DETERMINISTIC MODELLING

Delivery 3

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Exercise 11

Draw the bifurcation diagrams in a dynamic system that obeys the equation:

d) $\dot{x} = r \ln(x) + x - 1$ Which bifurcation type does it belong to?

We will consider $\dot{x} = y_1 - y_2$ where

$$y_1 = r \ln(x)$$

$$y_2 = 1 - x$$

$\underline{r \geq 0}$

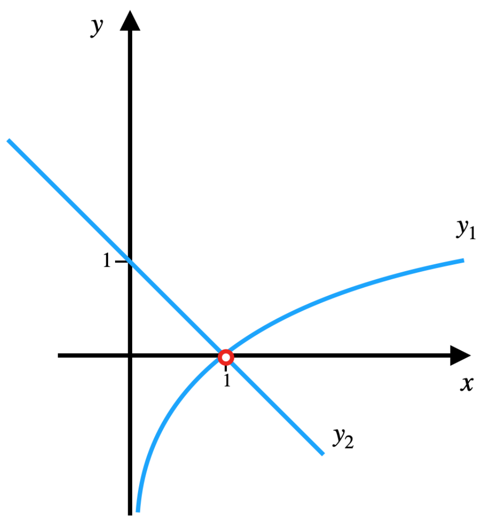

Figure 1: y_1 and y_2 for $r > 0$.

For all values of $r \geq 0$ we will have the stationary point (cross point between y_1 and y_2) at $y = 0$, $x = 1$ and we can see it's independent of r because $r \ln(x) = 0 \longrightarrow x = 1 \ \ \forall r > 0$.

This stationary point will be unstable because for $x > 1$ $y_1 > y_2 \implies \dot{x} > 0$ and for $x < 1$ $y_2 > y_1 \implies \dot{x} < 0$.

$\underline{r < 0}$

For $|r| >> 1$:

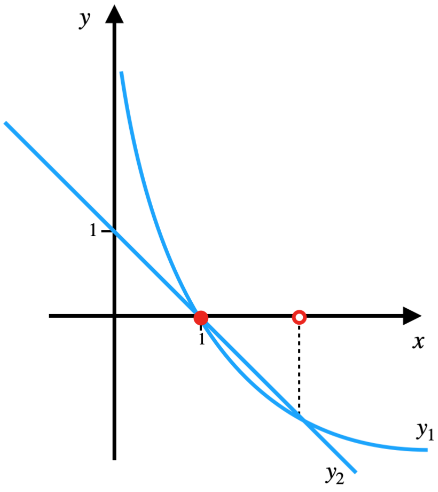

Figure 2: y_1 and y_2 for big values of $|r|$.

For $|r| << 1$:

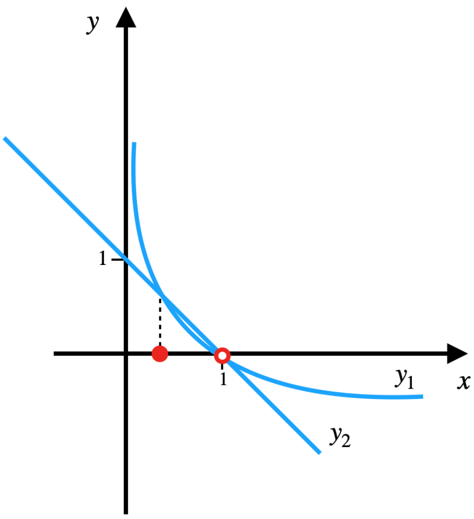

Figure 3: y_1 and y_2 for small values of $|r|$.

For $r = -1$:

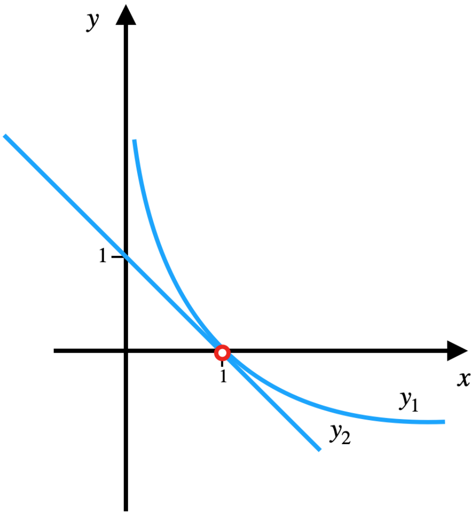

Figure 4: y_1 and y_2 for $r = -1$.

<u>Bifurcation diagram</u>:

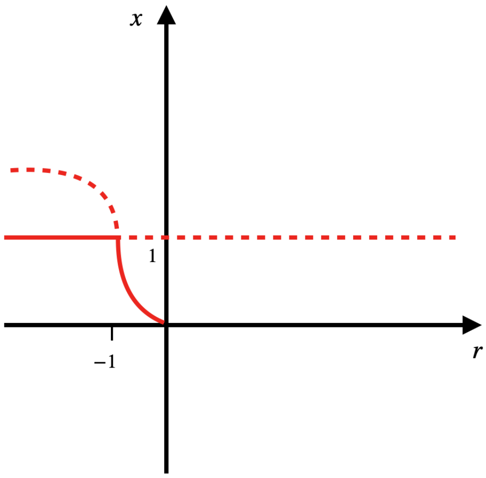

Figure 5: Bifurcation diagram with a transcritical bifurcation type.

As we can see in figure 5 there is a transcritical bifurcation at $r = -1$.

Exercise 14

Study graphically the following dynamic equation and find the phase portrait.

$$\frac{dx}{dt} = a - \frac{x^2}{1+x^2}$$

Prove that it shows a bifurcation, classify it and draw the bifurcation diagram.

Looking at the asymptotes of the function and noticing that for low values of x it is a parabola we can easily draw the phase portrait for the different possible values of a. And then with these phase portraits we are going to be able to draw the bifurcation diagram.

$a >> 1$

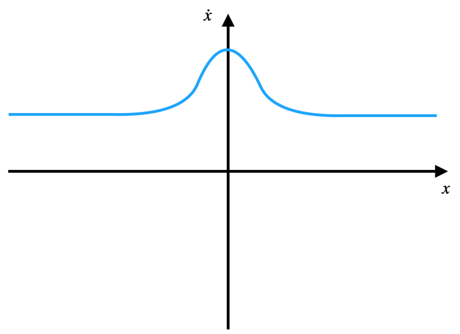

Figure 6: Phase portrait for $a >> 0$.

$a > 0$ and $a << 1$

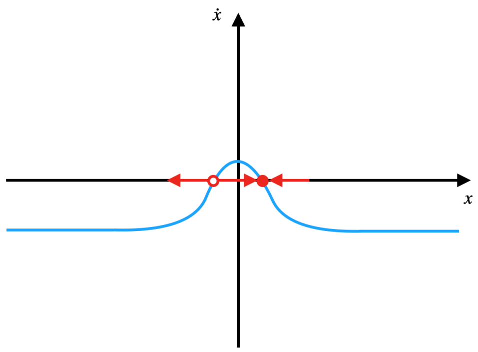

Figure 7: Phase portrait for $a > 0$ and $a << 1$.

$\underline{a = 0}$

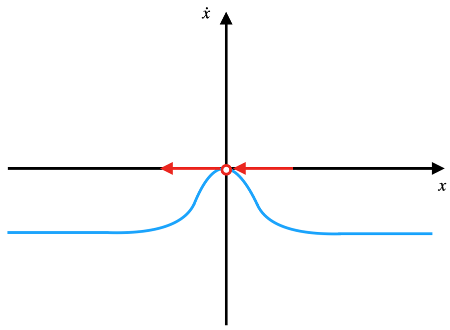

Figure 8: Phase portrait for $a = 0$.

$\underline{a < 0}$

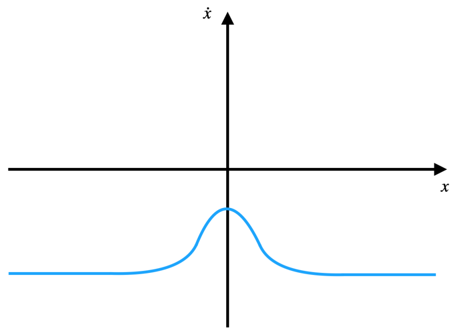

Figure 9: Phase portrait for $a < 0$

Bifurcation diagram

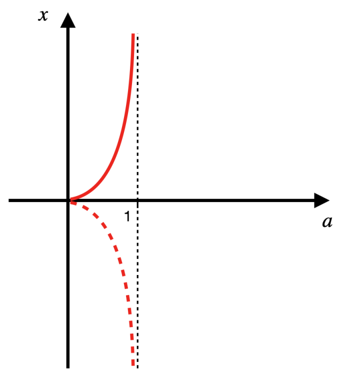

Figure 10: Bifurcation diagram with a saddle-node bifurcation type.

Exercise 18

Neurons are excitable cells: they can be in a state of rest and, if the difference in the membrane potential exceeds a threshold value, passes to an excited state. A simple model for the charge injection phase in an excitable cell (i.e, entry of Na+ inside a neuron) can be described by the equation:

$$\frac{dx}{dt} = kx(a-x)(1-x) \tag{1}$$

where x denotes the membrane potential difference (normalized), k>0 and 0<a<1, are model parameters.

a) Reduce the number of parameters as possible.

From equation 1 and the term $(1-x)$ we can deduce that x has no units.(Also because the problem says that x is normalized).

Therfore, a can't have units because of the term $(a-x)$.

So the only parameter that have units in the right is k, so comparing it with the left part we can deduce that

$$\left[\frac{dx}{dt}\right] = T^{-1} \Longrightarrow [k] = \left[\frac{dx}{dt}\right] = T^{-1}$$

So we can define a new non-dimensional parameter $\tau \equiv kt$ such that:

$$\frac{dx}{dt} = k\frac{dx}{d\tau} \Longrightarrow \boxed{\frac{dx}{d\tau} = x(a-x)(1-x)}$$

b) Study the model: equilibrium points, phase portrait, basins of attraction, bifurcation diagram.

From now on we will call $\dot{x} \equiv \frac{dx}{d\tau}$.

The equilibrium points are the ones that satisfy $\dot{x} = 0$:

$$x(a-x)(x-1) = 0 \begin{cases} \boxed{x_*^{(1)} = 0} \\ \boxed{x_*^{(2)} = a} \\ \boxed{x_*^{(3)} = 1} \end{cases}$$

And we can study the stability with the stability condition $f'(x_*) < 0$:

$$f'(x) = (a-x)(x-1) + x(a+1-2x) \begin{cases} f'(x_*^{(1)}) = f'(0) = -a < 0 & \boxed{x_*^{(1)} = 0 \ \text{ stable}} \\ f'(x_*^{(2)}) = f'(a) = a(1-a) > 0 & \boxed{x_*^{(2)} = a \ \text{ unstable}} \\ f'(x_*^{(3)}) = f'(1) = a-1 < 0 & \boxed{x_*^{(3)} = 1 \ \text{ stable}} \end{cases}$$

Now we can draw the phase portrait:

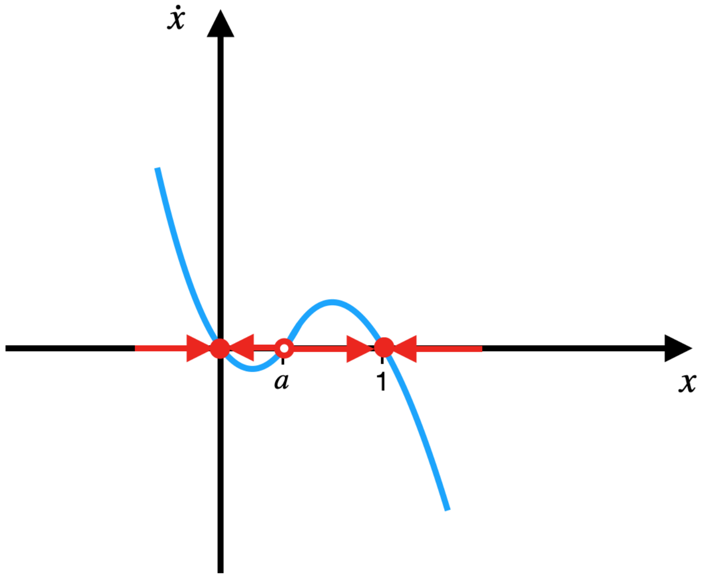

Figure 11: Phase portrait with the fixed points classified.

From Figure 11 we can see that due to the fact that $0 < a < 1$, the only thing that will change in the phase portrait will be the position of the $x_*^{(2)}$ equilibrium point, that will increase following $x_*^{(2)} = a$.

Considering this, we can draw the bifurcation diagram:

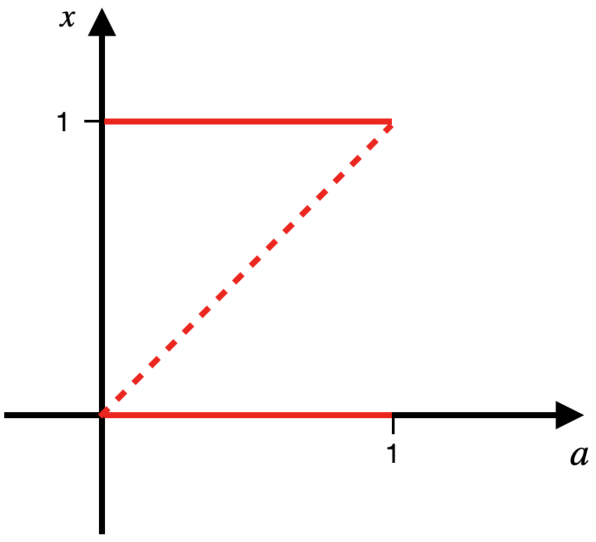

Figure 12: Caption

c) What is the interpretation of the equilibrium points?

After studying this model we can see that there are two equilibrium points for the equation

$$\frac{dx}{d\tau} = x(a - x)(1 - x).$$

We have defined $\tau \equiv kt$, where we can interpret k as the potential difference per unit of time "entering" to the neuron (using a dimension argument). And therefore, we can interpret $\frac{dx}{d\tau}$ as how the membrane potential change when potential (Na^+) "is entering" the neuron.

In addition, we can see that depending on which value of the membrane potential we have in relation to the a value, the system will evolve to one of the two equilibrium points. If $x > a$ x will evolve to the equilibrium point $x = 1$ and if $x < a$ x will evolve to the equilibrium point $x = 0$.

With this idea, we can relate this equilibrium points as the two possible states of the neuron (excited and at rest) with a as the threshold value that indicates the frontier of both equilibrium states. The state in which the membrane potential is $x = 1$ will be the excited state, because if the membrane potential exceeds the threshold ($x > a$) the neuron passes to an excited state and the system will tend to $x = 1$. And the state in which the membrane potential is $x = 0$ will be the rest state because that means that the membrane potential was not high enough to exceed the threshold ($x < a$) and therefore the neuron will go to the rest state ($x = 0$).

Finally, there is also the unstable state in which the membrane potential is exactly the same as the threshold ($x = a$) and it is unstable because from there, with a little perturbation the system can go either at the excited state or at the rest state.