st}$ is given, not only by the collisions with electrons, but also by the collisions of protons amongst themselves and with atoms of the residual gas. An assessment of the interaction between protons is beyond the scope of the present work. In any case, we shall assume that the density of the beam of protons is sufficiently small for us to be able to ignore their collisions during the time of attenuation.

Let us first examine collisions with electrons. In the Landau approximation[2,5,6], $\vec{j}^{st}$, in the space of usual momenta has the well-known form:

$$j_\alpha = 2\pi e^2 e'^2 L \int d^3p' \frac{u^2\delta_{\alpha\beta} - u_\alpha u_\beta}{u^3}\left(\frac{\partial f}{\partial p_\beta} f' - \frac{\partial f'}{\partial p'_\beta} f\right), \tag{I.1.5}$$

where e and e' are particle charges, $f(\vec{p}, \vec{r}, t)$ and $f'(\vec{p}', \vec{r}, t)$ are distribution functions, $\vec{u} = \vec{v} - \vec{v}'$ is the relative velocity and $L = \ln(\rho_{max}/\rho_{min})$ is the Coulomb logarithm. For the expression (I.1.5) to be correct it is necessary that $L \gg 1$. ρ_{min} is represented by the impact parameter at which the momentum transfer during the collision is $\Delta p \sim p$: $2|ee'|/\rho_{min}u \simeq \mu u$ (μ is the reduced mass). With a relative angular spread in the laboratory system of $10^{-3} - 10^{-2}$, $\rho_{min} \sim 5\ (\beta\gamma)^{-2} \times (10^{-9} - 10^{-7})$ cm. ρ_{max} may be represented by the Debye radius of the electron "plasma" d, the cross-sectional dimension of the beam b or the impact parameter $\rho_0 = u\tau_0$, which is determined by the time-of-flight of the proton through the beam τ_0:

$$\rho_{max} = \min\{d, b, \rho_0\}. \tag{I.1.6}$$

For typical parameters of the electron beam $\rho_{max} \sim 0.1 - 1$ cm. In this case, $L \simeq 20$. The large value of L makes it possible to use (I.1.5) even when spatial inhomogeneity is considerable.

The electron beam can be focused with a longitudinal magnetic field $\vec{H}$ in order to "compensate" the space charge. If the mean Larmor radius of the electrons ρ_Ω proves to be much less than ρ_{max}, the expression (I.1.5) will, generally speaking, be unusable, as it is necessary to allow for

Larmor rotation of electrons (with regard to the protons we shall assume that the distortion of their trajectories in the region of beam interactions can be ignored). In this case, it is necessary to use the general expression $\vec{j}^{st}$, which takes into account the effect of the external field on the collisions[6,7]. In the case of a homogeneous (or slightly inhomogeneous) magnetic field, $\vec{j}^{st}$ can be written in the form:

$$j_\alpha = e^2 e'^2 \int d^3p' d^3k \int_{-\infty}^{0} d\tau \exp\left[i\vec{k}\int_0^{\tau} \vec{u}_{\tau'} d\tau'\right] \frac{k_\alpha \vec{k}}{k^4}\left(\frac{\partial f}{\partial \vec{p}_\tau} f' - f\frac{\partial f'}{\partial \vec{p}'_\tau}\right) \qquad (I.1.7)$$

where $\vec{p}_\tau$, $\vec{p}'_\tau$ are the particle momenta in the magnetic field as a function of time with an initial condition of $\vec{p}_0 = \vec{p}$, $\vec{p}'_0 = \vec{p}'$. Integration over k is contained within the limits of ρ^{-1}_{max} and ρ^{-1}_{min}.

It is convenient to break up $\vec{p}_\tau = \vec{p}$ for protons, and $\vec{p}'_\tau$ for electrons into components which are longitudinal and transverse to the magnetic field:

$$p'_\tau = p'_H + p'_\perp(\tau) .$$

The phase of the exponent in (I.1.7) is

$$\sigma = \tau \vec{k}(\vec{v} - \vec{v}'_H) + \vec{k}_\perp \int_0^{\tau} \vec{v}'_\perp(\tau') d\tau' \equiv \vec{k}\vec{u}_H \tau + \vec{k}_\perp \vec{\tau}'_\perp(\tau). \qquad (I.1.8)$$

In order of magnitude, $|\vec{\tau}'_\perp(\tau)| \sim \rho_\Omega$. In the range $k\rho_\Omega \gg 1$ the small values of τ are substantial ($\omega_\Omega \tau \ll 1$); in this case $\sigma = \vec{k}\vec{u}\tau$; this means that the contribution of this range of impact parameters ($k = \rho^{-1}$) has the form (I.1.5) where $L = L_1 = \ln(\rho_\Omega/\rho_{min})$, $\vec{u} = \vec{v} - \vec{v}'$.

In the range $k\rho_\Omega < 1$ the second term in the expression (I.1.8) can be ignored, which corresponds to the substitution $\vec{u} \to \vec{u}_H = \vec{v} - \vec{v}'_H$. Thus, for this range, $\vec{j}^{st}$ can also be written in the form (I.1.5), where

$$L = L_2 = \ln(\rho_{max}/\rho_\Omega), \quad \vec{u} = \vec{v} - \vec{v}'_H , \quad \rho_{max} \gg \rho_\Omega . \qquad (I.1.9)$$

This structure of $\vec{j}^{st}$ has a simple physical meaning: in the range $k\rho_\Omega \ll 1$ the collisions occur adiabatically in relation to the Larmor rotation of electrons and the degrees of freedom which are transverse to $\vec{H}$ do not contribute to the momentum and energy exchange. (More accurately speaking, the condition of adiabaticity is $ku_H \ll \omega_\Omega$; if $u_H \gg v_T'$, then ρ_Ω in (I.1.9) must be replaced by $\rho_1 = u/\omega_\Omega$).

For future use, it is convenient to write (I.1.5) in the form

$$j_\alpha^{st} = \langle \Delta p_\alpha \rangle^e f - \frac{1}{2}\frac{\partial}{\partial p_\beta}\left(\langle \Delta p_\alpha \Delta p_\beta \rangle^e f\right) \tag{I.1.10}$$

where

$$\langle \Delta p_\alpha \rangle^e = F_\alpha^e + \frac{1}{2}\frac{\partial}{\partial p_\beta}\langle \Delta p_\alpha \Delta p_\beta \rangle \tag{I.1.11}$$

$$\vec{F}^e = \frac{\partial}{\partial \vec{v}}\int d^3p' \frac{u_\beta}{u}\frac{\partial f'}{\partial p'_\beta}\cdot 2\pi e^2 e'^2 L \tag{I.1.12}$$

$$\langle \Delta p_\alpha \Delta p_\beta \rangle^e \equiv d_{\alpha\beta}^e = 4\pi e^2 e'^2 L \int d^3p' f' \frac{u^2\delta_{\alpha\beta} - u_\alpha u_\beta}{u^3}. \tag{I.1.13}$$

In the physical sense, $\vec{F}^e$ is the mean momentum transferred to the particle by the medium in a unit of time, if no account is taken of scattering (to which corresponds $M \to \infty$). As a result, the force $\vec{F}$ is referred to as the dynamic frictional force[8,9]. The tensor of diffusion $d_{\alpha\beta}$, on the contrary, corresponds to the interaction with infinitely heavy moving Coulomb centres. The true mean force is equal to $\langle \Delta\vec{p} \rangle$. As can be seen from (I.1.12) the force $\vec{F}^e$ has a potential in the velocity space:

$$\vec{F}^e = -\frac{\partial}{\partial \vec{v}}\,\mathcal{U}(\vec{v}, \vec{r}) \tag{I.1.14}$$

where

$$\mathcal{U}(\vec{v},\vec{r}) = \frac{2\pi e^2 e'^2}{m} L \int d^3p' f' \left(\frac{\partial}{\partial \vec{v}'}, \frac{\vec{u}}{u}\right) = \frac{4\pi e^2 e'^2}{m} L \varphi_{\vec{v}} . \qquad (I.1.15)$$

For a current $\vec{j}_1$, when $\vec{u} = \vec{v} - \vec{v}'$, $\phi_{\vec{v}}$ as a function of the velocity $\vec{v}$ has the form of a potential of attraction, created by the distributed Coulomb sources:

$$\varphi_{\vec{v}} = -\int \frac{d^3p'}{u} f' ; \qquad \vec{\nabla}^2_{\vec{v}} \varphi_{\vec{v}} = 4\pi m^3 f' . \qquad (I.1.16)$$

This analogy, thanks to its familiarity, can conveniently be used for practical calculations and evaluations.

For a current $\vec{j}_2$, when $\vec{u} = \vec{v} - \vec{v}'_H$, $\phi_{\vec{v}}$ has no Coulomb analogue:

$$\varphi_{\vec{v}} = \frac{1}{2} \int f' d^3p' \left(\frac{\partial}{\partial \vec{v}'_H}, \frac{\vec{u}_H}{u_H}\right) = -\frac{1}{2} v_\perp^2 \int \frac{d^3p'}{u_H^3} f' = -\frac{\vec{v}_\perp}{2} \frac{\partial}{\partial \vec{v}_\perp} \varphi'_H , \qquad (I.1.17)$$

where ϕ_H is the potential (I.1.16) created by the δ-type distribution in a direction transverse to the magnetic field $f'(\vec{p}'_H,\vec{r}) = \delta(\vec{p}'_\perp)\int d^2p'_\perp f'(\vec{p}',\vec{r})$.

Let us now obtain $\vec{j}_0^{st}$ for collisions with atoms of residual gas. Its influence should be in lateral scattering of protons on nuclei and deceleration (entrainment in the accompanying system) on electrons. As the relative velocities of protons and atoms may be relativistic, it is necessary to use the expression $\vec{j}^{st}$ obtained in the relativistic case by Belyaev and Budker[4]:

$$j_\alpha = 2\pi e^4 \sum_a Z_a^2 L_a \int d^3p' S_{\alpha\beta} \cdot \left(\frac{\partial f}{\partial p_\beta} f_a - \frac{\partial f_a}{\partial p'_\beta} f\right)$$

where Z_a is the charge of type "a" particles, L_a is the corresponding Coulomb logarithm, $S_{\alpha\beta}$ is the Stoss-matrix, for which a cumbersome expression covering the general case is given in Ref. 4. In our case, we can obtain a simple expression for $S_{\alpha\beta}$, if we take advantage of the small velocity of the protons $\vec{v}$ in relation to the velocity of the gas v' and eliminate in $S_{\alpha\beta}$ the members of the order v/v', v/c and higher,

$$S_{\alpha\beta} = \frac{v'^2\delta_{\alpha\beta} - v'_\alpha v'_\beta}{v'^3} + O\left(\frac{v}{v'}, \frac{v}{c}\right) + \dots$$

If we take, for the particles of residual gas, the distribution function in the form $f_a = n_a\delta(\vec{p}' + \gamma m_a\vec{v}_0)$, where $\vec{v}_0$ is the mean proton velocity in the laboratory system, we will obtain

$$\langle\Delta p_\parallel\rangle_0 = F_{0\parallel} = -4\pi e^4 \sum_a \frac{L_a Z_a^2 n_a}{\gamma m_a v_0^2}\ ; \quad \langle\Delta\vec{p}_\perp\rangle = 0 \tag{I.1.18}$$

$$\begin{aligned} &\langle(\Delta p_z)^2\rangle_0 = \langle(\Delta p_{\bar z})^2\rangle_0 = 4\pi e^4 \sum_a L_a Z_a^2 \frac{n_a}{v_0}\ , \\ &\langle(\Delta p_\parallel)^2\rangle_0 = \langle\Delta p_\alpha \Delta p_\beta\rangle_{0\,\alpha\neq\beta} = 0\ ; \quad \left(\vec{p}_\perp = \{p_z, p_{\bar z}\}\right) \end{aligned} \tag{I.1.19}$$

as must be the case.

It is sufficient, in the expression $\langle\Delta p_\parallel\rangle_0$ to take into account only the electron component, in view of the electron's small mass. The scattering, on the contrary, occurs mainly on the nuclei.

2. SCATTERING ON COHERENT FLUCTUATIONS OF THE SPACE CHARGE

Coulomb scattering may not be the only mechanism of diffusion (or "heating") of the proton beam in an electron current. Apart from the basic, stationary part, it is also necessary to take into account, in the space charge field, an irregular "random" part which is associated with collective fluctuations in density and velocity, caused by sources of an "external" origin (oscillations in the controlling voltage, cathode scintillation, etc.).

We shall first make a formal deduction of the expression of the corresponding collision term, departing in (I.1.4) from the stationary part of the potential $\phi_c = \phi_c(\vec{r}) + \tilde{\phi}_c(\vec{r},t)$ and the Coulomb collisions. In accordance with the theory of disturbances, we shall expand $f(\vec{p},\vec{r},t)$ into a series by degrees of $\tilde{V} = e\tilde{\phi}_c$:

$$f = f^{(0)} + f^{(1)} + f^{(2)} + \dots \qquad \text{(I.2.1)}$$

Since the coherent fluctuation times may not be small in comparison with the period of movement in the external field, it is convenient to change over right away, in the kinetic equation, to the C_i variables:

$$\frac{\partial}{\partial t} f + [\tilde{V}; f] = 0 .$$

If we insert here the expansion (I.2.1) we will obtain:

$$\frac{\partial}{\partial t} f^{(1)} + [\tilde{V}; f^{(0)}] = 0 \;, \; \frac{\partial}{\partial t} f^{(2)} + [\tilde{V}; f^{(1)}] = 0$$

The average speed of variation of $f(C,t)$ is

$$\overline{\frac{\partial}{\partial t} f} = \overline{\frac{\partial}{\partial t} f^{(2)}} = \overline{\left[\tilde{V}; \left[\int_{-\infty}^{t} \tilde{V}(C,t')dt'; f\right]\right]} \qquad \text{(I.2.2)}$$

since, by definition

$$\overline{\frac{\partial}{\partial t} f^{(1)}} = \left[\overline{\tilde{V}}; f^{(0)}\right] = 0 .$$

We shall introduce the designation

$$\tilde{S} = \int_{-\infty}^{t} \tilde{V}(C,t')dt'$$

and transform (I.2.2), using the characteristics of the Poisson coefficient:

$$\frac{\partial}{\partial t} f - \frac{1}{2}\frac{d}{dt}\overline{[\tilde{S};[\tilde{S};f]]} - [U_c;f] = 0 \tag{I.2.3}$$

where $U_c = \frac{1}{2}[\tilde{V};\tilde{S}]$ is the correlative potential[10].

The last member in (I.2.3) is the usual Poisson coefficient and does not contribute to diffusion: the potential U_c can be considered as a minor correction to the regular part of $e\phi_c$. If we use the notation of the Poisson coefficient in the form of a divergence of the current vector in the phase space we will obtain the diffusion equation

$$\frac{\partial}{\partial t} f - \frac{1}{2}\frac{\partial}{\partial C_i}\overline{\left\{\frac{d}{dt}(\tilde{\Delta} C_i \tilde{\Delta} C_K)\right\}}\frac{\partial f}{\partial C_K} \tag{I.2.4}$$

where

$$\tilde{\Delta} C_i = [\tilde{S};C_i] \ .$$

If, as usual, we change over from time averaging to probability averaging, the scattering tensor in (I.2.4) can be written in the form

$$\tilde{D}_{iK} = \langle [\tilde{V};C_i][\tilde{S};C_K] + [\tilde{V};C_K][\tilde{S};C_i] \rangle.$$

$\tilde{D}_{ik}$ can be expressed explicitly by fluctuations in the electrical field:

$$[\tilde{V};C_i] = e\tilde{\Delta}\vec{E}\frac{\partial C_i}{\partial \vec{p}}; \quad [\tilde{S};C_i] = e\int_{-\infty}^{t}\tilde{\Delta}\vec{E}(\vec{r}_{t'},t')\frac{\partial C_i}{\partial p_{t'}}dt'$$

$$\tilde{D}_{iK} = e^2\int_{-\infty}^{t}\langle\tilde{\Delta}E_\alpha(\vec{r},t)\tilde{\Delta}E_\beta(\vec{r}_{t'},t')\rangle\left(\frac{\partial C_i}{\partial p_\alpha}\frac{\partial C_K}{\partial p_{\beta t'}} + \frac{\partial C_K}{\partial p_\alpha}\frac{\partial C_i}{\partial p_{\beta t'}}\right) \tag{I.2.5}$$

The mean < ... > in (I.2.5) is, by definition, a correlation function of fluctuations in the field $K_{\alpha\beta}(\vec{r}|\vec{r}',t|t')$. Provided there is a spatial and temporal homogeneity $K_{\alpha\beta}$ is a function of the difference of the arguments and may be expressed by the spectral density of the fluctuations[9,11]:

$$\mathcal{K}_{\alpha\beta}=\sum_{\vec{k},\omega}(E_\alpha E_\beta^*)_{\vec{k},\omega}\cdot \exp[i\vec{k}(\vec{r}-\vec{r}')-i\omega(t-t')] \qquad (I.2.6)$$

in this case

$$\langle \Delta E_\alpha(\vec{r},t)\Delta E_\beta(\vec{r},t)\rangle=\sum_{\vec{k},\omega}(E_\alpha E_\beta^*)_{\vec{k},\omega} \qquad (I.2.7)$$

In the spatially inhomogeneous case, $K_{\alpha\beta}$ is not a function only of the difference $\vec{r}-\vec{r}'$. The form of notation of (I.2.6) and (I.2.7) can be retained if the "local" spectral density is introduced.

$$(E_\alpha E_\beta^*)^{\vec{r}}_{\vec{k},\omega}\cdot\delta(\omega-\omega')=\int d^3k'\langle E_\alpha(\vec{k}',\omega')E_\beta^*(\vec{k},\omega)\rangle e^{i(\vec{k}'-\vec{k})\vec{r}} \qquad (I.2.8)$$

(the correlation of Fourier harmonics is now not proportional to $\delta(\vec{k}-\vec{k}')$. The diffusion tensor in the space of integrals C_i can thus be written in the form:

$$\tilde{D}_{ik}=e^2\sum_{\vec{k},\omega}(E_\alpha E_\beta^*)^{\vec{r}}_{\vec{k},\omega}\int_{-\infty}^{0}d\tau\cdot\exp[i\vec{k}(\vec{r}-\vec{r}_\tau)+i\omega\tau]A^{\alpha\beta}_{ik} \qquad (I.2.9)$$

where $A^{\alpha\beta}_{ik}$ is the expression in curved brackets in (I.2.5).

As we know[9], the integral of paired collisions (I.1.10) can also be obtained in the spirit of the general method for composing the Fokker-Planck equation, taking into account the "radiation" frictional force, calculated from the undisturbed movement of the charge, and scattering on thermodynamic fluctuations of the field of the medium. In the case of a homogeneous plasma the expressions $\vec{F}$ and $d_{\alpha\beta}$, which take into account dynamic polarization of the medium by interacting particles, have the form

$$F_\alpha=2e^2e'^2\sum_{\vec{k},\omega}\frac{k_\alpha k_\beta}{k^4|\varepsilon_{\parallel}|^2}\int d^3p'\frac{\partial f'}{\partial p'_\beta}\delta(\omega-\vec{k}\vec{v}')\cdot\delta(\omega-\vec{k}\vec{v}) \qquad (I.2.10)$$

$$d_{\alpha\beta} = \pi e^2 \sum_{\vec{k},\omega} (E_\alpha E_\beta^*)^{T'}_{\vec{k},\omega} \cdot \delta(\omega - \vec{k}\vec{v}) \qquad (I.2.11)$$

where

$$(E_\alpha E_\beta^*)^{T'}_{\vec{k},\omega} = \frac{2 K_\alpha K_\beta}{\pi K^4 |\varepsilon_{\parallel}|^2} \int d^3p' \, f' \cdot \delta(\omega - \vec{k}\vec{v}') \qquad (I.2.12)$$

$\varepsilon_{\parallel}(\vec{k},\omega)$ is the electrical permeability[6,9,12]. The factor $|\varepsilon_{\parallel}|^{-2}$ takes into account Debye screening ($|\varepsilon_{\parallel}| \sim 1 + (kd)^{-2}$). (I.2.10) and (I.2.11) coincide, with logarithmic accuracy, with (I.1.12) and (I.1.13).

It can be seen from a comparison of (I.2.9) and (I.2.11) that (I.2.9) may be a general expression of the tensor of diffusion through the total spectral density, including both (I.2.12) and an "epithermal" part.

In the case of a thermodynamically stable plasma, when $(E_\alpha E_\beta^*)_{\vec{k},\omega} = (E_\alpha E_\beta)^{T'}_{\vec{k},\omega}$, taking into account the absorption of the energy of the fluctuation field leads to thermal equilibrium of the group of test particles (protons) with the medium: T = T'. But if waves with random phases (coherent fluctuations) are now induced in the plasma, the spectral density may considerably exceed (I.2.12). Physically, this means that the energy of the waves is comparable or much greater than the Debye energy of the Coulomb interaction of charges of the plasma:

$$\langle (\tilde{\Delta} \vec{E})^2 \rangle \gtrsim \frac{T'}{d^3} \qquad (I.2.13)$$

i.e. it exceeds the thermodynamic equilibrium value[9,13] (at the same time, if the representation concerning the waves is to be applicable, it is necessary that $\langle (\tilde{\Delta}\vec{E})^2 \rangle \ll nT'$, i.e. when the waves are taken into account, there must be no variation in the "gross" characteristics of the medium). The diffusion rate increases accordingly. The rate of the Coulomb losses

$$-\left\langle \frac{dW}{dt} \right\rangle = -\vec{F} \cdot \vec{v}$$

practically does not change, since the action of the particle on the medium is due principally to paired collisions, on which the presence of "weak" coherent noise does not have any significant influence. In this way, the coherent fluctuations raise the final temperature of the proton beam, without affecting the attenuation rate. However, if attenuation is to take place at all, it is essential that the maximum value of Coulomb losses exceeds the diffusion rate: $|\vec{F}\vec{v}|_{v \sim v'_{T'}} \gg \langle (\Delta p)^2 \rangle / 2M$.

Let us now discuss certain special features of an interaction with coherent noises which are related to the specific problem. In an isolated volume of plasma, the spectral density of the waves is concentrated in the range $k \ll d^{-1}$, where a development of instabilities is possible, whilst in the range $k \gtrsim d^{-1}$ the waves are quickly attenuated. Consequently, the coherent part of the spectrum $(E^*_\alpha E_\beta)_{\vec{k},\omega}$ is sharply separated from the thermodynamic part [interaction with "plasmons" [13]]. In our case, when the time-of-flight of the proton through the beam may be comparable with the period of Langmuir oscillations $d/v'_{T'}$, and the beam and "waves" in it are constantly being renewed, the spectral density may be substantially different from (I.2.12) also in the range $k > d^{-1}$, since there is not enough time for the waves to be attenuated. For the same reason, possible plasma instabilities cannot play a substantial part either, since their times are normally $\tau \gg d/v'_{T'}$. The fluctuations caused by an "external" source may only be deformed substantially when spreading occurs in the electron current. Another special feature is due to the oscillatory character of proton motion near the equilibrium orbit: slow variations in current and density in the beam with frequencies of $\omega \ll \omega_i$ (ω_i are the partial frequencies of proton oscillations) are adiabatic in relation to the oscillation of protons and do not lead to an increase in amplitudes. For a given spectral density, this is automatically achieved by the structure of the tensor of diffusion (I.2.9).

3. AVERAGING BY PHASES

Let us now write, in C_i variables, a full kinetic equation which takes into account collisions with electrons of the beam, residual gas and scattering on coherent noise,

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial C_i}\left\{Q_i f - \frac{1}{2} D_{ik} \frac{\partial f}{\partial C_k}\right\} = 0 \,, \qquad (I.3.1)$$

where

$$Q_i = \frac{\partial C_i}{\partial \vec{p}}(\vec{F}^e + \vec{F}_o) \, ,$$

$$D_{ik} = \frac{\partial C_i}{\partial p_\alpha}\frac{\partial C_k}{\partial p_\beta}(d^e_{\alpha\beta} + d^o_{\alpha\beta}) + \widetilde{D}_{ik} \, . \qquad \text{(I.3.2)}$$

As canonical integrals of motion we shall take three pairs of conjugate variables, action I_i phase ϕ_i, through which radial, axial and longitudinal deviations of the coordinates and momenta from the equilibrium phase trajectory are expressed as follows[14]:

$$z = z_b + z_c \, ; \quad p_z = \frac{p_s}{R}\cdot\frac{\partial z}{\partial \theta_s}$$

$$z_b = \frac{R}{p_s}\sqrt{\tfrac{1}{2}I_z}\, f_z(\theta) \exp[i\nu_z\theta_s + i\varphi_z] + \text{K.C.}\, ; \quad z_c = \frac{\gamma R}{p_s}\Psi(\theta)\cdot p_{\parallel}$$

$$z = \frac{R}{p_s}\sqrt{\tfrac{1}{2}I_z}\, f_z(\theta) \exp[i\nu_z\theta_s + i\varphi_z] + \text{K.C.}\, ; \quad p_z = \frac{p_s}{R}\frac{\partial z}{\partial \theta_s} \qquad \text{(I.3.3)}$$

$$\vartheta = \theta - \theta_s = \vartheta_c + \vartheta_b \, ; \quad \vartheta_b = \frac{1}{R}\left(\Psi\frac{\partial z_b}{\partial \theta_s} - z_b\frac{d\Psi}{d\theta_s}\right),$$

$$\left|\frac{d\vartheta_c}{d\theta_s}\right| \sim \left|\frac{p_{\parallel}}{p_s}\right| \ll 1 \, .$$

Here, $\theta_s = \omega_s t$, $p_s = \gamma M\beta C$, $2\pi R$ are, respectively, the azimuth, momentum and length of the orbit of the equilibrium particle, f_r and f_z are Floquet functions, Ψ is the obligatory solution of the equation

$$\frac{d^2\Psi}{d\theta^2} + [1 - n(\theta)]\Psi = R/R(\theta),$$

$p_{\parallel}$ is the longitudinal momentum in the accompanying system. In the absence of an RF field $p_{\parallel} = I_{\parallel}$,

$$\frac{d\vartheta_c}{d\theta} = \sim \frac{p_{\parallel}}{p_s} = Const,$$

but in the self-phasing condition

$$p_{\parallel} = -\sqrt{2I_c}\,\sin(\nu_c\theta_s + \varphi_c)\, , \quad \frac{\partial I_c}{\partial p_{\parallel}} = p_{\parallel} \, . \qquad \text{(I.3.4)}$$

Together with the coordinates and momenta of particles, the coefficients of the kinetic equation (I.3.1) are periodic functions of "fast" phases $\Psi_i = \omega_{it} + \phi_i$. When considered as functions of I and ϕ they oscillate with time, in which case the oscillations of a relatively average level generally cannot be considered small. It is, however, possible to replace (I.3.1) by a much simpler equation with coefficients which are not time-dependent, if the variation of f(C,t) over a time of the order ω_i^{-1} is small. For this we shall apply to (I.3.1) the averaging method, and write the equation in the following form, for shortness:

$$\frac{\partial}{\partial t} f + \hat{L}(t) f = 0$$

The operator $\hat{L}(t)$, as a function with a line-spectrum, can be represented in the form

$$\hat{L}(t) = \bar{\hat{L}} + \tilde{\hat{L}}(t) ,$$

where $\bar{\hat{L}}$ represents the mean value of $\hat{L}$ over the period $T_0 \sim \omega_i^{-1}$, $\tilde{\hat{L}}$ is its oscillating part. (By the period T_0 we should understand a sufficiently large period of time such that

$$T_0^{-1} \left| \int_t^{t+T_0} \hat{L}\, dt' \right| \ll \left| \bar{\hat{L}} \right|$$

in this sense, the operator $\tilde{\hat{L}}$ can be treated as a periodic function of time, with a mean value equal to zero). The averaged equation, which gives the correct change in the function f over the period T_0 with an accuracy up to members of the second order, is

$$\frac{\partial}{\partial t} f + \bar{\hat{L}} f + \frac{1}{2} \overline{[\hat{L} \hat{M}]} f = 0 \qquad \text{(I.3.5)}$$

where $[\hat{L}\hat{M}]$ is the commutator of the operators $\hat{L}$ and $\hat{M}$ where $\hat{M}$ is given by

$$\hat{M} = \int_{t_0}^{t_0+t} \tilde{\hat{L}}\, dt' \quad .$$

If the third member in (I.3.5) represents a minor correction ($\hat{M} \ll 1$), taking it into account cannot change the nature of the solution even over great lengths of time, since the solution of the kinetic equation with time either tends to a stable stationary, or diverges. It is sufficient to use the equation

$$\frac{\partial}{\partial t} f = -\bar{\hat{L}} f = -\frac{\partial}{\partial C_i}\left(\bar{Q}_i f - \frac{1}{2}\bar{D}_{ik}\frac{\partial f}{\partial C_k}\right). \tag{I.3.6}$$

The condition $\hat{M} \ll 1$ means that the relaxation time $\tau \gg T_0$, since $\hat{M} \sim T_0\hat{L}$, and $\hat{L} \sim \tau^{-1}$, in a physical sense. An averaging of the coefficients of the kinetic equation for the time "on the trajectory" with this condition provides the basis for the so-called anti-diffusion approximation[10].

In practice, averaging for time can almost always be replaced by averaging for phases Ψ_i. The operator $\hat{L}$ as a function of phases can be represented by a Fourier series:

$$\hat{L}(I,\Psi) = \sum_{\{m\}} \hat{L}_{\{m\}} \exp\left[i \sum_i m_i \Psi_i\right].$$

The frequencies of variations in the harmonics with time are

$$\omega_{\{m\}} = \sum_i m_i \omega_i .$$

Apart from the "zero" harmonic $m_i = 0$, which represents simply the mean value for the phases $\hat{L}$, when averaging for time is effected, non-zero contributions to $\bar{\hat{L}}(C,t)$ may also come, generally speaking, from harmonics with frequencies of

$$\omega_{\{m\}} \lesssim \tau^{-1} \tag{I.3.7}$$

Formally speaking, it is always possible to select the combination of $\{m\}$ such that $\omega\{m\}$ can be made as small as desired. However, the condition (I.3.7) can in the general case be justified for quite high m only, if consideration is made for the fact that $\omega_i\tau \gg 1$, and the frequencies ω_i, as a rule, do not form, among themselves, rational relations of a low order. Consequently, the values of such harmonics will be small enough to be ignored. In reality, it is necessary to consider also the non-linearity of particle oscillations, which leads to the dependence of frequencies on amplitudes. In spite of the relatively small variation of frequency with amplitude:

$$|\Delta\omega_i| = \left|\frac{\partial\omega_i}{\partial I_K}\Delta I_K\right| << \omega_i ,$$

this dependence leads to the violation of the "resonance" condition (I.3.7) when, under the effect of collisions, I will receive an increment ΔI such that

$$|\Delta\omega_{\{m\}}| > \tilde{\tau}^{-1} .$$

This circumstance is particularly substantial when estimating the part played by resonating harmonics of a low order, which may be comparable in size with the zero harmonic. An exception to this may be those cases when the resonance condition is maintained by self-phasing, which occurs as a result of an "external" distrubance, for example, the electron beam field. In the case of non-linear resonances this phenomenon may be ignored if the phase volume inside the separatrix of the resonance is relatively small.

Thus, provided that $\tau >> T_0$, it may almost always be considered that the kinetic coefficients $\overline{Q}_i$ and $\overline{D}_{ik}$ do not depend on the phases ϕ_i. This makes it possible to change over to a much simpler equation for the function of three variables I_i, after having integrated equation (I.3.6) for phases:

$$\frac{\partial}{\partial t}f + \frac{\partial}{\partial I_i}\left\{\overline{Q}_i - \frac{1}{2}\overline{D}_{ik}\frac{\partial f}{\partial I_K}\right\} . \qquad \text{(I.3.8)}$$

The sign (-) now indicates averaging for phases of transverse and longitudinal oscillations, and for the azimuth on the equilibrium proton orbit.

The equation (I.3.8) can be conveniently written in the standard form of Fokker-Planck[15,16]:

$$\frac{\partial}{\partial t}f + \frac{\partial}{\partial I_i}\left\{\langle\Delta I_i\rangle f - \frac{1}{2}\frac{\partial}{\partial I_K}\left(\langle\Delta I_i\Delta I_K\rangle f\right)\right\} = 0 , \qquad \text{(I.3.9)}$$

where

$$\langle\Delta I_i\rangle = \overline{Q}_i + \frac{1}{2}\frac{\partial}{\partial I_K}\langle\Delta I_i\Delta I_K\rangle \equiv \langle\Delta I_i\rangle_d + \langle\Delta I_i\rangle_{fl} . \qquad \text{(I.3.10)}$$

$$\langle \Delta I_i \Delta I_k \rangle = \overline{D}_{ik} . \qquad \text{(I.3.11)}$$

In the physical sense $< \Delta I_i >_d = \overline{Q}_i$ gives the speed of $\dot{I}$ on account of dissipative processes ($\overline{Q}_i$ is the "power" of the frictional force) and $< \Delta I_i >_{f\ell}$ describes the average increment of I_i in a unit of time on account of absorption of the energy of the fluctuation field of the "medium". In the oscillation condition, when I_i defines the energy of the oscillator, the values of $< \Delta I_i >$ characterize, for a group of particles with close values of I_i, the direction of the kinetic process as a whole. In the case of infinity of motion, when $I \sim p$, the moments of (I.3.11) also are important in this sense.

The coefficients $\partial I_i/\partial p_\alpha$, which are necessary for composing (I.3.10-I.3.11), can be found directly from the expressions (I.3.3-I.3.4) by making use of the characteristic of canonical transformation of $(\partial I_i/\partial p_\alpha) = (\partial q_\alpha/\partial \Psi_i)$. Let us, furthermore put forward the usual expressions of $< \Delta I_i >$ in terms of the moments $< \Delta p_\alpha >$ and $< \Delta p_\alpha \Delta p_\beta >$ [these are, naturally, identical to (I.3.10)]. Taking (I.3.3) into account we obtain

$$\langle \Delta I_r \rangle = \overline{\frac{\partial I_r}{\partial p_r}\langle \Delta p_r \rangle} + \overline{\frac{\partial I_r}{\partial p_\parallel}\langle \Delta p_\parallel \rangle} + \frac{1}{2}\overline{\frac{\partial^2 I_r}{\partial p_r^2}\langle (\Delta p_r)^2 \rangle} + \frac{1}{2}\overline{\frac{\partial^2 I_r}{\partial p_\parallel^2}\langle (\Delta p_\parallel)^2 \rangle} + \overline{\frac{\partial^2 I_r}{\partial p_r \partial p_\parallel}\langle \Delta p_r \Delta p_\parallel \rangle} ; \qquad \text{(I.3.12)}$$

$$\langle \Delta I_z \rangle = \overline{\frac{\partial I_z}{\partial p_z}\langle \Delta p_z \rangle} + \frac{1}{2}\overline{\frac{\partial^2 I_z}{\partial p_z^2}\langle (\Delta p_z)^2 \rangle} ; \qquad \text{(I.3.13)}$$

$$\langle \Delta I_c \rangle = \overline{\frac{\partial I_c}{\partial p_\parallel}\langle \Delta p_\parallel \rangle} + \frac{1}{2}\overline{\frac{\partial^2 I_c}{\partial p_\parallel^2}\langle (\Delta p_\parallel)^2 \rangle} . \qquad \text{(I.3.14)}$$

The amplitude of the radial betatron oscillations $a_r \sim \sqrt{I_r}$, as can be seen from (I.3.12), changes under the effect of impacts not only in a radial, but also in an azimuthal direction, since the position of equilibrium, determined by the total energy (momentum) of the proton, changes, in this case, by successive jumps (in contra-distinction to the adiabatically slow oscillations in synchrotron motion).

PART II THE KINETICS OF ELECTRON COOLING

Using Eq. (I.3.9) we shall now investigate the basic characteristics of the kinetic process, associated with deviations from thermodynamic equilibrium in an electron current. Here the most important factors seem to be the attenuation rate and stabilized distribution on the shape of the electron velocity distribution and spatial inhomogeneity of the beam of electrons, and heating of the proton beam by coherent "noise" in the electron current.

1. THE "MONOCHROMATIC" INSTABILITY

Let us first establish the general nature of the dependence of the attenuation rate and stabilized mean amplitudes on the shape of the velocity distribution of electrons.

For greater clarity, we shall disregard scattering on coherent noise and residual gas, and consider the electron current as being spatially homogeneous. We shall also ignore, for simplicity, the azimuthal inhomogeneity of proton focusing ($|f_r|$ = Const, $|f_z|$ = Const).

In these conditions, the expressions of the kinetic coefficients (I.3.12-I.3.14) take the form:

$$\langle \Delta I_z \rangle = \overline{p_z F_z^e} + \frac{1}{2}\overline{d_{zz}^e} \,, \qquad \text{(II.1.1)}$$

$$\langle \Delta I_r \rangle = \overline{p_r F_r^e} + \frac{1}{2}\overline{d_{rr}^e} + \frac{\gamma^2}{2\nu_r^2}\overline{d_{\parallel}} \,, \qquad \text{(II.1.2)}$$

$$\langle \Delta I_c \rangle = \overline{p_{\parallel} F_{\parallel}^e} + \frac{1}{2}\overline{d_{\parallel}^e} \,, \qquad (d_{\parallel} = \langle (\Delta p_{\parallel})^2 \rangle) . \qquad \text{(II.1.3)}$$

As has been pointed out (I.1.14-I.1.16), the frictional force $\vec{F}^e(\vec{v})$ is the analogue of the field of attraction, created by the distribution of Coulomb sources $f'(\vec{v})$. From this we can immediately obtain the behaviour of $\vec{F}^e(\vec{v})$, when the distribution in the accompanying system is close to the Maxwellian $f'(\vec{v}) = f'_{T'}(\vec{v})$:

$$\vec{F}^e \sim -\frac{1}{m}\begin{cases} \dfrac{\vec{v}}{v'^3_{T'}} & v < v'_{T'} \\ \dfrac{\vec{v}}{v^3} & v > v'_{T'} \end{cases} \tag{II.1.4}$$

$d_{\alpha\alpha}$ can also be easily evaluated:

$$d_{\alpha\alpha} \sim \begin{cases} (v'_{T'})^{-1} & v < v'_{T'} \\ v^{-1} & v > v'_{T'} \end{cases} \tag{II.1.5}$$

The stabilized value of v_i^2 is found from the condition $\langle \Delta I_i \rangle = 0$:

$$\overline{v_i^2} \sim \frac{m}{M} v'^2_{T'} \quad ,$$

as must be the case.

Let us now examine the case where the Maxwellian distribution is "shifted" in the accompanying system by $\langle \vec{v}' \rangle = \vec{\Delta}$: $f' = f_{T'}(\vec{v}' - \vec{\Delta})$, (the error in the mean velocity of the electrons). If $\Delta < v'_{T'}$, then (II.1.4) and (II.1.5) remain correct, since the mean value of the frictional force $\langle \vec{F} \rangle = \vec{\Delta}/mv'_{T'}$ does not contribute to $\overline{Q}_i$ [the shift $\Delta < v'_{T'}$ does not alter the characteristic of friction which determines the decrement of the attenuation].

A quite different situation occurs if $\Delta > v'_{T'}$. Let the error $\vec{\Delta}$ be directed along the normal degree of freedom 1. Let us find the mean power of friction on this degree for small oscillations:

$$\overline{Q}_1 \sim -\frac{M}{m} \overline{v_1 \frac{v_1 - \Delta}{\left[(v_1-\Delta)^2 + v'^2_{T'}\right]^{3/2}}} \simeq 2\frac{M}{m}\frac{\overline{v_1^2}}{\Delta^3} \tag{II.1.6}$$

if

$$\Delta \gg v'_{T'} \text{ and } |\Delta| \gg |v_1| .$$

Thus, $\overline{Q}_i$ reverses sign, and the oscillations start to build up for this degree. The "transverse" degrees of freedom remain stable:

$$\overline{(Q_{i\alpha})}_\perp \sim -\frac{M}{m}\frac{\overline{v_\alpha^2}}{\Delta^3} \quad , \quad \langle \Delta I_i \rangle_{fl} \sim \frac{1}{\Delta} \ . \tag{II.1.7}$$

The reason for the appearance of the instability in the direction $\vec{\Delta}$ (here it is a substantial fact, however, that $\vec{\Delta}$ is directed along the normal degree of freedom) is that when there is a large shift in the mean velocity $\Delta > v'_{T'}$, small oscillations enter into the region where the characteristic friction is negative, and a build-up of oscillation energy occurs (Fig. 1). In spite of the evaluation (II.1.6) being based upon the condition $\Delta >> v'_{T'}$, it is clear that for the occurrence of the instability of small oscillations it is sufficient that the sign of the characteristic of friction changes, when the shift takes place. For degrees of freedom which are lateral to $\vec{\Delta}$, the appearance of an error is equivalent to a rise in the temperature of the electron beam in the relation $(\Delta/v'_{T'})^2$, whereas the characteristic of friction remains positive.

In the condition $\Delta >> v'_{T'}$, if small oscillations ($v < \Delta$) are considered, the non-monochromaticity of the electron beam may be ignored. The instability which occurs then has a simple interpretation: there is a pendulum around which flows a "wind", the frictional force having a negative characteristic and being small in relation to the elastic force. The oscillations then occur around an almost unchanged position of equilibrium ($x = 0$), but become unstable: the energy build-up over the half-period of movement "with the wind" exceeds the losses during movement "against it".

Let us now evaluate the stabilized mean amplitudes. As can be seen from Fig. 1 the build-up continues, in any case, until the amplitude value of the velocity v_1^0 approaches Δ:

$$\frac{\Delta - v_1^0}{v'_{T'}} \sim \frac{v'^2_{T'}}{\Delta^2} << 1$$

(as shown below, $u_L \simeq v'_{T'}$).

As the amplitude continues to grow the phase section $v_1^0 \sin \Psi_1 > \Delta$, begins to play a part, in which the frictional force has a different sign, and, when averaged, compensates the section $0 \leq v_1^0 \sin \Psi_1 \leq \Delta$. The stable amplitude v_1^S lies, apparently, in the region $\Delta < v_1^0 \lesssim 2\Delta$. Large amplitudes are attenuated to v_1^S.

It is an important fact that the amplitude spread is not of the order of an equilibrium value, as in the case of stability, but much less, since the mean amplitude in this case is not determined by the diffusion rate. If $v_1^S > \Delta$ then, approximately

$$\bar{Q}_1 \sim -\frac{M}{m} \overline{v_1 \frac{v_1 - \Delta}{\left[(v_1 - \Delta)^2 + v_{T'}'^2\right]^{3/2}}} \sim -\frac{M}{\pi m v_1^0} \ln \frac{v_1^0}{v_{T'}'} \qquad (v_1^0 > v_1^S) \qquad \text{(II.1.8)}$$

$$\langle \Delta I_1 \rangle_{fl} \sim \overline{v_{T'}'^2 \left[(v_1 - \Delta)^2 + v_{T'}'^2\right]^{-\frac{3}{2}}} \simeq \frac{1}{\pi v_1^S} \qquad \text{(II.1.9)}$$

i.e.

$$\left(\overline{|I_1 - \bar{I}_1|}/\bar{I}_1\right) \sim \left(\langle \Delta I_1 \rangle_{fl} / |\bar{Q}_1|\right) \sim \frac{m}{M} \ll 1.$$

This feature distinguishes dissipative "heating" from thermal heating. Energy is transferred to the oscillator, not from the thermal motion of the electron current but from its orderly motion. Consequently the amplitude distribution is also concentrated in a narrow range near to the mean value.

For oscillations in a direction transverse to $\vec{\Delta}$,

$$(Q_\alpha)_\perp \sim -\frac{M}{m} \overline{v_\alpha^2 \left[(v_1 - \Delta)^2 + v_{T'}'^2\right]^{-\frac{3}{2}}} \simeq \frac{M}{m} \cdot \frac{v_\alpha^2}{\pi v_{T'}'^2 v_1^S},$$

$$\langle \Delta I_\alpha \rangle_{fl} \sim \overline{(v_1 - \Delta)^2 \left[(v_1 - \Delta)^2 + v_{T'}'^2\right]^{-\frac{3}{2}}} \simeq \frac{1}{\pi v_1^S} \ln \frac{v_1^S}{v_{T'}'}$$

whence

$$\left(\overline{v_\alpha^2}/v_{T'}'^2\right) \sim \frac{m}{M} \ln \frac{v_1^S}{v_{T'}'} \ll 1.$$

Thus for lateral degrees practically the same amplitudes are established as in the case of thermodynamic equilibrium (the logarithm cannot be large). In comparison with the "start" of the process when $v^2 \sim v'^2_{T'}$, the attenuation time decreases in the relation $v'^2_{T'}/\Delta^2$.

Let us note that from (II.1.6) and (II.1.7) it follows that the sum of oscillation decrements is zero if they are defined as

$$\tau_i^{-1} = -(\bar{Q}_i/I_i) \; ;$$
$$\tau_1^{-1}+\tau_2^{-1}+\tau_3^{-1} \sim \left(-2\Delta^{-3}+\Delta^{-3}+\Delta^{-3}\right) = 0 . \qquad \text{(II.1.10)}$$

This approximate result is a specific instance of the general theorem established in Section 4.

Let us now consider the case in which $\vec{\Delta}$ has projections of the same order of magnitude on to two or three normal degrees of freedom. For small oscillations

$$\bar{Q}_\alpha \sim -\frac{M}{m}\overline{v_\alpha \frac{v_\alpha - \Delta_\alpha}{\left[(\vec{v}-\vec{\Delta})^2+v'^2_{T'}\right]^{3/2}}} \simeq -\frac{M}{m}\,\frac{\Delta^2 - 3\Delta_\alpha^2}{\Delta^5}\,\overline{v_\alpha^2} \; ; \qquad \text{(II.1.11)}$$

the appearance of an error in other degrees of freedom may thus compensate instability in the degree of freedom considered, since it is equivalent in its action to the increase in the temperature of the electron gas, as was pointed out above. The characteristic of force remains positive if $3\Delta_\alpha^2 < \Delta^2$, in spite of the fact that $\Delta_\alpha^2 > v'^2_{T'}$. At the same time, the sum of the decrements, as can be seen from (II.1.11) remains equal to zero. In reality, as will be shown later, when there is a large error $\Delta \gg v'^2_{T'}$ it is difficult to avoid the instability (although theoretically this is possible) but if Δ exceeds $v'_{T'}$ in an insignificant manner (but in such a way that the small oscillations lie in the region where the characteristic is negative, $\vec{\Delta}$ is directed along the normal degree of freedom), then all of the oscillations will be attenuated on condition that $\Delta_1 \simeq \Delta_2 \simeq \Delta_3$. This anisotropy of attenuation in the direction $\vec{\Delta}$ is explained by the existence of discrete directions of normal oscillations (undegenerated three-dimensional oscillator).

The occurrence of instability in the case of a distribution of the form $\sim \exp[-(\vec{v} - \vec{\Delta})^2/v'^2_{T'}]$ is a specific characteristic of oscillatory motion. If the motion is infinite (absence of stable synchrotron motion) instability does not occur, but there is an entrainment of one current by the other.

Although we have considered, here, the case in which the mean velocity of the electron current in the accompanying system is different from zero, this is not essential for the occurence of instability. For example, if we take $f'(\vec{v}')$ in the form of two extended Maxwellian distributions $\exp[-(\vec{v}' \pm \vec{\Delta})^2/v'^2_{T'}]$, then $\langle\vec{v}'\rangle = 0$, but the characteristic of friction, when $\vec{v} = 0$ in the direction of the shift, will be negative if $\Delta > v'_{T'}$, which leads to instability. In general, for the occurrence of instability it is essential that the energy of orderly motion in the electron current exceeds the thermal energy*), i.e. the distribution must be qualitatively different from the Maxwellian. An important characteristic, here, is that the stabilized value of the energy of the oscillator is M/m times greater, in order of magnitude, than the energy of "orderly motion of the electron", since there is an equalization of speeds, but not of temperatures: $(v^2)_{stab} \approx \Delta^2$.

S.T. Belyaev and G.I. Budker[4] also pointed out the case of spherical distribution

$$f'(\vec{v}') \sim v_0^{-1}\,\delta(v'^2 - v_c^2) \quad ; \qquad \text{(II.1.12)}$$

in this case, the frictional force and momentum transfer are equal to zero if $v < v^0$ (field of a charged sphere) and heating of the proton beam occurs: $v^2_{stab} \sim v_0^2$. This case clearly demonstrates the characteristics of Coulomb interaction, although it appears, in practice, to be exceptional.

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*) This condition justifies, in fact, the designation of instability.

Let us also evaluate the attenuation rate in a case which is important in practice, when the error $\vec{\Delta}$ oscillates with time. The oscillations may, for example, be due to oscillations in the control voltage. Let $w(\vec{\Delta})$ be the distribution of probability of error:

$$\int w(\vec{\Delta})\, d^3\Delta = 1 .$$

The mean probable value of the frictional force may be written in the form:

$$\langle \vec{F} \rangle = -\frac{1}{m} \int \frac{\vec{v} - \vec{\Delta} - \vec{v}'}{|\vec{v} - \vec{\Delta} - \vec{v}'|^3} f'(\vec{v}')\, w(\vec{\Delta})\, d^3v'\, d^3\Delta \qquad \text{(II.1.13)}$$

where $f'(v')$ is the velocity distribution of the electrons with regard to the mean velocity $\langle \vec{v}' \rangle = \vec{\Delta}$, which is close in form to the Maxwellian: $\langle v'^2 \rangle \simeq v'^2_{T'}$. The case $\langle \Delta^2 \rangle \gg v'^2_{T'}$ is of interest to us. If the oscillations occur in three dimensions then, obviously, for all degrees of freedom this is equivalent to the increase in thermal spread of the electrons up to a value of $\langle \Delta^2 \rangle$. (It is assumed that the distribution $w(\vec{\Delta})$ is bell-shaped). In the case of one-dimensional oscillations directed along a normal degree, for small oscillations of protons

$$\overline{Q}_1 \sim M\overline{v_1^2}\, \frac{\partial \langle F_1 \rangle}{\partial v_1} \sim -\frac{M\overline{v_1^2}}{m} \int d^3\Delta\, w(\vec{\Delta}) \frac{\partial}{\partial \Delta_1} \frac{\Delta_1}{[\Delta^2 + v'^2_{T'}]^{3/2}} \simeq -\frac{M\overline{v_1^2}}{m\langle \Delta^2 \rangle^{3/2}}$$

The characteristic of friction thus remains positive although the effective temperature of the electron beam increases just as in the case of three-dimensional oscillations. For the remaining degrees:

$$\overline{(Q_\alpha)}_\perp \sim -\frac{M}{m} \int \frac{\overline{v_\alpha^2}\, w(\vec{\Delta})\, d^3\Delta}{(\Delta^2 + v'^2_{T'})^{3/2}} \simeq -\frac{M\overline{v_\alpha^2}}{m v'^2_{T'} \langle |\Delta| \rangle} .$$

This result is explained simply: in view of the sharp dependence of the "lateral" frictional force on the error $\sim \Delta^{-3}$, the basic contribution to the power is given by the section $\Delta \sim v'_{T'}$: the fraction of time "passed" by the error on this section is equal to $v'_{T'}/\langle |\Delta| \rangle$.

Let us note that for δ-type oscillations $w(\vec{\Delta}) = \frac{1}{2}\Delta_0\delta(\Delta_1^2 - \Delta_0^2) \cdot \delta(\vec{\Delta}_\perp)$ the integral in (II.1.13) gives the previous result (II.1.6), i.e. an instability, as must be the case.

Thus, the appearance of a variable error with a distribution $w(\vec{\Delta})$ in the case $\langle\Delta^2\rangle \gg v'^2_{T'}$ is equivalent to the establishment of a standard distribution of electrons $w(\vec{v}')$ [in the case of one-dimensional or two-dimensional oscillations the electron spread which is lateral to them remains equal to $v'_{T'}$]. Bearing this analogy in mind, we may extend the above qualitative criterion of instability (or warming up) also to the case of non-stationary velocity distribution of electrons

$$\overline{\overline{E}^2} > \overline{(\Delta E)^2} = \overline{(E-\overline{E})^2} \;, \qquad \left(E = \frac{mv'^2}{2}\right)$$

where (-) designates the averaging for the "instantaneous" distribution and for time. This condition is necessary but not sufficient. A strict, necessary and sufficient condition is the formal requirement of negativity (equality to zero) of the characteristic of friction in the direction of a normal oscillation.

2. EFFECTS OF SPATIAL INHOMOGENEITY

Let us now examine the effect of the spatial inhomogeneity of the distribution of electrons $f'(\vec{p}',\vec{r})$ on the attenuation rate of small amplitudes. We will define spatial inhomogeneity by the gradients of the average velocity, temperature and density in the electron current assuming that in the absence of gradients the proton motion is attenuated (in the accompanying system). For this, it is sufficient to represent

$$f'(\vec{p}',\vec{r}) = f'_{T'}(\vec{v}'-\vec{\Delta}(\vec{r}))\,n(\vec{r}), \qquad \text{(II.2.1)}$$

where $f'_{T'}$ is a distribution of the Maxwellian type with a temperature $T' = T'(\vec{r})$, and the error $\vec{\Delta}(\vec{r})$ does not exceed in order of magnitude, the velocity

$$|\vec{\Delta}| < v'_{T'}\,. \qquad \text{(II.2.2)}$$

In this case, the frictional force for low proton velocities $v < v'_{T'}$ has the form (see II.1.4)

$$\vec{F}(\vec{v},\vec{r}) \sim -\frac{\vec{v}-\vec{\Delta}}{T'^{3/2}}\, n\,. \qquad \text{(II.2.3)}$$

If we assume focusing to be homogeneous, we obtain, with the aid of expressions (I.1.13), (I.3.2) and (I.3.3):

$$\begin{aligned} \bar{Q}_z &\sim -\overline{v_z \frac{v_z - \Delta_z}{T'^{3/2}}\, n} = -\overline{\frac{v_z^2 n}{T'^{3/2}}} < 0\,; \\ \bar{Q}_r &\sim -\overline{v_r F_r} - \omega_s \overline{r_b F_{\parallel}} \simeq -\overline{\frac{v_r^2 n}{T'^{3/2}}} - \omega_s \overline{r_b^2 \frac{\partial}{\partial r}\left[\frac{n \Delta_{\parallel}}{T'^{3/2}}\right]} \end{aligned} \qquad \text{(II.2.4)}$$

where

$$r = r_c + r_b\,, \qquad r_c = \frac{v_{\parallel}}{\nu_r^2 \omega_s}\,.$$

$$r_b = a_r \cos\Psi_r\,, \qquad v_r = -a_r \nu_r \omega_s \sin\Psi_r\,. \qquad \text{(II.2.5)}$$

In a longitudinal direction, it is sufficient to obtain a force $F_{\parallel}$ averaged for betatron oscillations:

$$\bar{F}_{\parallel} \sim -\overline{\frac{v_{\parallel} - \Delta_{\parallel}^{0}}{T'^{3/2}}\, n} + r_c \overline{\frac{\partial}{\partial r}\left[\frac{n \Delta_{\parallel}}{T'^{3/2}}\right]} \qquad \text{(II.2.6)}$$

where $\Delta_{\parallel}^{0} = \Delta_{\parallel}/_{r=0}$ (in the working conditions without an RF field it is necessary to assume $\Delta_{\parallel}^{0} = 0$, since the equilibrium velocity is determined from the condition $F_{\parallel} = 0$).

Taking (II.2.5) into account, the expressions (II.2.4) and (II.2.6) assume the form

$$\bar{Q}_r \sim -\overline{v_r^2}\left[\overline{\left(\frac{n}{T'^{3/2}}\right)_0} + \frac{1}{\nu_r^2 \omega_s} \overline{\frac{\partial}{\partial r}\left(\frac{n \Delta_{\parallel}}{T'^{3/2}}\right)_0}\right] \qquad \text{(II.2.7)}$$

$$F_{\parallel} \sim - v_{\parallel}\left[\overline{\left(\frac{n}{T'^{3/2}}\right)_c} - \frac{1}{\nu_r^2 \omega_s}\overline{\frac{\partial}{\partial r}\left(\frac{n\Delta_{\parallel}}{T'^{3/2}}\right)_c}\right] \tag{II.2.8}$$

In this way, the axial oscillations are always attenuated if the condition (II.2.2) is fulfilled, wheareas in (II.2.4) and (II.2.6) members appear which are proportional to the gradient $F_{\parallel}$ in a radial direction on the equilibrium orbit. These members are due to the coupling of radial and longitudinal motion, or, as is said, to the fact that the equilibrium orbit is closed, and produce decrements of attenuation which are identical in value but of different sign. From a comparison (II.2.7) and (II.2.8) the condition of stability can be obtained

$$\left|\overline{\frac{\partial}{\partial r}\left(\frac{n r_{\Delta}}{T'^{3/2}}\right)_0}\right| < \overline{\left(\frac{n}{T'^{3/2}}\right)_0} \quad , \qquad \left(r_{\Delta} \equiv \frac{\Delta_{\parallel}}{\nu_r^2 \omega_s}\right). \tag{II.2.9}$$

Although the influence of spatial inhomogeneity disappears if $\Delta_{\parallel} \equiv 0$, the condition $|\Delta_{\parallel}| > v'_{T'}$, is not at all essential for the occurrence of a "gradient" instability, as in the case of "monochromatic" instability which was examined above. Let, for example, $(\partial/\partial r)(nT'^{-3/2}) = 0$. Then, it follows from (II.2.9) that instability is possible on condition that

$$\left|\overline{\frac{\partial}{\partial r}\Delta_{\parallel}}\right| > \nu_r^2 \omega_s = \frac{dv_{\parallel}(r)}{dr} \tag{II.2.10}$$

where $v_{\parallel}(r)$ is the azimuthal velocity as a function of the radial deviation on the trajectory of the proton. If $|v_{\parallel}| < v'_{T'}$, then (II.2.10) and (II.2.2) may be compatible, since intrinsically (II.2.10) denotes that $|\Delta_{\parallel}| > |v_{\parallel}|$. In the case of $\Delta_{\parallel}$ = Const the condition of instability is

$$|\Delta_{\parallel}| > \left|\overline{\left(\frac{n}{T'^{3/2}}\right)_c} \Big/ \overline{\frac{\partial}{\partial r}\left(\frac{n}{T'^{3/2}}\right)}\right| \cdot \frac{dv_{\parallel}}{dr} \equiv b_r \frac{dv_{\parallel}}{dr} \tag{II.2.11}$$

where b_r is the dimension of inhomogeneity if the relative variation $\delta \ln(n/T'^{3/2}) \sim 1$. If the last two conditions are united, a general qualitative criterion of gradient instability can be formulated: on the dimension of radial inhomogeneity the mean radius value of the error $|\Delta_{\parallel}|$ must exceed the variation of $v_{\parallel}(r)$. If there is no RF field instability may arise only if there is a mean velocity gradient; in the self-phasing condition the gradients of density and temperature also contribute to the decrements if there is an error in the velocity on the equilibrium orbit. It would appear, in practice, that the velocity gradient is the most dangerous.

Let us evaluate the maximum amplitudes achieved with gradient instability. For this, without assuming the smallness of $|\vec{v} - \vec{\Delta}|/v'_{T'}$, we shall take the force $\vec{F}$ in the form

$$\vec{F} \sim - \frac{\vec{v} - \vec{\Delta}}{[(\vec{v} - \vec{\Delta})^2 + v'^2_{T'}]^{3/2}}\, n \; ; \qquad \vec{\Delta} = \{0, \Delta_{\parallel}, 0\}$$

If the radial betatron oscillations are unstable we may consider that $v^2_{\parallel} \ll v'^2_{T'}$ since the longitudinal motion will be attenuated (let $\Delta^0_{\parallel} < v'_{T'}$). Then,

$$\overline{Q}_r \sim - \overline{(v_r^2 + \omega_s r_b \Delta_{\parallel})(\Delta^2_{\parallel} + v_r^2 + v'^2_{T'})^{-\frac{3}{2}}} \, , \qquad \Delta_{\parallel} \simeq \Delta^0_{\parallel} + r_b \frac{\partial \Delta_{\parallel}}{\partial r} \, .$$

Let us pose, for concreteness,

$$\frac{\partial}{\partial r} v'_{T'} = 0 \, , \quad \frac{\partial}{\partial r} \Delta_{\parallel} = \frac{\Delta'}{r_0} \, ,$$

where r_0 is the radial dimension of the beam. Bearing in mind the conditions (II.2.10) and (II.2.11) we may conclude that the stabilized amplitude $a_r > r_0$ is determined in the general case by the equality

$$\overline{v_r^2} \simeq \langle \Delta^2_{\parallel} \rangle_r \, .$$

In the case it is not essential that $\Delta^2_{\parallel} < v'^2_{T'}$. In the case of instability of the synchrotron motion

$$F_{\parallel} \sim - (v_{\parallel} - \Delta_{\parallel})\left[(v_{\parallel} - \Delta_{\parallel})^2 + v'^2_{T'}\right]^{-\frac{3}{2}} \; ; \quad \Delta_{\parallel} = \Delta^0_{\parallel} + \frac{v_{\parallel}}{\nu_r^2 \omega_s} \frac{\partial \Delta_{\parallel}}{\partial r} .$$

If the motion is infinite the "anti-attenuation" ceases, when the radial deviation will exceed the dimension of the beam:

$$|v_{\parallel}| \sim \tau_0 \nu_i^2 \omega_s \,.$$

In the self-phasing condition, the oscillations build up indefinitely. This is obvious if $\Delta^0_{\parallel} = 0$ and the condition (II.2.10) is fulfilled. If, however $(\partial\Delta_{\parallel}/\partial r) = 0$, and the condition (II.2.11) is fulfilled then for all amplitudes $\overline{Q}_c > 0$, since the energy build-up occurs only at "small" velocities $|v_{\parallel}| < |\Delta_{\parallel}|$, when the particle trajectory passes through the beam.

3. THE CRITICAL LEVEL OF COHERENT FLUCTUATIONS AND OTHER QUESTIONS

Let us evaluate the permissible level of fluctuations of the space charge of an electron beam, starting from the condition that the diffusion growth of the amplitude does not exceed the maximum power of Coulomb losses:

$$\langle \Delta I_i \rangle_{fl} < |\overline{Q}_i|_{max}$$

where

$$\langle \Delta I_i \rangle_{fl} = \frac{1}{2}\langle (\Delta p_i)^2 \rangle \quad , \quad \overline{Q}_i = \overline{p_i F_i} \;.$$

Let us consider scattering on the fluctuations of an electric field (in the accompanying system), the correlation time of which is $\tau_{corr} < w_i$, and the spatial dimension (wavelength) is $k^{-1} > v\tau_0$, where v is the velocity of the protons, $\tau_0 = \ell/\beta c$ is the time taken by the proton to pass through the section of the orbit occupied by the electron beam. "Collisions" then occur instantaneously in relation to the periodic motion of the protons and the scattering cross-section does not depend on the momentum. In a single passage a proton gathers the momentum

$$\Delta\vec{p} = e\Delta\vec{E}\cdot\tau_0 \;;$$

whence

$$\langle (\Delta p_i)^2 \rangle = e^2 \left[(\Delta E_i)^2 \right] \frac{\tau_0^2}{T_0} \quad , \qquad (T_0 = 2\pi/\omega_s)$$

where $[(\Delta E_i)^2]$ denotes the statistical mean.

As the frictional force in the region $v > v'_{T'}$ decreases as v^{-2}, the maximum $|\overline{Q}_i|$ is reached when $v \sim v'_{T'}$ and, in order of magnitude, is equal to

$$|\overline{Q}|_{max} \simeq (4\pi e^4 L n \tau_0 M / m v'_{T'} T_0) \tag{II.3.1}$$

For a relative fluctuation we obtain

$$\left([(\Delta \vec{E})^2]/E^2\right)^{\frac{1}{2}}_{cr} \sim (\gamma \theta_e n V m) L M)^{-\frac{1}{2}}$$

where L is the Coulomb logarithm, θ_e the angular spread of electrons in the laboratory system, γ the relativistic factor, n and V the density and volume of the electron beam. Let us take, for example, the numerical values

$$L \simeq 20, \; \gamma = 2, \; \theta_e = 3 \cdot 10^{-3}, \; n = 10^{8}\,\text{cm}^{-3}, \; V = 3 \cdot 10^{3}\,\text{cm}^{3}.$$

we have then

$$(\Delta E/E) \sim 5 \cdot 10^{-3}.$$

The type of fluctuation under consideration is, apparently, the most dangerous, as far as the size of the scattering cross-section is concerned.

Let us also make here an evaluation of the critical value of the density of the residual gas n_0 [1], scattering on which can be included in the over-all "fluctuation background", (I.1.18):

$$(n_0^{\kappa p})_{scatt} \sim (L M n \tau_0 / L_Z m Z^2 \gamma^3 \theta_e)$$

where Z is the charge of the nucleus, L_Z the corresponding Coulomb logarithm and the densities refer to the laboratory system.

For the condition without the RF field, the question may arise as to whether the deceleration on the electrons of residual gas will not prevail over the "entrainment" of the protons by the electron beam. With the aid of the expression for the force of friction on the residual gas (I.1.18) it is simple to obtain the relation:

$$\left[(n_0^{KP})_{dec} / (n_0^{KP})_{scatt}\right] \sim \left(Z L_Z m / \theta_e L_e M\right)$$

where L_e is the Coulomb logarithm for scattering on the electrons of the gas. In practice, this relation does not differ markedly from unity.

Finally, let us also make an evaluation of the life of the protons with regard to the process of recombination (formation of atoms) in the electron beam. The relationship of τ_{rec} to the relaxation time τ is equal to:

$$\frac{\tilde{\tau}_{rec}}{\tilde{\tau}} \sim \frac{m}{M} L \frac{\sigma_R}{\sigma_{rec}} \sim \frac{m}{M} L \left\{\alpha(\gamma\beta\theta)^2 \ln\left[1+\left(\frac{\alpha}{\gamma\beta\theta}\right)^2\right]\right\}^{-1}, \qquad \text{(II.3.2)}$$

where σ_R is the Rutherford cross-section of wide angle scattering:

$$\sigma_R \sim \frac{4 r_e^2}{(\gamma\beta\theta)^4},$$

σ_{rec} is the total recombination cross-section for all levels of the hydrogen atom, θ the relative angular spread of particles in the laboratory system, α the fine structure constant (II.3.2) is an interpolation formula[17] and gives the correct behaviour for small velocities $(\gamma\beta\theta)^2/\alpha^2 << 1$ and large ones $1 < (\gamma\beta\theta/\alpha)^2 < \simeq 10$. For example, for $\theta = 3 \cdot 10^{-3}$, $\gamma = 2$, $(\tau_{rec}/\tau) \sim 3 \cdot 10^4$.

4. THE KINETICS OF SMALL AMPLITUDES

We shall now investigate the solution of the kinetic equation (I.3.9), assuming the condition that the total diffusion rate is much less than the critical level at which attenuation does not occur at all. Let us first consider the region of low velocities $v < v'_T$, assuming also that the spatial inhomogeneity for the corresponding range of amplitudes can be characterized with a high degree of accuracy by the gradient $f'(\vec{p}', \vec{r})$. As the characteristic scale of variation of the frictional force $\vec{F}(\vec{v}, \vec{r})$ is of the order of v'_T, the latter can be expanded into the series:

$$F_\alpha(\vec{v}, \vec{r}) = (F_\alpha)_0 + \left(\frac{\partial F_\alpha}{\partial \vec{v}}\right)_0 \vec{v} + \left(\frac{\partial F_\alpha}{\partial \vec{r}_\perp}\right)_0 \vec{r}_\perp + \ldots$$

We shall make a similar assumption for the quadratic fluctuations:

$$\langle \Delta p_\alpha \Delta p_\beta \rangle = \langle \Delta p_\alpha \Delta p_\beta \rangle_0 + \dots$$

If we confine ourselves, in these expansions, to members of the lower order which make a non-vanishing contribution where Q_i and D_{ik} are averaged for phases, we obtain, after averaging

$$\bar{Q}_i = -2\lambda_i I_i \tag{II.4.1}$$

$$\bar{D}_{ik} = 2\mu_i I_i \equiv D_i \ , \quad i = k \quad ; \qquad D_{ik} = 0 \ , \quad i \neq k \tag{II.4.2}$$

where (I.3.3)

$$\lambda_r = \frac{1}{2M}\left[-\overline{\left(\frac{\partial F_r}{\partial v_r}\right)_0} + \overline{\gamma \frac{d\Psi}{d\theta_s}\left(\frac{\partial F_{\parallel}}{\partial v_r}\right)_0} + \overline{\frac{\Psi}{\omega_s}\left(\frac{\partial F_{\parallel}}{\partial r}\right)_0} \right], \tag{II.4.3}$$

$$\lambda_{\parallel} = \frac{1}{2M}\left[-\overline{\left(\frac{\partial F_{\parallel}}{\partial v_{\parallel}}\right)_0} - \overline{\gamma \frac{d\Psi}{d\theta_s}\left(\frac{\partial F_{\parallel}}{\partial v_r}\right)_0} - \overline{\frac{\Psi}{\omega_s}\left(\frac{\partial F_{\parallel}}{\partial r}\right)_0} \right], \tag{II.4.4}$$

$$\lambda_z = -\frac{1}{2M}\overline{\left(\frac{\partial F_z}{\partial v_z}\right)_0} \ ;$$

$$\mu_r = \frac{1}{2}\overline{|f_r|^2 \langle (\Delta p_r)^2 \rangle_0} + \frac{\gamma^2}{2}\overline{|\zeta|^2 \langle (\Delta p_{\parallel})^2 \rangle_0} + \gamma\,\overline{\mathrm{Im}\,(f_r^* \zeta) \langle \Delta p_r \Delta p_{\parallel} \rangle_0} \ , \tag{II.4.5}$$

$$\mu_{\parallel} = \frac{1}{2}\overline{\langle (\Delta p_{\parallel})^2 \rangle_0} \ , \qquad \mu_z = \frac{1}{2}\overline{|f_z|^2 \langle (\Delta p_z)^2 \rangle_0} \ .$$

Here

$$\zeta = \nu_r f_r \Psi - i\left(\Psi \frac{df_r}{d\theta_s} - f_r \frac{d\Psi}{d\theta_s}\right).$$

The physical meaning of the coefficients μ_i is clear from the relation $\langle \Delta I_i \rangle_{f\ell} = \frac{1}{2}(\partial D_{ik}/\partial I_k)$, whence $\langle \Delta I_i \rangle_{f\ell} = \mu_i$. In the condition without the RF field it is necessary to pose $D_{\parallel} = 2\mu_{\parallel}$, since $I_{\parallel} = p_{\parallel}$.

The values of λ_i represent decrements in attenuation of the oscillations for normal degrees of freedom. (II.4.4) coincides with the usual determination of the decrement and in the absence of self-phasing:

$$\langle \Delta p_{\parallel} \rangle = -2\lambda_{\parallel} p_{\parallel} \quad ;$$

the energy spread, however, is attenuated in this case twice as fast as in the oscillation condition:

$$\frac{d}{dt} p_{\parallel}^2 = -4\lambda_{\parallel} p_{\parallel}^2$$

This difference has a simple physical meaning: the specific heat of the oscillators is twice that of the free particles.

The decrements of radial and longitudinal motion include members which are proportional to the derivative $F_{\parallel}$ in velocity and coordinate in the radial direction. The influence of the members $\sim (\partial F_{\parallel}/\partial r)$ has been discussed in detail above. The appearance of the members $\sim (\partial F_{\parallel}/\partial v_r)$ is associated to the modulation of the velocity v_r by synchrotron motion, in the case of azimuthal inhomogeneity of focusing and the curvature of the equilibrium orbit. Qualitatively, their role may be substantial, provided the error $\vec{\Delta}$ has non-zero components $\Delta_r \sim \Delta_{\parallel} \gtrsim v'_{T'}$.

As can be seen from the expressions of the decrements, the sum $\lambda_r + \lambda_{\parallel}$ does not depend on the coupling of radial and longitudinal motion. We also know of a similar result for decrements of radiation damping in the theory of accelerators[14]. For a full sum, the remarkable relation occurs

$$\lambda_r + \lambda_{\parallel} + \lambda_z = -\frac{1}{2M} \mathrm{div}_{\vec{v}} \vec{F} = \frac{8\pi^2 e^4 L}{mM} m^3 \overline{f'_0} \quad , \tag{II.4.6}$$

(I.1.14 - I.1.16) where $\overline{f'_0}$ is the mean value of the function of distribution of the electrons $f'(\vec{p}', \vec{r})$ on the equilibrium trajectory of protons.

In this way, the sum of decrements does not depend either on the "orientation" of the anisotropic velocity distribution and is determined only by its size. This theorem is a general postulate irrespective of the value and sign of the individual decrements and the shape of $f'(\vec{p}', \vec{r})$.

For the isotropic spatially homogeneous distribution of electrons

$$\lambda_z = \lambda_{\parallel} = \lambda_z = \lambda = \frac{8\pi^2 e^4 L}{3mM} m^3 \overline{f_0'} \simeq \frac{8\pi^2 e^4 L n}{3mM v_{T'}^3} \qquad \text{(II.4.7)}$$

which corresponds to the usual formula for the relaxation time of the plasma, when the velocities of the ions v_T are less than those of the electrons $v'_{T'}$ [1,2,3].

From (II.4.6) in particular, it follows that all λ_i can remain positive even if the error $\Delta \gg v'_{T'}$. In this case, however, their value becomes quite small, since it is proportional to the "tail" of the distribution. Thus, for the Maxwellian "spectrum" $\Sigma\lambda_i \sim \exp(-\Delta^2/v'^2_{T'})$. This conclusion agrees with the results of the approximate investigation performed in Section 1.

Let us also note that in conditions of spatial homogeneity the values of the decrements cannot, in order of magnitude, exceed the value (II.4.7). But in the case of a strong spatial inhomogeneity, as follows from the results of Section 2, the value $|\lambda_r - \lambda_{\parallel}|$ can become much greater than λ. Naturally, the frictional power cannot, in any circumstances, exceed the maximum value (II.3.1).

Let us construct the solution of the kinetic equation

$$\frac{\partial}{\partial t} f - \sum_i \frac{\partial}{\partial I_i}\left\{2\lambda_i I_i f + \mu_i I_i \frac{\partial f}{\partial I_i}\right\} = 0 \qquad \text{(II.4.8)}$$

assuming the condition that all of the $\lambda_i > 0$. By the method of separating the variables $f = \pi_i e^{-\kappa_i t} f_i(I_i)$ we obtain the equation (for brevity we shall leave out the index i)

$$\varkappa f + \frac{d}{dI}\left(2\lambda I f + \mu I \frac{\partial f}{\partial I}\right) = 0$$

which, by means of the substitutions $I = (\mu/2\lambda)x$, $f = ye^{-x}$, is brought to the equation for Laguerre polynomials[18]:

$$xy'' + (1-x)y' + \alpha y = 0 \quad , \qquad \alpha = \frac{\varkappa}{2\lambda} = n = 0, 1, 2, \ldots$$

The normal solution of the equation (II.4.8) is

$$f_{\{n\}} = \prod_i L_{n_i}(x_i) \exp(-x_i - 2n_i\lambda_i t) .$$

The general solution can be expressed by the fundamental solution, or by the Green function

$$G(I|I',t) = \prod_i g_i(x_i|x_i',t)$$

where

$$g_i(x|x';t) = e^{-x} \sum_{n=0}^{\infty} L_n(x) L_n(x') e^{-2n\lambda t} = \frac{e^{-\frac{x - x' e^{-2\lambda_i t}}{1 - e^{-2\lambda_i t}}}}{1 - e^{-2\lambda_i t}} I_0\left(\frac{\sqrt{xx'}}{\operatorname{sh}\lambda_i t}\right) \qquad \text{(II.4.9)}$$

(I_0 is the Bessel function of the imaginary argument):

$$f(I,t) = \int d^3 I' \, G(I|I',t) f(I',0) .$$

A direct check will confirm that

$$g(x|x',0) = \delta(x-x') \quad , \qquad g(x|x',\infty) = e^{-x}$$

$$\int g(x|x',t) dx' = 1 \quad .$$

The equilibrium distribution and evolution of the mean amplitudes can also be obtained directly from (II.4.8):

$$f_s = \prod_i \exp\left(-\frac{I}{I_s}\right) \quad , \quad I_{is} = \frac{\mu_i}{2\lambda_i} \quad ;$$

$$\frac{d}{dt}\langle I_i \rangle = -2\lambda_i I_i + \mu_i \quad ,$$

as must be the case.

In the conditions without an RF field, the solution of the equation for $f(p_{\parallel})$

$$\varkappa_{\parallel} f + \frac{d}{dp_{\parallel}}\left(2\lambda_{\parallel} p_{\parallel} f + \mu_{\parallel}\frac{df}{dp_{\parallel}}\right) = 0$$

is $f_n = e^{-x^2} H_n(x)$ [18], $x = (\lambda_{\parallel}/\mu_{\parallel})p$, where H_n is a Hermite polynomial. The fundamental solution[19] is

$$g(x|x',t) = e^{-x^2}\sum_n H_n(x) H_n(x') e^{-2\lambda n t} =$$

$$= \frac{1}{2\sqrt{\pi}}\left\{\sqrt{\operatorname{cth}\lambda t}\, e^{-\frac{(x-x'e^{-2\lambda t})^2}{1-e^{-4\lambda t}}} + \sqrt{\operatorname{th}\lambda t}\, e^{-\frac{(x+x'e^{-2\lambda t})^2}{1-e^{-4\lambda t}}}\right\}. \tag{II.4.10}$$

The equilibrium solution is e^{-x^2}, and the evolution of $\langle p^2{}_{\parallel}\rangle$ is determined by the equation

$$\frac{d}{dt}\langle p_{\parallel}^2\rangle = -4\lambda_{\parallel}\langle p_{\parallel}^2\rangle + 2\mu_{\parallel}.$$

Knowledge of the fundamental solution makes it possible, if necessary, to obtain directly the evolution of the initial distribution to a state of equilibrium. Let us take, for example,

$$f(I,0) = I_c^{-1}\exp(-I/I_0).$$

If we integrate $f(I,0)$ with the Green function (II.4.9) we obtain[19]:

$$f(I,t) = I^{-1}(t)\exp(-I/I(t)), \quad I(t) = I_s(1-e^{-2\lambda t}) + I_0 e^{-2\lambda t},$$

i.e. the shape of the distribution is retained, and only $\langle I\rangle$ is changed. A similar result is obtained by means of (II.4.10) in the case of the initial distribution

$$f(p_{\parallel},0) = (p_0\sqrt{2\pi})^{-1}\exp(-p_{\parallel}^2/2p_0^2) \quad ;$$

$$f(p_{\parallel},t) = (p(t)\sqrt{2\pi})^{-1}\exp(-p_{\parallel}^2/2p^2(t)), \quad p^2(t) = \frac{\mu_{\parallel}}{\lambda_{\parallel}}(1-e^{-4\lambda_{\parallel}t}) + p_0^2 e^{-4\lambda_{\parallel}t}.$$

5. EVALUATION OF A SOLUTION IN THE REGION OF LARGE AMPLITUDES

By the term "large amplitudes" we shall understand the general case, when the kinetic coefficients cannot be linearized for the variables I_i. This may, basically, be due to the non-linear behaviour of the frictional force at velocities $v > v'_{T'}$, or to a strong spatial inhomogeneity of the electron beam (for example, when the amplitudes of the oscillations exceed the lateral dimension of the beam).

We shall first examine the nature of the kinetic process in conditions of spatial homogeneity, ignoring the azimuthal inhomogeneity of focusing. In the range $v^2 > v'^2_{T'}$, the coefficients $\bar{Q}_i$ decrease as v^{-1} (or faster than $\sim v_i^2/v^3$, if $v_i^2 \ll v^2$). In the presence of a fluctuation background [we shall assume the diffusion speed on this to be constant $\langle \Delta I_i \rangle_{f\ell} =$ Const $= \mu_i$] which considerably exceeds the thermodynamic level of fluctuations in the electron beam, for sufficiently large amplitudes:

$$|\bar{Q}_i| < \mu_i$$

and the particles are not captured in the damping conditions. Let us evaluate the region of captured amplitudes in the condition $|\bar{Q}|_{max} \gg \mu$. For one-dimensional oscillations,

$$|\bar{Q}_1| \simeq 4\pi L e^4 n \frac{M}{m} (v_0 |\sin\psi_1|)^{-1} \simeq \frac{2}{\pi} \frac{v'_{T'}}{v_0} |\bar{Q}|_{max} \cdot \ln \frac{v_0}{v'_{T'}}$$

where v_0 is the amplitude value of the velocity,

$$|\bar{Q}|_{max} \sim \frac{4\pi e^4 L n M}{m v'_{T'}} \cdot \frac{\tilde{\tau}}{T_c}$$

In this way

$$(v_0^{\kappa p}/v'_{T'}) \simeq \frac{2}{\pi} \frac{|\bar{Q}|_{max}}{\mu} \ln \frac{|\bar{Q}|_{max}}{\mu} \quad .$$

For two- or three-dimensional oscillations the sub-integral expression, in the case of averaging for phases, has no singularities, consequently

$$(v_0^{\kappa p}/v'_{T'}) \simeq \frac{|\bar{Q}|_{max}}{\mu} \quad .$$

Accordingly, the time taken to pass through the region of large amplitudes in the damping conditions $v'_{T'} < v_0 < v_0^{kp}$ is equal to

$$\tilde{\tau}_{(1)} = \tilde{\tau}\, \ln^{-1}\left(\frac{v_H}{v'_{T'}}\right)$$

in the one-dimensional case also

$$\hat{\tau}_{(2)} = (\varkappa/\ln(1+\sqrt{2}))\,\tilde{\tau} \qquad \left(\frac{2}{\pi} < \varkappa < 1\right)$$

for two-dimensional oscillations, if $v_H^1 = v_H^2$, where

$$\tau = \frac{m M v_H^3}{24 e^4 L n}$$

and $v_H \gg v'_{T'}$ is the initial amplitude of the velocity along one degree of freedom. In the three-dimensional case τ_3 also differs from τ by a numerical factor which is close to unity.

The over-all picture of the movement of the amplitudes, which is described approximately by the equations

$$\frac{dI_i}{dt} = \overline{Q}_i + \mu_i \equiv \langle \Delta I_i \rangle$$

is fairly complex, but accessible to a qualitative analysis. The nature of the process is illustrated by Fig. 2, showing the trajectories of two-dimensional movement

$$(dI_1/\langle \Delta I_1 \rangle) = (dI_2/\langle \Delta I_2 \rangle)$$

when I_3 is constant (or equal to zero). The dotted curves 1 and 2 correspond to the equations $\langle \Delta I_1 \rangle = 0$ and $\langle \Delta I_2 \rangle = 0$. A simultaneous damping of the amplitudes occurs only in the region of D, which is bounded by these curves. The curves Γ_1 and Γ_2 bound the region of captured amplitudes. As can be seen from the diagram, when there is a strong excitation of one degree in the region of capture the other degree at first "warms up", after which the trajectory passes into the region D, where both amplitudes are damped. The points S and $\bar{S}$ correspond to the stable and unstable position of equilibrium. When the third degree of freedom is "switched on" the figure can be considered as a projection of the three-dimensional picture on to a plane. The region D is transformed into a "cocoon", and the overall nature of the movement is unchanged.

Strictly speaking, stationary distribution does not exist, since the region of captured amplitudes is limited. It is, however, possible to talk of a quasi-stationary distribution and life of the particles in the region of capture [or in the region of permissible amplitudes $I < I_{perm}$], if $|\bar{Q}|_{max} >> \mu$. The "equilibrium" distribution is found from the equation

$$\sum_i \frac{\partial j_i}{\partial I_i} = \sum_i \frac{\partial}{\partial I_i} \left\{ \bar{Q}_i f - \mu_i I_i \frac{\partial f}{\partial I_i} \right\} = 0 \ .$$

The solution can be found in the general form, if $\mu_1 = \mu_2 = \mu_3 = \mu$, if advantage is taken of the property of the frictional force $\vec{F} = -(\partial U/\partial \vec{v})$ (I.1.14). As $v_i = (\sqrt{2I_i}/M) \sin \Psi_i$, then in the conditions considered here,

$$\bar{Q}_i = -\overline{p_i \frac{\partial \mathcal{U}}{\partial v_i}} = -2MI_i \frac{\partial}{\partial I_i} \bar{\mathcal{U}} \ .$$

If we assume $j_i = 0$, we obtain the equations

$$2M \frac{\partial \mathcal{U}}{\partial I_i} f + \mu \frac{\partial f}{\partial I_i} = 0$$

which have a joint solution

$$f = C \exp\left(-\frac{2M}{\mu} \bar{\mathcal{U}}\right). \qquad \text{(II.5.1)}$$

In accordance with what has been said above, this solution cannot be normalized, since $\bar{U} \to$ Const at infinity. Its use has a meaning if, in the interval $0 \le I \le I_{perm}$, the large majority of the particles are concentrated in the region $I << I_{perm}$. The index of the exponential can be written in the form

$$-\frac{2M}{\mu} \bar{\mathcal{U}} = 2 \frac{|\bar{Q}|_{max}}{\mu} \left\langle \frac{v'_{\tau}}{u} \right\rangle$$

where $u = |\vec{v} - \vec{v}'|$, and $\langle \ldots \rangle$ denotes averaging for the distribution of the electrons. If the distribution is close to the Maxwellian, the solution in the region $v < v'_{T'}$ is

$$f \sim \exp\left[-\frac{2\lambda}{\mu}(I_1 + I_2 + I_3)\right]$$

when λ coincides with the expression (II.4.7). The "normalized" solution when $|\bar{Q}|_{max} \gg \mu$ has the form

$$f = (2\lambda/\mu)^3 \exp\left[2\frac{|\bar{Q}|_{max}}{\mu}\left\langle \frac{v'_{T'}}{u} - \frac{v'_{T'}}{v'} \right\rangle\right] . \qquad \text{(II.5.2)}$$

The solution (II.5.2) can in fact be used for evaluating the "tail" of the distribution also when the μ_i are sharply different in value, if we simply assume for degrees of freedom with small μ_i, $v_i = 0$. Let us examine, for example, scattering on the residual gas. In this case $\mu_r = \mu_z = \mu$, $\mu_{\parallel} \ll \mu$, so that $(v_{\perp}^2)_{stab} \gg (v_{\parallel}^2)_{stab} \sim (m/M)\, v'^2_{T'}$, and, for an evaluation of the lateral distribution and life we can pose $v_{\parallel} = 0$ in (II.5.2). Let $v'_{T'} \ll (v_1)_{perm} \ll v_0^{kp}$. The "tail" is extended mainly in the directions of normal oscillations, seeing that here

$$\left\langle \frac{1}{u} \right\rangle_{(1)} \simeq \frac{2}{\pi v_c} \ln \frac{v_c}{v'_{T'}} > \frac{1}{v_c} \simeq \left\langle \frac{1}{u} \right\rangle_{(2)} .$$

The first member of the expansion $\langle 1/u \rangle$ according to $v_{20}^2 \ll v'^2_{T'} \ll v_{10}^2$ is $\simeq -(v_{20}^2/2\pi v_{10} v'^2_{T'})$. From this, it can be seen that for $v_1 = (v_1)_{perm}$

$$(v_{20}^2)_{stab} \simeq \pi \frac{(v_1)_{perm}}{v'_{T'}} \frac{\mu}{|\bar{Q}|_{max}} v'^2_{T'} \ll v'^2_{T'} .$$

If we integrate the distribution over I_2 when $I_1 = (I_1)$ for the probability of "emergence" into the region $I_1 > (I_1)$ of time:

$$w = \frac{4\lambda^2}{\mu}(M v'_{T'})^2 \ln\frac{(v_1)_{perm}}{v'_{T'}} \exp\left[2\frac{|\bar{Q}|_{max}}{\mu}\left(\frac{2 v'_{T'}}{\pi (v_1)_{perm}} \ln\frac{(v_1)_{perm}}{v'_{T'}} - \left\langle \frac{v'_{T'}}{v'} \right\rangle\right)\right] .$$

Let us note that from (II.5.1) it follows that, in accordance with the evaluation in Section 1, when an error $|\vec{\Delta}| > v'_{T'}$ is present, the distribution near $I = I_s$ is Gaussian.

$$f \sim \exp\left[-\frac{1}{2I_s}\left(\frac{\partial\bar{Q}}{\partial I}\right)_s (I-I_s)^2\right]; \qquad \left(\bar{Q}(I_s)=0\right)$$

It is easy to evaluate that $\langle (I - I_s)^2 \rangle \ll I_s^2$ if $v'_{T'} \ll \Delta \ll v_0^{kp}$, i.e. the distribution in the case of "monochromatic" instability is concentrated in the proximity of I_s.

Finally, let us discuss the dependence of the attenuation rate on the transverse dimensions of the electron beam. Let the beam be situated symmetrically in relation to the equilibrium orbit of the protons. In the case of excitation of two-dimensional betatron oscillations the decrease in the transverse dimensions b_r, b_z always proves to be advantageous, since the power, here, "builds up" independently to the dimensions, at velocities $|v_i| \sim v_{i0}$, and the product of the density n by the fraction of the phases, when the particles are situated in the beam, does not, in any case, decrease. Consequently, at a fixed current, the integral time of attenuation decreases with a reduction of the dimensions.

For one-dimensional oscillations $v_0 \gg v'_{T'}$ when the dimension is reduced in the direction of the oscillations from a value b = a (oscillation amplitude) to a certain b < a, the power decreases in the relation $\ln(v_0/v'_{T'})$, since at small velocities $|v| \sim v'_{T'}$, which give the basic contribution for $b \gtrsim a$, the particle is outside the beam ($v = - v_c \sin \Psi$, $r = a \cos \Psi$).

The situation is different when the synchrotron motion is excited. In the condition without self-phasing the effect of bypassing the beam is manifest when synchrotron deflection of the radius exceeds b_r. This very effect leads to a sharp decrease in the power $\bar{Q}_c$ in the condition of oscillations too since $r_c \sim v_{\parallel}$. It can easily be evaluated that for amplitudes of $v_{\parallel 0} \lesssim v'_{T'}$ the power decreases in the relation $\eta \sim r_c^2/b_r^2$, when b_r is reduced from the value $b_r^2 = r_c^2$. If, however,

$$1 < (v_{\parallel 0}^2/v'^2_{T'}) < \frac{r_c^2}{b_r^2},$$

then

$$\eta = (v_{\parallel 0}/v'_{T'})^2 \frac{b_r^2}{r_c^2} > 1 .$$

The radial dimension of the beam of electrons must thus be kept at a level of $b_r^2 \sim \overline{r_c^2}$.

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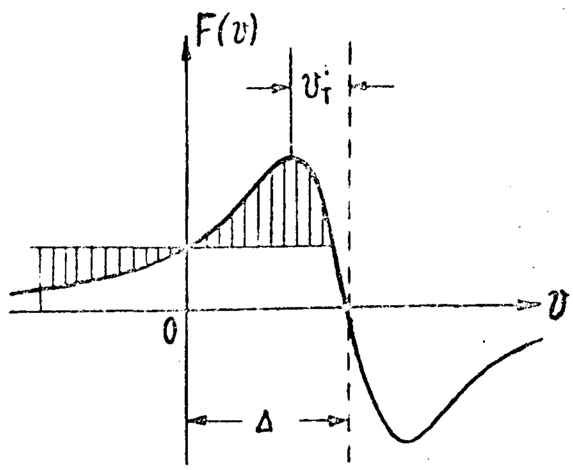

Fig. 1 Graph of the frictional force, corresponding to the distribution $f'(\vec{v}') \sim \exp[-(\vec{v}' - \vec{\Delta})^2/v'^2_{T'}]$. The power $\bar{Q}_1 = \overline{F_1 v_1}$ is positive for all amplitudes $v_1^0 < \Delta$.

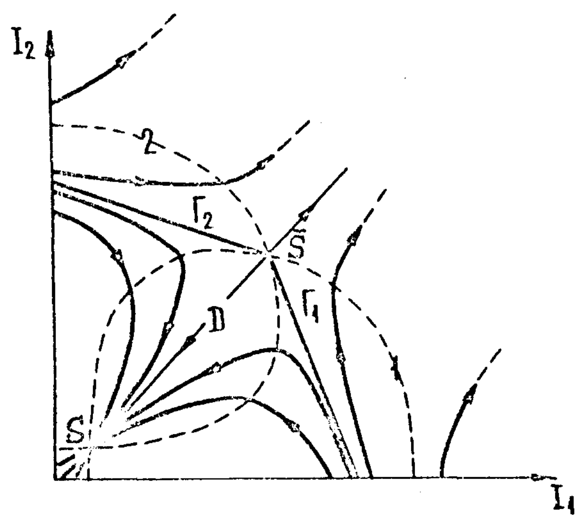

Fig. 2 Trajectory of movement of amplitudes

$$\dot{I}_1 = \bar{Q}_1 + \langle \Delta I_1 \rangle_{fl},$$
$$\dot{I}_2 = \bar{Q}_2 + \langle \Delta I_2 \rangle_{fl}.$$