# DETERMINISTIC MODELLING

# Delivery 7

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## Exercise 4

### (a) Find the 3 fixed points in the Lorenz system.

The Lorenz system:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(r - z) - y \\ \dot{z} = xy - bz \end{cases}$$

A fixed point $(x_*, y_*, z_*)$ will satisfy $\dot{x} = \dot{y} = \dot{z} = 0$.

$$\dot{x} = 0 \Longrightarrow y = x = \tau$$

$$\dot{y} = 0 \Longrightarrow \tau(r - z) - \tau = 0 \xRightarrow{\tau \neq 0} z = r - 1$$

$$\dot{z} = 0 \Longrightarrow \tau^2 - bz \begin{cases} \tau = \pm\sqrt{b(r-1)} & \tau \neq 0 \\ z = 0 & \tau = 0 \end{cases}$$

Therefore, the fixed points of the system are:

<u>If $r < 1$</u>

$$\boxed{P_0 = (0, 0, 0)}$$

<u>If $r > 1$</u>

$$\boxed{P_0 = (0, 0, 0)}$$

$$\boxed{C^+ = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r - 1)}$$

$$\boxed{C^- = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r - 1)}$$

## (b) Study the stability of the origin.

Near the origin, we can omit the quadratic terms:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y \\ \dot{z} = -bz \end{cases}$$

With this system, we can see that in the $z$ axis the system will be stable in the origin since $\dot{z} < 0$ for $z > 0$ and $\dot{z} > 0$ for $z < 0$.

So now we can study the $x - y$ axes as a two dimensional system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

Since the trace is $\tau = -\sigma - 1 < 0$ and the determinant is $\Delta = \sigma(1 - r)$. We will have a:

- saddle-point if $r > 1$ ($\Delta < 0$).
- stable-node if $r < 1$ ($\Delta > 0$)

We can check that there won't be centers or spirals since $\tau^2 - 4\Delta = (\sigma - 1)^2 + 4\sigma r > 0$.

## (c) Find the characteristic equation for $C^+$ and $C^-$.

We can compute the jacobian of the system:

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

And evaluate it to $C^+$:

$$J|_{C^+} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{b(r-1)} \\ -\sqrt{b(r-1)} & -\sqrt{b(r-1)} & -b \end{pmatrix}$$

And now we can find the characteristic equation by computing the following determinant:

$$\begin{vmatrix} -\sigma - \lambda & \sigma & 0 \\ 1 & -1 - \lambda & \sqrt{b(r-1)} \\ -\sqrt{b(r-1)} & -\sqrt{b(r-1)} & -b - \lambda \end{vmatrix} = ... = -\lambda^3 - \lambda^2(\sigma + b + 1) - \lambda b(\sigma + r) - 2\sigma b(r - 1)$$

So we obtain the characteristic equation:

$$\boxed{\lambda^3 + \lambda^2(\sigma + b + 1) + \lambda b(\sigma + r) + 2\sigma b(r - 1) = 0}$$

One can see that the same result is obtained when doing the same procedure with $C^-$.

**(d) A Hopf bifurcation takes place when the eigenvalues cross the imaginary line, ie, they are purely imaginary. Replace $\lambda = i\omega$ in the equation below, and find the value of r where the Hopf bifurcation occurs. Are C$\pm$ stable for values of r greater than this?**

If we substitute $\lambda = i\omega$ in the previous equation:

$$-\omega^2(\sigma + b + 1) + 2\sigma b(r-1) + i\omega[b(\sigma + r) - \omega^2] = 0$$

This implies that

$$-\omega^2(\sigma + b + 1) + 2\sigma b(r-1) = 0 \longrightarrow \omega^2 = \frac{2\sigma b(r-1)}{\sigma + b + 1} \tag{2}$$

$$\omega(b(\sigma + r) - \omega^2) = 0 \longrightarrow \begin{cases} \omega = 0 \\ \omega^2 = b(\sigma + r) \end{cases} \tag{3}$$

Now equating both equations (2), the non-zero $\omega$ of (3) and resolving for $r$ we obtain:

$$\boxed{r = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right)}$$

Where we have to impose that $r > 1$ (in order to have $C^{\pm}$ as fixed points) and therefore: $b+1 < \sigma$. As we have seen in the theory classes, for higher values of this $r_H$ it appears chaos, so $C^{\pm}$ won't be stable.