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New Algorithm for Dynamical Friction of Ions in a Magnetized Electron Beam

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Abstract. Relativistic magnetized electron cooling in untested parameter regimes is essential to achieve the ion luminosity requirements of proposed electron-ion collider designs. Therefore, accurate calculations of magnetized dynamic friction are required, with the ability to include all relevant physics that might increase the cooling time, including space charge forces, field errors and complicated phase space distributions of imperfectly magnetized electron beams. We present a fundamentally new analytic treatment of momentum transfer from a single magnetized electron to a drifting ion, with arbitrary initial conditions and arbitrary interaction time. The map describing such collisions, which is derived from a Hamiltonian and conserves momentum, removes any need to numerically integrate the complicated Larmor trajectories. This important result, made possible by effectively exploiting the perturbative nature of the problem, will enable rapid semi-analytic calculations of dynamic friction on ions in magnetized cooling systems for arbitrary electron distributions.

INTRODUCTION

We present a fresh look at the long-standing challenge (see Ref. [1] for a review) of calculating the dynamic friction on a single ion through electron cooling systems relevant to high-luminosity electron-ion colliders [2, 3, 4]. Early work [5, 6, 7] considered nonrelativistic beams. Because the ion and electron beams propagate with the same velocity, the dynamical friction can be simulated in the beam frame, where the dynamics is nonrelativistic; however, the relativistic case is fundamentally different due to finite interaction time effects. Unfortunately, the cooling time increases as γ^2, with one γ factor due to Lorentz expansion of the beams (i.e. reduction of the electron density) and the other due to Lorentz contraction of the interaction time. Due to this scaling with γ, the unmagnetized cooling time for ~100 GeV/nucleon ions is ~1 hour [8], depending on many aspects of the cooling system design. Any electron-ion collider design requires a thorough understanding of the single-pass dynamics. It has been shown in the context of unmagnetized cooling [9] that a fast analytic treatment of single electron-ion scattering events can yield a speedup factor of 10^4 as compared to full simulations. We present here the required analytic single-scattering map for magnetized electrons.

THE HAMILTONIAN FOR 2-BODY MAGNETIZED COLLISIONS

We consider a single drifting ion and a single magnetized electron in a longitudinal magnetic field $\mathbf{B} = B_0\,\mathbf{z}$ that is constant in space and time. Our first approximation is that the motion of the more massive ion is not perturbed by the magnetic field. It is convenient [10] to choose the vector potential $\mathbf{A} = -B_0\,y\,\mathbf{x}$, which leads to a difference between the electron kinetic and canonical momentum along the x-axis. For the ion momentum vector, and the other electron momentum components, the canonical momentum is equal to the product of mass and velocity. In Eq. (1), we write the Hamiltonian as the sum of a zeroth-order term H_0, which can be solved analytically, and a collisional term H_C. Details are provided in Eqs. (2). The subscripts *ion* and *e* are used to distinguish the ion from the electron, respectively. We use MKS units, and the electron Larmor frequency (also known as the gyrofrequency) is written as $\Omega_e = eB_0/m_e$.

$$H\left(\vec{x}_{ion},\vec{p}_{ion},\vec{x}_e,\vec{p}_e\right)=H_0\left(\vec{p}_{ion},y_e,\vec{p}_e\right)+H_C\left(\vec{x}_{ion},\vec{x}_e\right) \tag{1}$$

$$H_0\left(\vec{p}_{ion},y_e,\vec{p}_e\right)=\frac{1}{2m_{ion}}\left(p_{ion,x}^2+p_{ion,y}^2+p_{ion,z}^2\right)+\frac{1}{2m_e}\left[\left(p_{ex}+eB_0y_e\right)^2+p_{ey}^2+p_{ez}^2\right] \tag{2a}$$

$$H_C\left(\vec{x}_{ion},\vec{x}_e\right)=\left.-Ze^2\middle/4\pi\varepsilon_0\right.\sqrt{\left(x_{ion}-x_e\right)^2+\left(y_{ion}-y_e\right)^2+\left(z_{ion}-z_e\right)^2} \tag{2b}$$

$$p_{e,x}=m_e\left(v_{e,x}-\Omega_e y_e\right) \tag{2c}$$

The standard 2nd-order symplectic leapfrog integrator for Eqs. (2) is constructed from a symmetric application of the maps associated with the two terms of the splittable Hamiltonian, M_0 and M_c. These maps are simply the analytic solution to the corresponding partial Hamiltonians, for a sufficiently small time interval. If we choose the time step Δt such that $\Omega_e\Delta t<<2\pi$, then we can accurately advance the trajectories via the composite map shown in Eq. (3). Use of the Hamiltonian in Eqs. (2) requires that we resolve the Larmor oscillations of the magnetized electron, which implies $\sim 10^4$ time steps, so symplecticity is essential for obtaining valid dynamics.

$$M\left(\Delta t\right)=M_0\left(\Delta t/2\right)M_C\left(\Delta t\right)M_0\left(\Delta t/2\right) \tag{3}$$

THE GUIDING CENTER HAMILTONIAN

To obtain equations of motion that can be integrated with orders of magnitude less computational effort, we transform to guiding center coordinates via Lichtenberg and Lieberman's canonical generating function of the 1st kind [10], given by Eq. (4). Here (x_{gc}, y_{gc}) are the guiding center coordinates of the electron, with y_{gc} serving as a canonical coordinate and $p_{gc}=m_e\Omega_e x_{gc}$ the corresponding canonical momentum. The magnetic moment of the electron also serves as a canonical momentum, $p_\phi=m_e v_p^2/2\Omega_e$, where $v_p^2=v_{ex}^2+v_{ey}^2$. The gyrophase ϕ is the corresponding coordinate, defined by $tan(\phi)=v_{ex}/v_{ey}$. These canonical pairs replace (x_e,p_{ex}) and (y_e,p_{ey}). The other canonical pairs in Eqs. (2) are unchanged. The Hamiltonian is written formally in Eq. (5) as two solvable pieces, H_0 and H_C. The functional forms of the partial Hamiltonians are shown in Eqs. (6). The Larmor radius is defined by $r_L=p_\phi/m_e\Omega_e$.

$$F\left(x_e,y_e,\varphi,y_{gc}\right)=m_e\Omega_e\left[\frac{1}{2}\left(y_e-y_{gc}\right)^2\cot(\varphi)-x_e y_{gc}\right] \tag{4}$$

$$H\left(\vec{x}_{ion},\vec{p}_{ion},\varphi,y_{gc},z_e,p_\varphi,p_{gc},p_{ez}\right)=H_0\left(\vec{p}_{ion},p_\varphi,p_{ez}\right)+H_C\left(\vec{x}_{ion},\varphi,y_{gc},z_e,p_\varphi,p_{gc}\right) \tag{5}$$

$$H_0\left(\vec{p}_{ion},p_\varphi,p_{e,z}\right)=\frac{1}{2m_{ion}}\vec{p}_{ion}\cdot\vec{p}_{ion}+\Omega_e p_\varphi+\frac{1}{2m_e}p_{e,z}^2 \tag{6a}$$

$$H_C\left(\vec{x}_{ion},\varphi,y_{gc},z_e,p_\varphi,p_{gc},p_{ez}\right)=\frac{-Ze^2/4\pi\varepsilon_0}{\sqrt{\begin{array}{l}\left(x_{ion}-p_{gc}/m_e\Omega_e\right)^2+\left(y_{ion}-y_{gc}\right)^2+\left(z_{ion}-z_e\right)^2+r_L^2\\+2\left(x_{ion}-p_{gc}/m_e\Omega_e\right)r_L\cos(\varphi)-2\left(y_{ion}-y_{gc}\right)r_L\sin(\varphi)\end{array}}} \tag{6b}$$

The rapidly changing gyrophase ϕ still appears explicitly in Eqs. (6); hence, the electron Larmor oscillations must still be resolved when integrating the equations of motion. We remove the phase through the standard action-angle transformation of 1D classical perturbation theory [11]. To do so we must assume that H_C is a perturbation, which is very well satisfied for all electron trajectories that stay at least one Larmor radius away from the ion. It's also necessary to assume that r_L satisfies Eq. (7) for all times during the interaction, a condition that is approximately valid for most

relevant trajectories and has the useful property of failing gracefully as r_L becomes large, a limit that doesn't contribute strongly to the integrated dynamical friction coefficient.

$$r_L << \left[\left(x_{ion} - x_{gc}\right)^2 + \left(y_{ion} - y_{gc}\right)^2 + \left(z_{ion} - z_e\right)^2 + r_L^2 \right]^{1/2} \tag{7}$$

Following Lichtenberg and Lieberman [11], expanding to first order in r_L, we transform from (ϕ,p_ϕ) to (Θ,J) through Eq. (8), which shows how to calculate J – a constant of the motion that is very close in magnitude to p_ϕ. Likewise, the new phase Θ differs only perturbatively from ϕ. We don't present the functional form of Θ, because it does not affect the dynamics of the electron-ion scattering event.

$$J = p_\varphi + \frac{Ze^2}{4\pi\varepsilon_0} \frac{r_L}{\Omega_e} \frac{\left(x_{ion} - x_{gc}\right)\cos(\varphi) - \left(y_{ion} - y_{gc}\right)\sin(\varphi)}{\left(\left(x_{ion} - x_{gc}\right)^2 + \left(y_{ion} - y_{gc}\right)^2 + \left(z_{ion} - z_e\right)^2 + r_L^2\right)^{3/2}} \tag{8}$$

The new Hamiltonian, shown in Eqs. (9) and (10), differs very little in value from the previous one; however, the perturbative collisional term has a much simpler form. Except for the new action-angle pair, the same conjugate pairs from Eqs. (6) are used here. We emphasize that the coordinate Θ has been removed from the problem, effectively removing two degrees of freedom from the 12-dimensional phase space.

$$H\left(\vec{x}_{ion}, y_{gc}, z_e, \vec{p}_{ion}, p_{gc}, p_{ez}, J\right) = H_0\left(\vec{p}_{ion}, p_{ez}, J\right) + H_C\left(\vec{x}_{ion}, y_{gc}, z_e, p_{gc}, J\right) \tag{9}$$

$$H_0\left(\vec{p}_{ion}, J, p_{e,z}\right) = \frac{1}{2m_{ion}} \vec{p}_{ion} \cdot \vec{p}_{ion} + \Omega_e J + \frac{1}{2m_e} p_{e,z}^2 \tag{10a}$$

$$H_C\left(\vec{x}_{ion}, y_{gc}, z_e, J, p_{gc}\right) = \frac{-Ze^2}{4\pi\varepsilon_0} \Bigg/ \left[\left(x_{ion} - \frac{p_{gc}}{m_e\Omega_e}\right)^2 + \left(y_{ion} - y_{gc}\right)^2 + \left(z_{ion} - z_e\right)^2 + \frac{2}{m_e\Omega_e} J \right]^{1/2} \tag{10b}$$

The 2nd-order symplectic integrator for Eqs. (10) is again a symmetric application of the maps associated with the two terms of the splittable Hamiltonian, M_0 and M_c, as specified in Eq. (3). The time step Δt can now be orders of magnitude larger, chosen for example to be much smaller than the longitudinal ion-electron separation divided by the corresponding difference in longitudinal velocities. Care is required for cases where the ion-electron pair closely approach each other; however, this scenario does not contribute to the integrated dynamical friction coefficient and can be ignored in practice.

Calculation of M_0 is straightforward and is not shown here, while calculation of M_c is more difficult. We ignore Θ and the following momenta are constant: x_{ion}, z_e and J. The other momenta are modified as shown in Eqs. (11), where $\alpha = Ze^2/4\pi\varepsilon_0$ and $\Delta p_{ez} = -\Delta p_{ion,z}$. We emphasize that Eqs. (11) define M_c, which is to be used with M_0 in Eq. (3). We note that symplectic integration is not required here, because we are considering only a single interaction, the fast oscillations have been removed, and the number of required time steps has been reduced dramatically.

$$\Delta p_{ion,x} = \alpha\left(p_{gc}/m_e\Omega_e - z_{ion}\right)\Delta t \Big/ b^3\left(\vec{x}_{ion}, y_{gc}, z_e, J, p_{gc}\right) \tag{11a}$$

$$\Delta p_{ion,y} = \alpha\left(y_{gc} - z_{ion}\right)\Delta t \Big/ b^3\left(\vec{x}_{ion}, y_{gc}, z_e, J, p_{gc}\right) \tag{11b}$$

$$\Delta p_{ion,z} = \alpha\left(z_e - z_{ion}\right)\Delta t \Big/ b^3\left(\vec{x}_{ion}, y_{gc}, z_e, J, p_{gc}\right) \tag{11c}$$

$$b\left(\vec{x}_{ion}, y_{gc}, z_e, J, p_{gc}\right) = \left[\left(x_{ion} - \frac{p_{gc}}{m_e\Omega_e}\right)^2 + \left(y_{ion} - y_{gc}\right)^2 + \left(z_{ion} - z_e\right)^2 + \frac{2}{m_e\Omega_e} J \right]^{1/2} \tag{11d}$$

ANALYTIC MAP FOR MAGNETIZED ELECTRON-ION SCATTERING

We now use the Magnus expansion [12] to perturbatively obtain a factored map treatment of Eqs. (10), which will remove any need for numerical integration. The factored map has the form $M_{total}(T) = M_I(T) M_0(T)$, where T is the total interaction time, the interaction map $M_I(T)$ is applied first, and we again make the valid approximation that the effects of the Coulomb interaction between the ion and electron are perturbative. The unperturbed or 'drift' map $M_0(T)$ is just the analytic solution to the unperturbed Hamiltonian H_0 of Eq. (10a), which is straightforward to calculate and so is not presented here. We define the interaction Hamiltonian H_I as $H_I(t) = M_0(t) H_C$, meaning that the unperturbed solutions for the coordinates and momenta are inserted into the partial Hamiltonian H_C. The interaction map $M_I(t)$ is then defined by Eq. (12), where the colons indicate a Lie operator and the integral is a partial time integration over $H_I(t)$, including the explicit time dependence of all the unperturbed coordinates and momenta.

$$M_I(t) = \exp\left[-: \int_0^t d\sigma \, H_I(\sigma): \right] \tag{12}$$

The integral in Eq. (12) can be evaluated for the approximate, averaged Hamiltonian of Eqs. (10); however, it is then problematic to analytically evaluate the exponentiated Lie operator. Because H_I is perturbative and the effects of the map M_I are small, we can expand the exponential to obtain Eq. (13), where the symbol I represents the identity operator and the colons indicate a Lie operator. Also, we have let t=T, the full interaction time. If the partial integration over time of H_I yields a Hamiltonian with analytic solutions, then the effect of the map M_I is to increment the coordinates and momenta by the negative of these solutions, evaluated at t=T. Expanding the exponential breaks the Hamiltonian nature of the map, so that our approximate form in Eq. (13) is not symplectic; however, loss of symplecticity is not important for our problem, and we will see that momentum is still conserved.

$$M_I(T) \approx I - : \int_0^T d\sigma \, H_I(\sigma): \tag{13}$$

The integral of Eq. (13) can be evaluated and the resulting operator can be calculated analytically. We present in Eqs. (14) some intermediate quantities that are used in the final result. The coordinates and momenta here are initial conditions, which are used to obtain the perturbative kicks of M_I, after which the drift motion of M_0 is to be applied.

$$C_1 = \left(x_{ion} - \frac{p_{gc}}{m_e \Omega_e} \right)^2 + \left(y_{ion} - y_{gc} \right)^2 + \left(z_{ion} - z_e \right)^2 + \frac{2}{m_e \Omega_e} J \tag{14a}$$

$$C_2 = 2\left(x_{ion} - x_{gc} \right) v_{ion,x} + 2\left(y_{ion} - y_{gc} \right) v_{ion,y} + 2\left(z_{ion} - z_e \right)\left(v_{ion,z} - v_{ez} \right) \tag{14b}$$

$$C_3 = v_{ion,x}^2 + v_{ion,y}^2 + \left(v_{ion,x} - v_{ez} \right)^2 \tag{14c}$$

$$b = \left[C_1 + C_2 T + C_3 T^2 \right]^{1/2} \qquad \Delta = 4C_1 C_3 - C_2^2 \tag{14d}$$

$$D_1 = \left[\frac{2C_3 T + C_2}{b} - \frac{C_2}{\sqrt{C_1}} \right] \tag{14e}$$

$$D_2 = \left[\frac{2C_1 + C_2 T}{b} - 2\sqrt{C_1} \right] \tag{14f}$$

We ignore Θ and J is constant, as are the coordinates x_{ion} and z_e. The other coordinates and momenta are modified as shown in Eqs. (15), where $\alpha = Z e^2 / 4\pi \varepsilon_0$ and $\Delta p_{ez} = -\Delta p_{ion,z}$.

$$\Delta p_{ion,x} = \frac{-2\alpha}{\Delta} \left[\left(x_{ion} - p_{gc}/m_e \Omega_e \right) D_1 - \left(p_{ion,x}/m_{ion} \right) D_2 \right] \tag{15a}$$

$$\Delta p_{ion,y} = \frac{-2\alpha}{\Delta}\left[\left(y_{ion} - y_{gc}\right)D_1 - \left(p_{ion,y}/m_{ion}\right)D_2\right] \tag{15b}$$

$$\Delta p_{ion,z} = \frac{-2\alpha}{\Delta}\left[\left(z_{ion} - z_e\right)D_1 - \left(\frac{p_{ion,z}}{m_{ion}} - \frac{p_{ez}}{m_e}\right)D_2\right] \tag{15c}$$

$$\Delta p_{gc} = -\Delta p_{ion,x} \qquad \Delta y_{gc} = -\Delta p_{ion,y}/m_e\Omega_e \tag{15d}$$

FUTURE WORK

Our primary result is presented in Eqs. (14) and (15); however, this mapping must be combined with the analytic solutions to the partial Hamiltonian of Eq. (10a) according to $M_{total}(T) = M_I(T)\, M_0(T)$, where T is the total interaction time. Next, we will numerically test these results by comparing three solutions for a variety of initial conditions: a) direct application of Eq. (3) to Eqs. (2); b) application of Eq. (3) to Eqs. (10); c) use of Eqs. (14) and (15).

The dynamical friction force on a single ion involves an integral of the momentum kicks in Eqs. (15) over some distribution of electrons. Assuming simple electron distributions, perhaps with other approximations, we can evaluate these integrals analytically, which in turn will enable direct comparison with previous results [13, 14, 15]. It will also be fast and simple to numerically integrate Eqs. (15) over Gaussian electron distributions or simulated distributions obtained from particle accelerator codes. Numerical results with arbitrary electron distributions will be obtained orders of magnitude more quickly than is possible via direct simulation Monte Carlo (DSMC), particle-in-cell (PIC) or other direct particle-based methods.

We plan to extend our analytic method to include other perturbative effects that are essential to understand. Space charge forces cause an **E** x **B** drift of the electron gyrocenters, resulting in an effective transverse velocity that reduces dynamic friction. This is a perturbative effect that, if treated as an external radial force, can be represented by another simple term in what would now be a Hamiltonian that splits into three pieces. In the beam frame of a relativistic cooler, magnetic field perturbations manifest as time-varying transverse electric fields [9], which could be treated in a similar way to the space charge fields. Hence, we could obtain quantitative calculations of negative effects that to date have been either ignored or else added via ad hoc parameterizations.

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