DETERMINISTIC MODELLING

Delivery 1

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Problem 3

a) Find the fixed points on the map:

$$x_{n+1} = x_n \exp(\mu(1 - x_n)).$$

Fixed points (x_*) satisfy $x_* = f(x_*)$:

$$x_* = x_* e^{\mu(1-x_*)} \Longrightarrow \begin{cases} \forall x_* \in \Re & \mu = 0 \\ x_*^{(1)} = 0 & \forall \mu \\ x_*^{(2)} = 1 & \forall \mu \end{cases}$$

b) Analyze the stability of the fixed points for different values of $\mu > 0$ and $x_n \geq 0$.

Knowing the stability condition for fixed points $|f'(x_*)| < 1$:

$$f'(x) = (1 - \mu x)e^{\mu(1-x)} \begin{cases} f'(0) = e^{\mu} \Longrightarrow x_*^{(1)} \text{ Unstable } \mu > 0 \\ \\ f'(1) = 1 - \mu \Longrightarrow \begin{cases} x_*^{(2)} \text{ Stable} & 0 < \mu < 2 \\ x_*^{(2)} \text{ Unstable} & \mu > 2 \end{cases} \end{cases}$$

c) Make a code and draw the orbit diagram up to $\mu = 4$. Compare it with the orbit diagram for the logistic map.

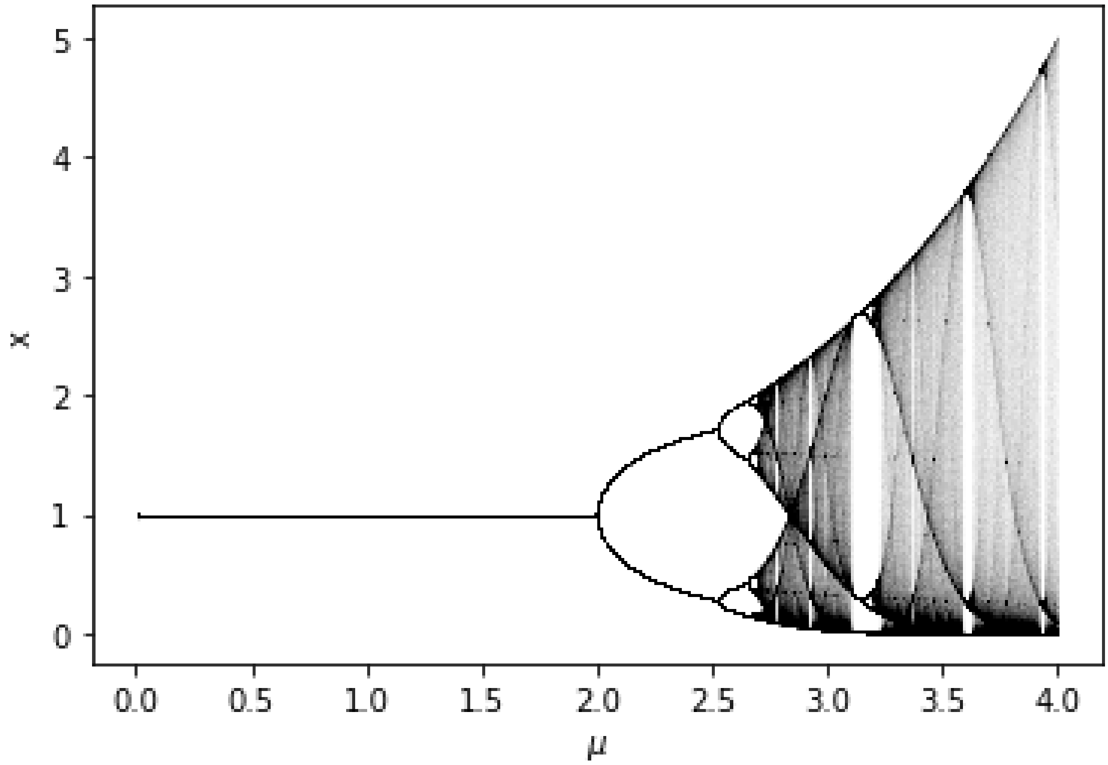

Figure 1: Orbit diagram for the map $x_{n+1} = x_n \exp(\mu(1 - x_n))$.

When comparing 1 with the orbit diagram for the logistic map, we can observe that both diagrams present qualitatively the same change from normal to chaotic dynamics: each of the bifurcation points is a period doubling bifurcation, and the ratio between consecutive μ where bifurcation occurs converges to a constant.

This is the behaviour we expected considering the universality of the features of these maps: Period doubling route to chaos applies to general unimodal maps (meaning smooth, concave and with one maximum).

Problem 4

$$x_{n+1} = \begin{cases} 2x_n & 0 \le x_n \le \frac{1}{2} \\ \\ 2 - 2x_n & \frac{1}{2} < x_n \le 1 \end{cases}$$

a) Guess why it is called a "tent map".

If we represent x_{n+1} in terms of x_n we can see it looks like a tent.

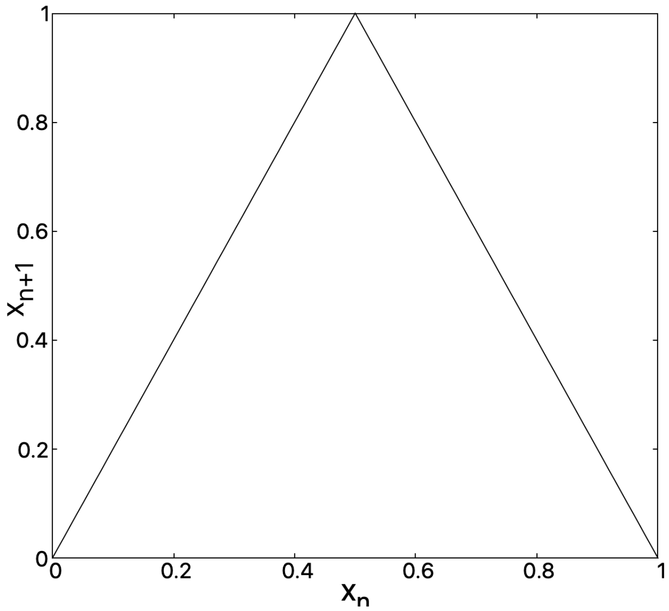

Figure 2: x_{n+1} in terms of x_n for the tent map

b) Find the fixed points and classify their stability.

Fixed points satisfy $x_* = f(x_*)$.

$$x_* = f(x_*) \begin{cases} x_* = 2x_* & 0 \le x \le \frac{1}{2} \Longrightarrow \boxed{x_*^{(1)} = 0} \\ \\ z_* = 2 - 2x_* & \frac{1}{2} < x \le 1 \Longrightarrow \boxed{x_*^{(2)} = \dfrac{2}{3}} \end{cases}$$

The stability of the fixed points depends on (the absolut value of) the slope of the function evaluated in these points. But in our case, $|f'(x)| = 2$ for all x. This means that the condition $|f'(x)| > 1$ is satisfied for all points and therefore: $\boxed{\text{all fixed points are unstable.}}$

c) Show graphically and analytically that the map displays a period 2 orbit. Prove analytically that the cycle is not stable.

Graphically:

$$(f \circ f)(x) = \begin{cases} 4x & 0 \leq x \leq \frac{1}{4} \\ 2 - 4x & \frac{1}{4} < x \leq \frac{1}{2} \\ 4x - 2 & \frac{1}{2} < x < \frac{3}{4} \\ 4 - 4x & \frac{3}{4} \leq x \leq 1 \end{cases}$$

Now, representing $(f \circ f)(x)$, $f(x)$ and $y(x) = x$ we can find the points where $(f \circ f)(x) = x$ (that don't satisfy $f(x) = x$) and therfore the period 2 orbit that changes between these points.

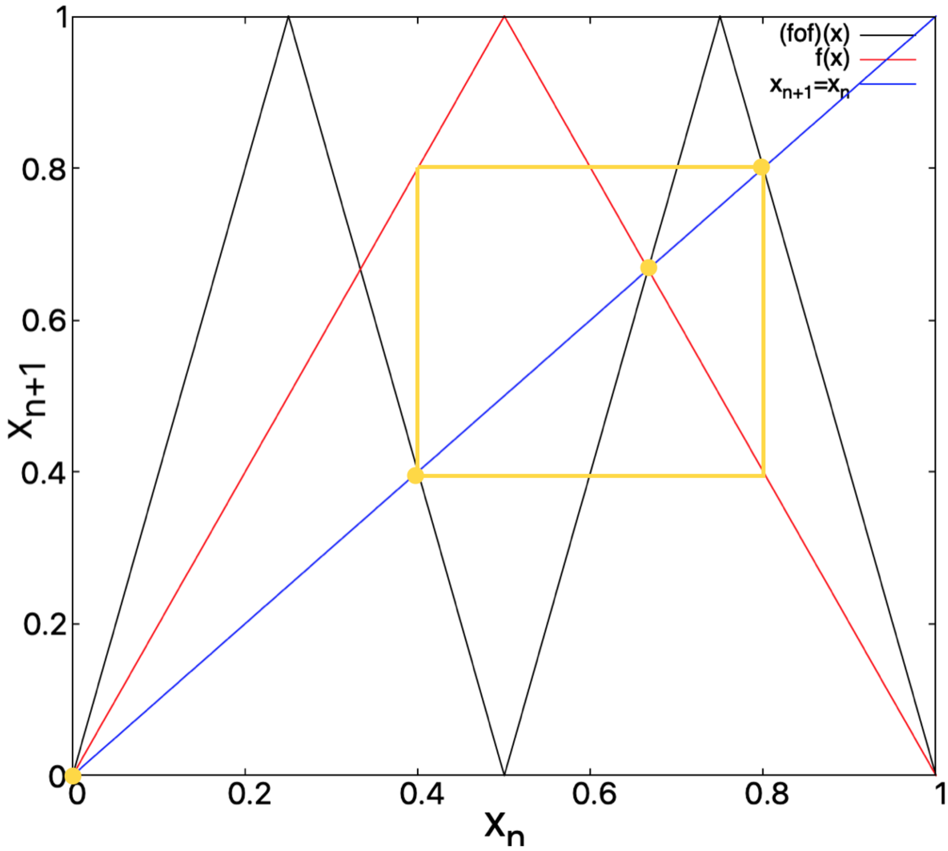

Figure 3: $(f \circ f)(x)$, $f(x)$ and $y(x) = x$. In yellow points the points where $(f \circ f)(x) = x$ and in yellow lines the orbit of period 2.

Analytically:

Period 2 orbits fulfill $(f \circ f)(x_{**}) = x_{**}$

$$(f \circ f)(x_{**}) = x_{**} \begin{cases} 4x_{**} = x_{**} & 0 \le x \le \frac{1}{4} \Longrightarrow \boxed{x_{**}^{(1)} = 0} \\ 2 - 4x_{**} = x_{**} & \frac{1}{4} < x \le \frac{1}{2} \Longrightarrow \boxed{x_{**}^{(2)} = \dfrac{2}{5}} \\ 4x_{**} - 2 = x_{**} & \frac{1}{2} < x < \frac{3}{4} \Longrightarrow \boxed{x_{**}^{(3)} = \dfrac{2}{3}} \\ 4 - 4x_{**} = x_{**} & \frac{3}{4} \le x \le 1 \Longrightarrow \boxed{x_{**}^{(4)} = \dfrac{4}{5}} \end{cases}$$

As we have seen, the points $x = 0$ and $x = \frac{2}{3}$ are fixed points (period 1 orbits) so the period 2 orbit points will be the other two $\boxed{x_{**}^{(2)} = \dfrac{2}{5}}$ and $\boxed{x_{**}^{(4)} = \dfrac{4}{5}}$.

Again, we have the same (absolut value of the) slope for all the points $|(f \circ f)'(x)| = 4$ so in particular for the period 2 orbit points the condition $|(f \circ f)'(x_{**})| > 1$ will be fulfilled and therfore $\boxed{\text{the cycle will be unstable.}}$