Construction and implementation of multivariate dispersion models

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Introduction
Motivation

- Generalized linear models: usual in statistical modelling;
  $\longrightarrow$ *mostly univariate cases*.
- There is no analogous multivariate framework for GLM.
- Most multivariate techniques are based on the Multivariate Normal distribution;
  - Suitable only for continuous and symmetrical data.

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- Statistical models are realistic when can describe the dependency structure, when it exists:
  - Temporal;
  - Spatial;
  - Spatio-temporal;
  - Genetic;
  - Longitudinal and repeated measures.
- We can be interested in more than one response variable, possibly correlated.

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The main goals of this work are:

- To build probability distributions for multivariate, non-normal random variables;
  - discrete, strong asymmetrical and heavy tailed data.
- Multivariate regression models;
- Implement the models in R.

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The Normal distribution is expressed by

$$p(y; \mu, \sigma^2) = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}. \qquad (1)$$

where μ is a location parameter and σ^2 a dispersion parameter. This can be generalized as **dispersion model**

$$p(y; \mu, \sigma^2) = a(y; \sigma^2) \exp\left\{-\frac{1}{2\sigma^2} d(y; \mu)\right\}, \quad y \in C, \qquad (2)$$

where $a \geq 0$ is an adequate function, C is the smallest interval containing the realizable values of y, d is a unit deviance in $C \times \Omega$, $\mu \in \Omega$ and $\sigma^2 \in \Re_{+}$.

Materials & Methods

Plots

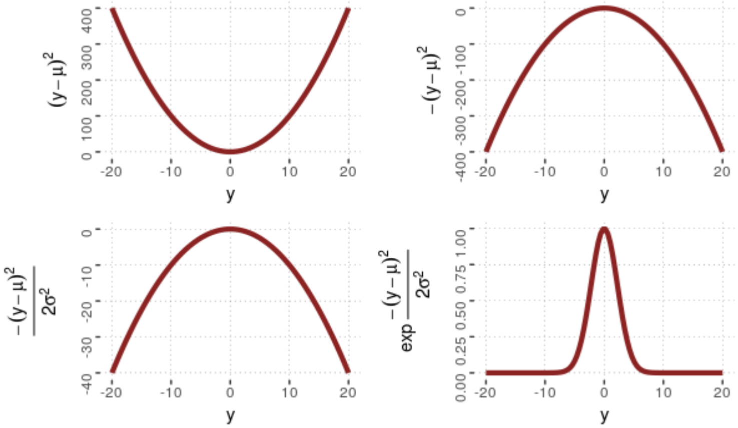

Figure: Core of a normal distribution.

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In 1987, JØRGENSEN introduced the theory of the dispersion models, that are based on deviance residuals.

- A function is called a unit deviance if it satisfies:

$$d(y; y) = 0 \quad \forall \ y \in \Omega \tag{3}$$

$$d(y; \mu) > 0 \quad \forall \ y \neq \mu. \tag{4}$$

Being Ω the parametric space for μ, $\Omega \subseteq \Re$. On a log-likelihood "point-of-view", the deviance can be obtained as:

$$d(y; \mu) = c\{l(y; y) - l(y; \mu)\} \tag{5}$$

for a constant c, given that (3) and (4) are satisfied.

Distribution	Deviance	C	Ω
Binomial	$2\left\{ylog\frac{y}{\mu} + (n-y)log\frac{n-y}{n-\mu}\right\}$	{0,1...n}	(0, 1)
Poisson	$2\left(ylog\frac{y}{\mu} - y + \mu\right)$	{0,1...}	$(0, \infty)$
Gamma	$2\left(\frac{y}{\mu} - log\frac{y}{\mu} - 1\right)$	$(0, \infty)$	$(0, \infty)$
Inverse Normal	$(y-\mu)^2/y\mu^2$	$(0, \infty)$	$(0, \infty)$

Table: Unit deviances.

Materials & Methods

Multivariate Dispersion Models

The multivariate extension of the dispersion model was proposed by JØRGENSEN; LAURITZEN, in 2000

$$p(\boldsymbol{y}; \boldsymbol{\mu}, \Sigma) = a(\boldsymbol{y}; \Sigma) \exp\left\{-\frac{1}{2} t(\boldsymbol{y}; \boldsymbol{\mu})^\top \Sigma^{-1} t(\boldsymbol{y}; \boldsymbol{\mu})\right\}, \qquad (6)$$

where $\boldsymbol{\mu} \in \Omega$ is a open interval in $\Re^p$, Σ is a positive-definite symmetric matrix $p \times p$, and $t(\boldsymbol{y}; \boldsymbol{\mu})$ is a vector of deviance residuals, given by

$$t(\boldsymbol{y}; \boldsymbol{\mu}) = sign(\boldsymbol{y} - \boldsymbol{\mu})\sqrt{d(\boldsymbol{y}; \boldsymbol{\mu})},$$

and $t(\boldsymbol{\mu}; \boldsymbol{\mu}) = \mathbf{0}$, for $\boldsymbol{\mu} \in \Omega$.

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Challenges

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Obtain the normalizing constant a($\boldsymbol{y}$;Σ)

- It can involve integrals of dimension p or infinite sums.

Possible approaches:

- Edgeworth and *saddle-point* (BARNDORFF-NIELSEN; COX);
- Laplace approximation (TIERNEY; KASS; KADANE);
- Numerical integration.

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Computational Implementation

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- Software *R* (R Core Team, 2018)

Results
Overall

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Main results, so far:

- Construction of non-normalized distributions.
- Characterizing the probability distributions.
- Parameter interpretation.

Results

Discrete Cases - Binomial

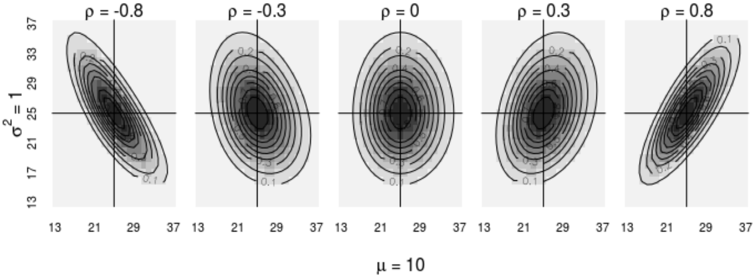

Figure: Core of the non-normalized bivariate Binomial distribution.

Results
Discrete Cases - Poisson

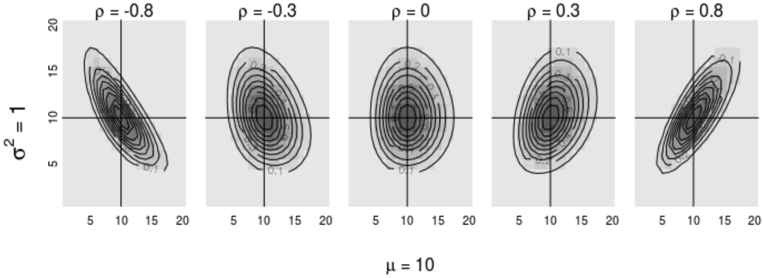

Figure: Core of the non-normalized bivariate Poisson distribution

Results
Continuous Cases

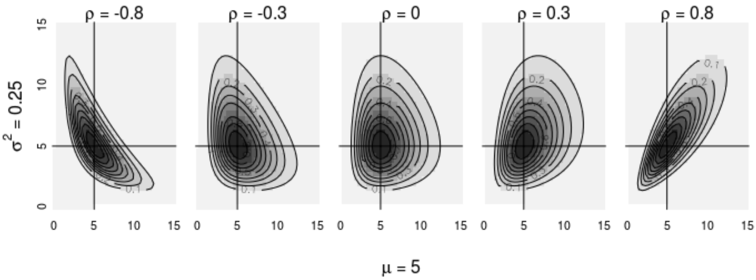

Figure: Core of the non-normalized bivariate Gamma distribution.

Results
Continuous Cases

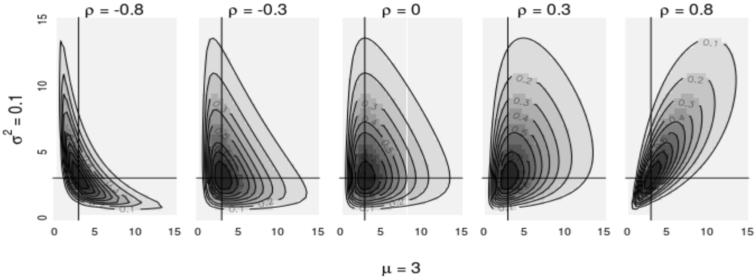

Figure: Core of the non-normalized bivariate inverse Normal distribution.

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- The parameter ρ controls the correlation.
- The dispersion parameters control the variability and shape.
- Similar to the bivariate normal distribution.
- μ is not necessarily a vector of expectations:

$\rightarrow$ *better interpreted as a vector of modes*.

Discussion

Topics

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- The method is relatively simple and the interpretation of the parameters is intuitive.
- Results about the normalizing constants do not influence directly on the construction of regression models for the location parameters.

Future work:

- Evaluate the performance of approximations to the normalizing constants.
- Perform inference.
- Provide computational implementation.

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Thank You!

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