DETERMINISCTIC MODELLING

Delivery 2

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Problem 6

Use linear stability and graphical representation methods to classify the fixed points of the following equations:

a) $\dot{x} = x(1-x)(2-x)$

Fixed points satisfy $\dot{x} = f(x) = 0$:

$$\dot{x} = 0 \Longrightarrow x(1-x)(2-x) = 0 \begin{cases} x_*^{(1)} = 0 \\ x_*^{(2)} = 1 \\ x_*^{(3)} = 2 \end{cases}$$

The stability condition for the fixed points is $f'(x_*) < 0$ so:

$$f'(x) = (1-x)(2-x) + 2x^2 - 3x \begin{cases} f'(x_*^{(1)}) = f'(0) = 2 > 0 \longrightarrow \boxed{x_*^{(1)} = 0 \quad \text{unstable}} \\ f'(x_*^{(2)}) = f'(1) = -1 < 0 \longrightarrow \boxed{x_*^{(2)} = 1 \quad \text{stable}} \\ f'(x_*^{(3)}) = f'(2) = 2 > 0 \longrightarrow \boxed{x_*^{(3)} = 2 \quad \text{unstable}} \end{cases}$$

b) $\dot{x} = \ln(x)$

$$\dot{x} = 0 \Longrightarrow \ln(x) = 0 \Longrightarrow x_* = 1$$

$$f'(x) = \frac{1}{x} \Longrightarrow f'(x_*) = f'(1) = 1 > 0 \longrightarrow \boxed{x_* = 1 \quad \text{Unstable}}$$

c) $\dot{x} = \cos(x)$

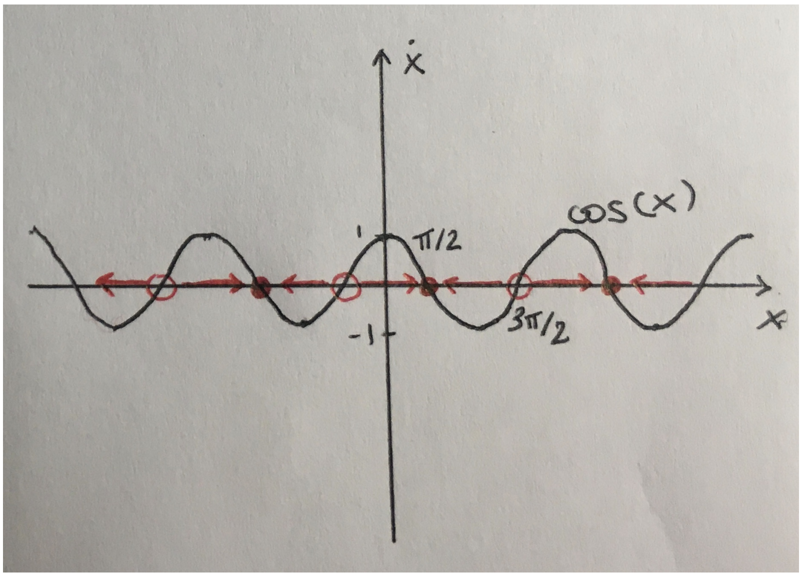

Figure 1: Phase portrait

d) $\dot{x} = x^2(4 - x)$

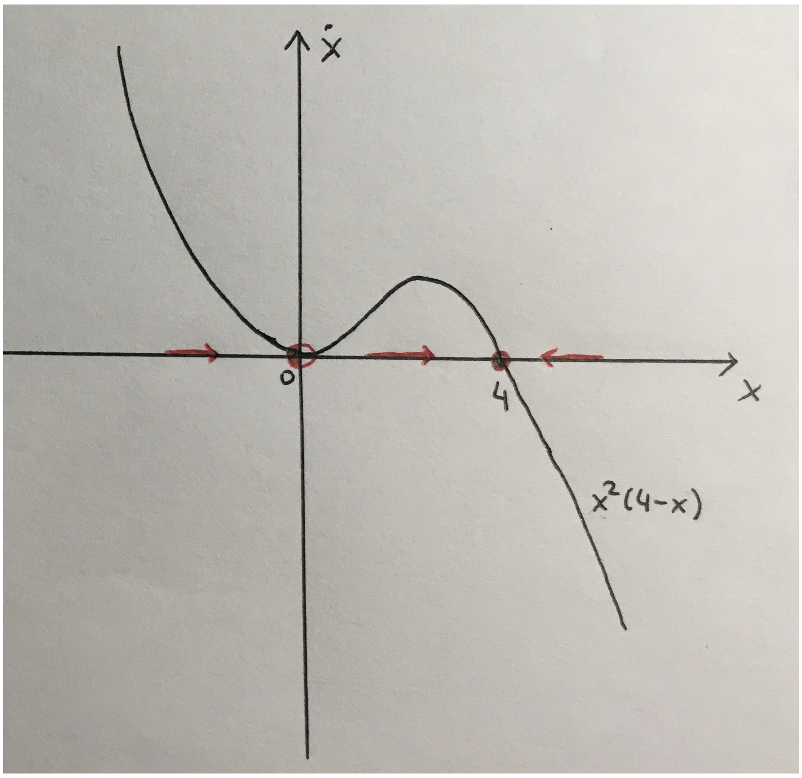

Figure 2: Phase portrait

Problem 8

Logistic model. In the study of population dynamics in ecology, the logistic model is widely used to model the limitation of resources:

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right),$$

where r and K are positive parameters.

a) Find the stationary states, sudy their stability and draw phase portrait.

$$rX\left(1 - \frac{X}{K}\right) = 0 \begin{cases} X_*^{(1)} = 0 \\ X_*^{(2)} = K \\ \forall X \in \Re \qquad r = 0 \quad \text{Can't happen}(r > 0) \end{cases}$$

$$f(x) = rX\left(1 - \frac{X}{K}\right) = r\left[1 - \frac{X}{K} - \frac{X}{K}\right] = r\left(1 - \frac{2X}{K}\right)$$

$$f'(X_*^{(1)} = 0) = r > 0 \rightarrow \boxed{X_*^{(1)} = 0 \ \text{ unstable}}$$

$$f'(X_*^{(2)} = K) = -r < 0 \rightarrow \boxed{X_*^{(2)} = K \ \text{ stable}}$$

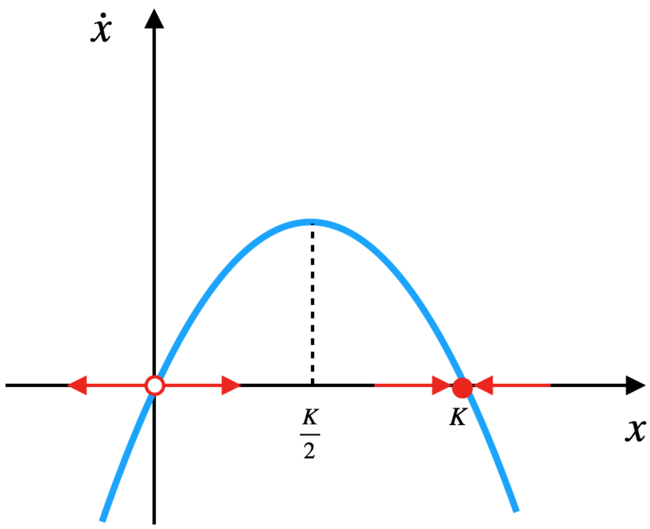

Figure 3: Phase portrait with the fixed points classified.

b) What are the physical meanings of these parameters?

The parameter r is the growth rate, meaning the normalized difference between births and deaths of the population.

The parameter K is the carrying capacity, which indicates the maximum population size that the environment can support given the resources available.

Problem 10

Tumor growth. The growth of cancerous tumors can be modeled by the Gompertz law $dN/dt = -aNln(N/B)$, where $N(t)$ is the number of tumor cells, and a and B are positive constants.

a) Interpret a and B biologically.

We can interpret the parameter a as the growth rate of the tumorous cells, meaning the normalized difference between the number of new tumorous cells and the number of regressed tumorous cells. This means it is a constant related to the proliferative ability of the cells. B is the carrying capacity, which indicates the maximum size that the tumor can reach with the available nutrients in the system.

b) Draw the phase portrait.

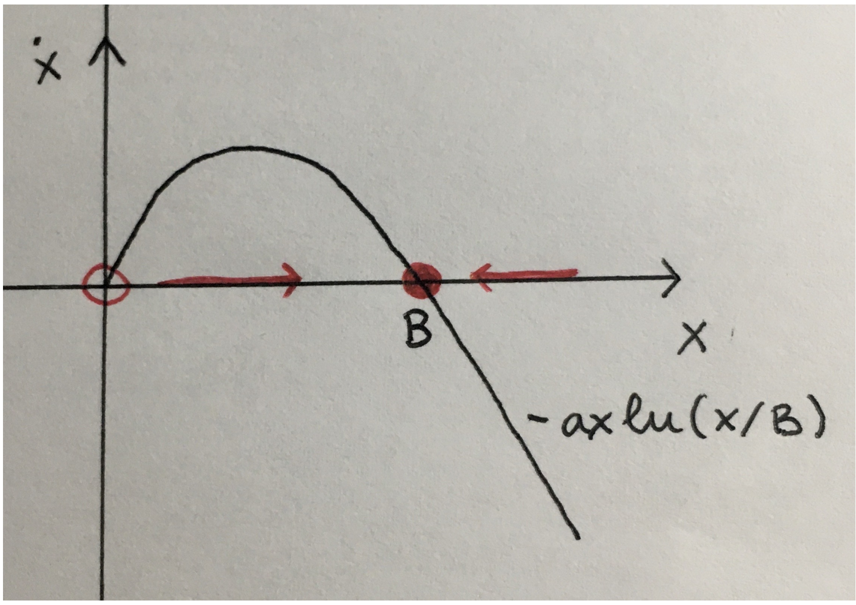

Figure 4: Phase portrait

c) Show that the time evolution is:

$$N(t) = B \exp(\ln(N_0/B) \exp(-at))$$

$$\frac{dN}{dt} = -aN\ln\left(\frac{N}{B}\right) \to dN = -aN\ln\left(\frac{N}{B}\right)dt \to \int_{N_0}^{N_t} dN = -a\int_{t=0}^{t} N\ln\left(\frac{N}{B}\right)dt$$

$$\int_{N_0}^{N_t} \frac{dN}{N\ln\left(\frac{N}{B}\right)} = -a\int_{t=0}^{t} dt \xrightarrow{y=ln(N/B)} \int_{y_0}^{y_t} \frac{dy}{y} = -at$$

$$\ln(y)\Big|_{y_0}^{y_t} = -at \to \ln(y_n) = -at + \ln(y_0) \to y_n = \exp(-at + \ln(y_0)) \to y_n = y_0\exp(-at)$$

$$\ln\left(\frac{N_t}{B}\right) = \ln\left(\frac{N_0}{B}\right)\exp(-at) \to \boxed{N(t) = B\exp\left(\ln\left(\frac{N_0}{B}\right)\exp(-at)\right)}$$

Problem 11

Draw the bifurcation diagrams in a dynamic system that obeys the equation:

a) $\dot{x} = r - x^2$ (Saddle-node bifurcation type).

First we are going to find the fixed points that obey $f(x_*) = 0$:

$$r - x_*^2 = 0 \Longrightarrow x_*^{\pm} = \pm\sqrt{r}$$

So r must be a positive number.

Now we can see the stability of each point analytically or graphically:

Analytically:

$$f'(x) = -2x \begin{cases} f'(x_*^+) < 0 \\ f'(x_*^-) > 0 \end{cases}$$

Graphically:

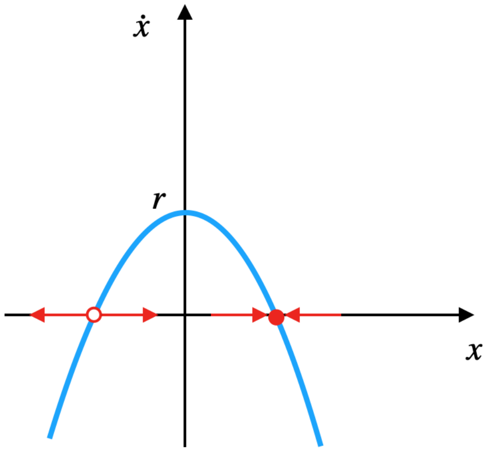

Figure 5: Phase portrait with the fixed points classified.

So finally the bifurcation diagram will be:

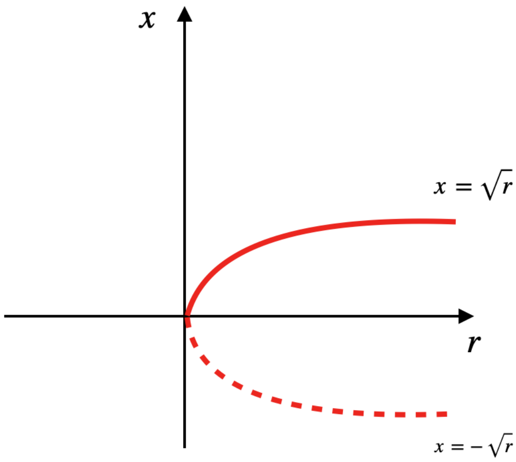

Figure 6: Bifurcation diagram. As in theory class, doted line means unstable and the full line means stable.