Control and Optimization 2023/2024 (2^{nd} semester)

U.PORTO
FEUP FACULDADE DE ENGENHARIA UNIVERSIDADE DO PORTO

Master in Electrical and Computer Engineering

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Notebook #04: Lyapunov Stability

1- Rotated Rigid Spacecraft

Activity 1

The Euler equations of a rotating rigid spacecraft are given by

$$
\begin{aligned}
J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\
J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3
\end{aligned}
$$

where ω_1 to ω_3 are the components of the angular velocity vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ along the principal axes, u_1 to u_3 are the torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

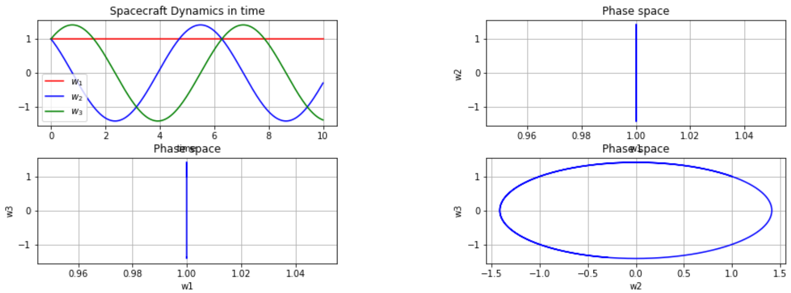
1.1 Prove that with $u_1 = u_2 = u_3 = 0$ the origin $\boldsymbol{\omega} = \mathbf{0}$ is globally stable. Is it globally asymptotically stable?

1.2 Confirm the results through simulation by plotting the **time-evolution** of the state and in the **phase space** for different initial conditions with $u = 0$.

In [ ]:

```
# Solution

import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt
# show plots in notebook
# matplotlib inline
```

```
# parameters
J1, J2, J3 = 1, 0.5, 0.5

# vector field
def Sys_f(x, t=0):
  return np.array([(J2-J3)/J1*x[1]*x[2],
                   (J3-J1)/J2*x[0]*x[2],
                   (J1-J2)/J3*x[0]*x[1]
                   ])

# generate 1000 linearly spaced points for t
t_end=10
t = np.linspace(0, t_end,  1000)

# initial values:
x0 = np.array([1.0, 1.0, 1.0])

# type "help(integrate.odeint)" if you want more information about integr
x, infodict = integrate.odeint(Sys_f, x0, t, full_output=True)
# infodict['message']                    # integration successful

w1,w2, w3 = x.T

#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(2,2,1)
ax2 = fig.add_subplot(2,2,2)
ax3 = fig.add_subplot(2,2,3)
ax4 = fig.add_subplot(2,2,4)

ax1.plot(t, w1, 'r-', label='$w_1$')
ax1.plot(t, w2, 'b-', label='$w_2$')
ax1.plot(t, w3, 'g-', label='$w_3$')
ax1.set_title("Spacecraft Dynamics in time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(w1, w2, color="blue")
ax2.set_xlabel("w1")
ax2.set_ylabel("w2")
ax2.set_title("Phase space")
ax2.grid()

ax3.plot(w1, w3, color="blue")
ax3.set_xlabel("w1")
ax3.set_ylabel("w3")
ax3.set_title("Phase space")
ax3.grid()

ax4.plot(w2, w3, color="blue")
ax4.set_xlabel("w2")
ax4.set_ylabel("w3")
ax4.set_title("Phase space")
ax4.grid()
```



1.3 Suppose now that the torque inputs are given by the feedback control law

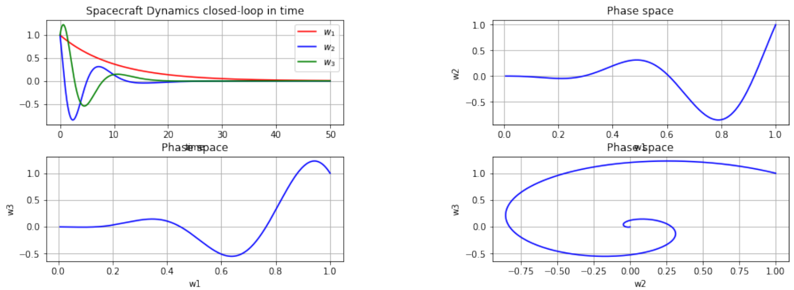
$$u_i = -k_i \omega_i, \quad k_i > 0, \quad i = 1, 2, 3$$

Prove that the origin of the close loop system is Globally Asymptotically Stable (GAS).

1.4 Confirm the results through simulation by plotting the **time-evolution** of the state and in the **phase space** for different initial conditions. Check also what happens when the feedback gains increase and/or decrease.

In [ ]:

```python
# Solution

import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt
# show plots in notebook
# matplotlib inline

# parameters
J1, J2, J3 = 1, 0.5, 0.5

k1, k2, k3 = 0.1, 0.1, 0.1
#k1, k2, k3 = 0.5, 0.5, 0.5

# vector field
def Sys_f(x, t=0):
  return np.array([(J2-J3)/J1*x[1]*x[2]-k1/J1*x[0],
                   (J3-J1)/J2*x[0]*x[2]-k2/J2*x[1],
                   (J1-J2)/J3*x[0]*x[1]-k3/J3*x[2]
                   ])

# generate 1000 linearly spaced points for t
t_end=50
t = np.linspace(0, t_end,  1000)

# initial values:
x0 = np.array([1.0, 1.0, 1.0])

# type "help(integrate.odeint)" if you want more information about integr
x, infodict = integrate.odeint(Sys_f, x0, t, full_output=True)
# infodict['message']                     # integration successful

w1,w2, w3 = x.T

#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(2,2,1)
ax2 = fig.add_subplot(2,2,2)
ax3 = fig.add_subplot(2,2,3)
ax4 = fig.add_subplot(2,2,4)

ax1.plot(t, w1, 'r-', label='$w_1$')
ax1.plot(t, w2, 'b-', label='$w_2$')
ax1.plot(t, w3, 'g-', label='$w_3$')
ax1.set_title("Spacecraft Dynamics closed-loop in time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(w1, w2, color="blue")
ax2.set_xlabel("w1")
ax2.set_ylabel("w2")
ax2.set_title("Phase space")
ax2.grid()

ax3.plot(w1, w3, color="blue")
ax3.set_xlabel("w1")
ax3.set_ylabel("w3")
ax3.set_title("Phase space")
ax3.grid()

ax4.plot(w2, w3, color="blue")
ax4.set_xlabel("w2")
ax4.set_ylabel("w3")
ax4.set_title("Phase space")
ax4.grid()
```



1.5 Consider now the case

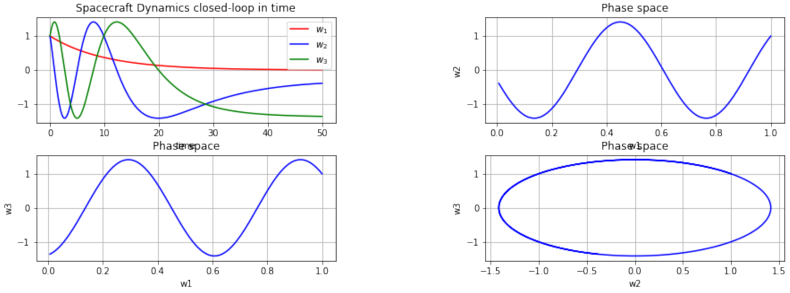
$$\begin{aligned} u_1 &= -k_1 \omega_1, \quad k_1 > 0 \\ u_2 &= 0 \\ u_3 &= 0 \end{aligned}$$

What can you say about the stability of the origin? Use LaSalle's theorem to analyze the convergence to other points.

1.6 Confirm the results through simulation by plotting the **time-evolution** of the state and in the **phase space** for different initial conditions.

In [ ]:

```
# Solution

import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt
# show plots in notebook
## matplotlib inline

# parameters
J1, J2, J3 = 1, 0.5, 0.5
#J1, J2, J3 = 1, 1.0, 0.5  #now with J2 not equal to J3 which implies tha

k1, k2, k3 = 0.1, 0, 0
#k1, k2, k3 = 0.5, 0.5, 0.5

# vector field
def Sys_f(x, t=0):
  return np.array([(J2-J3)/J1*x[1]*x[2]-k1/J1*x[0],
                   (J3-J1)/J2*x[0]*x[2]-k2/J2*x[1],
                   (J1-J2)/J3*x[0]*x[1]-k3/J3*x[2]
                   ])

# generate 1000 linearly spaced points for t
t_end=50
t = np.linspace(0, t_end,  1000)

# initial values:
x0 = np.array([1.0, 1.0, 1.0])

# type "help(integrate.odeint)" if you want more information about integr
x, infodict = integrate.odeint(Sys_f, x0, t, full_output=True)
# infodict['message']                     # integration successful

w1,w2, w3 = x.T

#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(2,2,1)
ax2 = fig.add_subplot(2,2,2)
ax3 = fig.add_subplot(2,2,3)
ax4 = fig.add_subplot(2,2,4)

ax1.plot(t, w1, 'r-', label='$w_1$')
ax1.plot(t, w2, 'b-', label='$w_2$')
ax1.plot(t, w3, 'g-', label='$w_3$')
ax1.set_title("Spacecraft Dynamics closed-loop in time")
ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(w1, w2, color="blue")
ax2.set_xlabel("w1")
ax2.set_ylabel("w2")
ax2.set_title("Phase space")
ax2.grid()

ax3.plot(w1, w3, color="blue")
ax3.set_xlabel("w1")
ax3.set_ylabel("w3")
ax3.set_title("Phase space")
ax3.grid()

ax4.plot(w2, w3, color="blue")
ax4.set_xlabel("w2")
ax4.set_ylabel("w3")
ax4.set_title("Phase space")
ax4.grid()
```



2- Other exercises

2.1 Show that the origin of the system

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -x_2 - x_2^3$$

is GAS.

In [ ]:

```python
# Solution

import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt

# vector field
def Sys_f(x, t=0):
  return np.array([- x[0]+x[1],
                 - x[1]-x[1]**3
                 ])

# generate 1000 linearly spaced points for t
t_end=10
t = np.linspace(0, t_end,  1000)

# initial values:
x0 = np.array([1.0, 1.0])

# integrate.odeint inputs and outputs.
x, infodict = integrate.odeint(Sys_f, x0, t, full_output=True)

x1,x2 = x.T

#plot
fig = plt.figure(figsize=(15,5))
fig.subplots_adjust(wspace = 0.5, hspace = 0.3)
ax1 = fig.add_subplot(2,1,1)
ax2 = fig.add_subplot(2,1,2)


ax1.plot(t, x1, 'r-', label='$x_1$')
ax1.plot(t, x2, 'b-', label='$x_2$')


ax1.set_xlabel("time")
ax1.grid()
ax1.legend(loc='best')

ax2.plot(x1, x2, color="blue")
ax2.set_xlabel("1")
ax2.set_ylabel("x2")
ax2.set_title("Phase space")
ax2.grid()
```