1 Consider the process $x_{k+1} = 0.5x_k + w_k$, where $\{w_k\}$ is a white gaussian sequence with zero mean and variance σ^2, and x_0 is a null constant.

(a) Obtain the equation for the evolution of the variance of x_k.

(b) Determine the steady state variance.

(c) Use MATLAB/Python to obtain realizations of the process x_k when $\sigma^2 = 1$.

2 Consider the process $x_{k+2} = \frac{a}{2}x_k + \frac{a}{2}x_{k+1} + (1-a)w_k$, where $a \neq 1$ is constant and $\{w_k\}$ is a white gaussian sequence with zero mean and variance σ^2, and x_0 and x_1 are null constants.

(a) Rewrite the process in a vector form, taking $X_k = [x_k \quad x_{k+1}]^T$.

(b) Conclude that X_k has zero mean.

(c) Obtain the equation for the evolution of the covariance of X_k.

(d) Determine the steady state covariance matrix.

(e) Use MATLAB/Python to obtain realizations of the process X_k for $a = 0.1$ and for $a = 0.95$, when $\sigma^2 = 0.01$. Compare in each case the corresponding realizations of x_k and w_k.

3 Consider the process defined in the previous problem, with $a = 0.9$.

(a) Compute the covariance matrices of X_{50}, X_{100}, X_{500}. Also obtain the steady state covariance.

(b) Implement a Monte Carlo simulation for the evolution of the process.

(c) Perform a set of simulations to estimate the covariance matrices of X_{50}, X_{100}, X_{500}. Consider different numbers of runs and present the results.

4 Consider the linear motion of a point mass given by $\ddot{x} = a$

(a) Taking the position x and the velocity $\dot{x}$ of the point mass as state variables and the acceleration a as the input, obtain a state space model.

(b) Assume the acceleration is constant for intervals of width T. Obtain a discrete time equivalent model, taking T as the sampling period.

(c) Assume now that the acceleration measurement in each interval is corrupted by an error that follows a normal distribution with zero mean and variance σ_a^2, and that such errors for different intervals are independent.

i. Obtain a linear model in discrete time with a random input that describes such behavior.
ii. Obtain the equations that give the mean and the covariance of the state.
iii. Taking $T = 0.1$, $a_k = 0$ and $\sigma_a^2 = 0.01$ and assuming the system in initially at rest, use MATLAB/Python to obtain several realizations of the system evolution. Use Monte Carlo simulation estimate the expected value and covariance of x_{100}. Compare the results with the the exact value of such covariance.