Control and Optimization 2023/2024 (2^{nd} semester)

Master in Electrical and Computer Engineering

Department of Electrical and Computer Engineering

Professors:

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Project - Part 1

Note: This is to be done in group of **3** elements. Use this notebook to answer all the questions. At the end of the work, you should **send** the **notebook** and a **pdf file** with a printout of the notebook with all the results.

Deadlines: Present the state of your work (and answer questions) on the week of **April 1st** in your corresponding practical class. Send the files until 23:59 of **May 31, 2024**.

Project of Control and Optimization (Part I and Part II)

Note: This is to be done in group of **3** elements. Use this notebook to answer all the questions (note that you should include your computations in a picture format or in latex). At the end of the work, you should **send** the **notebook** and a **pdf file** with a printout of the notebook with all the results.

Deadlines: Present the state of your work (and answer questions) on the week of **May 20** in your corresponding practical class. Send the files until 23:59 of **May 31, 2024**.

In [ ]:
```
# To make a nice pdf file of this file, you have to do the following:
# - upload your file to print into the running folder (click on the corre
# Then run this (which will make a html file into the current folder):
!jupyter nbconvert --to html "name_of_the_file.ipynb"
# Then just download the html file and print it to pdf!
```

```
[NbConvertApp] WARNING | pattern 'name_of_the_file.ipynb' matched no files
This application is used to convert notebook files (*.ipynb)
        to various other formats.

        WARNING: THE COMMANDLINE INTERFACE MAY CHANGE IN FUTURE RELEASES.

Options
=======
The options below are convenience aliases to configurable class-options,
as listed in the "Equivalent to" description-line of the aliases.
To see all configurable class-options for some <cmd>, use:
    <cmd> --help-all

--debug
    set log level to logging.DEBUG (maximize logging output)
    Equivalent to: [--Application.log_level=10]
--show-config
    Show the application's configuration (human-readable format)
    Equivalent to: [--Application.show_config=True]
--show-config-json
    Show the application's configuration (json format)
    Equivalent to: [--Application.show_config_json=True]
--generate-config
    generate default config file
    Equivalent to: [--JupyterApp.generate_config=True]
-y
    Answer yes to any questions instead of prompting.
    Equivalent to: [--JupyterApp.answer_yes=True]
--execute
    Execute the notebook prior to export.
    Equivalent to: [--ExecutePreprocessor.enabled=True]
--allow-errors
    Continue notebook execution even if one of the cells throws an error a
nd include the error message in the cell output (the default behaviour is
to abort conversion). This flag is only relevant if '--execute' was specif
ied, too.
    Equivalent to: [--ExecutePreprocessor.allow_errors=True]
--stdin
    read a single notebook file from stdin. Write the resulting notebook w
ith default basename 'notebook.*'
    Equivalent to: [--NbConvertApp.from_stdin=True]
--stdout
    Write notebook output to stdout instead of files.
    Equivalent to: [--NbConvertApp.writer_class=StdoutWriter]
--inplace
    Run nbconvert in place, overwriting the existing notebook (only
            relevant when converting to notebook format)
    Equivalent to: [--NbConvertApp.use_output_suffix=False --NbConvertApp.
export_format=notebook --FilesWriter.build_directory=]
--clear-output
    Clear output of current file and save in place,
            overwriting the existing notebook.
    Equivalent to: [--NbConvertApp.use_output_suffix=False --NbConvertApp.
export_format=notebook --FilesWriter.build_directory= --ClearOutputPreproc
essor.enabled=True]
--coalesce-streams
    Coalesce consecutive stdout and stderr outputs into one stream (within
each cell).
    Equivalent to: [--NbConvertApp.use_output_suffix=False --NbConvertApp.
export_format=notebook --FilesWriter.build_directory= --CoalesceStreamsPre
```

```
processor.enabled=True]
--no-prompt
    Exclude input and output prompts from converted document.
    Equivalent to: [--TemplateExporter.exclude_input_prompt=True --Templat
eExporter.exclude_output_prompt=True]
--no-input
    Exclude input cells and output prompts from converted document.
            This mode is ideal for generating code-free reports.
    Equivalent to: [--TemplateExporter.exclude_output_prompt=True --Templa
teExporter.exclude_input=True --TemplateExporter.exclude_input_prompt=Tru
e]
--allow-chromium-download
    Whether to allow downloading chromium if no suitable version is found
on the system.
    Equivalent to: [--WebPDFExporter.allow_chromium_download=True]
--disable-chromium-sandbox
    Disable chromium security sandbox when converting to PDF..
    Equivalent to: [--WebPDFExporter.disable_sandbox=True]
--show-input
    Shows code input. This flag is only useful for dejavu users.
    Equivalent to: [--TemplateExporter.exclude_input=False]
--embed-images
    Embed the images as base64 dataurls in the output. This flag is only u
seful for the HTML/WebPDF/Slides exports.
    Equivalent to: [--HTMLExporter.embed_images=True]
--sanitize-html
    Whether the HTML in Markdown cells and cell outputs should be sanitize
d..
    Equivalent to: [--HTMLExporter.sanitize_html=True]
--log-level=<Enum>
    Set the log level by value or name.
    Choices: any of [0, 10, 20, 30, 40, 50, 'DEBUG', 'INFO', 'WARN', 'ERRO
R', 'CRITICAL']
    Default: 30
    Equivalent to: [--Application.log_level]
--config=<Unicode>
    Full path of a config file.
    Default: ''
    Equivalent to: [--JupyterApp.config_file]
--to=<Unicode>
    The export format to be used, either one of the built-in formats
            ['asciidoc', 'custom', 'html', 'latex', 'markdown', 'noteboo
k', 'pdf', 'python', 'qtpdf', 'qtpng', 'rst', 'script', 'slides', 'webpd
f']
            or a dotted object name that represents the import path for an
            ``Exporter`` class
    Default: ''
    Equivalent to: [--NbConvertApp.export_format]
--template=<Unicode>
    Name of the template to use
    Default: ''
    Equivalent to: [--TemplateExporter.template_name]
--template-file=<Unicode>
    Name of the template file to use
    Default: None
    Equivalent to: [--TemplateExporter.template_file]
--theme=<Unicode>
    Template specific theme(e.g. the name of a JupyterLab CSS theme distri
buted
    as prebuilt extension for the lab template)
```

```
        Default: 'light'
        Equivalent to: [--HTMLExporter.theme]
    --sanitize_html=<Bool>
        Whether the HTML in Markdown cells and cell outputs should be sanitize
d.This
        should be set to True by nbviewer or similar tools.
        Default: False
        Equivalent to: [--HTMLExporter.sanitize_html]
    --writer=<DottedObjectName>
        Writer class used to write the
                                            results of the conversion
        Default: 'FilesWriter'
        Equivalent to: [--NbConvertApp.writer_class]
    --post=<DottedOrNone>
        PostProcessor class used to write the
                                            results of the conversion
        Default: ''
        Equivalent to: [--NbConvertApp.postprocessor_class]
    --output=<Unicode>
        Overwrite base name use for output files.
                    Supports pattern replacements '{notebook_name}'.
        Default: '{notebook_name}'
        Equivalent to: [--NbConvertApp.output_base]
    --output-dir=<Unicode>
        Directory to write output(s) to. Defaults
                                      to output to the directory of each noteb
ook. To recover
                                      previous default behaviour (outputting t
o the current
                                      working directory) use . as the flag val
ue.
        Default: ''
        Equivalent to: [--FilesWriter.build_directory]
    --reveal-prefix=<Unicode>
        The URL prefix for reveal.js (version 3.x).
                This defaults to the reveal CDN, but can be any url pointing t
o a copy
                of reveal.js.
                For speaker notes to work, this must be a relative path to a l
ocal
                copy of reveal.js: e.g., "reveal.js".
                If a relative path is given, it must be a subdirectory of the
                current directory (from which the server is run).
                See the usage documentation
                (https://nbconvert.readthedocs.io/en/latest/usage.html#reveal-
js-html-slideshow)
                for more details.
        Default: ''
        Equivalent to: [--SlidesExporter.reveal_url_prefix]
    --nbformat=<Enum>
        The nbformat version to write.
                Use this to downgrade notebooks.
        Choices: any of [1, 2, 3, 4]
        Default: 4
        Equivalent to: [--NotebookExporter.nbformat_version]

    Examples
    --------

        The simplest way to use nbconvert is
```

```
            > jupyter nbconvert mynotebook.ipynb --to html

            Options include ['asciidoc', 'custom', 'html', 'latex', 'markd
own', 'notebook', 'pdf', 'python', 'qtpdf', 'qtpng', 'rst', 'script', 'sli
des', 'webpdf'].

            > jupyter nbconvert --to latex mynotebook.ipynb

            Both HTML and LaTeX support multiple output templates. LaTeX i
ncludes
            'base', 'article' and 'report'.  HTML includes 'basic', 'lab'
and
            'classic'. You can specify the flavor of the format used.

            > jupyter nbconvert --to html --template lab mynotebook.ipynb

            You can also pipe the output to stdout, rather than a file

            > jupyter nbconvert mynotebook.ipynb --stdout

            PDF is generated via latex

            > jupyter nbconvert mynotebook.ipynb --to pdf

            You can get (and serve) a Reveal.js-powered slideshow

            > jupyter nbconvert myslides.ipynb --to slides --post serve

            Multiple notebooks can be given at the command line in a coupl
e of
            different ways:

            > jupyter nbconvert notebook*.ipynb
            > jupyter nbconvert notebook1.ipynb notebook2.ipynb

            or you can specify the notebooks list in a config file, contai
ning::

                c.NbConvertApp.notebooks = ["my_notebook.ipynb"]

            > jupyter nbconvert --config mycfg.py

To see all available configurables, use `--help-all`.
```

In [ ]:

```
#!pip install control
```

Identification

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An Autonomous Underwtare Vehicle (UAV) model in the vertical plan

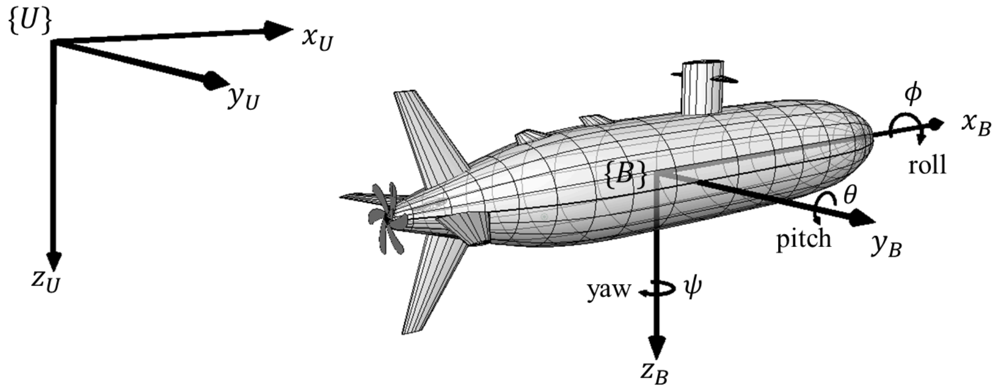


Fig. 1. Coordinate frames, position and orientation variables of an AUV.

Consider an Autonomous Underwater Vehicle (AUV) illustrated in Fig. 1 that can only generate force in x_B-direction by means of an actuator composed of an electric motor with a propeller coupled to the shaft.

In this work, the final goal is to design a tracking controller in the vertical plane so that that the vehicle will move according to a desired depth trajectory.

In the vertical plane, the kinematic equations take the form

$$\dot{x} = u\cos\theta + w\sin\theta \tag{1}$$
$$\dot{z} = -u\sin\theta + w\cos\theta \tag{2}$$
$$\dot{\theta} = q \tag{3}$$

where u, w and q are the linear and angular velocities of the vehicle, respectively, in surge (x_B), heave (z_B) and pitch (θ) direction of the body-fixed coordinates $\{B\}$. The Cartesian coordinates of the vehicle's center of mass is denoted by x and z, and θ is the pitch angle.

The simplified equations of motion for surge, heave, and pitch rate when there is no actuated force in Z_B direction (that is, the vehicle is underactuated) yield

$$m_u\dot{u} + m_w wq + d_u(u)u = \tau_u \tag{4}$$
$$m_w\dot{w} - m_u uq + d_w(w)w = 0 \tag{5}$$
$$m_q\dot{q} + m_{uw}uw + d_q(q)q - z_B B\sin\theta = \tau_q \tag{6}$$

where $m_u = m - X_{\dot{u}}$, $m_w = m - Z_{\dot{w}}$, $m_q = I_y - M_{\dot{q}}$ and $m_{uw} = m_u - m_w$ are mass and hydrodynamic added mass terms, B denotes the buoyancy, and the hydrodynamic damping effects are considered to be of the form

$$d_u(u) = -X_u - X_{u|u|}|u| \tag{7}$$
$$d_w(w) = -Z_w - Z_{w|w|}|w| \tag{8}$$
$$d_q(q) = -M_q - M_{q|q|}|q| \tag{9}$$

In the above equations, it is assumed that the AUV is neutrally buoyant and that the center of buoyancy can be expressed as $(x_B, y_B, z_B) = (0, 0, z_B)$, where z_B is the metacentric height. The symbols τ_u and τ_q denote the actuated force in surge direction and torque around the y-axis of the vehicle, respectively.

Part 1: Stability analysis

We take the practical situation that there exist autopilots controllers in charge of tracking reference signals in u and q. Thus, we consider at this stage that the actuation signals are u and q.

1.1 Show that the speed controller given by

$$u = \frac{v_d - w \sin\theta}{\cos\theta} \tag{10}$$

forces the AUV to move with a constant horizontal velocity v_d, that is, $\dot{x} = v_d$. Show also that in this case the equations of motion in the vertical plane of the AUV reduces to

$\textit{AUV model}$

$$\begin{aligned}
\dot{x} &= v_d \\
\dot{z} &= -v_d \tan\theta + \frac{1}{\cos\theta} w \\
\dot{w} &= d_1 w + d_2 w|w| + \bar{m}\left(\frac{v_d}{\cos\theta} - w\tan\theta\right) q \\
\dot{\theta} &= q
\end{aligned}$$

where z is the vertical position (depth) of the AUV, w is the linear velocity along the axis z_B (*heave*), θ is the angle of *pitch*, and q is the angular velocity around the axis y_B.

1.1 Solution

$$u = \frac{v_d - \omega \sin(\theta)}{cos(\theta)} \iff \dot{x} = ucos(\theta) + \omega sin(\theta)$$

$$\dot{x} = (\frac{v_d - \omega sin(\theta)}{cos(\theta)})cos(\theta) + \omega sin(\theta) = v_d - \omega sin(\theta) + \omega sin(\theta)$$

$$\iff \dot{x} = v_d$$

$$\dot{z} = -usin(\theta) + \omega cos(\theta) = -\frac{(v_d - \omega sin(\theta))}{cos(\theta)} sin(\theta) + \omega cos(\theta)$$

$$= (-v_d + \omega sin(\theta))tan(\theta) + \omega cos(\theta) = -v_d tan(\theta) + \omega sin(\theta) tan(\theta) + \omega cos(\theta)$$

$$= -v_d tan(\theta) + \omega(\frac{sin^2(\theta)}{cos(\theta)} + cos(\theta)) = -v_d tan(\theta) + \omega(\frac{cos^2(\theta) + sin^2(\theta)}{cos(\theta)})$$

$$\iff \dot{z} = -v_d tan(\theta) + \frac{1}{cos(\theta)} \omega$$

$$m_\omega \dot{\omega} - m_u uq + d_\omega(\omega)\omega = 0$$

$$\iff \dot{\omega} = \frac{m_\omega}{m_\omega}(\frac{v_d}{cos(\theta)} - \omega tan(\theta)) - \frac{1}{m_\omega}(-Z_\omega - Z_{w|w|}|w|)\omega$$

$$= \tilde{m}(\frac{v_d}{cos(\theta)} - \omega tan(\theta)) + \frac{Z_\omega}{m_\omega}\omega + \frac{Z_{w|w|}}{m_\omega}|\omega|\omega$$

$$\iff \dot{\omega} = d_1 \omega + d_2 \omega|\omega| + \tilde{m}(\frac{v_d}{cos(\theta)} - \omega tan(\theta))$$

In the sequel, we will consider that system parameters have the following values (in appropriate units): $v_d = 1$; $d_1 = -3$; $d_2 = -12$, $\bar{m} = 0.9$

1.2. Let z_d be a given desired depth. Defining the state $\mathbf{x} = (z - z_d, w, \theta)^\top$, input $\mathbf{u} = q$, output $\mathbf{y} = z - z_d$, write the system in state-space form and linearize it around the origin $\mathbf{x} = \mathbf{0}$.

1.2 Solution

Showing that $\mathbf{x} = \mathbf{0}$ is an equilibrium point

$$X = (z - zd, \omega, \theta)^T = (0, 0, 0)^T$$

$$X = \begin{cases} \tilde{z} = -v_d tan(\theta) + \frac{\omega}{cos(\theta)} \\ \dot{\omega} = d_1\omega + d_2\omega|\omega| + \tilde{m}(\frac{v_d}{cos(\theta)} - \omega tan(\theta))q \\ \theta = q \end{cases} = \begin{cases} \tilde{z} = 0 \\ \dot{\omega} = 0 \\ \theta = 0 \end{cases}$$

Linearization around $\mathbf{x} = \mathbf{0}$

$$X = (z - zd, \omega, \theta)^T$$

$$\begin{cases} x_1 = \tilde{z} \\ x_2 = \omega \\ x_3 = \theta \end{cases}$$

$$X = \begin{cases} \tilde{z} = -v_d tan(\theta) + \frac{\omega}{cos(\theta)} \\ \dot{\omega} = d_1\omega + d_2\omega|\omega| + \tilde{m}(\frac{v_d}{cos(\theta)} - \omega tan(\theta))q \\ \theta = q \end{cases}$$

$$\begin{cases} \dot{x}_1 = -v_d tan(x_3) + x_2/cos(x_3) = f_1 \\ \dot{x}_2 = d_1 x_2 + d2x_2|x_2| + \tilde{m}(v_d/cos(x_3) - x_2 tan(x_3))q = f_2 \\ \dot{x}_3 = q = f_3 \end{cases}$$

$$Y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \frac{df_1}{dx_3} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \frac{df_2}{dx_3} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_2} & \frac{df_3}{dx_3} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{cos(x_3)} & -\frac{v_d}{cos^2(x_3)} + x_2\frac{sin(x_3)}{cos^2(x_3)} \\ 0 & d_1 + 2d_2x_2 - \tilde{m}qtan(x_3) & \frac{\tilde{m}v_d sin(x_3)}{cos^2(x_3)} - \frac{x_2 q}{cos^2(x_3)} \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -v_d \\ 0 & d_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{df_1}{dq} \\ \frac{df_2}{dq} \\ \frac{df_3}{dq} \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{m}(\frac{v_d}{cos(x_3)} - x_2 tan(x_3)) \\ 1 \end{bmatrix}, X = 0$$

$$= \begin{bmatrix} 0 \\ \tilde{m}v_d \\ 1 \end{bmatrix}$$

In [ ]:

```
# Confirmation with Code

# System Linearization
import sympy as sp

# Define symbols
x1, x2, x3 = sp.symbols('x1 x2 x3')
vd, d1, d2, m, q = sp.symbols('v_d d_1 d_2 m q')

# Define state vector and input
x = sp.Matrix([x1, x2, x3])
u = sp.Matrix([q])

# Define system equations
f = sp.Matrix([-vd * sp.tan(x3) + 1/sp.cos(x3) * x2,
               d1 * x2 + d2 * x2 * x2 + m * (vd/sp.cos(x3) - x2 * sp.tan(
               q])

# Define output equation
g = sp.Matrix([x1])

# Compute Jacobians
A = f.jacobian(x)
B = f.jacobian(u)
C = g.jacobian(x)

# Substitute X = 0
```

```
A = f.jacobian(x).subs({x1: 0, x2: 0, x3: 0})
B = f.jacobian(u).subs({x1: 0, x2: 0, x3: 0})
C = g.jacobian(x).subs({x1: 0, x2: 0, x3: 0})

# Print the Jacobians
print("A:")
sp.pprint(A)
print("\nB:")
sp.pprint(B)
print("\nC:")
sp.pprint(C)
```

```
A:
⎡0  1   -v_d⎤
⎢           ⎥
⎢0  d₁   0  ⎥
⎢           ⎥
⎣0  0    0  ⎦

B:
⎡  0  ⎤
⎢     ⎥
⎢m⋅v_d⎥
⎢     ⎥
⎣  1  ⎦

C:
[1  0  0]
```

1.3 Solution

Since matrix A is triangular, the eigenvalues are the values that form the diagonal:

$$A = \begin{bmatrix} 0 & 1 & -v_d \\ 0 & d_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues of matrix (A) are:

$$\lambda_1 = 0$$

$$\lambda_2 = d_1 = -3$$

$$\lambda_3 = 0$$

Once we have eigenvalues located in the imaginary axis we cannot conclude about the nonlinear system stability

1.3. Analyze the stability of the origin $\mathbf{x} = \mathbf{0}$ with $\mathbf{u} = \mathbf{0}$ of the linear and nonlinear system using the **Lyapunov indirect method**.

In [ ]:
```
# Confirmation with code

# Lypaunov indirect method
import numpy as np


# Constants
```

```
vd = 1
d1 = -3
d2 = -12
m = 0.9
k1 = 1
k2 = 1
zd = 1

# Matrix A
A = np.array([[0.0, 1.0, -vd],
              [0.0, d1, 0.0],
              [0.0, 0.0, 0.0]])

# Matrix B
B = np.array([[0.0],
              [m*vd],
              [1.0]])

# Matrix C
C = np.array([1.0, 0.0, 0.0])

print('---Stability---')
# Eigenvalues of A
eigenvalues = np.linalg.eigvals(A)
print('Eigenvalues of A:', eigenvalues)

# Check if eigenvalues are all negative
if all(eigenvalue < 0 for eigenvalue in eigenvalues):
    print("Stable")
elif any(eigenvalue == 0 for eigenvalue in eigenvalues):
    print("On imaginary axis (cannot conclude stability of the nonlinear
else:
    print("Unstable")
```

```
---Stability---
Eigenvalues of A: [ 0. -3.  0.]
On imaginary axis (cannot conclude stability of the nonlinear system)
```

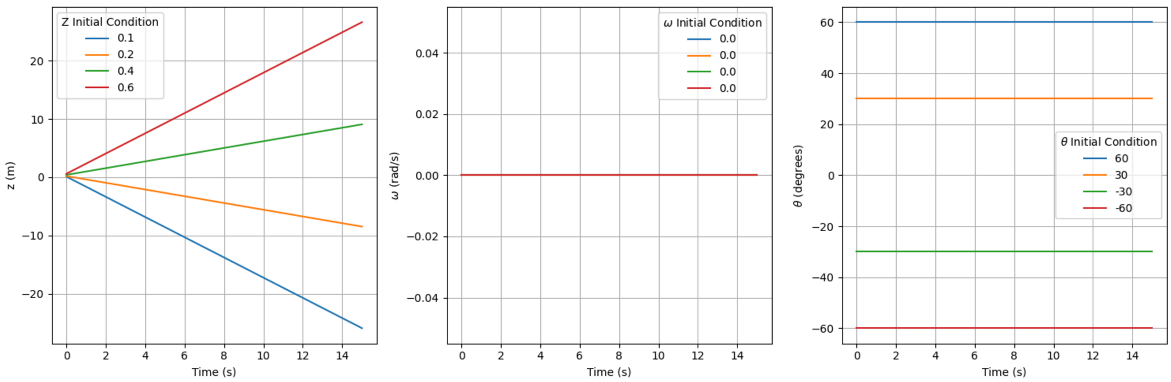
1.4 For $z_d = 1\,m$, plot the **time-evolution** of the state for the nonlinear and linear systems (with $q = 0$) for different initial conditions.

Use the numerical integrator `integrate.odeint` of `scipy`.

In [ ]:

```
import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt

# Nonlinear System
def Sys_f(x, t=0):
    z = x[0]
    w = x[1]
    th = x[2]
    q = 0
    dx1 = -vd*np.tan(th) + w/np.cos(th)
    dx2 = d1*w + d2*w*np.abs(w) + q*m*(vd/np.cos(th) - w*np.tan(th))
    dx3 = q
    return np.array([dx1, dx2, dx3])

# Generate 1000 linearly spaced points for t
t_end = 15
```

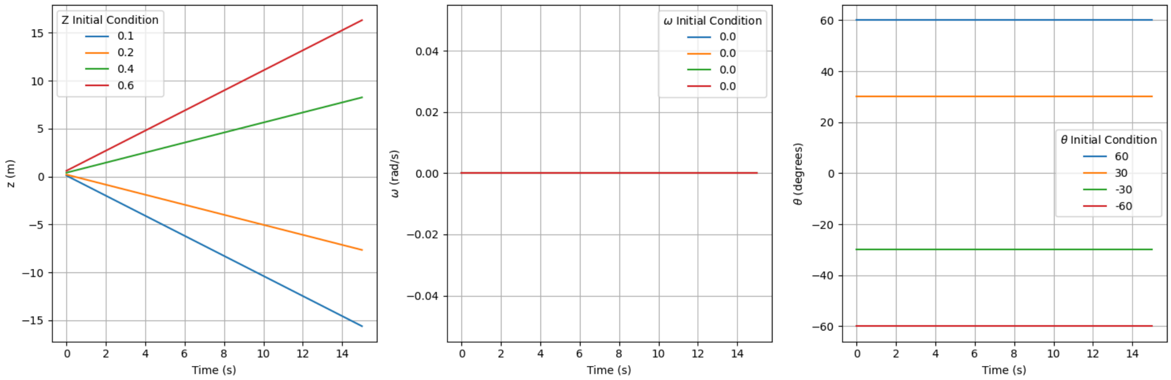
```
t = np.linspace(0, t_end, 1000)


# Different initial conditions to test
initial_conditions = [
    np.array([0.1, 0.0, np.pi/3]),
    np.array([0.2, 0.0, np.pi/6]),
    np.array([0.4, 0.0, -np.pi/6]),
    np.array([0.6, 0.0, -np.pi/3]),
    # Add more initial conditions as needed
]

# Plot
fig, axs = plt.subplots(1, 3, figsize=(15, 5))

for i, var in enumerate([r'z (m)', r'$\omega$ (rad/s)', r'$\theta$ (degre
    axs[i].set_xlabel('Time (s)')
    axs[i].set_ylabel(var)
    axs[i].grid(True)

    for x0 in initial_conditions:
        x_nl, infodict = integrate.odeint(Sys_f, x0, t, full_output=True)
        if var == r'$\theta$ (degrees)':
            axs[i].plot(t, x_nl[:, i]*180/np.pi, label=f'{round(x0[i]*180
        else :
            axs[i].plot(t, x_nl[:, i], label=f'{x0[i]}')

axs[0].legend(loc='best', title=r'Z Initial Condition')
axs[1].legend(loc='best', title=r'$\omega$ Initial Condition')
axs[2].legend(loc='best', title=r'$\theta$ Initial Condition')
plt.tight_layout()
plt.show()
```



In [ ]:

```
import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt


# Linear System
def Sys_f_Linear(x, t=0):
  z = x[0]
  w = x[1]
  th = x[2]
  q = 0
  dx1 = w - vd*th
  dx2 = d1*w + m*vd*q
  dx3 = q
  return np.array([ dx1, dx2, dx3
```

```
                    ])

# Generate 1000 linearly spaced points for t
t_end = 15
t = np.linspace(0, t_end, 1000)


# Different initial conditions to test
initial_conditions = [
    np.array([0.1, 0.0, np.pi/3]),
    np.array([0.2, 0.0, np.pi/6]),
    np.array([0.4, 0.0, -np.pi/6]),
    np.array([0.6, 0.0, -np.pi/3]),
    # Add more initial conditions as needed
]

# Plot
fig, axs = plt.subplots(1, 3, figsize=(15, 5))

for i, var in enumerate([r'z (m)', r'$\omega$ (rad/s)', r'$\theta$ (degre
    axs[i].set_xlabel('Time (s)')
    axs[i].set_ylabel(var)
    axs[i].grid(True)

    for x0 in initial_conditions:
        x_l, infodict = integrate.odeint(Sys_f_Linear, x0, t, full_output
        if var == r'$\theta$ (degrees)':
            axs[i].plot(t, x_l[:, i]*180/np.pi, label=f'{round(x0[i]*180/
        else :
            axs[i].plot(t, x_l[:, i], label=f'{x0[i]}')

axs[0].legend(loc='best', title=r'Z Initial Condition')
axs[1].legend(loc='best', title=r'$\omega$ Initial Condition')
axs[2].legend(loc='best', title=r'$\theta$ Initial Condition')
plt.tight_layout()
plt.show()
```



Nonlinear System vs Linear System

From both simulations plotted above we can see specially in the z that the linear system only aproximates well the Nonlinear system when close to the origin.

1.5 Consider now the nonlinear subsystem $(\tilde{z}, \theta)$ with $\tilde{z} = z - z_d$, q as input and assume that $w = 0$.

Prove that the origin of the closed-loop system with control law

$$q = k_1(z - z_d) - k_2\theta \qquad (1)$$

with positive gains k_1 and k_2 (and $v_d > 0$) is asymptotically stable. Use the Lyapunov function

$$V(\tilde{z}, \theta) = \frac{k_1}{2v_d}\tilde{z}^2 + \int_0^\theta \tan(\phi)d\phi \qquad (11)$$

1.5 Solution

$$V(\tilde{z},\theta) = \frac{k_1\tilde{z}^2}{2v_d} + \int_0^\theta tan(\phi)\,d\phi = \frac{k_1\tilde{z}^2}{2v_d} + [ln|\frac{1}{cos(\theta)}|]\Big|_0^\theta\Big| = \frac{k_1\tilde{z}^2}{2v_d} + ln|\frac{1}{cos(\theta$$

V(z̃, θ) is positive definite.

$$V(\tilde{z},\theta) = \frac{dV}{dz}\dot{Z} + \frac{dV}{d\theta}\theta = \frac{2k_1\tilde{z}\dot{\tilde{z}}}{2v_d} + tan(\theta)\theta = -k_1\tilde{z}tan(\theta) + tan(\theta)q = -k_1\tilde{z}t$$

V̇(z̃, θ) is semi negative.

La Salle's Theorem

$$E = \{x \in \mathbb{R}^2 : V(\tilde{z},\theta) = 0\} \Leftrightarrow \{x \in \mathbb{R}^2 : \theta = 0\}$$

$\theta = 0; \dot{\theta} = 0$

$$\theta = q \Leftrightarrow \theta = k_1\tilde{z} - k_2\theta \Leftrightarrow \tilde{z} = 0$$

M ⊂ E

$$M = \{x \in \mathbb{R}^2 : x = 0\}$$

$\mathbf{x} = \mathbf{0}$ is A.S

$$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$$

1.6 For the above item, confirm the results through simulation by plotting the **time-evolution** of the state and in the **phase space** for different initial conditions with $k_1 = k_2 = 1$.

In [ ]:

```
import control

# Matrix A
A1 = np.array([[0.0, -vd],
              [0.0, 0.0]])

# Matrix B
B1 = np.array([[0.0],
              [1.0]])

# Controllability
print ('---Controllability---')
print ('rank of ctrb(A,b):' , np.linalg.matrix_rank( control.ctrb( A1, B1
```

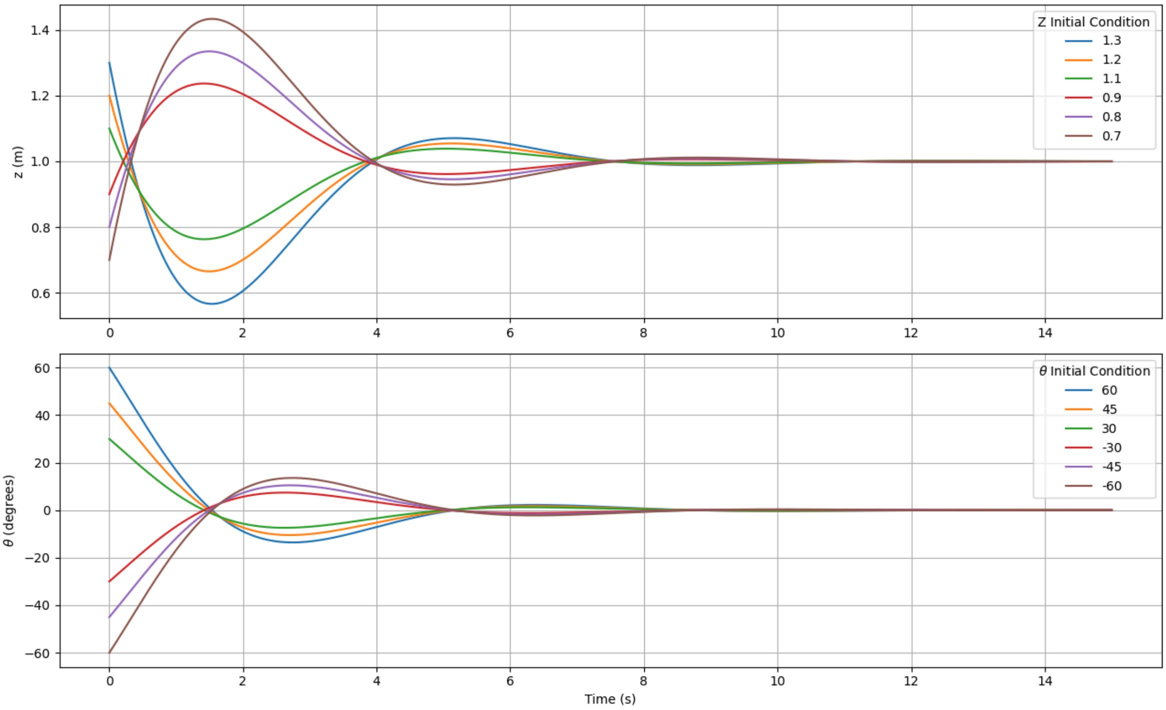
```
print ('Eigenvalues of A:', np.linalg.eig( A1 )[0])


# Pole Placement
K = control.place( A1, B1, [-1, -2] )
K = np.array([ [-k1], [k2]]).T
print ('\n---Pole Placement\nK=:', K)

# Verification of Eigen values of A-BK
print ('\n---Verification of Eigenvalues of A-BK---')
Acl = A1 - B1 @ K
#print(Acl)
eig_Acl, eig_vect = np.linalg.eig( Acl )
print ('Eigenvalues of A-BK:', eig_Acl)
```

```
---Controllability---
rank of ctrb(A,b): 2
Eigenvalues of A: [0. 0.]

---Pole Placement
K=: [[-1  1]]

---Verification of Eigenvalues of A-BK---
Eigenvalues of A-BK: [-0.5+0.8660254j -0.5-0.8660254j]
```

In [ ]:

```
import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt

# Linear System
def Sys_f_Linear(x, t=0):
    z = x[0]
    th = x[1]
    q = k1*(z-zd) - k2*th
    return np.array([-vd*th, q])

# Generate 1000 linearly spaced points for t
t_end = 15
t = np.linspace(0, t_end, 1000)


# Different initial conditions to test
initial_conditions = [
    np.array([1.3, np.pi/3]),
    np.array([1.2, np.pi/4]),
    np.array([1.1, np.pi/6]),
    np.array([0.9, -np.pi/6]),
    np.array([0.8, -np.pi/4]),
    np.array([0.7, -np.pi/3]),
    # Add more initial conditions as needed
]

# Plot
fig, axs = plt.subplots(2, 1, figsize=(13, 8))

for x0 in initial_conditions:
    x, infodict = integrate.odeint(Sys_f_Linear, x0, t, full_output=True)

    axs[0].plot(t, x[:, 0], label=f'{x0[0]}')
    axs[0].set_ylabel(r'z (m)')
```

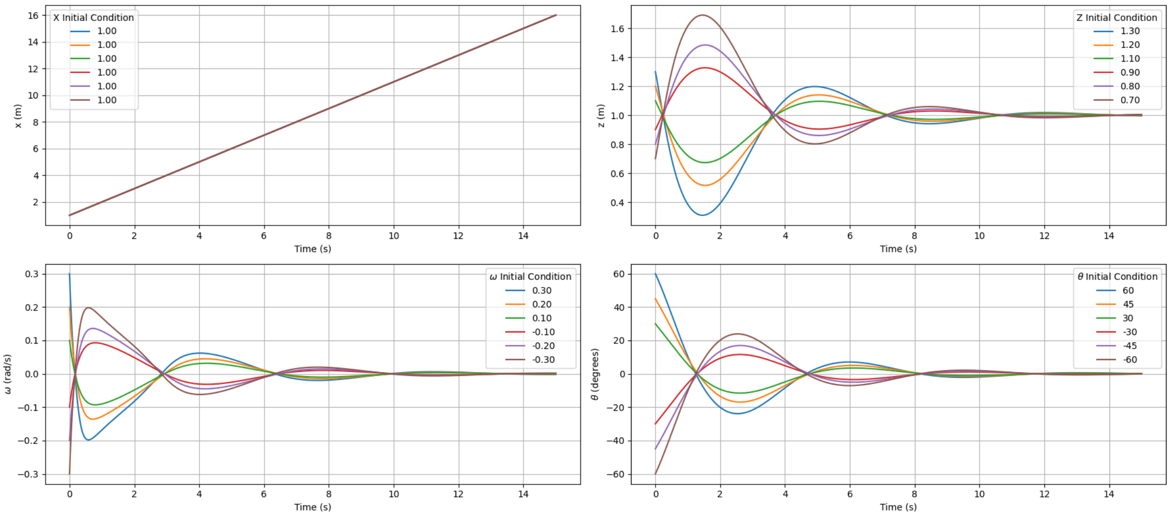
```
    axs[0].grid(True)
    axs[0].legend()

    axs[1].plot(t, x[:, 1]*180/np.pi, label=f'{round(x0[1]*180/np.pi)}')
    axs[1].set_ylabel(r'$\theta$ (degrees)')
    axs[1].grid(True)
    axs[1].legend()


axs[0].legend(loc='best', title=r'Z Initial Condition')
axs[1].legend(loc='best', title=r'$\theta$ Initial Condition')
plt.xlabel('Time (s)')
plt.tight_layout()
plt.show()
```



1.7 Consider now the *AUV model* (x, z, w, θ) in closed-loop with the control law (1). Plot the **time-evolution** of the state for different initial conditions.

In [ ]:
```
import numpy as np
from scipy import integrate
import matplotlib.pyplot as plt

# Constants
vd = 1
d1 = -3
d2 = -12
m = 0.9
k1 = 1
k2 = 1
zd = 1

# Nonlinear System
def Sys_f(x, t=0):
  z = x[1]
  w = x[2]
  th = x[3]
  q = k1*(z-zd) - k2*th
```

```
  dx1 = vd
  dx2 = -vd*np.tan(th) + w/np.cos(th)
  dx3 = d1*w + d2*w*np.abs(w) + m*(vd/np.cos(th) - w*np.tan(th))*q
  dx4 = q
  return np.array([ dx1, dx2, dx3, dx4
                  ])

# Generate 1000 linearly spaced points for t
t_end = 15
t = np.linspace(0, t_end, 1000)

# Initial conditions
initial_conditions = [
    np.array([vd, 1.3, 0.3, np.pi/3]),
    np.array([vd, 1.2, 0.2, np.pi/4]),
    np.array([vd, 1.1, 0.1, np.pi/6]),
    np.array([vd, 0.9, -0.1, -np.pi/6]),
    np.array([vd, 0.8, -0.2, -np.pi/4]),
    np.array([vd, 0.7, -0.3, -np.pi/3]),
    # Add more initial conditions as needed
]

# Integrate the system and plot
fig, axs = plt.subplots(2, 2, figsize=(18, 8))

for x0 in initial_conditions:
    x = integrate.odeint(Sys_f, x0, t, full_output=False)
    axs[0, 0].plot(t, x[:, 0], label=f'{x0[0]:.2f}')
    axs[0, 1].plot(t, x[:, 1], label=f'{x0[1]:.2f}')
    axs[1, 0].plot(t, x[:, 2], label=f'{x0[2]:.2f}')
    axs[1, 1].plot(t, x[:, 3]*180/np.pi, label=f'{x0[3]*180/np.pi:.0f}')

# Set labels and legends
for ax_row in axs:
    for ax in ax_row:
        ax.set_xlabel('Time (s)')
        ax.grid(True)
        ax.legend()

# Set y-axis labels for each column
axs[0, 0].set_ylabel(r'x (m)')
axs[0, 1].set_ylabel(r'z (m)')
axs[1, 0].set_ylabel(r"$\omega$ (rad/s)")
axs[1, 1].set_ylabel(r"$\theta$ (degrees)")

axs[0][0].legend(loc='best', title=r'X Initial Condition')
axs[0][1].legend(loc='best', title=r'Z Initial Condition')
axs[1][0].legend(loc='best', title=r'$\omega$ Initial Condition')
axs[1][1].legend(loc='best', title=r'$\theta$ Initial Condition')
plt.tight_layout()
plt.show()
```



Part 2: Control Design

2.1 Consider now the nonlinear subsystem $(\tilde{z}, \theta)$ with $\tilde{z} = z - z_d$, where z_d is a constant desired depth. Assuming q as input and $w = 0$, design a **Backstepping** Lyapunov based feedback law such that $z(t)$ converges to z_d as $t \to \infty$ and the tracking error system at the origin is AS.

To this end, in the first step of the methodology assume that the **virtual control signal** is $\tan(\theta)$ (and not θ).

$$X = (\tilde{z}, \theta)$$

$$\begin{cases} \tilde{z} = -v_d tan(\theta) \\ \theta = q \end{cases}$$

$$\eta = tan(\theta)$$

$$\dot{\eta} = \frac{1}{cos^2(\theta)}\theta = (1+\eta^2)\theta = (1+\eta^2)q$$

$$\begin{cases} \dot{z} = -v_d\eta \\ \dot{\eta} = (1+\eta^2)q \end{cases}$$

$$\tilde{z} = z - z_d$$

$$\tilde{\eta} = \eta - \phi_1 \iff \eta = \tilde{\eta} + \phi_1$$

$$V_1(\tilde{z}) = \frac{1}{2}\tilde{z}^2 \to V_1(\tilde{z}) = \tilde{z}\tilde{z} = \tilde{z}(-v_d\eta) = \tilde{z}(-\frac{v_d}{v_d}k_1\tilde{z}) = -k_1\tilde{z}^2 < 0$$

$$\phi_1 = \eta = \frac{1}{v_d}k_1\tilde{z}$$

$$\dot{\phi}_1 = \frac{k_1}{v_d}\tilde{z} = -\frac{k_1}{v_d}v_d\eta = -k_1\eta$$

$\dot{V}1(\tilde{z})$ → Negative definite

$$V_2(\tilde{z},\tilde{\eta}) = \frac{1}{2}\tilde{z} + \frac{1}{2}\tilde{\eta} \rightarrow \dot{V}_2(\tilde{z},\tilde{\eta}) = \tilde{z}\dot{\tilde{z}} + \tilde{\eta}\dot{\tilde{\eta}} = \tilde{z}(\dot{z} - \dot{z}_d) + \tilde{\eta}(\dot{\eta} - \dot{\phi}_1)$$

$$= \tilde{z}(-v_d\eta - 0) + \tilde{\eta}((1+\eta^2)q - \dot{\phi}_1)$$

$$= -v_d\tilde{z}(\tilde{\eta} + \phi_1) + \tilde{\eta}((1+\eta^2)q - \dot{\phi}_1)$$

$$= -v_d\tilde{z}(\tilde{\eta} + \frac{1}{v_d}k_1\tilde{z}) + \tilde{\eta}(1+\eta^2)q - \tilde{\eta}\dot{\phi}_1$$

$$= -v_d\tilde{z}\tilde{\eta} - k_1\tilde{z}^2 + \tilde{\eta}(1+\eta^2)q - \tilde{\eta}\dot{\phi}_1$$

$$q = \frac{1}{(1+\eta^2)}(v_d\tilde{z} - k_2\tilde{\eta} + \dot{\phi}_1)$$

$$\dot{V}_2(\tilde{z},\tilde{\eta}) = -k_1\tilde{z}^2 - k_2\tilde{\eta}^2 < 0$$

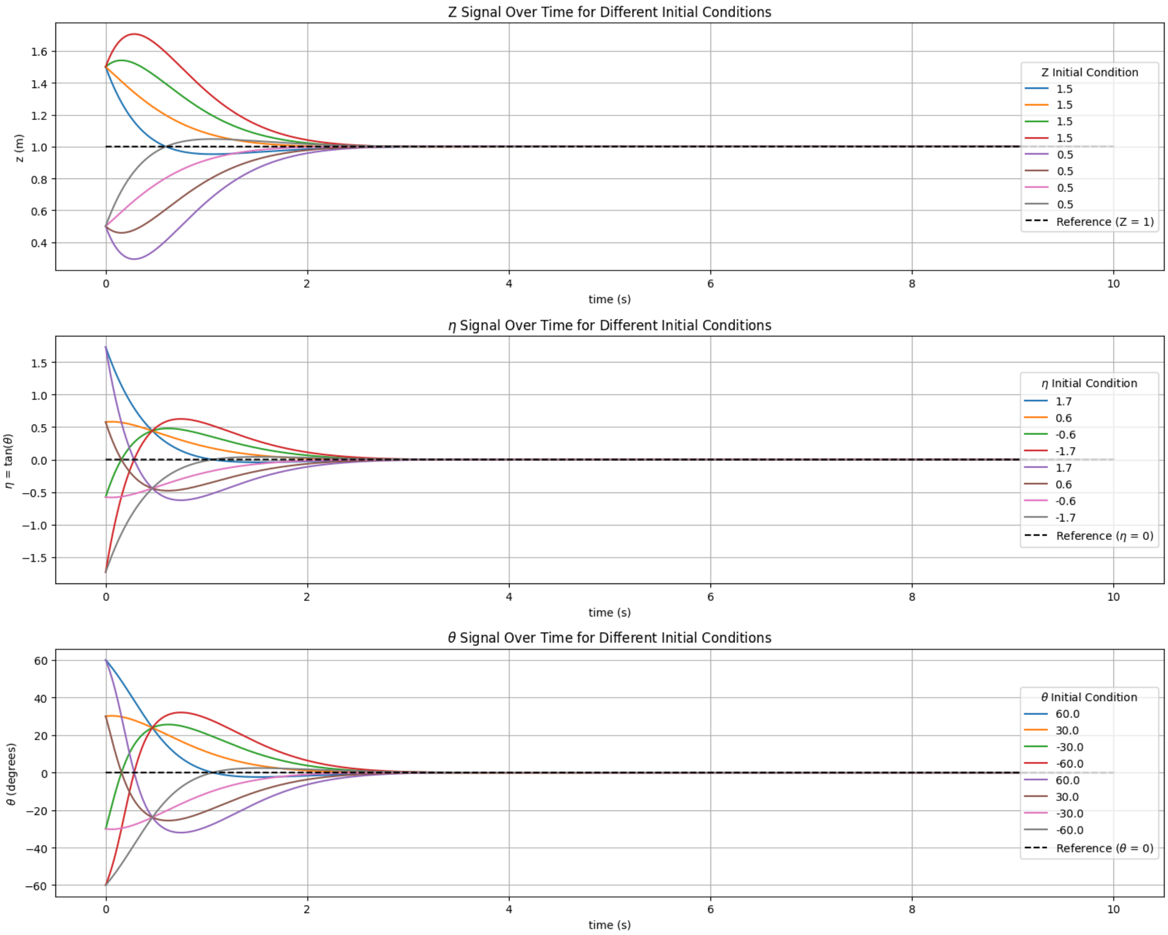
V̇2(z̃,η̃) → Negative definite

2.2 Confirm the results through simulation by plotting the **time-evolution** of the state $z(t)$, the tracking error $\tilde{z}(t)$, the pitch angle θ and the input signal q.

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt

# Define the step function for Euler Method
def step_f(z, eta, zd, vd, K1, K2, dt):
    # Calculate tilde_z
    tilde_z = z - zd
    # Calculate phi
    phi = K1 * tilde_z / vd
    # Calculate tilde_eta
    tilde_eta = eta - phi
    # Calculate derivative of phi
    dot_phi = - (K1 * eta)
    # Calculate control u
    u = (-K2 * tilde_eta + vd * tilde_z + dot_phi )/ (1 + eta**2)
    # Return updated state variables using Euler Method
    return z + (-vd * eta) * dt, eta + (1 + eta**2) * u * dt, np.arctan(e

# Function to simulate the system
def simulate_system(vd, K1, K2, zd, dt, t_end, z_initial_conditions, eta_
    # Create time samples
    t_signal = np.arange(0, t_end, dt)
    zd_signal = zd * np.ones_like(t_signal)
    z_signals = []
    eta_signals = []
    theta_signals = []
    # Iterate over initial conditions
    for z_initial in z_initial_conditions:
        for eta_initial in eta_initial_conditions:
            z_signal = np.zeros_like(t_signal)
            eta_signal = np.zeros_like(t_signal)
            theta_signal = np.zeros_like(t_signal)
            # Initialize state variables
            z_signal[0] = z_initial
            eta_signal[0] = eta_initial
            theta_signal[0] = np.arctan(eta_initial)
```

```
            # Iterate over time samples
            for i in range(t_signal.shape[0] - 1):
                # Update state variables using Euler Method
                z_signal[i+1], eta_signal[i+1], theta_signal[i+1]= step_f
            # Append new simulations
            z_signals.append(z_signal)
            eta_signals.append(eta_signal)
            theta_signals.append(theta_signal)
    return t_signal, z_signals, eta_signals, theta_signals, zd_signal

# Function to plot results
def plot_results(t_signal, z_signals, zd_signal, eta_signals, theta_signa
    # Create a figure
    fig, axs = plt.subplots(3, figsize=(15, 12))
    # Plot z and zd for different initial conditions
    for z_signal in z_signals:
        axs[0].plot(t_signal, z_signal, label=f'{round(z_signal[0],1)}')
    axs[0].plot(t_signal, zd_signal, 'k--', label=f'Reference (Z = {zd})'
    axs[0].set_xlabel("time (s)")
    axs[0].set_ylabel("z (m)")
    axs[0].set_title("Z Signal Over Time for Different Initial Conditions
    axs[0].grid()
    axs[0].legend(loc='right', title=r'Z Initial Condition')

    # Plot eta for different initial conditions
    for eta_signal in eta_signals:
        axs[1].plot(t_signal, eta_signal, label=f'{round(eta_signal[0],1)
    axs[1].plot(t_signal, t_signal*0, 'k--', label=r'Reference ($\eta$ =
    axs[1].set_xlabel("time (s)")
    axs[1].set_ylabel(r"$\eta$ = tan($\theta$)")
    axs[1].set_title(r"$\eta$ Signal Over Time for Different Initial Cond
    axs[1].grid()
    axs[1].legend(loc='right', title=r'$\eta$ Initial Condition')

    # Plot theta for different initial conditions
    for theta_signal in theta_signals:
        axs[2].plot(t_signal, theta_signal * 180 / np.pi, label=f'{round(
    axs[2].plot(t_signal, t_signal*0, 'k--', label=r'Reference ($\theta$
    axs[2].set_xlabel("time (s)")
    axs[2].set_ylabel(r"$\theta$ (degrees)")
    axs[2].set_title(r"$\theta$ Signal Over Time for Different Initial Co
    axs[2].grid()
    axs[2].legend(loc='right', title=r'$\theta$ Initial Condition')

    plt.tight_layout()
    plt.show()

# Parameters
dt = 0.001
t_end = 10
vd = 1
zd = 1
K1 = 2
K2 = 2
z_initial_conditions = [1.5, 0.5]  # Initial conditions for z
eta_initial_conditions = [np.tan(np.pi/3),   # Initial conditions for eta
                          np.tan(np.pi/6),
                          np.tan(-np.pi/6),
                          np.tan(-np.pi/3)]
```

```
# Simulate and plot
t_signal, z_signals, eta_signals, theta_signals, zd_signal = simulate_sys
plot_results(t_signal, z_signals, zd_signal, eta_signals, theta_signals)
```

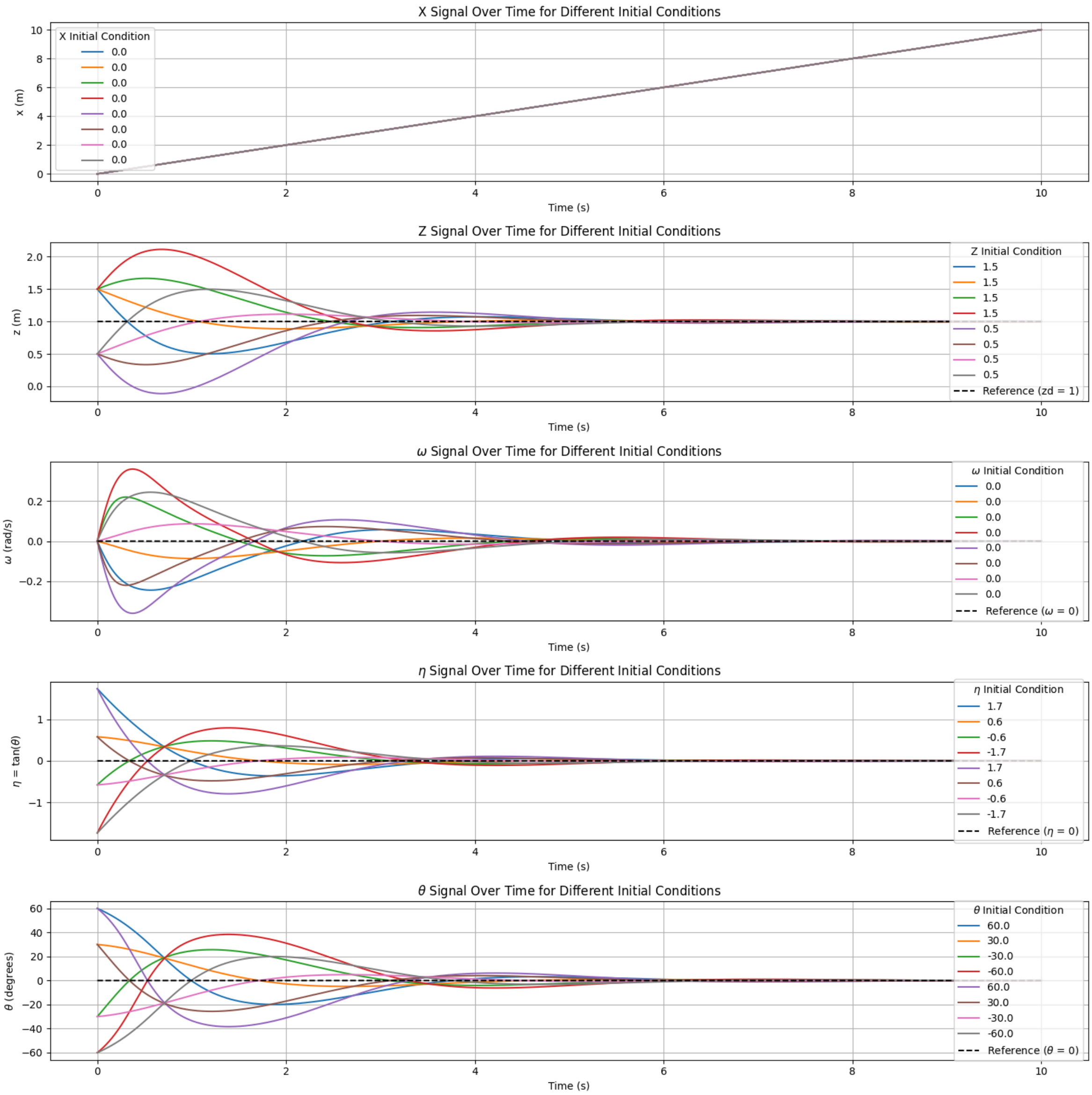


2.3 Consider now the $AUV\ model$ (x, z, w, θ) in closed-loop with the backstepping control law. Plot the **time-evolution** of the state for different initial conditions.

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt

# Define the step function for Euler Method
def step_f(x, z, w, eta, zd, m, vd, K1, K2, dt):
    d1 = -3
    d2 = -12
    m = 0.9
    # Calculate tilde_z
    tilde_z = z - zd
    # Calculate phi
    phi = K1 * tilde_z / vd
    # Calculate tilde_eta
    tilde_eta = eta - phi
    # Calculate derivative of phi
    dot_phi = - (K1 * eta)
    # Calculate control u
    u = (-K2 * tilde_eta + vd * tilde_z + dot_phi )/ (1 + eta**2)
    # Calculate Theta
    Theta = np.arctan(eta)
    # Return updated state variables using Euler Method
    return x + vd * dt, z + (-vd * np.tan(Theta) + w/np.cos(Theta)) * dt,

# Function to simulate the system
```

```
def simulate_system(m, vd, K1, K2, zd, dt, t_end, z_initial_conditions, e
    # Create time samples
    t_signal = np.arange(0, t_end, dt)
    zd_signal = zd * np.ones_like(t_signal)
    x_signals, z_signals, w_signals, eta_signals, theta_signals = [], [],

    # Iterate over initial conditions
    for z_initial in z_initial_conditions:
        for eta_initial in eta_initial_conditions:
            x_signal = np.zeros_like(t_signal)
            z_signal = np.zeros_like(t_signal)
            w_signal = np.zeros_like(t_signal)
            eta_signal = np.zeros_like(t_signal)
            theta_signal = np.zeros_like(t_signal)
            # Initialize state variables
            z_signal[0] = z_initial
            eta_signal[0] = eta_initial
            theta_signal[0] = np.arctan(eta_initial)
            # Iterate over time samples
            for i in range(t_signal.shape[0] - 1):
                # Update state variables using Euler Method
                x_signal[i + 1], z_signal[i + 1], w_signal[i + 1], eta_si
                    x_signal[i], z_signal[i], w_signal[i], eta_signal[i],
            # Append new simulations
            x_signals.append(x_signal)
            z_signals.append(z_signal)
            eta_signals.append(eta_signal)
            w_signals.append(w_signal)
            theta_signals.append(theta_signal)

    return t_signal, x_signals, z_signals, w_signals, eta_signals, theta_

# Function to plot results
def plot_results(t_signal, x_signals, z_signals, zd_signal, w_signals, et
    # Create a figure
    fig, axs = plt.subplots(5, figsize=(15, 15))

    # Plot z and zd for different initial conditions
    for z_signal in z_signals:
        axs[1].plot(t_signal, z_signal, label=f'{round(z_signal[0],1)}')
    axs[1].plot(t_signal, zd_signal, 'k--', label=f'Reference (zd = {zd})
    axs[1].set_xlabel("Time (s)")
    axs[1].set_ylabel("z (m)")
    axs[1].set_title("Z Signal Over Time for Different Initial Conditions
    axs[1].legend(loc='right', title=r'Z Initial Condition')
    axs[1].grid()

    # Plot eta for different initial conditions
    for eta_signal in eta_signals:
        axs[3].plot(t_signal, eta_signal, label=f'{round(eta_signal[0],1)
    axs[3].plot(t_signal, np.zeros_like(t_signal), 'k--', label=r'Referen
    axs[3].set_xlabel("Time (s)")
    axs[3].set_ylabel(r"$\eta$ = tan($\theta$)")
    axs[3].set_title(r"$\eta$ Signal Over Time for Different Initial Cond
    axs[3].legend(loc='right', title=r'$\eta$ Initial Condition')
    axs[3].grid()

    # Plot theta for different initial conditions
    for theta_signal in theta_signals:
        axs[4].plot(t_signal, theta_signal * 180 / np.pi, label=f'{round(
```

```
    axs[4].plot(t_signal, np.zeros_like(t_signal), 'k--', label=r'Referen
    axs[4].set_xlabel("Time (s)")
    axs[4].set_ylabel(r"$\theta$ (degrees)")
    axs[4].set_title(r"$\theta$ Signal Over Time for Different Initial Co
    axs[4].legend(loc='right', title=r'$\theta$ Initial Condition')
    axs[4].grid()

    # Plot x for different initial conditions
    for x_signal in x_signals:
        axs[0].plot(t_signal, x_signal, label=f'{round(x_signal[0], 1)}')
    axs[0].set_xlabel("Time (s)")
    axs[0].set_ylabel("x (m)")
    axs[0].set_title("X Signal Over Time for Different Initial Conditions
    axs[0].legend(loc='best', title=r'X Initial Condition')
    axs[0].grid()

    # Plot w for different initial conditions
    for w_signal in w_signals:
        axs[2].plot(t_signal, w_signal, label=f'{round(w_signal[0], 1)}')
    axs[2].plot(t_signal, np.zeros_like(t_signal), 'k--', label=r'Referen
    axs[2].set_xlabel("Time (s)")
    axs[2].set_ylabel(r"$\omega$ (rad/s)")
    axs[2].set_title(r"$\omega$ Signal Over Time for Different Initial Co
    axs[2].legend(loc='right', title=r'$\omega$ Initial Condition')
    axs[2].grid()

    plt.tight_layout()
    plt.show()

# Parameters
dt = 0.001
t_end = 10
vd = 1
m = 0.9
zd = 1
K1 = 1
K2 = 1
z_initial_conditions = [1.5, 0.5]  # Initial conditions for z
eta_initial_conditions = [np.tan(np.pi/3),   # Initial conditions for eta
                          np.tan(np.pi/6),
                          np.tan(-np.pi/6),
                          np.tan(-np.pi/3)]

# Simulate and plot
t_signal, x_signals, z_signals, w_signals, eta_signals, theta_signals, zd
plot_results(t_signal, x_signals, z_signals, zd_signal, w_signals, eta_si
```



2.4 For the same conditions stated in **2.1**, design a **sliding mode controller** and confirm the results through simulation. For the sliding surface use

$$s = \dot{\tilde{z}} + \lambda \tilde{z}, \quad \lambda > 0$$

Sliding mode demonstration:

$$\begin{cases} \dot{z} = -v_d tan(\theta) \\ \ddot{z} = -v_d \frac{d}{dt}(tan(\theta)) = -\frac{v d\theta}{cos^2(\theta)} \\ \theta = q \end{cases}$$

Having:

$$\tilde{z} = z - z_d$$

And:

$$\dot{\tilde{z}} = \dot{z} - \dot{z}_d = \dot{z}$$
$$\ddot{\tilde{z}} = \ddot{z} - \ddot{z}_d = \ddot{z}$$

Using the Sliding surface we get:

$$\ddot{s} = \dddot{\tilde{z}} + \lambda\dot{\tilde{z}} = \ddot{z} + \lambda\dot{z}$$

And choosing the following V(s):

$$V(s) = \frac{1}{2}s^2$$

Then we get:

$$\dot{V} = s\dot{s} = s(\ddot{z} + \lambda\dot{z}) = s(-\frac{vdq}{cos^2(\theta)} - \lambda vdtan(\theta))$$

$$\iff q = -\frac{cos^2(\theta)}{vd}(\lambda vdtan(\theta) + \mu 1)$$

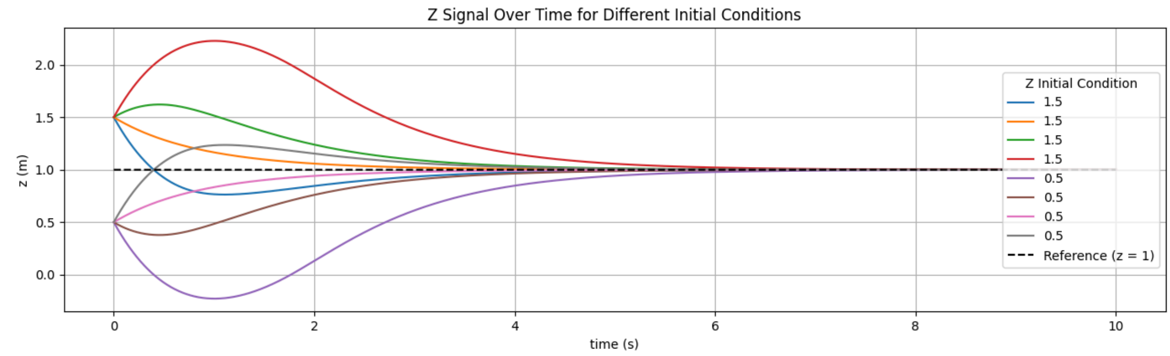
where:

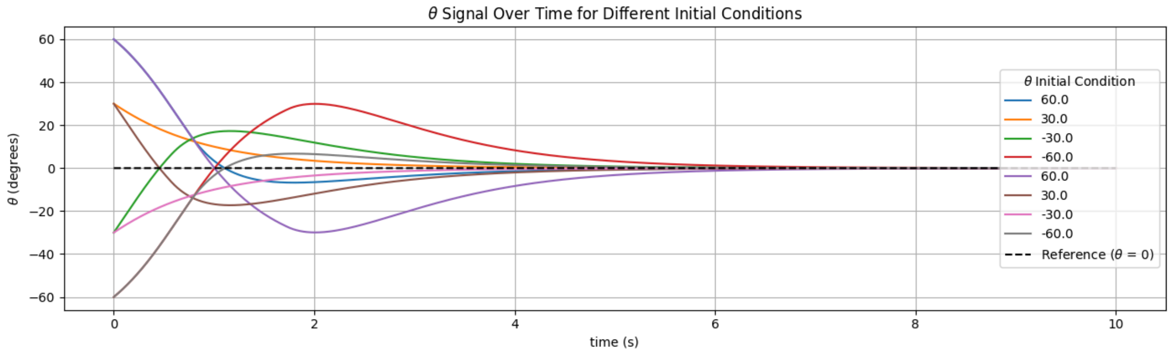
$$\mu 1 = -ksat(s/\epsilon)$$

In [ ]:

```
import numpy as np
import matplotlib.pyplot as plt


# Define the step function for Euler Method
def step_f(z, theta, zd, vd, LAMBDA, K, dt):
    epsilon = 0.5
    # Calculate tilde_z
    tilde_z = z - zd
    # Calculate tilde_z first derivative
    tilde_z_dev = - vd * np.tan(theta) - 0
    # Calculate Sliding Surface
    s = tilde_z_dev + LAMBDA * tilde_z
    # Calculate u1 Switching Control
    if abs(s) < epsilon:
        u1 = -K * s / epsilon
    else:
        u1 = -K * np.sign(s / epsilon)
    # Calculate control u
    u = -(np.cos(theta) ** 2 / vd) * (LAMBDA * vd * np.tan(theta) + u1)
    # Return updated state variables using Euler Method
    return z + (-vd * np.tan(theta)) * dt, theta + u * dt

# Function to simulate the system
def simulate_system(vd, LAMBDA, k, zd, dt, t_end, z_initial_conditions, t
    # Create time samples
    t_signal = np.arange(0, t_end, dt)
    zd_signal = zd * np.ones_like(t_signal)
    z_signals = []
    theta_signals = []
    # Iterate over initial conditions
    for z_initial in z_initial_conditions:
        for theta_initial in theta_initial_conditions:
            z_signal = np.zeros_like(t_signal)
            theta_signal = np.zeros_like(t_signal)
            # Initialize state variables
            z_signal[0] = z_initial
```

```
            theta_signal[0] = theta_initial
            # Iterate over time samples
            for i in range(t_signal.shape[0] - 1):
                # Update state variables using Euler Method
                z_signal[i + 1], theta_signal[i + 1] = step_f(z_signal[i]
            # Append new simulations
            z_signals.append(z_signal)
            theta_signals.append(theta_signal)
    return t_signal, z_signals, theta_signals, zd_signal

# Function to plot results
def plot_results(t_signal, z_signals, zd_signal, theta_signals):
    # Create a figure
    fig, axs = plt.subplots(2, figsize=(13, 8))
    # Plot z and zd for different initial conditions
    for z_signal in z_signals:
        axs[0].plot(t_signal, z_signal, label=f'{z_signal[0]}' )
    axs[0].plot(t_signal, zd_signal, 'k--', label=f"Reference (z = {zd})"
    axs[0].set_xlabel("time (s)")
    axs[0].set_ylabel("z (m)")
    axs[0].set_title("Z Signal Over Time for Different Initial Conditions
    axs[0].grid()
    axs[0].legend(loc='right', title=r'Z Initial Condition')

    # Plot theta for different initial conditions
    for theta_signal in theta_signals:
        axs[1].plot(t_signal, theta_signal * 180 / np.pi, label=f'{round(
    axs[1].plot(t_signal, t_signal * 0, 'k--', label=r"Reference ($\theta
    axs[1].set_xlabel("time (s)")
    axs[1].set_ylabel(r"$\theta$ (degrees)")
    axs[1].set_title(r"$\theta$ Signal Over Time for Different Initial Co
    axs[1].grid()
    axs[1].legend(loc='right', title=r'$\theta$ Initial Condition')

    plt.tight_layout()
    plt.show()

# Parameters
dt = 0.001
t_end = 10
vd = 1
zd = 1
LAMBDA = 1
K = 1
z_initial_conditions = [1.5, 0.5]  # Initial conditions for z
theta_initial_conditions = [np.pi / 3,
                            np.pi / 6,
                            -np.pi / 6,
                            -np.pi / 3]  # Initial conditions for theta

# Simulate and plot
t_signal, z_signals, theta_signals, zd_signal = simulate_system(vd, LAMBD
plot_results(t_signal, z_signals, zd_signal, theta_signals)
```



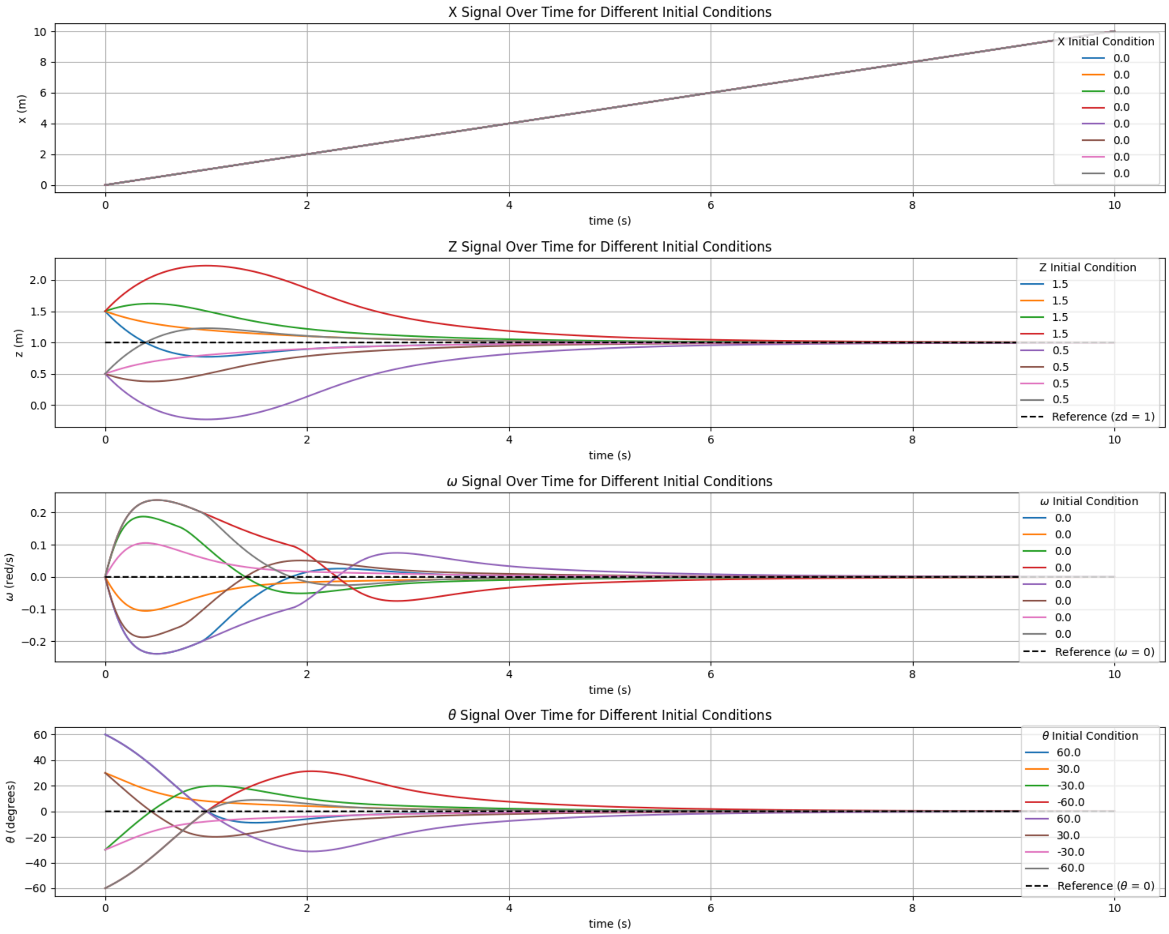


In [ ]:

```
import numpy as np
import matplotlib.pyplot as plt

# Define the step function for Euler Method
def step_f(x, z, w, theta, zd, vd, LAMBDA, K, dt):
    d1 = -3
    d2 = -12
    m = 0.9
    epsilon = 0.5
    # Calculate tilde_z
    tilde_z = z - zd
    # Calculate tilde_z first derivative
    tilde_z_dev = -vd * np.tan(theta) + w/np.cos(theta)
    # Calculate Sliding Surface
    s = tilde_z_dev + LAMBDA * tilde_z
    # Calculate u1 Switching Control
    if abs(s) < epsilon:
        u1 = -K * s / epsilon
    else:
        u1 = -K * np.sign(s / epsilon)
    # Calculate control u
    u = -(np.cos(theta) ** 2 / vd) * (LAMBDA * vd * np.tan(theta) + u1)
    # Return updated state variables using Euler Method
    x_new = x + vd * dt
    z_new = z + (-vd * np.tan(theta)) * dt
    w_new = w + (d1 * w + d2 * w * abs(w) + (m * u) * (vd/np.cos(theta) -
    theta_new = theta + u * dt
    return x_new, z_new, w_new, theta_new

# Function to simulate the system
def simulate_system(vd, LAMBDA, K, zd, dt, t_end, z_initial_conditions, t
    # Create time samples
    t_signal = np.arange(0, t_end, dt)
    zd_signal = zd * np.ones_like(t_signal)
    x_signals = []
    z_signals = []
    w_signals = []
    theta_signals = []
```

```
    # Iterate over initial conditions
    for z_initial in z_initial_conditions:
        for theta_initial in theta_initial_conditions:
            x_signal = np.zeros_like(t_signal)
            z_signal = np.zeros_like(t_signal)
            w_signal = np.zeros_like(t_signal)
            theta_signal = np.zeros_like(t_signal)
            # Initialize state variables
            x_signal[0] = 0  # Assume x initial condition as 1
            z_signal[0] = z_initial
            w_signal[0] = 0  # Assume w initial condition as 1
            theta_signal[0] = theta_initial
            # Iterate over time samples
            for i in range(t_signal.shape[0] - 1):
                # Update state variables using Euler Method
                x_signal[i + 1], z_signal[i + 1], w_signal[i + 1], theta_
                    x_signal[i], z_signal[i], w_signal[i], theta_signal[i
                )
            # Append new simulations
            x_signals.append(x_signal)
            z_signals.append(z_signal)
            w_signals.append(w_signal)
            theta_signals.append(theta_signal)
    return t_signal, x_signals, z_signals, w_signals, theta_signals, zd_s

# Function to plot results
def plot_results(t_signal, x_signals, z_signals, zd_signal, w_signals, th
    # Create a figure
    fig, axs = plt.subplots(4, figsize=(15, 12))

    # Plot x for different initial conditions
    for x_signal in x_signals:
        axs[0].plot(t_signal, x_signal, label=f'{x_signal[0]}')
    axs[0].set_xlabel("time (s)")
    axs[0].set_ylabel("x (m)")
    axs[0].set_title("X Signal Over Time for Different Initial Conditions
    axs[0].grid()
    axs[0].legend(loc='right', title=r'X Initial Condition')

    # Plot z and zd for different initial conditions
    for z_signal in z_signals:
        axs[1].plot(t_signal, z_signal, label=f'{z_signal[0]}')
    axs[1].plot(t_signal, zd_signal, 'k--', label=f"Reference (zd = {zd})
    axs[1].set_xlabel("time (s)")
    axs[1].set_ylabel("z (m)")
    axs[1].set_title("Z Signal Over Time for Different Initial Conditions
    axs[1].grid()
    axs[1].legend(loc='right', title=r'Z Initial Condition')

    # Plot w for different initial conditions
    for w_signal in w_signals:
        axs[2].plot(t_signal, w_signal, label=f'{w_signal[0]}')
    axs[2].plot(t_signal, t_signal * 0, 'k--', label=r"Reference ($\omega
    axs[2].set_xlabel("time (s)")
    axs[2].set_ylabel(r"$\omega$ (red/s)")
    axs[2].set_title(r"$\omega$ Signal Over Time for Different Initial Co
    axs[2].grid()
    axs[2].legend(loc='right', title=r'$\omega$ Initial Condition')

    # Plot theta for different initial conditions
```

```
    for theta_signal in theta_signals:
        axs[3].plot(t_signal, theta_signal * 180 / np.pi, label=f'{round(
    axs[3].plot(t_signal, t_signal * 0, 'k--', label=r"Reference ($\theta
    axs[3].set_xlabel("time (s)")
    axs[3].set_ylabel(r"$\theta$ (degrees)")
    axs[3].set_title(r"$\theta$ Signal Over Time for Different Initial Co
    axs[3].grid()
    axs[3].legend(loc='right', title=r'$\theta$ Initial Condition')

    plt.tight_layout()
    plt.show()

# Parameters
dt = 0.001
t_end = 10
vd = 1
zd = 1
LAMBDA = 1
K = 1
z_initial_conditions = [1.5, 0.5]  # Initial conditions for z
theta_initial_conditions = [np.pi / 3, np.pi / 6, -np.pi / 6, -np.pi / 3]

# Simulate and plot
t_signal, x_signals, z_signals, w_signals, theta_signals, zd_signal = sim
plot_results(t_signal, x_signals, z_signals, zd_signal, w_signals, theta_
```



2.5 (Extra) For the linearized system in **1.2**, design a **LQR controller** and confirm the results through simulation.

Note: This question is optional. If you solve it, you get extra 15 points (in 100).

The linearized system is given by:

$$\dot{X} = AX + BX$$

Where:

$$A = \begin{bmatrix} 0 & 1 & -v_d \\ 0 & d_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \tilde{m}v_d \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} \tilde{z} \\ \omega \\ \theta \end{bmatrix}$$

First, we need to choose the (Q) and (R) matrices.

For (Q), we have:

$$Q = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix}$$

By choosing appropriate values in the diagonal based on the importance of each state variable in our control objective. Higher values indicate a higher cost for deviations in that state.

For (R), it's a single scalar since it's a one-dimensional input system:

$$R = r$$

Similarly, by choosing an appropriate value for (r) based on the desired control effort. A higher value will penalize larger control inputs more heavily.

Now, we need to solve the Continuous Algebraic Riccati Equation to find the optimal gain matrix (K):

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

where:

- (A) is the system matrix,
- (B) is the input matrix,
- (Q) is the state cost matrix,
- (R) is the control cost matrix, and
- (P) is the solution matrix.

The optimal gain matrix (K) is then given by:

$$K = R^{-1}B^T P$$

Finally, we can use the control law (u = -Kx) in our control system to stabilize the linearized system.

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.linalg import solve_continuous_are, inv

# Define the closed-loop step function using Euler Method
def step_f_lqr(x, A_cl, dt):
    return x + np.dot(A_cl, x) * dt

# Function to simulate the closed-loop system
def simulate_system_lqr(A_cl, dt, t_end, x_initial_conditions):
    t_signal = np.arange(0, t_end, dt)
    x_signals = []

    for x_initial in x_initial_conditions:
        x_signal = np.zeros((len(t_signal), len(x_initial)))
        x_signal[0] = x_initial

        for i in range(len(t_signal) - 1):
            x_signal[i+1] = step_f_lqr(x_signal[i], A_cl, dt)

        x_signals.append(x_signal)

    return t_signal, x_signals

# Function to plot results with specific state names
def plot_results(t_signal, x_signals):
    # Define state names
    state_names = ['z (m)', r'$\omega$ (rad/s)', r'$\theta$ (degrees)']
    names = ['Z', r'$\omega$', r'$\theta$']

    # Create a figure
    fig, axs = plt.subplots(3, figsize=(15, 10))

    # Plot each state
    for idx, state_name in enumerate(state_names):
        for x_signal in x_signals:
            if idx == 2:  # Convert theta (state 3) from radians to degre
                axs[idx].plot(t_signal, (x_signal[:, idx]) * 180 / np.pi,
            else:
                axs[idx].plot(t_signal, x_signal[:, idx], label=f'{x_sign
        axs[idx].plot(t_signal, t_signal*0, 'k--', label=f'Reference ({na
        axs[idx].set_xlabel("time (s)")
        axs[idx].set_ylabel(state_name)
        axs[idx].set_title(f"{names[idx]} Over Time for Different Initial
        axs[idx].legend(loc='right', title=f'{names[idx]} Initial Conditi
        axs[idx].grid()

    plt.tight_layout()
    plt.show()
```

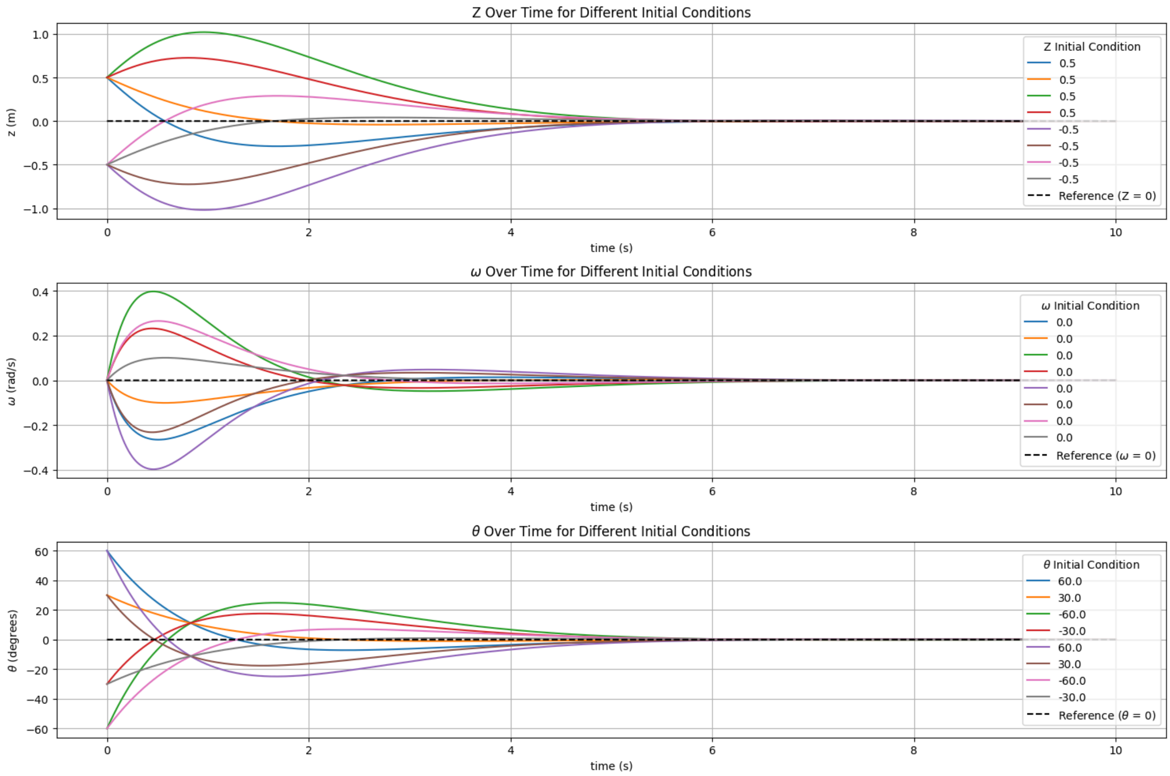
In [ ]:
```
# System parameters
vd = 1
d1 = -3
m = 0.9
zd = 0
```

```python
# Define system matrices
A = np.array([[0, 1, -vd],
              [0, d1, 0],
              [0, 0, 0]])

B = np.array([[0],
              [m * vd],
              [1]])

# Define weighting matrices
Q = np.array([[1, 0, 0],
              [0, 1, 0],
              [0, 0, 1]]) * 0.01

R = np.array([[1]]) * 0.01

# Solve Riccati equation to get P
P = solve_continuous_are(A, B, Q, R)

# Compute the LQR gain matrix K
K = np.dot(inv(R), np.dot(B.T, P))

print("Gain matrix K:")
print(K)
```

```
Gain matrix K:
[[-1.         -0.23303826  2.01760794]]
```

In [ ]:

```python
# Compute closed-loop system matrix A_cl
A_cl = A - np.dot(B, K)

# Parameters for simulation
dt = 0.001
t_end = 10
x_initial_conditions = [
    np.array([0.5, 0.0, np.pi/3]),
    np.array([0.5, 0.0, np.pi/6]),
    np.array([0.5, 0.0, -np.pi/3]),
    np.array([0.5, 0.0, -np.pi/6]),
    np.array([-0.5, 0.0, np.pi/3]),
    np.array([-0.5, 0.0, np.pi/6]),
    np.array([-0.5, 0.0, -np.pi/3]),
    np.array([-0.5, 0.0, -np.pi/6]),
]

# Simulate and plot
t_signal, x_signals = simulate_system_lqr(A_cl, dt, t_end, x_initial_cond
plot_results(t_signal, x_signals)
```


Z Over Time for Different Initial Conditions
z (m)
time (s)
Z Initial Condition
0.5
0.5
0.5
0.5
-0.5
-0.5
-0.5
-0.5
Reference (Z = 0)
ω Over Time for Different Initial Conditions
ω (rad/s)
time (s)
ω Initial Condition
0.0
0.0
0.0
0.0
0.0
0.0
0.0
0.0
Reference (ω = 0)
θ Over Time for Different Initial Conditions
θ (degrees)
time (s)
θ Initial Condition
60.0
30.0
-60.0
-30.0
60.0
30.0
-60.0
-30.0
Reference (θ = 0)