Control and Optimization 2023/2024 (2^{nd} semester)

Master in Electrical and Computer Engineering

Department of Electrical and Computer Engineering

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Notebook #05: Stability of Nonautonomous Systems and convergence to limit cycles

1 - Nonautonomous Systems

Consider the following nonautonomous system with a limit cycle:

$\dot{x} = -x - e^{-2t}y$

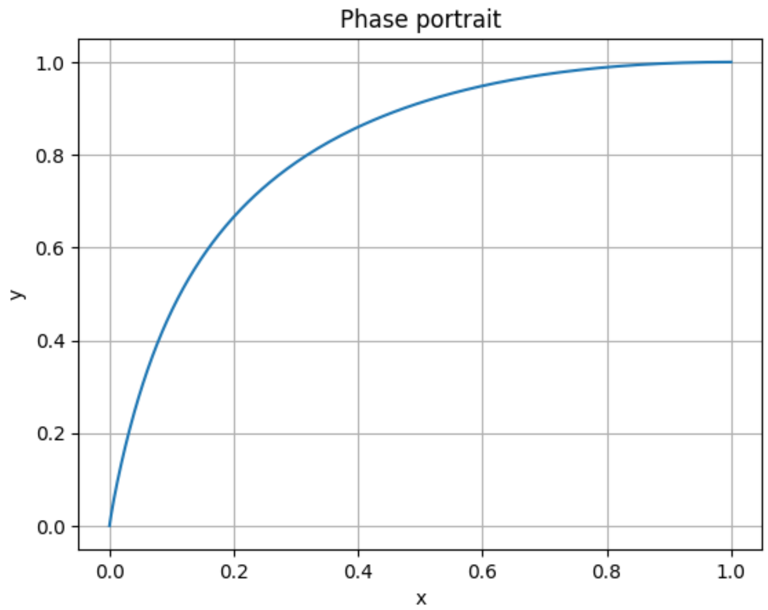
$\dot{y} = x - y$

1.1

Simulate this system.

In [ ]:
```
#to complete
import numpy as np
from scipy.integrate import odeint


def system(z, t):
  x, y = z[0], z[1]
  dxdt = #to complete
  dydt = #to complete
  return [dxdt, dydt]


t = np.linspace(0, 10, 1000)
z0 = [1, 1]

sol = odeint(system, z0, t)
x, y = sol[:, 0], sol[:, 1]

#plot
import matplotlib.pyplot as plt
```

In [5]:
```
#Solution
import numpy as np
from scipy.integrate import odeint


def system(z, t):
  x, y = z[0], z[1]
  dxdt = - x -np.exp(-2*t)*y
  dydt = x - y
  return [dxdt, dydt]


t = np.linspace(0, 10, 1000)
#t = np.linspace(-1, 9, 1000)
z0 = [1, 1]

sol = odeint(system, z0, t)
x, y = sol[:, 0], sol[:, 1]


import matplotlib.pyplot as plt

plt.plot(x, y)
plt.xlabel('x')
plt.ylabel('y')
plt.grid()
plt.title('Phase portrait')
plt.show()
```



1.2

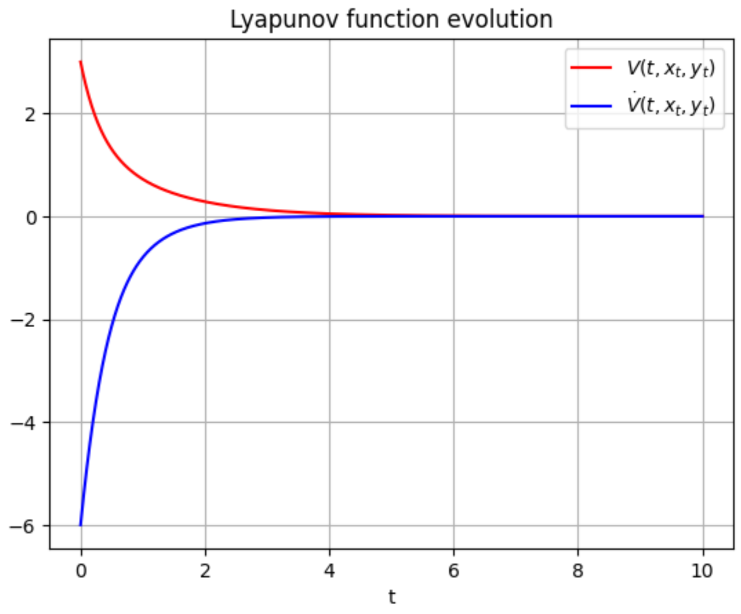
To show that the origin is GUAS, we can use a Lyapunov theory for nonautonomous systems. Consider the function

$V(t, x, y) = x^2 + (1 + e^{-2t})y^2.$

Plot the functions $t \to V(t, x(t), y(t))$ and $t \to \dot{V}(t, x(t), y(t))$ for the trajectories already computed.

In [ ]:
```
# to complete

def fV(t,x,y):
  return x*x + (1 + np.exp(-2*t)) * y


def dV(t,x,y):
  return # to complete

fV_t=fV(t,x,y)
dV_t=dV(t,x,y)

plt.plot(t, fV_t, 'r-', label='$V(t,x_t,y_t)$')
plt.plot(t, dV_t, 'b-', label='$\dot V(t,x_t,y_t)$')
plt.xlabel('t')
plt.title('Lyapunov function evolution')
plt.legend(loc='best')
plt.show()
```

In [6]:
```
#Solution
def fV(t,x,y):
  return x*x + (1 + np.exp(-2*t)) * y


def dV(t,x,y):
  return -2 * (x*x - x* y + y*y * (1 + 2*np.exp(-2*t)) )


fV_t=fV(t,x,y)
dV_t=dV(t,x,y)

plt.plot(t, fV_t, 'r-', label='$V(t,x_t,y_t)$')
plt.plot(t, dV_t, 'b-', label='$\dot V(t,x_t,y_t)$')
plt.xlabel('t')
plt.title('Lyapunov function evolution')
plt.legend(loc='best')
plt.grid()
plt.show()
```



1.3

What can you say about the stability of this system?

2- Satellite in planar orbit: convergence to a circular orbit

The equations of motion of a satellite in a planar orbit about a point mass M are:

$$\ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2} + T\sin\phi$$
$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{T}{r}cos\phi$$

where:

r= radial distance from mass M \ θ = angle from a fixed reference point in the orbit \ μ = GM, the gravitational constant of the attracting mass M \ and the control inputs are: T= thrust ϕ = thrust angle.

Take the state as $x = [r\ \dot{r}\ \theta\ \dot{\theta}]^T$,and the control input vector as $u = [\phi\ T]^T$.

Activity 1

2.1

Write the system in state space form

Solution

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 x_4^2 - \frac{\mu}{x_1^2} + u_2 \sin u_1$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -\frac{2x_2x_4}{x_1} + \frac{u_2}{x_1}\cos u_1$$

2.2

One solution of the satellite equations is a circular orbit, which has values of: $r = r_0, \dot{r} = 0, \dot{\theta} = w, T = 0$ and $\mu = r_0^3 w^2$(this is Kepler's third law). The constant angular velocity is equal to w, and the constant radius of the circular orbit is r_0. Assume that the thrust angle is 90 degrees at nominal.

Simulate the nominal trajectory x_0 and small perturbations around it.

In [ ]:

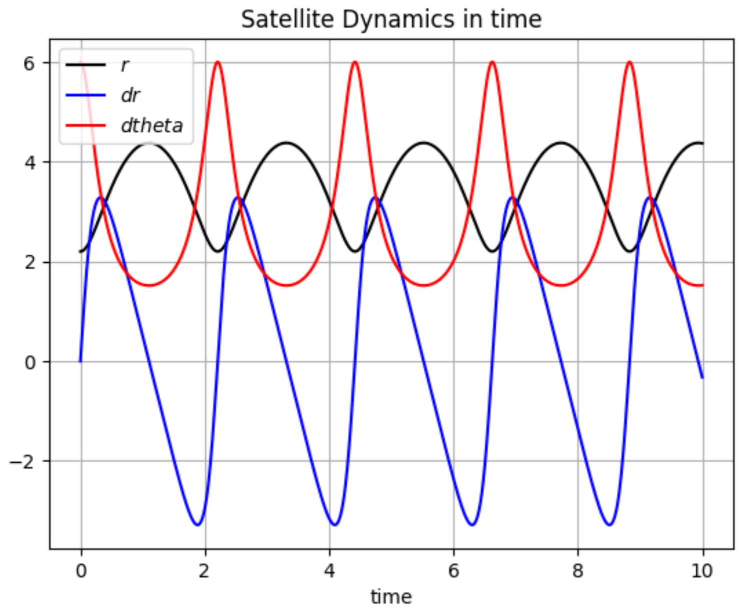
```python
# To complete


import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt
# show plots in notebook
# matplotlib inline

# parameters
r0=2
w=6
mu=(r0**3) * (w**2)

# vector field
def Sys_f(x, t=0):
  r=x[0]
  dr=x[1]
  th=x[2]
  dth=x[3]
  dx1= ...
  dx2= ...
  dx3= ...
  dx4=  ...
  return np.array([ dx1, dx2, dx3, dx4
                   ])

# generate 1000 linearly spaced points for t
t_end=10
t = np.linspace(0, t_end,  1000)

# initial values:
x0 = np.array([r0, 0.0, 0, w])
#x0 = np.array([r0+0.2, 0.0, 0, w])

#  integrate.odeint
x = integrate.odeint(Sys_f, x0, t, full_output=False)

x1,x2, x3, x4 = x.T

#plot

plt.figure(1)
plt.plot(t, x1, 'k-', label='$r$')
plt.plot(t, x2, 'b-', label='$dr$')
#plt.plot(t, x3*360/6.28, 'g-', label='$theta$')
plt.plot(t, x4, 'r-', label='$d theta$')
plt.title("Satellite Dynamics in time")
plt.xlabel("time")
plt.grid()
plt.legend(loc='best')

plt.figure(2)
# setting the axes projection as polar
plt.axes(projection = 'polar')
plt.polar(x3, x1, 'r-')
```

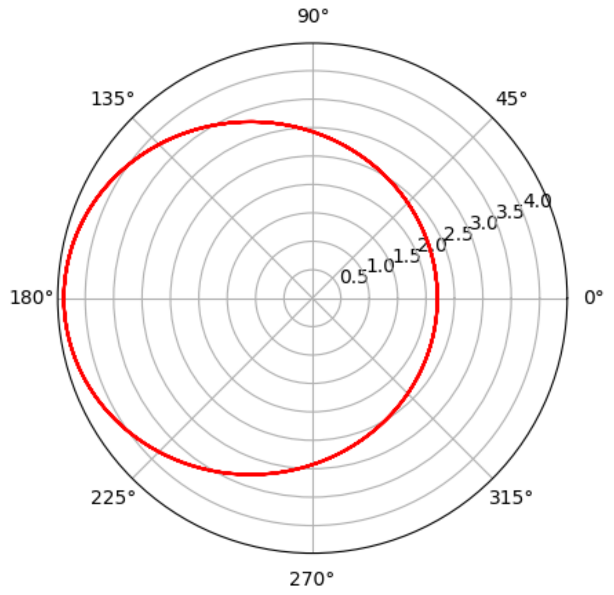
In [12]:

```python
# Solution


import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt
# show plots in notebook
# matplotlib inline

# parameters
r0=2
w=6
mu=(r0**3) * (w**2)

# vector field
def Sys_f(x, t=0):
  r=x[0]
  dr=x[1]
  th=x[2]
  dth=x[3]
  dx1=dr
  dx2= r * (dth*dth)  - mu/(r*r)
  dx3=dth
  dx4=  - 2 * dr * dth /r
  return np.array([ dx1, dx2, dx3, dx4
                   ])

# generate 1000 linearly spaced points for t
t_end=10
t = np.linspace(0, t_end,  1000)

# initial values:
x0 = np.array([r0, 0.0, 0, w])
#x0 = np.array([r0+0.2, 0.0, 0, w])

#  integrate.odeint
x = integrate.odeint(Sys_f, x0, t, full_output=False)

x1,x2, x3, x4 = x.T

#plot

plt.figure(1)
plt.plot(t, x1, 'k-', label='$r$')
plt.plot(t, x2, 'b-', label='$dr$')
#plt.plot(t, x3*360/6.28, 'g-', label='$theta$')
plt.plot(t, x4, 'r-', label='$d theta$')
plt.title("Satellite Dynamics in time")
plt.xlabel("time")
plt.grid()
plt.legend(loc='best')

plt.figure(2)
# setting the axes projection as polar
plt.axes(projection = 'polar')
plt.polar(x3, x1, 'r-')
```

Out[12]:

```
[<matplotlib.lines.Line2D at 0x7ddcd01b19c0>]
```





2.3

Consider the dynamics of the error to the nominal trajectory x_0, that is $y = x - x_0$

Linearize the system. Notice that although x_0 is time-varying, the linearized error dynamics is autonomous. Why?

2.4

Simulate the linearized system. Compare the original nonlinear systems with the linearization plus the nominal system.

In [ ]:

```
# To complete


import numpy as np
#import control

r0 = 10
omega= 1
A = np.array([\
            [, , , ],\
            [, , , ],\
            [, , , ],\
            [, , , ] ] )

B = np.array([\
            [0.0, 0.0],\
            [0.0, 1.0],\
            [0.0, 0.0],\
            [0.0, 0.0] ] )


print ('A\n', A)
print ('B\n', B)

eig_open_loop, eig_vect = np.linalg.eig( A )
print ('Eigenvalues of A \n', eig_open_loop)
```

In [ ]:
```
# Solution

import numpy as np
#import control

r0 = 10
omega= 1
A = np.array([\
            [0.0, 1.0, 0.0, 0.0],\
            [3*omega**2, 0.0, 0.0, 2*r0*omega],\
            [0.0, 0.0, 0.0, 1],\
            [0.0, -2*omega/r0, 0.0, 0.0] ] )

B = np.array([\
            [0.0, 0.0],\
            [0.0, 1.0],\
            [0.0, 0.0],\
            [0.0, 0.0] ] )


print ('A\n', A)
print ('B\n', B)

eig_open_loop, eig_vect = np.linalg.eig( A )
print ('Eigenvalues of A \n', eig_open_loop)
```

```
A
 [[ 0.    1.    0.    0. ]
 [ 3.    0.    0.   20. ]
 [ 0.    0.    0.    1. ]
 [ 0.   -0.2   0.    0. ]]
B
 [[0. 0.]
 [0. 1.]
 [0. 0.]
 [0. 0.]]
Eigenvalues of A
 [0.+0.j 0.+1.j 0.-1.j 0.+0.j]
```

2.5

Where are the open-loop poles of the linearized system. Is it controllable?

Consider its subsystem $\dot{z} = \bar{A}z + \bar{B}v$ with $z = [r\ \dot{r}]$ and $v = T$. Check that the subsystem is controllable. Find a linear feedback matrix K such that $v = -Kz$ places the poles at -1 and -2.

In [ ]:
```
pip install control
```

```
Collecting control
  Downloading control-0.9.4-py3-none-any.whl (455 kB)
     ━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━ 455.1/455.1 kB 4.0 MB/s eta
0:00:00
Requirement already satisfied: numpy in /usr/local/lib/python3.10/dist-pa
ckages (from control) (1.25.2)
Requirement already satisfied: scipy>=1.3 in /usr/local/lib/python3.10/di
st-packages (from control) (1.11.4)
Requirement already satisfied: matplotlib in /usr/local/lib/python3.10/di
st-packages (from control) (3.7.1)
Requirement already satisfied: contourpy>=1.0.1 in /usr/local/lib/python
3.10/dist-packages (from matplotlib->control) (1.2.0)
Requirement already satisfied: cycler>=0.10 in /usr/local/lib/python3.10/
dist-packages (from matplotlib->control) (0.12.1)
Requirement already satisfied: fonttools>=4.22.0 in /usr/local/lib/python
3.10/dist-packages (from matplotlib->control) (4.49.0)
Requirement already satisfied: kiwisolver>=1.0.1 in /usr/local/lib/python
3.10/dist-packages (from matplotlib->control) (1.4.5)
Requirement already satisfied: packaging>=20.0 in /usr/local/lib/python3.
10/dist-packages (from matplotlib->control) (23.2)
Requirement already satisfied: pillow>=6.2.0 in /usr/local/lib/python3.1
0/dist-packages (from matplotlib->control) (9.4.0)
Requirement already satisfied: pyparsing>=2.3.1 in /usr/local/lib/python
3.10/dist-packages (from matplotlib->control) (3.1.2)
Requirement already satisfied: python-dateutil>=2.7 in /usr/local/lib/pyt
hon3.10/dist-packages (from matplotlib->control) (2.8.2)
Requirement already satisfied: six>=1.5 in /usr/local/lib/python3.10/dist
-packages (from python-dateutil>=2.7->matplotlib->control) (1.16.0)
Installing collected packages: control
Successfully installed control-0.9.4
```

```
In [ ]: # To complete
import control

# Controllability
print ('---Controllability')
print ('rank of ctrb(A,b)' , np.linalg.matrix_rank( control.ctrb( A, B )
print ('Eigenvalues of A ', np.linalg.eig( A )[0])


#Sun-system
r0 = 2
omega= 6
A1 = np.array([\
            [ , ],\
            [ , ] ] )

B1 = np.array([\
            [0.0],\
            [1.0] ] )

print ('A1\n', A1)
print ('B1\n', B1)

eig_open_loop, eig_vect = np.linalg.eig( A1 )
print ('Eigenvalues of A1 \n', eig_open_loop)

# Pole Placement
K = control.place( A1, B1, [-1, -2] )
print ('---Pole Placement\nK=\n', K)

# Verification of Eigen values of A-BK
print ('\n---Verification of Eigenvalues of A-BK\n')
Acl = A1-B1@K
#print(Acl)
eig_Acl, eig_vect = np.linalg.eig( Acl )
print ('Eigenvalues of A-BK:', eig_Acl)
```

```
In [ ]: # Solution
import control

# Controllability
print ('---Controllability')
print ('rank of ctrb(A,b)' , np.linalg.matrix_rank( control.ctrb( A, B )
print ('Eigenvalues of A ', np.linalg.eig( A )[0])


#Sun-system
r0 = 2
omega= 6
A1 = np.array([\
            [0.0, 1.0],\
            [3*omega**2, 0.0] ] )

B1 = np.array([\
            [0.0],\
            [1.0] ] )

print ('A1\n', A1)
print ('B1\n', B1)

eig_open_loop, eig_vect = np.linalg.eig( A1 )
print ('Eigenvalues of A1 \n', eig_open_loop)

# Pole Placement
K = control.place( A1, B1, [-1, -2] )
print ('---Pole Placement\nK=\n', K)

# Verification of Eigen values of A-BK
print ('\n---Verification of Eigenvalues of A-BK\n')
Acl = A1-B1@K
#print(Acl)
eig_Acl, eig_vect = np.linalg.eig( Acl )
print ('Eigenvalues of A-BK:', eig_Acl)
```

```
---Controllability
rank of ctrb(A,b) 3
Eigenvalues of A  [0.+0.j 0.+1.j 0.-1.j 0.+0.j]
A1
 [[  0.   1.]
 [108.   0.]]
B1
 [[0.]
 [1.]]
Eigenvalues of A1
 [ 10.39230485 -10.39230485]
---Pole Placement
K=
 [[110.   3.]]

---Verification of Eigenvalues of A-BK

Eigenvalues of A-BK: [-1. -2.]
```

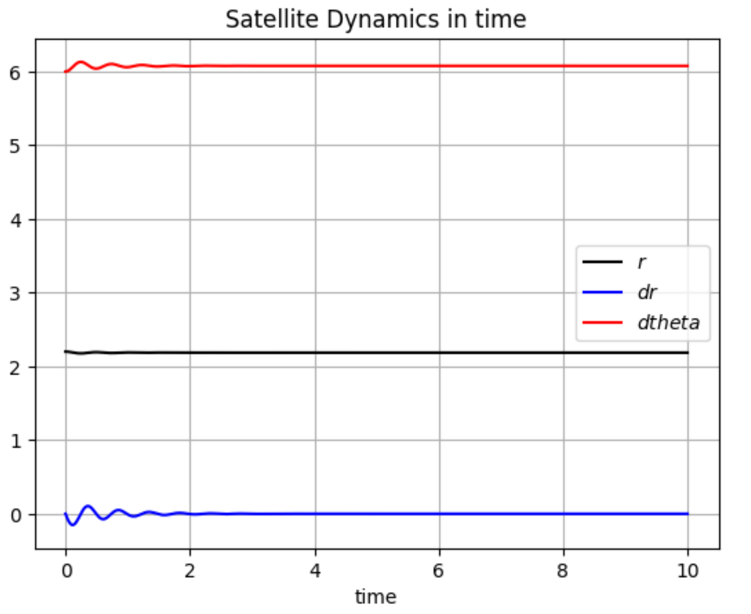
2.6

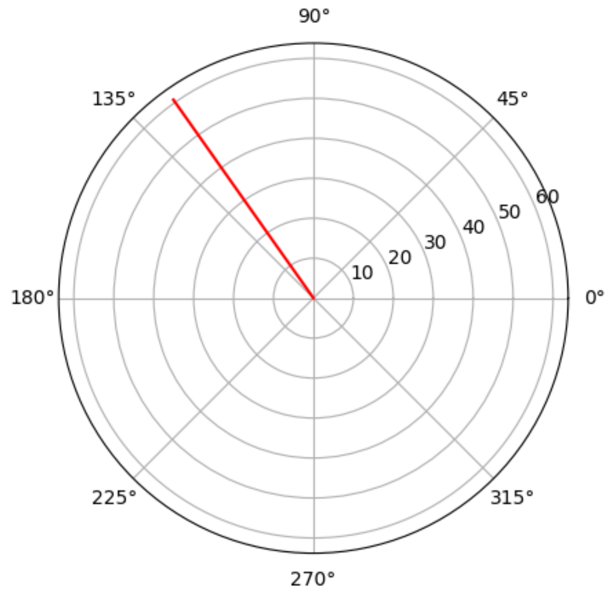
Simulate the complete system with the designed controller. Start from initial points outside the nominal trajectory.

In [ ]:
```python
# To complete


import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt
# show plots in notebook
# matplotlib inline

# parameters
r0=2
w=6
mu=(r0**3) * (w**2)

# vector field
def Sys_f(x, t=0):
  r=x[0]
  dr=x[1]
  th=x[2]
  dth=x[3]
  T = -K[0][0]*(r-r0)-K[0][1]*dr
  dx1=...
  dx2= ...
  dx3= ...
  dx4=  ...
  return np.array([ dx1, dx2, dx3, dx4
                  ])

# generate 1000 linearly spaced points for t
t_end=10
t = np.linspace(0, t_end,  1000)

# initial values:
x0 = np.array([r0+0.2, 0.0, 0, w])

#  integrate.odeint
x = integrate.odeint(Sys_f, x0, t, full_output=False)

x1,x2, x3, x4 = x.T

#plot

plt.figure(1)
plt.plot(t, x1, 'k-', label='$r$')
plt.plot(t, x2, 'b-', label='$dr$')
#plt.plot(t, x3*360/6.28, 'g-', label='$theta$')
plt.plot(t, x4, 'r-', label='$d theta$')
plt.title("Satellite Dynamics in time")
plt.xlabel("time")
plt.grid()
plt.legend(loc='best')

plt.figure(2)
# setting the axes projection as polar
plt.axes(projection = 'polar')
plt.polar(x1, x3, 'r-')
```

In [ ]:
```
# Solution


import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt
# show plots in notebook
# matplotlib inline

# parameters
r0=2
w=6
mu=(r0**3) * (w**2)

# vector field
def Sys_f(x, t=0):
  r=x[0]
  dr=x[1]
  th=x[2]
  dth=x[3]
  T = -K[0][0]*(r-r0)-K[0][1]*dr
  dx1=dr
  dx2= r * (dth*dth)  - mu/(r*r)+T
  dx3=dth
  dx4=  - 2 * dr * dth /r
  return np.array([ dx1, dx2, dx3, dx4
                   ])

# generate 1000 linearly spaced points for t
t_end=10
t = np.linspace(0, t_end,  1000)

# initial values:
x0 = np.array([r0+0.2, 0.0, 0, w])

#  integrate.odeint
x = integrate.odeint(Sys_f, x0, t, full_output=False)

x1,x2, x3, x4 = x.T

#plot

plt.figure(1)
plt.plot(t, x1, 'k-', label='$r$')
plt.plot(t, x2, 'b-', label='$dr$')
#plt.plot(t, x3*360/6.28, 'g-', label='$theta$')
plt.plot(t, x4, 'r-', label='$d theta$')
plt.title("Satellite Dynamics in time")
plt.xlabel("time")
plt.grid()
plt.legend(loc='best')

plt.figure(2)
# setting the axes projection as polar
plt.axes(projection = 'polar')
plt.polar(x1, x3, 'r-')
```

Out[ ]:
```
[<matplotlib.lines.Line2D at 0x7be546878520>]
```





2.7

Consider the function $V(x) = r^2 - r\dot{r} + \frac{1}{2}\dot{r}^2$. Use it to show that the linear subsystem is G.A.S.

What can you conclude about the stability of the complete linearized system using V? What can you conclude about the original nonlinear system?