A New Approach to Calculating Magnetized Dynamical Friction for JLEIC

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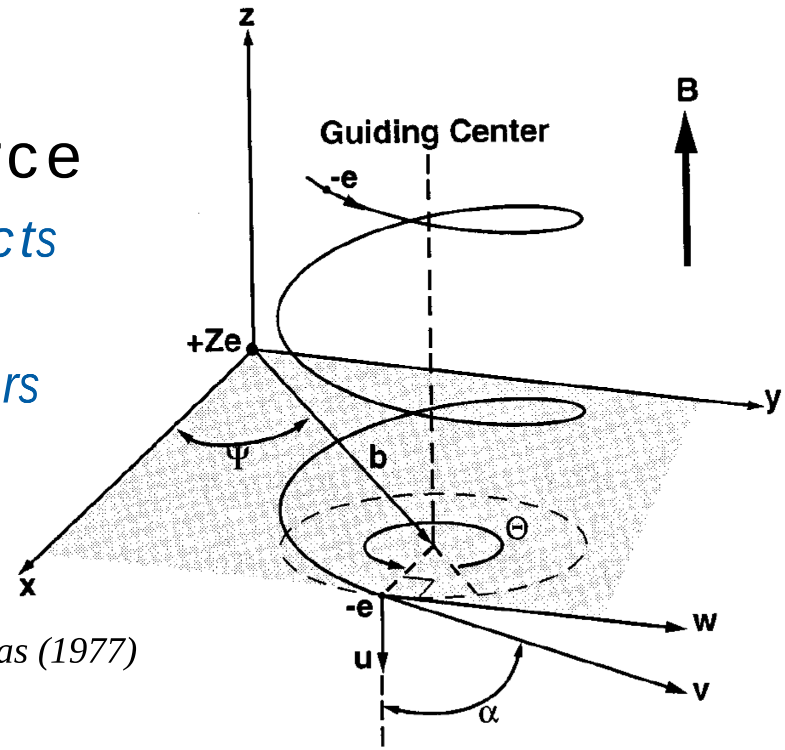
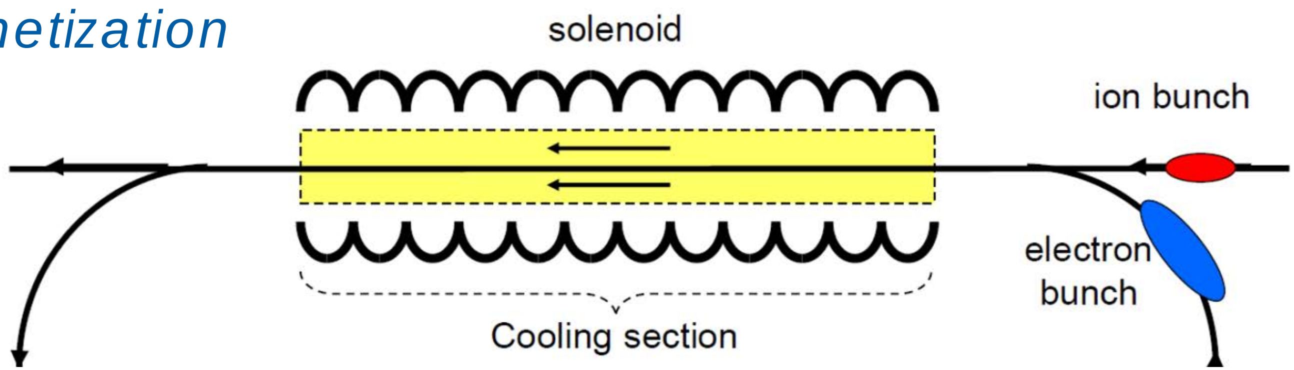
Goals

- Simulate magnetized friction force
  - *include all relevant real world effects*
    - e.g. incoming beam distribution
  - *include a wide range of parameters*
  - *cannot succeed via brute force*
    - new theory is required



from Geller & Weisheit, Phys. Plasmas (1977)

- Include key aspects of magnetized e- beam transport
  - *imperfect magnetization*
  - *space charge*
  - *field errors*



from Zhang et al., MEIC design, arXiv (2012)

Hamiltonian for 2-body magnetized collision

$$H\left(\vec{x}_{ion},\vec{p}_{ion},\vec{x}_e,\vec{p}_e\right)=H_0\left(\vec{p}_{ion},y_e,\vec{p}_e\right)+H_C\left(\vec{x}_{ion},\vec{x}_e\right)$$

$$\vec{B}=B_0\,\hat{z} \qquad \vec{A}=-B_0 y\,\hat{x} \qquad p_{e,x}=m_e\left(v_{e,x}-\Omega_L y_e\right)$$

$$H_0\left(\vec{p}_{ion},y_e,\vec{p}_e\right)=\frac{1}{2m_{ion}}\left(p_{ion,x}^2+p_{ion,y}^2+p_{ion,z}^2\right)+\frac{1}{2m_e}\left[\left(p_{e,x}+eB_0 y_e\right)^2+p_{e,y}^2+p_{e,z}^2\right]$$

$$H_C\left(\vec{x}_{ion},\vec{x}_e\right)=\frac{-Ze^2}{4\pi\varepsilon_0}\Bigg/\sqrt{\left(x_{ion}-x_e\right)^2+\left(y_{ion}-y_e\right)^2+\left(z_{ion}-z_e\right)^2}$$

Resulting equations of motion, in the standard drift-kick symplectic form:

$$M\left(\Delta t\right)=M_0\left(\Delta t/2\right)M_C\left(\Delta t\right)M_0\left(\Delta t/2\right)$$

D.L. Bruhwiler and S.D. Webb, “New algorithm for dynamical friction of ions in a magnetized electron beam,” in *AIP Conf. Proc.* **1812**, 050006 (2017); http://aip.scitation.org/doi/abs/10.1063/1.4975867

Symplectic drift map for 2-body system

$$M_0(\Delta t): \qquad \vec{p}_{ion} = constant \qquad p_{e,x} = constant \qquad p_{e,z} = constant$$

$$\vec{x}_{ion}(t+\Delta t) = \vec{x}_{ion}(t) + \frac{\vec{p}_{ion}(t)}{m_{ion}}\Delta t \qquad z_e(t+\Delta t) = z_e(t) + \frac{p_{e,z}(t)}{m_e}\Delta t$$

$$x_e(t+\Delta t) = x_e(t) + r_L\left[\cos(\varphi_0 + \Omega_e \Delta t) - \cos(\varphi_0)\right]$$

$$y_e(t+\Delta t) = y_e(t) - r_L\left[\sin(\varphi_0 + \Omega_e \Delta t) - \sin(\varphi_0)\right]$$

$$\varphi_0 = \tan^{-1}\left(v_{e,x}/v_{e,y}\right) \qquad v_{e,\perp}^2 = v_{e,x}^2 + v_{e,y}^2 \qquad \Omega_L = |eB_0|/m_e \qquad r_L = V_{e,\perp}/\Omega_L$$

Symplectic kick for 2-body system

$$M_C(\Delta t): \quad \vec{x}_{ion} = constant \qquad \vec{x}_e = constant$$

$$\Delta\vec{p}_{ion} = \frac{\alpha(\vec{x}_e - \vec{x}_{ion})\Delta t}{b^3(\vec{x}_{ion}, \vec{x}_e)} \qquad \Delta\vec{p}_e = \frac{\alpha(\vec{x}_{ion} - \vec{x}_e)\Delta t}{b^3(\vec{x}_{ion}, \vec{x}_e)}$$

$$\alpha = \frac{Ze^2}{4\pi\varepsilon_0} \qquad b(\vec{x}_{ion}, \vec{x}_e) = \left[(x_{ion} - x_e)^2 + (y_{ion} - y_e)^2 + (z_{ion} - z_e)^2\right]^{1/2}$$

These 2nd-order equation of motion are simple and robust.

They can be made 4th-order via standard Yoshida algorithm.

However, they require resolution of the gyroperiod and, hence, are slow:

$$\Delta t_{\max} \approx \frac{1}{8}\frac{2\pi}{\Omega_e}$$

Transform to Action-Angle variables

We follow Lichtenberg and Lieberman, *Regular & Chaotic Dynamics* (1992).
We use their canonical generating function of the 2nd kind:

$$F\left(x_e, y_e, \varphi, y_{gc}\right) = m_e\Omega_e\left[\frac{1}{2}\left(y_e - y_{gc}\right)^2 \cot(\varphi) - x_e y_{gc}\right]$$

which yield the following Hamiltonian:

$$H\left(\vec{x}_{ion}, \vec{p}_{ion}, \varphi, y_{gc}, z_e, p_\varphi, p_{gc}, p_{ez}\right) = H_0\left(\vec{p}_{ion}, p_\varphi, p_{ez}\right) + H_C\left(\vec{x}_{ion}, \varphi, y_{gc}, z_e, p_\varphi, p_{gc}\right)$$

$$H_0\left(\vec{p}_{ion}, p_\varphi, p_{e,z}\right) = \frac{1}{2m_{ion}}\vec{p}_{ion}\cdot\vec{p}_{ion} + \Omega_e p_\varphi + \frac{1}{2m_e}p_{e,z}^2$$

$$H_C\left(\vec{x}_{ion}, \varphi, y_{gc}, z_e, p_\varphi, p_{gc}, p_{ez}\right) = \frac{-Ze^2/4\pi\varepsilon_0}{\sqrt{\begin{array}{l}\left(x_{ion} - x_{gc}/m_e\Omega_e\right)^2 + \left(y_{ion} - y_{gc}\right)^2 + \left(z_{ion} - z_e\right)^2 + r_L^2 + \cdots \\ \qquad \cdots + 2\left(x_{ion} - x_{gc}\right)r_L\cos(\varphi) + 2\left(y_{ion} - y_{gc}\right)r_L\sin(\varphi)\end{array}}}$$

$$x_{gc} = p_{gc}/m_e\Omega_e$$

$$r_L = \left(2p_\varphi/m_e\Omega_e\right)^{1/2}$$

Zero'th-order dynamics is now very simple,
but H_C is problematic…

Transform to next-order Action-Angle variables

We follow Lichtenberg and Lieberman, *Regular & Chaotic Dynamics* (1992).
We use standard secular perturbation theory, requiring two approximations:

1) H_C is a perturbation. This requires $E_{kinetic} >> E_{potential}$.

2) $r_L << \left[\left(x_{ion} - x_{gc}\right)^2 + \left(y_{ion} - y_{gc}\right)^2 + \left(z_{ion} - z_e\right)^2 + r_L^2 \right]^{1/2}$

This is approximately satisfied for relevant trajectories and fails gracefully.

The result is to remove the fast ϕ-dependence from the Hamiltonian:

$$H\left(\vec{x}_{ion}, y_{gc}, z_e, \vec{p}_{ion}, p_{gc}, p_{ez}, J\right) = H_0\left(\vec{p}_{ion}, p_{ez}, J\right) + H_C\left(\vec{x}_{ion}, y_{gc}, z_e, p_{gc}, J\right)$$

$$J = p_\varphi + \frac{Ze^2}{4\pi\varepsilon_0} \frac{r_L}{\Omega_e} \frac{\left(x_{ion} - x_{gc}\right)\cos(\varphi) + \left(y_{ion} - y_{gc}\right)\sin(\varphi)}{\left(\left(x_{ion} - x_{gc}\right)^2 + \left(y_{ion} - y_{gc}\right)^2 + \left(z_{ion} - z_e\right)^2 + r_L^2\right)^{3/2}}$$

$$H_0\left(\vec{p}_{ion}, J, p_{e,z}\right) = \frac{1}{2m_{ion}} \vec{p}_{ion} \cdot \vec{p}_{ion} + \Omega_e J + \frac{1}{2m_e} p_{e,z}^2$$

$$H_C\left(\vec{x}_{ion}, y_{gc}, z_e, J, p_{gc}\right) = \frac{-Ze^2}{4\pi\varepsilon_0} \Bigg/ \left[\left(x_{ion} - \frac{p_{gc}}{m_e\Omega_e} \right)^2 + \left(y_{ion} - y_{gc}\right)^2 + \left(z_{ion} - z_e\right)^2 + \frac{2}{m_e\Omega_e} J \right]^{1/2}$$

Symplectic maps for averaged 2-body system

Equations of motion are still in the standard drift-kick symplectic form:

$$M(\Delta t) = M_0(\Delta t/2) M_C(\Delta t) M_0(\Delta t/2)$$

$M_0(\Delta t)$:

$$\vec{p}_{ion} = constant \qquad J = constant \qquad p_{gc} = constant$$

$$p_{ez} = constant \qquad \theta \text{ is ignored} \qquad y_{gc} = constant$$

$$\vec{x}_{ion}(t+\Delta t) = \vec{x}_{ion}(t) + \frac{\vec{p}_{ion}(t)}{m_{ion}} \Delta t \qquad z_e(t+\Delta t) = z_e(t) + \frac{p_{e,z}(t)}{m_e} \Delta t$$

Much larger time steps are now possible:

$$\Delta t_{\max} \approx \frac{1}{8} \frac{|z_{ion} - z_e|}{|v_{ion,z} - v_{e,z}|}$$

Symplectic kick for averaged 2-body system

$$M_C(\Delta t):$$

$$\vec{x}_{ion} = \mathit{constant}$$

$$\Delta p_{ion,x} = \alpha\left(p_{gc}/m_e\Omega_e - z_{ion}\right)\Delta t/b^3\left(\vec{x}_{ion}, y_{gc}, z_e, J, p_{gc}\right)$$

$$J = \mathit{constant}$$

θ is ignored

$$\Delta p_{ion,y} = \alpha\left(y_{gc} - z_{ion}\right)\Delta t/b^3\left(\vec{x}_{ion}, y_{gc}, z_e, J, p_{gc}\right)$$

$$z_e = \mathit{constant}$$

$$\Delta p_{ion,z} = \alpha\left(z_e - z_{ion}\right)\Delta t/b^3\left(\vec{x}_{ion}, y_{gc}, z_e, J, p_{gc}\right)$$

$$\Delta p_{ez} = -\Delta p_{ion,z}$$

$$\alpha = \frac{Ze^2}{4\pi\varepsilon_0}$$

$$b\left(\vec{x}_{ion}, y_{gc}, z_e, J, p_{gc}\right) = \left[\left(x_{ion} - \frac{p_{gc}}{m_e\Omega_e}\right)^2 + \left(y_{ion} - y_{gc}\right)^2 + \left(z_{ion} - z_e\right)^2 + \frac{2}{m_e\Omega_e}J\right]^{1/2}$$

Time-explicit vs Averaged – approx. agreement

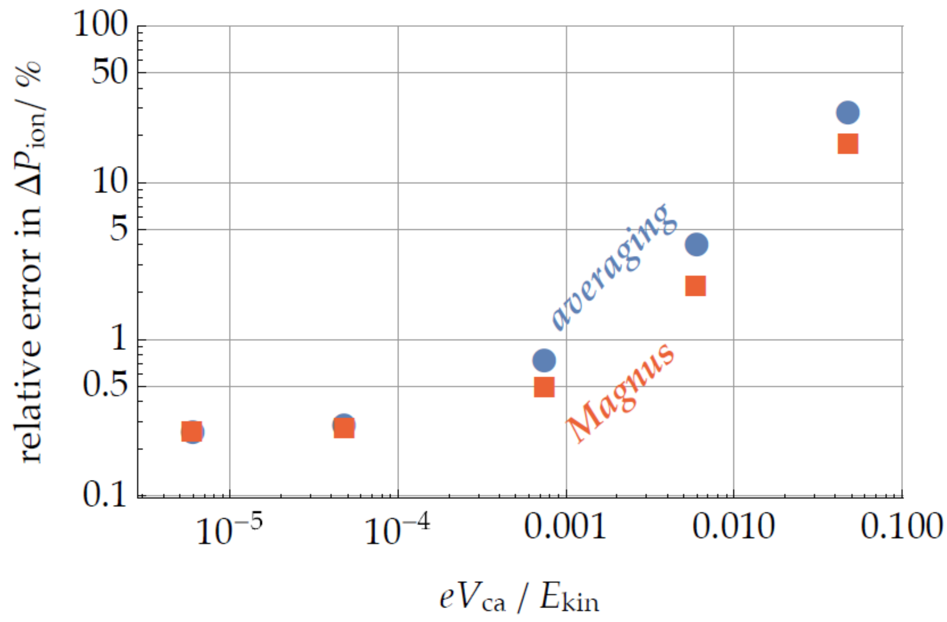
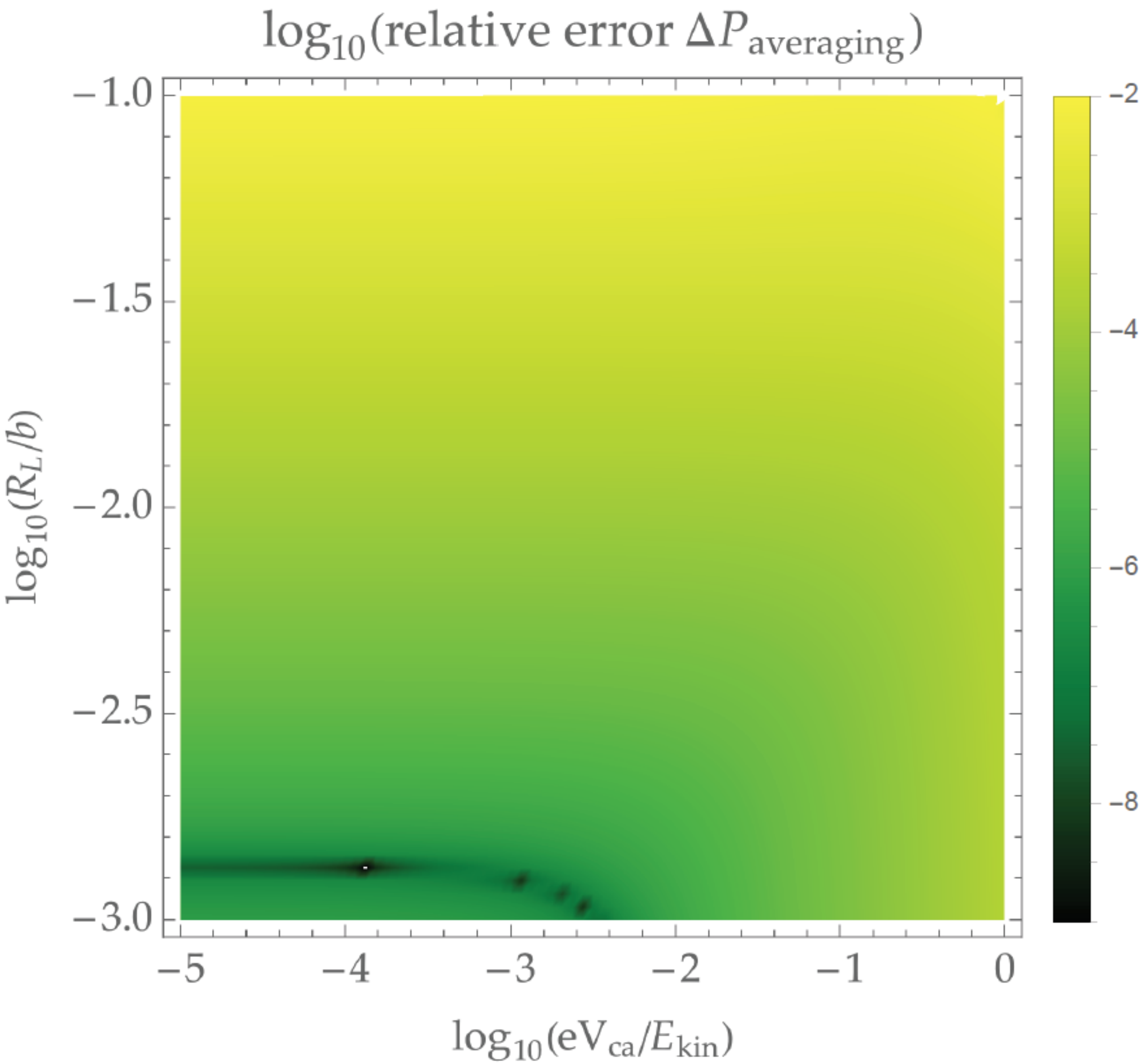


Figure 1: Relative error in the value of ΔP_{ion} computed using four-turn averaging (blue circles) and the Magnus expansion (red squares).

Time-explicit vs Averaged – approx. agreement



This graphic shows the relative error made by our averaging computation of the ion momentum kick ΔP. The average is taken over one full gyrotron period. The scale is logarithmic, so the largest errors here are about 1 %.

Magnus expansion yields analytic result

We follow Alex Dragt, ***Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics*** (Version of 22 June 2016), p. 861:

http://www.physics.umd.edu/dsat/dsatliemethods.html

The result is an analytic calculation of the ion momentum change!

$$M(t) = M_I(t) M_0(t)$$

$$M_I(t) = \exp\left[- : \int_0^t d\sigma \, H_I(\sigma) : \right] \qquad H_I(t) = M_0(t) H_C$$

$$M_I(T) \approx I - : \int_0^T d\sigma \, H_I(\sigma) :$$

We can evaluate this approximate expression analytically.
It is valid, when the Coulomb interaction is a perturbation.

Analytic calculation of $\Delta \mathbf{p}_{ion}$ (1)

$$C_1 = \left(x_{ion} - \frac{p_{gc}}{m_e \Omega_e} \right)^2 + \left(y_{ion} - y_{gc} \right)^2 + \left(z_{ion} - z_e \right)^2 + \frac{2}{m_e \Omega_e} J \tag{14a}$$

$$C_2 = 2\left(x_{ion} - x_{gc} \right) v_{ion,x} + 2\left(y_{ion} - y_{gc} \right) v_{ion,y} + 2\left(z_{ion} - z_e \right)\left(v_{ion,z} - v_{ez} \right) \tag{14b}$$

$$C_3 = v_{ion,x}^2 + v_{ion,y}^2 + \left(v_{ion,x} - v_{ez} \right)^2 \tag{14c}$$

$$b = \left[C_1 + C_2 T + C_3 T^2 \right]^{1/2} \qquad \Delta = 4 C_1 C_3 - C_2^2 \tag{14d}$$

$$D_1 = \left[\frac{2 C_3 T + C_2}{b} - \frac{C_2}{\sqrt{C_1}} \right] \tag{14e}$$

$$D_2 = \left[\frac{2 C_1 + C_2 T}{b} - 2\sqrt{C_1} \right] \tag{14f}$$

Analytic calculation of $\Delta\mathbf{p}_{\text{ion}}$ (2)

$$\Delta p_{ion,x} = \frac{-2\alpha}{\Delta}\left[\left(x_{ion} - p_{gc}/m_e\Omega_e\right)D_1 - \left(p_{ion,x}/m_{ion}\right)D_2\right] \tag{15a}$$

$$\Delta p_{ion,y} = \frac{-2\alpha}{\Delta}\left[\left(y_{ion} - y_{gc}\right)D_1 - \left(p_{ion,y}/m_{ion}\right)D_2\right] \tag{15b}$$

$$\Delta p_{ion,z} = \frac{-2\alpha}{\Delta}\left[\left(z_{ion} - z_e\right)D_1 - \left(\frac{p_{ion,z}}{m_{ion}} - \frac{p_{ez}}{m_e}\right)D_2\right] \tag{15c}$$

$$\Delta p_{gc} = -\Delta p_{ion,x} \qquad \Delta y_{gc} = -\Delta p_{ion,y}/m_e\Omega_e \tag{15d}$$

Time-explicit vs Magnus – approx. agreement

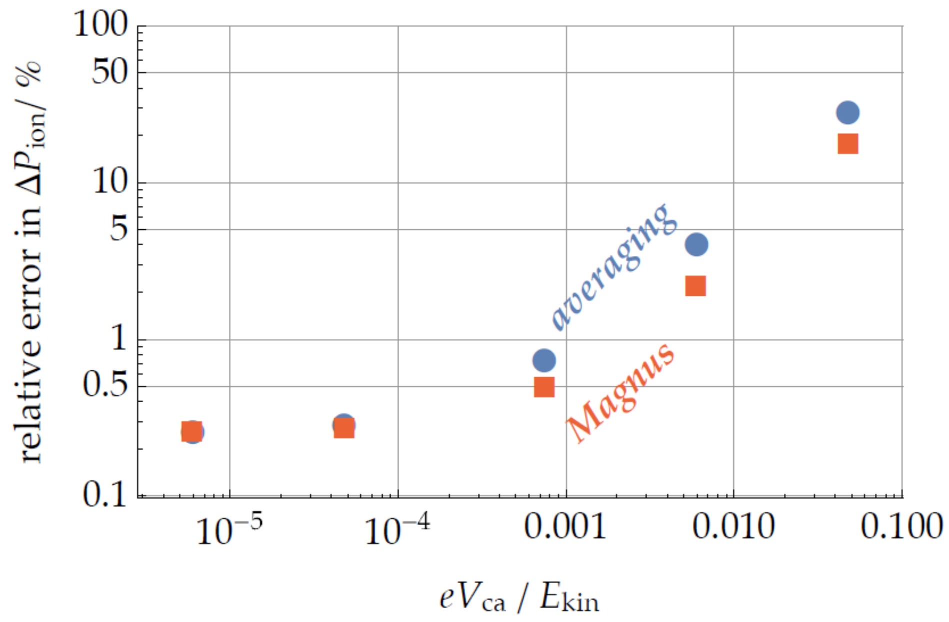
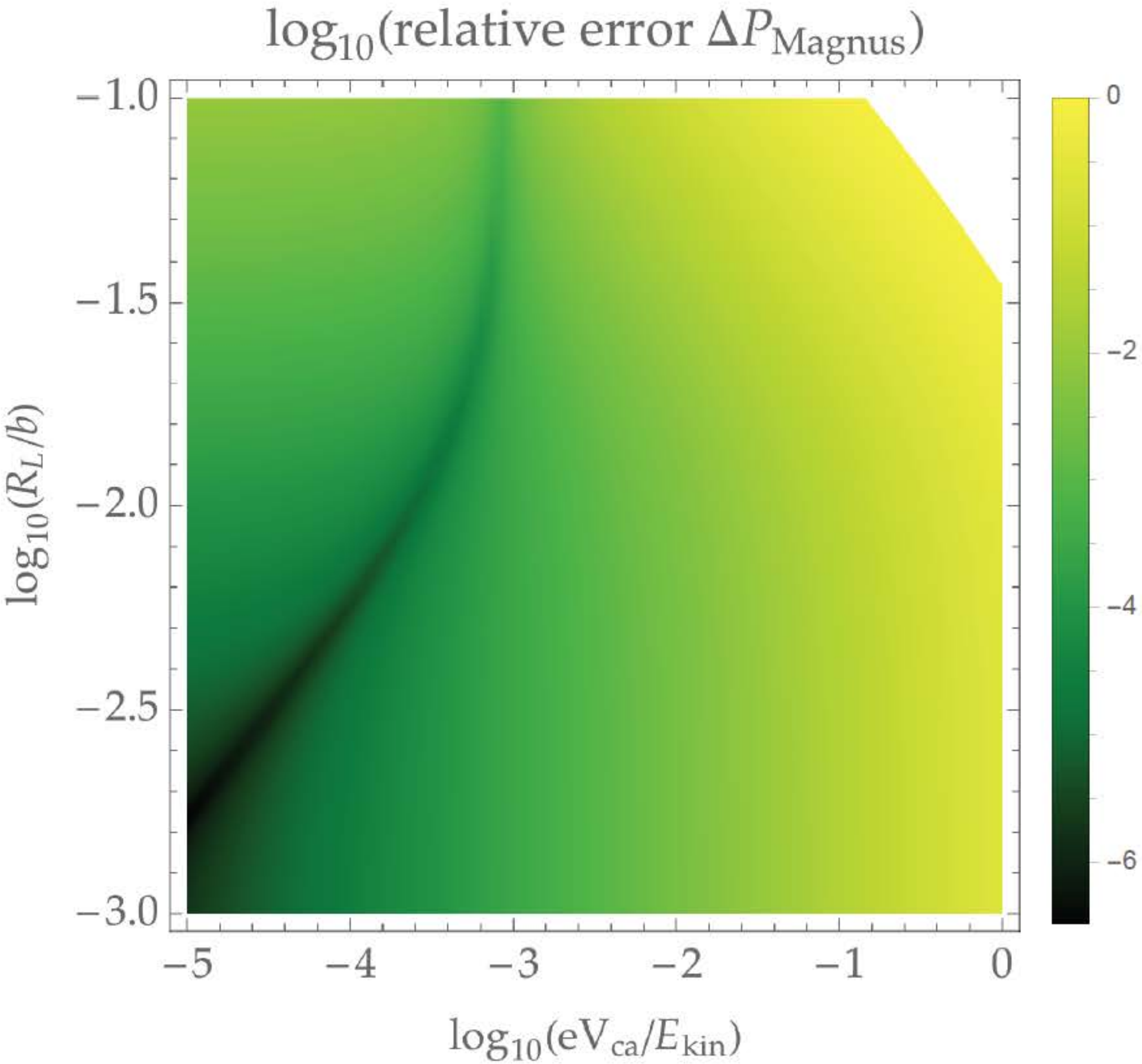


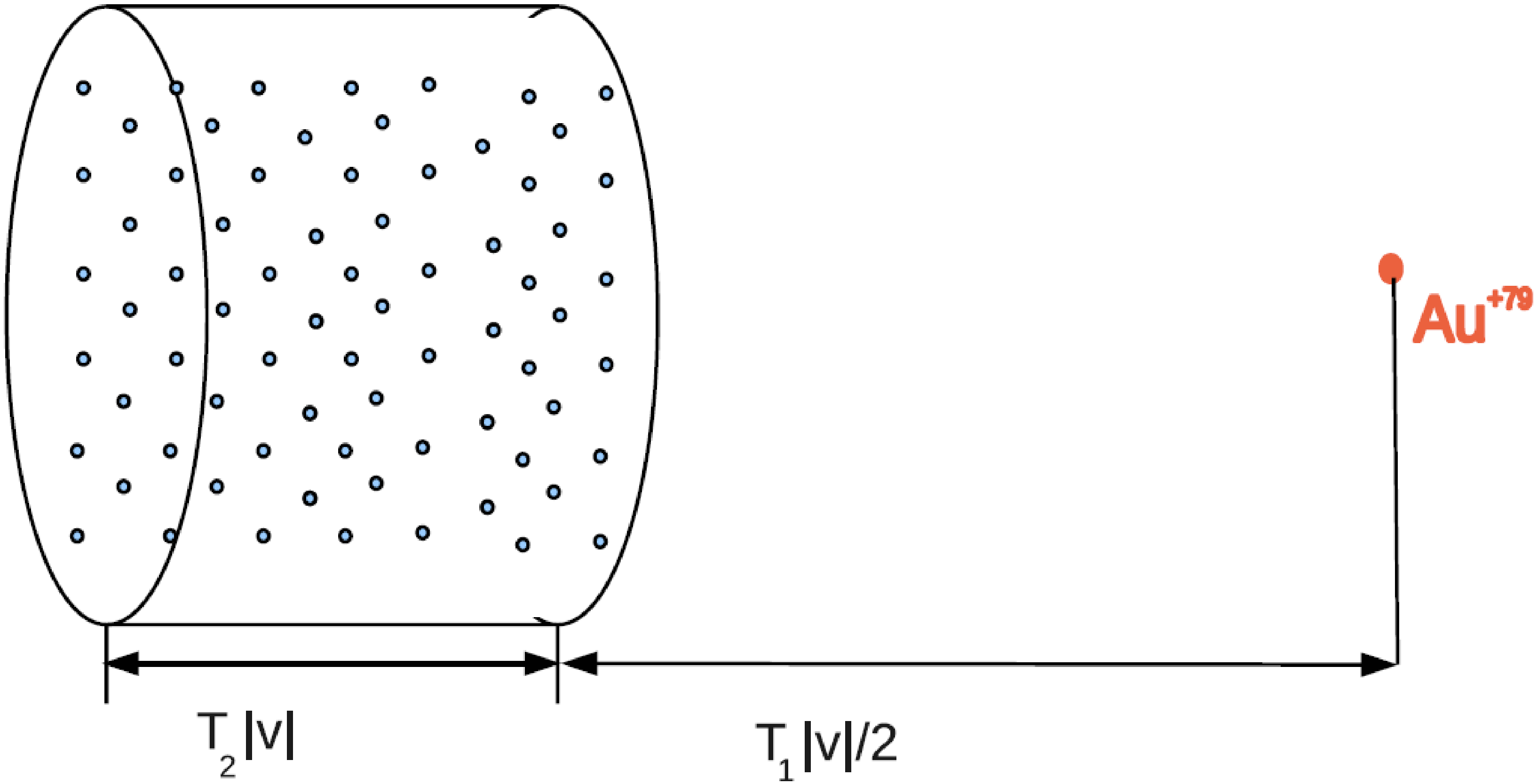
Figure 1: Relative error in the value of $\Delta P_{\rm ion}$ computed using four-turn averaging (blue circles) and the Magnus expansion (red squares).

Time-explicit vs Magnus – approx. agreement



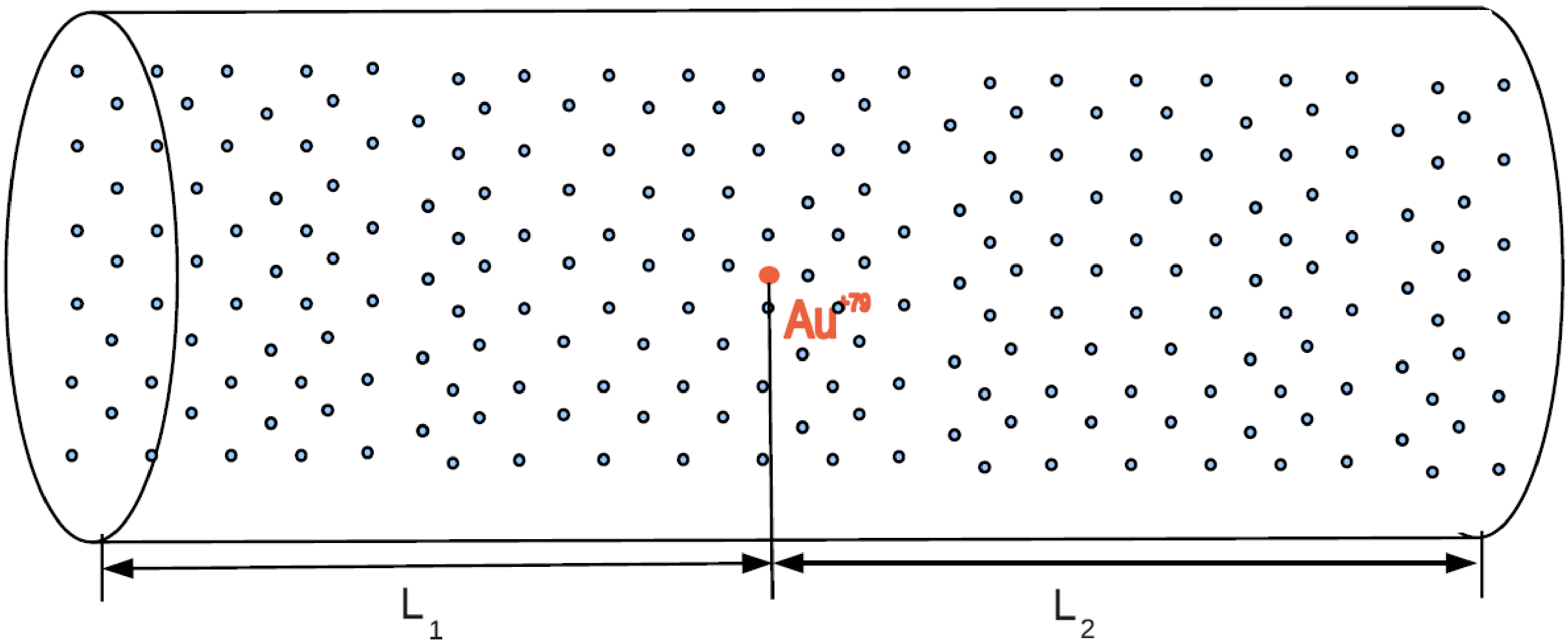
This graphic shows the relative error made by our Magnus computation of ΔP.

Integrate to obtain Friction force



$$F = -\frac{n_\mathrm{e} m_\mathrm{e}}{T} \iiint_{\mathbb{R}^3} \mathrm{d}^3 v \iiint_{V} \mathrm{d}r\, \mathrm{d}z\, \mathrm{d}\varphi\, \Delta v(T, r, \varphi, z, v) r p(v)$$

Integrate to obtain Friction force



$$F = -\frac{n_e m_e}{T} \iiint_{\mathbb{R}^3} \mathrm{d}^3 v \iiint_V \mathrm{d}r\, \mathrm{d}z\, \mathrm{d}\varphi\, \Delta v(T, r, \varphi, z, v) r p(v)$$

Include other effects in Magnus Expansion

$$H\left(\vec{x}_{ion}, \vec{p}_{ion}, \vec{x}_e, \vec{p}_e\right) = H_0\left(\vec{p}_{ion}, y_e, \vec{p}_e\right) + H_C\left(\vec{x}_{ion}, \vec{x}_e\right) + H_{space-charge}\left(\vec{x}_{ion}, \vec{x}_e\right) + H_{solenoid-field-errors}(??)$$

- Quantitative treatment of space charge & field errors?
  - *space charge should work*
  - *field errors are more challenging*
- Requires generalization of Magnus expansion
  - *we are optimistic this can be done*