Lagrangian:
$$\mathcal{L} = \frac{m\vec{v}_e^{\,2}}{2} + \frac{M\vec{v}_i^{\,2}}{2} - e\vec{A}(\vec{r}_e)\vec{v}_e + Ze\vec{A}(\vec{r}_i)\vec{v}_i + \frac{Ze^2}{|\vec{r}_e-\vec{r}_i|} \qquad (3.3)$$

magnetic field along $\vec{e}_z$ $\Rightarrow$ $\vec{B}=B\vec{e}_z$ and vector-potential for that is

$$\vec{A}(\vec{r}) = \frac{1}{2}[\vec{B}\vec{r}] = \frac{1}{2}\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ B_x & B_y & B_z \\ x & y & z \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 0 & 0 & \\ x & y & z \end{vmatrix} = \frac{1}{2}(yB\,\vec{e}_x - Bx\vec{e}_y) = \frac{B}{2}(y\vec{e}_x - x\vec{e}_y)$$

Checking:
$$\vec{B} = \mathrm{rot}\vec{A} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yB}{2} & -\frac{xB}{2} & 0 \end{vmatrix} = \vec{e}_x\left(0 - \frac{1}{2}\frac{\partial(xB)}{\partial z}\right) + \vec{e}_y\left(\frac{1}{2}\frac{\partial(yB)}{\partial z} - 0\right) + \vec{e}_z\left(\frac{1}{2}\frac{\partial(xB)}{\partial x} - \frac{\partial(-xy)}{\partial y}\right)$$
$$= 0\cdot\vec{e}_x + 0\cdot\vec{e}_y + B\vec{e}_z \quad \text{OK!}$$

Let ion moves uniformly with velocity

$$\vec{V}_i = \begin{pmatrix} V_{i\perp} \\ 0 \\ V_{i\parallel} \end{pmatrix} \quad (3.1) \longrightarrow \vec{r}_i(t) = \vec{V}_i\cdot t$$

Input relative coordinate of electron $\vec{r}(t) = \vec{r}_e(t) - \vec{r}_i(t) = \vec{r}_e(t) - \vec{V}_i\cdot t$ (3.2a)

So, $\vec{r}(t)$ — coordinate ~~of~~ electron in the ion's frame. Farther

(3.2b) $\vec{V}(t) = \vec{V}_e(t) - \vec{V}_i(t) = \vec{V}_e(t) - \vec{V}_i$ — relative electron's velocity in the ion's frame

Let's define the radius and velocity of the center of mass:

$$\begin{cases} \vec{r}_{c.m} = (m_e\vec{r}_e + M_i\vec{r}_i)/(m_e+M_i) \\ \vec{r} = \vec{r}_e - \vec{r}_i \end{cases} \text{ and } \begin{cases} \vec{V}_{c.m} = (m_e\vec{V}_e + M_i\vec{V}_i)/(m_e+M_i) \\ \vec{V} = \vec{V}_e - \vec{V}_i \end{cases}$$

$$\begin{cases} m\vec{r}_e + M\vec{r}_i = (m_e+M_i)\vec{r}_{cm} \\ \vec{r}_e - \vec{r}_i = \vec{r} \end{cases} \longrightarrow \Delta = \begin{vmatrix} m_e & M_i \\ 1 & -1 \end{vmatrix} = -(m_e+M_i) \longrightarrow \vec{r}_e = \frac{1}{\Delta}\begin{vmatrix} (m_e+M_i)\vec{r}_{cm} & M_i \\ \vec{r} & -1 \end{vmatrix} = \vec{r}_{cm} + \frac{M_i\vec{r}}{m_e+M_i}$$

and $\vec{r}_i = \frac{1}{\Delta}\begin{vmatrix} m_e & (m_e+M_i)\vec{r}_{cm} \\ 1 & \vec{r} \end{vmatrix} = \vec{r}_{cm} - \frac{m_e}{m_e+M_i}\vec{r} \simeq \vec{r}_{cm} - \frac{m}{M}\vec{r}$

$\vec{V}_i t$

$\vec{r}_{cm} \simeq \vec{V}_i t + \frac{m}{M}\vec{r}$

and analogusly

$$\vec{V}_e = \vec{V}_{cm} + \frac{M_i}{m_e+M_i}\vec{V}$$
$$\vec{V}_i = \vec{V}_{cm} - \frac{m_e}{m_e+M_i}\vec{V} \simeq \vec{V}_{cm} - \frac{m}{M}\vec{V}$$

$$\mu = 1\Big/\left(\frac{1}{m_e}+\frac{1}{M_i}\right) = \frac{m_e M_i}{m_e+M_i}$$

So

$$\begin{cases}\vec r_e = \vec r_{cm} + \frac{\mu}{m_e}\vec r \\ \vec r_i = \vec r_{cm} - \frac{\mu}{M_i}\vec r\end{cases} \quad \text{and} \quad \begin{cases}\vec V_e = \vec V_{cm} + \frac{\mu}{m_e}\vec V \\ \vec V_i = \vec V_{cm} - \frac{\mu}{M_i}\vec V\end{cases}$$

$$\vec A(\vec r_e) = \frac{1}{2}[\vec B\cdot \vec r_e] = \frac{1}{2}\left[\vec B\cdot\left(\vec r_{cm} + \frac{\mu}{m}\vec r\right)\right]$$

$$\vec A(\vec r_i) = \frac{1}{2}[\vec B\cdot\vec r_i] = \frac{1}{2}\left[\vec B\cdot\left(\vec r_{cm} - \frac{\mu}{M}\vec r\right)\right]$$

Then

$$\mathcal{L} = \frac{m}{2}\left(\vec V_{cm} + \frac{\mu}{m}\vec V\right)^2 + \frac{M}{2}\left(\vec V_{cm} - \frac{\mu}{M}\vec V\right)^2 - e\frac{1}{2}\left[\vec B\left(\vec r_{cm} + \frac{\mu}{m}\vec r\right)\right]\left(\vec V_{cm} + \frac{\mu}{m}\vec V\right) - \frac{Ze}{2}\left[\vec B\left(\vec r_{cm} - \frac{\mu}{M}\vec r\right)\right]\left(\vec V_{cm} - \frac{\mu}{M}\vec V\right) + \frac{Ze^2}{r} =$$

$$= \underset{(1)}{\frac{m}{2}\vec V_{cm}^2} + \cancel{\mu\vec V\vec V_{cm}} + \underset{(2)}{\frac{\mu^2}{2m}\vec V^2} + \underset{(3)}{\frac{M}{2}\vec V_{cm}^2} - \cancel{\mu\vec V\vec V_{cm}} + \underset{(4)}{\frac{\mu^2}{2M}\vec V^2} - \frac{e}{2}\Big\{\underset{(5)}{([\vec B\vec r_{cm}]\vec V_{cm})} + \underset{(6)}{\frac{\mu}{m}([\vec B\vec r]\vec V_{cm})} + \underset{(7)}{([\vec B\vec r_{cm}]\frac{\mu}{m}\vec V)} +$$

$$+ \underset{(8)}{\frac{\mu^2}{m^2}([\vec B\vec r]\vec V)}\Big\} + \frac{Ze}{2}\Big\{\underset{(9)}{([\vec B\vec r_{cm}]\vec V_{cm})} - \underset{(10)}{\frac{\mu}{M}([\vec B\vec r]\vec V_{cm})} - \underset{(11)}{\frac{\mu}{M}([\vec B\vec r_{cm}]\vec V)} + \underset{(12)}{\frac{\mu^2}{M^2}([\vec B\vec r]\vec V)}\Big\} + \frac{Ze^2}{r} =$$

$$= \underset{(1)+(3)\ \leftarrow\ \text{this is an constant}}{\frac{m+M}{2}\vec V_{cm}^2} + \underset{(2)\quad(4)}{\frac{\mu^2}{2}\vec V^2\left(\frac{1}{m}+\frac{1}{M}\right)} + \frac{Ze^2}{r} + \underset{(5)\quad(9)\ \leftarrow\ \text{this is an constant}}{\frac{(Z-1)e}{2}([\vec B\cdot\vec r_{cm}]\vec V_{cm})} + \underset{(12)\quad(8)}{\frac{\mu^2}{2}\left(\frac{Ze}{M^2} - \frac{e}{m^2}\right)([\vec B\vec r]\vec V)} +$$

$$- \frac{\mu}{2}([\vec B\vec r]\vec V_{cm})\underset{(6)\quad(10)}{\left(\frac{e}{m} + \frac{Ze}{M}\right)} - \frac{\mu}{2}\underset{(7)\quad(11)}{\left(\frac{e}{m} + \frac{Ze}{M}\right)}([\vec B\vec r_{cm}]\vec V) \simeq$$

this is (3.5) and taking into account $\mu \simeq m \ll M$

$$\simeq \underset{(2)+(4)}{\frac{m\vec V^2}{2}} + \frac{Ze^2}{r} - \underset{(12)+(8)}{\frac{e}{2}([\vec B\vec r]\vec V)} - \left(\frac{e}{m} + \frac{Ze}{M}\right)\frac{\mu}{2}\underset{(6)+(10)+(7)+(11)}{\Big(([\vec B\vec r]\overset{\vec V_i + \frac{m}{M}\vec V}{\vec V_{cm}}) + ([\vec B\vec r_{cm}]\vec V)\Big)} =$$

$$\simeq \frac{m\vec V^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec B\vec r]\vec V) - \frac{e}{2}\left\{\left([\vec B\vec r](\vec V_i + \frac{m}{M}\vec V)\right) + ([\vec B\vec r_{cm}]\vec V)\right\} = \frac{m\vec V^2}{2} + \frac{Ze^2}{r} - \frac{e}{2}([\vec B\vec r]\vec V) -$$

$$- \frac{e}{2}\left\{[\vec B\vec r]\vec V_i + \left[\vec B\left(\vec r_{cm} + \frac{m}{M}\vec r\right)\right]\vec V\right\} =$$

$$= \frac{m\vec{v}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{v}) - \frac{e}{2}\left\{([\vec{B}\vec{r}]\vec{v}_i) + ([\vec{B}\cdot\vec{v}_i t]\vec{v}) + 2\frac{m}{M}([\vec{B}\vec{r}]\vec{v})\right\} =$$

$$= \frac{m\vec{v}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{v}_i) - \frac{e}{2}([\vec{B}\cdot\vec{v}_i t]\vec{v}) - \frac{e}{2}\Big(1 + \underbrace{2\frac{m}{M}}_{2\frac{m}{M} \ll 1}\Big)([\vec{B}\vec{r}]\vec{v}) \Longrightarrow$$

$$\mathcal{L} = \frac{m\vec{v}^2}{2} + \frac{ze^2}{r} - \frac{e}{2}([\vec{B}\vec{r}]\vec{v}) - \frac{e}{2}\left\{([\vec{B}\cdot\vec{v}_i t]\vec{v}) + ([\vec{B}\vec{r}]\vec{v}_i)\right\}$$ this is (3.6)

§ Equation of motion:

$$m\frac{d\vec{v}}{dt} = -\frac{\partial \mathcal{L}}{\partial \vec{r}} = -\vec{\nabla}\frac{ze^2}{r}$$

From (3.14) for $R \to 0$ („Guiding" center approach):

$$\vec r^{\,2}(t) = b^2 + [(V_{e\parallel}-V_{i\parallel})^2 + V_{i\perp}^2]\,t^2 - 2V_{i\perp} b\, t \sin\theta \qquad (3.15)$$

Let's define the relative velocity $\vec V$ of the guiding center and ion:

$$\vec V = \vec V_\perp + \vec V_\parallel = \begin{pmatrix}0\\0\\V_{e\parallel}-V_{i\parallel}\end{pmatrix} + \begin{pmatrix}-V_{i\perp}\\0\\0\end{pmatrix} \qquad (3.16)$$

then $(V_{e\parallel}-V_{i\parallel})^2 + V_{i\perp}^2 = \vec V^2 \equiv \bar V^2$

Let's t_0 - time t_0 when electron reaches the minimal distance $\vec r_0$ to ion (this is impact parameter of collision); i.e.

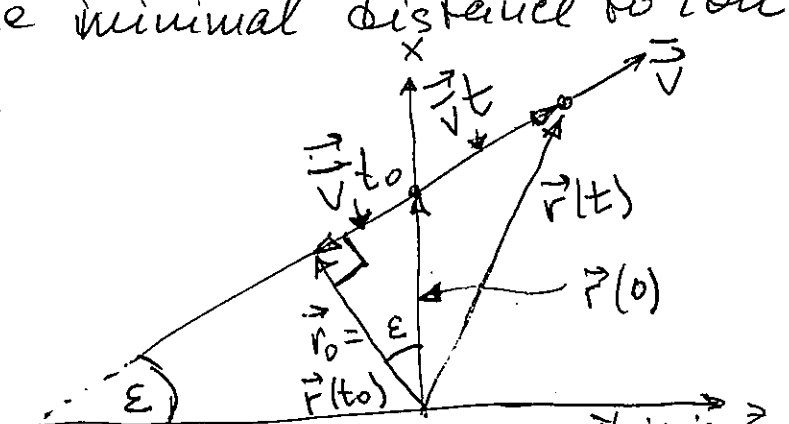


Fig. 2

$$r_0^2 = b^2 + \bar V^2 t_0^2 - 2V_{i\perp} b\, t_0 \sin\theta$$

Very important: $\vec r_0 \perp \vec V$ and from picture one has

$$\vec r(t) = \vec r_0 + \vec V (t - t_0) \qquad (3.20)$$

then

$$\vec r^{\,2} = r_0^2 + \underbrace{\vec V^2}_{\equiv \bar V^2}(t-t_0)^2 + 2\underbrace{\vec r_0 \vec V}_{=0}(t-t_0) = r_0^2 + \bar V^2 (t-t_0)^2$$

So

$$\vec r^{\,2} = b^2 + \bar V^2 t^2 - 2V_{i\perp} b\, t \sin\theta = r_0^2 + \bar V^2 (t-t_0)^2 = b^2 + \bar V^2 t_0^2 - 2V_{i\perp} b\, t_0 \sin\theta + \bar V^2 (t-t_0)^2$$

or

$$\cancel{b^2} + \cancel{\bar V^2 t^2} - 2V_{i\perp} b\, t \sin\theta = \cancel{b^2} + \underline{\bar V^2 t_0^2} - 2V_{i\perp} b\, t_0 \sin\theta + \cancel{\bar V^2 t^2} + \underline{\bar V^2 t_0^2} - 2\bar V^2 t\, t_0$$

$$\cancel{2\bar V^2 t_0^2 - 2t_0\,(V_{i\perp} b \sin\theta - \bar V^2 t) + 2V_{i\perp} b\, t \sin\theta = 0}$$

So

$$0 = 2v_{i\perp} b\sin\theta\cdot(t-t_0) + 2\bar{v}^2 t_0(t_0-t) \Longrightarrow t_0 = \frac{v_{i\perp} b\sin\theta}{\bar{v}^2} \quad (3.18)$$

$$\cancel{= \frac{\bar{v}_\perp b\sin\theta}{\bar{v}^2}} =$$

and for this reason

$$r_0^2 = b^2 + \bar{v}^2 t_0^2 - 2t_0\cdot\underbrace{\frac{v_{i\perp}\sin\theta}{\bar{v}^2}}_{=t_0}\bar{v}^2 = b^2+\bar{v}^2 t_0^2 - 2\bar{v}^2 t_0^2 = b^2 - \bar{v}^2 t_0^2 \quad (3.19)$$

and

$$\vec{r}_0 = \vec{r}(t_0) = \begin{pmatrix} b\sin\theta - v_{i\perp} t_0 \\ -b\cos\theta \\ (v_{e\parallel}-v_{i\parallel})t_0 \end{pmatrix} = \begin{pmatrix} b\sin\theta + \bar{v}_\perp t_0 \\ -b\cos\theta \\ \bar{v}_\parallel t_0 \end{pmatrix}$$

this is from (3.10) with $R=0$

and then

$$\vec{r}(t) = \begin{pmatrix} -b\sin\theta - v_{i\perp} t \\ -b\cos\theta \\ (v_{e\parallel}-v_{i\parallel})t \end{pmatrix} = \begin{pmatrix} -b\sin\theta + \bar{v}_\perp t \\ -b\cos\theta \\ \bar{v}_\parallel t \end{pmatrix} = \begin{pmatrix} -b\sin\theta + \bar{v}_\perp(t_0+t-t_0) \\ -b\cos\theta \\ \bar{v}_\parallel(t_0+t-t_0) \end{pmatrix} =$$

$$= \begin{pmatrix} -b\sin\theta + \bar{v}_\perp t_0 \\ -b\cos\theta \\ \bar{v}_\parallel t_0 \end{pmatrix} + \begin{pmatrix} \bar{v}_\perp(t-t_0) \\ 0 \\ \bar{v}_\parallel(t-t_0) \end{pmatrix} = \vec{r}_0 + \vec{\bar{v}}(t-t_0)$$

this is (3.20) again

Let's introduce the dimensionless variables:

$$\tau = \frac{\bar{v}t}{r_0} \quad (3.21) \qquad \tau_0 = \frac{\bar{v}t_0}{r_0}, \quad \sigma = \tau - \tau_0 = \frac{\bar{v}(t-t_0)}{r_0} \quad (3.22)$$

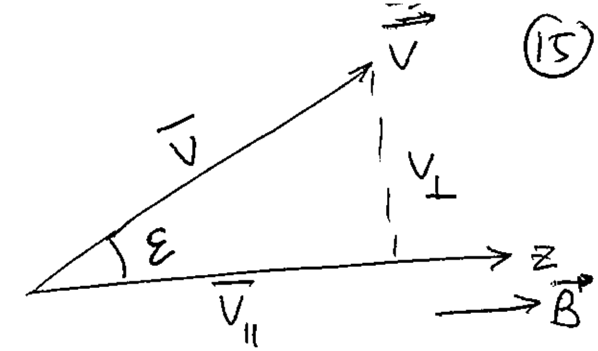
This is (3.29 left)

$$\gamma_\parallel = \frac{\bar{v}_\parallel}{\bar{v}} \quad (3.24), \qquad \gamma_\perp = \frac{\bar{v}_\perp}{\bar{v}} \quad (3.25) \qquad \beta = \frac{b}{r_0} \quad (3.26)$$

Then from (3.18) $\tau_0 = \frac{\bar{v}t_0}{r_0} = \frac{\bar{v}}{r_0}\,\frac{-\bar{v}_\perp b\sin\theta}{\bar{v}^2} =$

$= -\beta\gamma_\perp \sin\theta$

$$\frac{\vec r_0}{r_0} = \frac{1}{r_0}\begin{pmatrix} b\sin\theta + \bar V_\perp t_0 \\ -b\cos\theta \\ \bar V_\parallel t_0\end{pmatrix} = \begin{pmatrix}\beta\sin\theta+\gamma_\perp\tau_0 \\ -\beta\cos\theta \\ \gamma_\parallel \tau_0\end{pmatrix}$$



$$\cos\varepsilon = \frac{\bar V_\parallel}{\bar V} = \gamma_\parallel \quad (3.28)$$

$$\sin\varepsilon = \frac{\bar V_\perp}{\bar V} = \gamma_\perp \quad (3.28^*)$$

Instead (3.19) one has

$$b^2 = r_0^2 + \bar V^2 t_0^2 \to \beta^2 = 1+\tau_0^2 \quad (3.32)$$

and, of course, $\gamma_\parallel^2+\gamma_\perp^2=1$

Let's input

$$\sin\psi = -\beta\cos\theta \quad (3.30)$$

Then

$$\beta\sin\theta+\gamma_\perp\tau_0 = \left(\text{from 3.29 left } \beta\sin\theta = -\frac{\tau_0}{\gamma_\perp}\right) = -\frac{\tau_0}{\gamma_\perp}+\gamma_\perp\tau_0 = \tau_0\frac{\gamma_\perp^2-1}{\gamma_\perp} = -\frac{\tau_0\gamma_\parallel^2}{\gamma_\perp}$$

But

$$\tau_0 = -\beta\gamma_\perp\sin\theta = -\gamma_\perp\beta\sqrt{1-\cos^2\theta} = -\gamma_\perp\beta\sqrt{1-\frac{\sin^2\psi}{\beta^2}} = -\gamma_\perp\sqrt{\beta^2-\sin^2\psi} =$$

$$= -\gamma_\perp\sqrt{\beta^2-1+\cos^2\psi} = -\gamma_\perp\sqrt{\tau_0^2+\cos^2\psi} \quad \text{or}$$

$$\tau_0^2 \overset{\text{from (3.32)}}{=} \gamma_\perp^2\tau_0^2+\gamma_\perp^2\cos^2\psi \to \tau_0^2(1-\gamma_\perp^2) = \gamma_\perp^2\cos^2\psi \to$$

$$\tau_0^2\gamma_\parallel^2 = \gamma_\perp^2\cos^2\psi \to \tau_0 = \pm\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi$$ and it is neccesary to select sign "−"!

$$\tau_0 = -\frac{\gamma_\perp}{\gamma_\parallel}\cos\psi \ (3.29\text{ right}) \to \gamma_\parallel\tau_0 = -\underbrace{\gamma_\perp}_{\text{from (3.28*)}}\cos\psi = -\sin\varepsilon\cos\psi \quad (3.29^*)$$

So

$$\beta\sin\theta+\gamma_\perp\tau_0 = -\frac{\tau_0\gamma_\parallel^2}{\gamma_\perp} = \frac{\gamma_\perp\cos\psi\cdot\gamma_\parallel^2}{\gamma_\parallel\gamma_\perp} = \gamma_\parallel\cos\psi \quad (3.31)$$

There fore

$$\frac{\vec{r}_0}{r_0} = \begin{pmatrix} \beta \sin\theta + \gamma_\perp \tau_0 \\ -\beta \cos\theta \\ \gamma_\parallel \tau_0 \end{pmatrix} = \begin{pmatrix} \text{using } (3.31) \\ \text{using } (3.30) \\ \text{using } (3.29^*) \end{pmatrix} = \begin{pmatrix} \cos\varepsilon \cos\psi \\ \sin\psi \\ -\sin\varepsilon \cos\psi \end{pmatrix} \qquad (3.27)$$

Finaly:

(3.21) $\tau = \frac{\bar{v} t}{r_0} \qquad \tau_0 = \frac{\bar{v} t_0}{r_0}$

(3.22) $\sigma = \tau - \tau_0 = \frac{\bar{v}(t - t_0)}{r_0}$

(3.24) $\gamma_\parallel = \frac{\overline{v_\parallel}}{V_0}$

(3.25) $\gamma_\perp = \frac{\overline{v_\perp}}{V_0}$

$$\left.\begin{aligned} \gamma_\parallel &= \frac{\overline{v_\parallel}}{V_0} \\ \gamma_\perp &= \frac{\overline{v_\perp}}{V_0} \end{aligned}\right\} \quad \gamma_\parallel^2 + \gamma_\perp^2 = 1$$

(3.26) $\beta = \frac{b}{r_0}$

(3.27) $\frac{\vec{r}_0}{r_0} = \begin{pmatrix} \cos\varepsilon \cos\psi \\ \sin\psi \\ -\sin\varepsilon \cos\psi \end{pmatrix}$

(3.28) $\cos\varepsilon = \gamma_\parallel \longrightarrow \sin\varepsilon = \sqrt{1 - \cos^2\varepsilon} = \sqrt{1 - \gamma_\parallel^2} = \gamma_\perp$ (3.28*)

(3.29) $\tau_0 = -\beta \gamma_\perp \sin\theta = -\frac{\gamma_\perp}{\gamma_\parallel} \cos\psi \rightarrow \gamma_\parallel \tau_0 = -\sin\varepsilon \cos\psi$ (3.29*)

(3.30) $\beta \cos\theta = -\sin\psi$

(3.31) $\beta \sin\theta + \gamma_\perp \tau_0 = -\frac{\gamma_\parallel^2 \tau_0}{\gamma_\perp} = \gamma_\parallel \cos\psi$

(3.32) $\beta^2 = 1 + \tau_0^2$