Control and Optimization 2023/2024 (2^{nd} semester)

Master in Electrical and Computer Engineering

Department of Electrical and Computer Engineering

A. Pedro Aguiar (pedro.aguiar@fe.up.pt), **M. Rosário Pinho** (mrpinho@fe.up.pt)

FEUP, Fev. 2024

Notebook #02: State-Space Model

1- The Pendulum System

Consider the following nonlinear pendulum model

$$m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell\sin\theta = 0,$$

where

- θ is the pendulum angle measured from the vertical;
- m is the pendulum mass;
- ℓ is the pendulum rod length;
- g is the gravity acceleration;
- b is the coefficient of rotational friction.

Activity 1

1.1. State-Space Dynamics

Using as state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$, obtain the state-space dynamics.

Sol 1.1

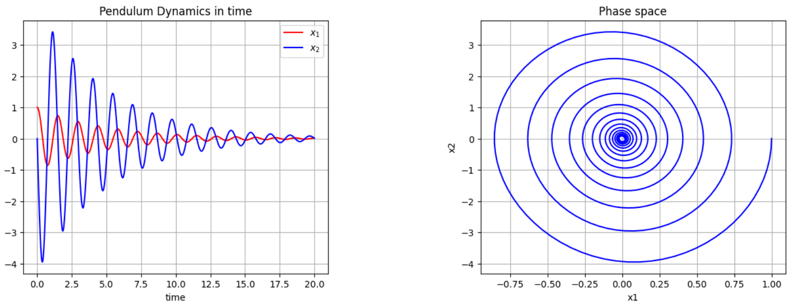
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{b}{ml^2}x_2 - \frac{g}{l}\sin(x_1)\end{aligned}$$

1.2. Time-evolution and Phase space

Plot the **time-evolution** of the state and the **phase space** for different initial conditions. Use the numerical integrator `integrate.odeint` of `scipy` or the `NonlinearIOSystem` of `control`

In [ ]:
```python
# To complete

import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt
# show plots in notebook
# matplotlib inline

# parameters
m = 1.0
l = 0.5
g = 9.81
b = 0.1

# vector field
def Sys_f(x, t=0):
  return np.array( #,
                   #
                   ])

# generate 1000 linearly spaced points for t
t_end=20
t = np.linspace(0, t_end,  1000)

# initial values:
x0 = np.array([1.0, 0])
#x0 = np.array([3.13, 0])

# type "help(integrate.odeint)" if you want more information about integr
x, infodict = integrate.odeint(Sys_f, x0, t, full_output=True)
# infodict['message']                      # integration successful

x1,x2 = x.T

#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(1,2,1)
ax2 = fig.add_subplot(1,2,2)

ax1.plot(t, x1, 'r-', label='$x_1$')
ax1.plot(t, x2, 'b-', label='$x_2$')
ax1.set_title("Pendulum Dynamics in time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(x1, x2, color="blue")
ax2.set_xlabel("x1")
ax2.set_ylabel("x2")
ax2.set_title("Phase space")
ax2.grid()
```

In [11]:
```python
# Solution 1.2

import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt
# show plots in notebook
# matplotlib inline

# parameters
m = 1.0
l = 0.5
g = 9.81
b = 0.1

# vector field
def Sys_f(x, t=0):
  return np.array([x[1],
                  -g/l*np.sin(x[0]) - b/(m*l**2)*x[1]
                  ])

# generate 1000 linearly spaced points for t
t_end=20
t = np.linspace(0, t_end,  1000)

# initial values:
x0 = np.array([1.0, 0])
#x0 = np.array([3.13, 0])

# type "help(integrate.odeint)" if you want more information about integr
x, infodict = integrate.odeint(Sys_f, x0, t, full_output=True)
# infodict['message']                      # integration successful

x1,x2 = x.T

#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(1,2,1)
ax2 = fig.add_subplot(1,2,2)

ax1.plot(t, x1, 'r-', label='$x_1$')
ax1.plot(t, x2, 'b-', label='$x_2$')
ax1.set_title("Pendulum Dynamics in time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(x1, x2, color="blue")
ax2.set_xlabel("x1")
ax2.set_ylabel("x2")
ax2.set_title("Phase space")
ax2.grid()
```



1.3.Euler and Runge-Kutta numerical methods

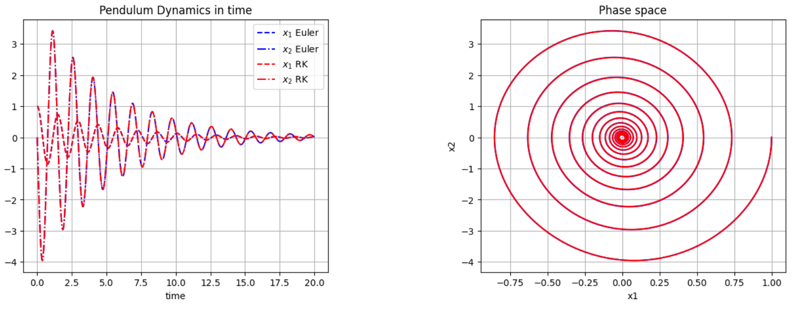
Do the same using an Euler discretization and Runge-Kutta 4th order. Compare the results for different sample-times.

In [ ]:
```
#To complete

#def Sys_f(x, t=0):
#  return np.array([x[1],
#                   -g/l*np.sin(x[0]) - b/(m*l**2)*x[1]
#                   ])


def euler_step_f(x, h):
  return np.array(#)

def rk_step_f(x, h, t):
  # k1, k2, k3, k4: intermediate values used in the Runge-Kutta method
  k1 = h * Sys_f( x, t)
  k2 = h * Sys_f( x + k1/2, t + h/2)
  k3 = h * Sys_f( x + k2/2, t + h/2)
  k4 = h * Sys_f( x + k3, t + h)
  # x_new: updated value of the solution using the Runge-Kutta method
  x_new = #
  return x_new



#Sample-time
dt = 0.0001

# Simulate the system
t_signal = np.arange(0,t_end,dt) # time samples

x1_signal = np.zeros_like(t_signal)
x2_signal = np.zeros_like(t_signal)

#Initial conditions of our system
x1_signal[0] = 1.0
x2_signal[0] = 0.0

x1_signalRK = x1_signal.copy()   # Copy Not Reference! Check what happens
x2_signalRK = x2_signal.copy()
```

```
for i in range(0,t_signal.shape[0]-1):
    x1_signal[i+1], x2_signal[i+1] = euler_step_f((x1_signal[i],x2_signal
    x1_signalRK[i+1], x2_signalRK[i+1] = rk_step_f((x1_signalRK[i],x2_sig


#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(1,2,1)
ax2 = fig.add_subplot(1,2,2)

ax1.plot(t_signal, x1_signal, 'b--', label='$x_1$ Euler')
ax1.plot(t_signal, x2_signal, 'b-.', label='$x_2$ Euler')
ax1.plot(t_signal, x1_signalRK, 'r--', label='$x_1$ RK')
ax1.plot(t_signal, x2_signalRK, 'r-.', label='$x_2$ RK')
ax1.set_title("Pendulum Dynamics in time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(x1_signal, x2_signal, color="blue",label=' Euler')
ax2.plot(x1_signalRK, x2_signalRK, color="red",label=' RK')
ax2.set_xlabel("x1")
ax2.set_ylabel("x2")
ax2.set_title("Phase space")
ax2.grid()
```

In [14]:
```
#Solution 1.3

#def Sys_f(x, t=0):
#  return np.array([x[1],
#                   -g/l*np.sin(x[0]) - b/(m*l**2)*x[1]
#                   ])


def euler_step_f(x, h):
  return np.array(x + Sys_f(x) * h)

def rk_step_f(x, h, t):
  # k1, k2, k3, k4: intermediate values used in the Runge-Kutta method
  k1 = h * Sys_f( x, t)
  k2 = h * Sys_f( x + k1/2, t + h/2)
  k3 = h * Sys_f( x + k2/2, t + h/2)
  k4 = h * Sys_f( x + k3, t + h)
  # x_new: updated value of the solution using the Runge-Kutta method
  x_new = x + (k1 + 2*k2 + 2*k3 + k4) / 6
  return x_new


#Sample-time
dt = 0.0001

# Simulate the system
t_signal = np.arange(0,t_end,dt) # time samples

x1_signal = np.zeros_like(t_signal)
x2_signal = np.zeros_like(t_signal)

#Initial conditions of our system
```

```python
x1_signal[0] = 1.0
x2_signal[0] = 0.0

x1_signalRK = x1_signal.copy()    # Copy Not Reference! Check what happens
x2_signalRK = x2_signal.copy()


for i in range(0,t_signal.shape[0]-1):
    x1_signal[i+1], x2_signal[i+1] = euler_step_f((x1_signal[i],x2_signal
    x1_signalRK[i+1], x2_signalRK[i+1] = rk_step_f((x1_signalRK[i],x2_sig


#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(1,2,1)
ax2 = fig.add_subplot(1,2,2)

ax1.plot(t_signal, x1_signal, 'b--', label='$x_1$ Euler')
ax1.plot(t_signal, x2_signal, 'b-.', label='$x_2$ Euler')
ax1.plot(t_signal, x1_signalRK, 'r--', label='$x_1$ RK')
ax1.plot(t_signal, x2_signalRK, 'r-.', label='$x_2$ RK')
ax1.set_title("Pendulum Dynamics in time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(x1_signal, x2_signal, color="blue",label=' Euler')
ax2.plot(x1_signalRK, x2_signalRK, color="red",label=' RK')
ax2.set_xlabel("x1")
ax2.set_ylabel("x2")
ax2.set_title("Phase space")
ax2.grid()
```



1.4. No friction solution

Try now with no friction, that is, $b = 0$.

In [ ]:

```python
b=0
#b=0.1

# run 1.3
```

1.5. Equilibrium points

Compute the equilibrium points. Use the `Sympy` module for symbolic mathematics.

In [15]:

```python
# Solution 1.5

import sympy as sm

x1, x2 = sm.symbols('x1, x2')
f1 = x2
f2 = -m*g*l*sm.sin(x1) - b*x2

# setting the vector field to zero
f1Equal = sm.Eq(f1, 0)
f2Equal = sm.Eq(f2, 0)

# compute the equilibrium points
equilibria = sm.solve( (f1Equal, f2Equal), x1, x2 )
print(equilibria)
```

```
[(0.0, 0.0), (3.14159265358979, 0.0)]
```

1.6. Quiver plot

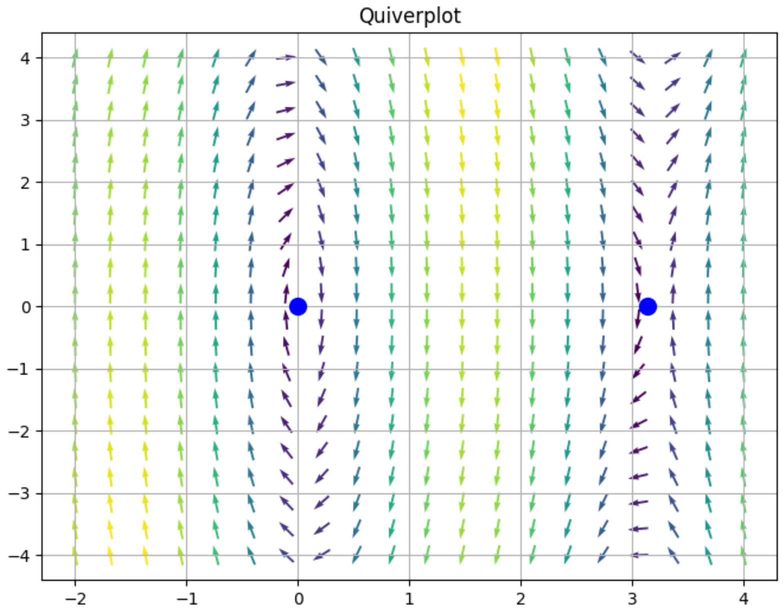
Draw the quiver plot.

In [ ]:

```
#To complete

#Sample-time
dt = 0.01

# Simulate the system
t_signal = np.arange(0,t_end,dt) # time samples
x1_signal = np.zeros_like(t_signal)
x2_signal = np.zeros_like(t_signal)

#Initial conditions of our system
x1_signal[0] = 1.0
#x1_signal[0] = np.pi
x2_signal[0] = 0.0

for i in range(0,t_signal.shape[0]-1):
    x1_signal[i+1], x2_signal[i+1] = rk_step_f((x1_signal[i],x2_signal[i]


#plot
fig2 = plt.figure(figsize=(8,6))
ax4 = fig2.add_subplot(1,1,1)

# plot equilibrium points
for point in equilibria:
  ax4.plot(point[0],point[1],"blue", marker = "o", markersize = 10.0)
ax4.set_title("Quiverplot")
#ax4.legend(loc='best')

# quiverplot
# define a grid and compute direction at each point
x1 = np.linspace(-2, 4, 20)
x2 = np.linspace(-4, 4, 20)

X1 , X2  = np.meshgrid(x1, x2)                  # create a grid
DX1, DX2 = #                      # compute growth rate on the grid
M = (np.hypot(DX1, DX2))                        # norm growth rate
M[ M == 0] = 1.                                 # avoid zero division err
DX1 /= M                                        # normalize each arrows
DX2 /= M

ax4.quiver(X1, X2, DX1, DX2, M, pivot='mid')
#ax4.plot(myx1, myx2, color="blue")
ax4.plot(x1_signal, x2_signal, color="blue")
#ax4.axis('Equal')
ax4.grid()
```

In [19]:

```
# Solution

#Sample-time
dt = 0.01

# Simulate the system
t_signal = np.arange(0,t_end,dt) # time samples
x1_signal = np.zeros_like(t_signal)
x2_signal = np.zeros_like(t_signal)

#Initial conditions of our system
#x1_signal[0] = 1.0
x1_signal[0] = np.pi
x2_signal[0] = 0.0

for i in range(0,t_signal.shape[0]-1):
    x1_signal[i+1], x2_signal[i+1] = rk_step_f((x1_signal[i],x2_signal[i]


#plot
fig2 = plt.figure(figsize=(8,6))
ax4 = fig2.add_subplot(1,1,1)

# plot equilibrium points
for point in equilibria:
  ax4.plot(point[0],point[1],"blue", marker = "o", markersize = 10.0)
ax4.set_title("Quiverplot")
#ax4.legend(loc='best')

# quiverplot
# define a grid and compute direction at each point
x1 = np.linspace(-2, 4, 20)
x2 = np.linspace(-4, 4, 20)

X1 , X2  = np.meshgrid(x1, x2)                  # create a grid
DX1, DX2 = Sys_f([X1, X2])                      # compute growth rate on
M = (np.hypot(DX1, DX2))                        # norm growth rate
M[ M == 0] = 1.                                 # avoid zero division err
DX1 /= M                                        # normalize each arrows
DX2 /= M

ax4.quiver(X1, X2, DX1, DX2, M, pivot='mid')
#ax4.plot(myx1, myx2, color="blue")
ax4.plot(x1_signal, x2_signal, color="blue")
#ax4.axis('Equal')
ax4.grid()
```



2- Other 2nd order systems

Activity 2

Repeat the previous points, that is, find all the equilibrium points, sketch the phase portrait, and determine the type of each isolated equilibrium point for the systems:

2.1. Nonlinear system 1

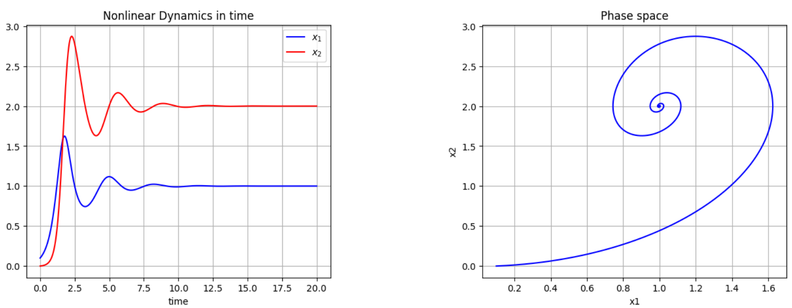
$$\dot{x}_1 = 2x_1 - x_1 x_2$$
$$\dot{x}_2 = 2x_1^2 - x_2$$

In [ ]:

```
# To complete

# vector field
def Sys_f(x, t=0):
  return np.array([#,
                   #
                   ])


#Sample-time
dt = 0.01

# Simulate the system
t_signal = np.arange(0,t_end,dt) # time samples

x1_signal = np.zeros_like(t_signal)
x2_signal = np.zeros_like(t_signal)

#Initial conditions of our system
x1_signal[0] = 0.1
x2_signal[0] = 0.0


for i in range(0,t_signal.shape[0]-1):
    x1_signal[i+1], x2_signal[i+1] = rk_step_f((x1_signal[i],x2_signal[i]


#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(1,2,1)
ax2 = fig.add_subplot(1,2,2)

ax1.plot(t_signal, x1_signal, 'b-', label='$x_1$')
ax1.plot(t_signal, x2_signal, 'r-', label='$x_2$')
ax1.set_title("Nonlinear Dynamics in time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(x1_signal, x2_signal, color="blue")
ax2.set_xlabel("x1")
ax2.set_ylabel("x2")
ax2.set_title("Phase space")
ax2.grid()
```

In [20]:

```
# Solution

# vector field
def Sys_f(x, t=0):
  return np.array([2*x[0] - x[0]*x[1],
                   2*x[0]**2 - x[1]
                   ])


#Sample-time
dt = 0.01

# Simulate the system
t_signal = np.arange(0,t_end,dt) # time samples

x1_signal = np.zeros_like(t_signal)
x2_signal = np.zeros_like(t_signal)

#Initial conditions of our system
x1_signal[0] = 0.1
x2_signal[0] = 0.0


for i in range(0,t_signal.shape[0]-1):
    x1_signal[i+1], x2_signal[i+1] = rk_step_f((x1_signal[i],x2_signal[i]


#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(1,2,1)
ax2 = fig.add_subplot(1,2,2)

ax1.plot(t_signal, x1_signal, 'b-', label='$x_1$')
ax1.plot(t_signal, x2_signal, 'r-', label='$x_2$')
ax1.set_title("Nonlinear Dynamics in time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(x1_signal, x2_signal, color="blue")
ax2.set_xlabel("x1")
ax2.set_ylabel("x2")
ax2.set_title("Phase space")
ax2.grid()
```



2.2. Nonlinear system 2 - Van der Pol oscillator

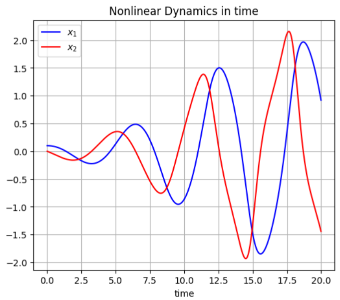
The Van der Pol oscillator

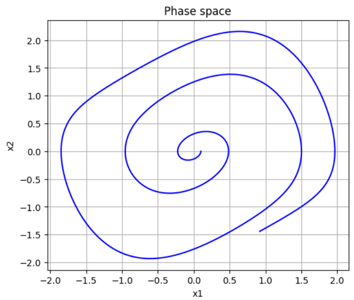
$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

with $\mu = 0.5$ and $x_0 = (1, 1)$.

In [24]:

```
#Solution

# vector field
mu=0.5

def Sys_f(x, t=0):
  return np.array([x[1],
                   mu*(1-x[0]**2)*x[1] - x[0]
                   ])

#Sample-time
dt = 0.01

# Simulate the system
t_signal = np.arange(0,t_end,dt) # time samples

x1_signal = np.zeros_like(t_signal)
x2_signal = np.zeros_like(t_signal)

#Initial conditions of our system
x1_signal[0] = 0.1
x2_signal[0] = 0.0


for i in range(0,t_signal.shape[0]-1):
    x1_signal[i+1], x2_signal[i+1] = rk_step_f((x1_signal[i],x2_signal[i]


#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(1,2,1)
ax2 = fig.add_subplot(1,2,2)

ax1.plot(t_signal, x1_signal, 'b-', label='$x_1$')
ax1.plot(t_signal, x2_signal, 'r-', label='$x_2$')
ax1.set_title("Nonlinear Dynamics in time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(x1_signal, x2_signal, color="blue")
ax2.set_xlabel("x1")
ax2.set_ylabel("x2")
ax2.set_title("Phase space")
ax2.grid()
```





2.3. Nonlinear System 3

$$\dot{x}_1 = x_1(2 - x_1 - x_2)$$
$$\dot{x}_2 = x_2(x_1 - 1)$$

In [ ]:

```
# vector field
def Sys_f(x, t=0):
  return np.array([  x[0] * (2 - x[0] - x[1]),
                   x[1] * (x[0] - 1)
                   ])


  #run 1.3
```