Reducing a polynomial matrix to an equivalent second order polynomial matrix

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November 19, 2019

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Outline

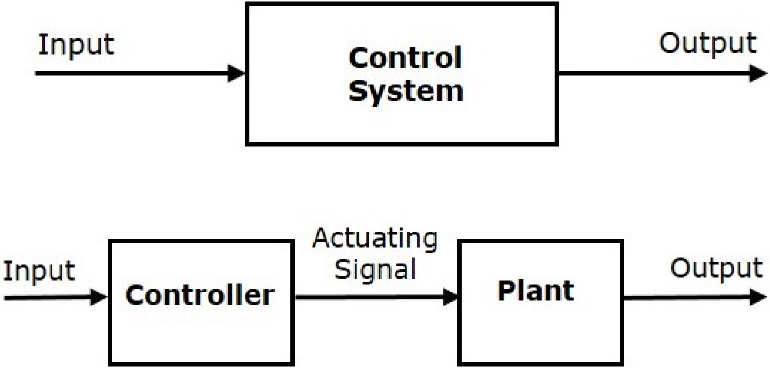
1. Basic notions, definitions and notations
2. Structure of polynomial matrices
3. Constructing strong quadratifications from dual minimal bases
4. Quadratification of polynomial matrices with symmetries
5. Conclusions and Extensions

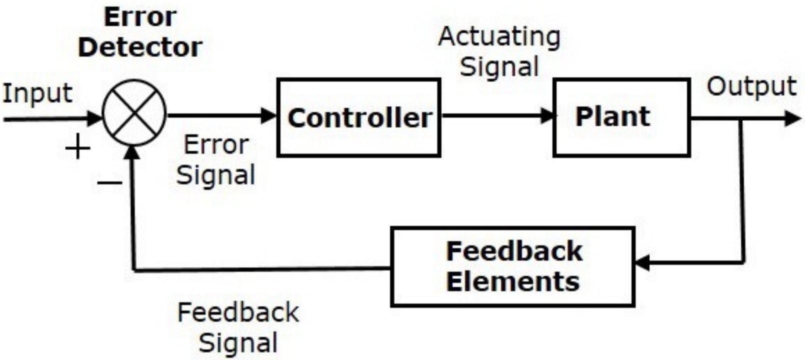
Outline

1. Basic notions, definitions and notations

Control Systems

Control Systems




Error
Detector
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Controller
Actuating
Signal
Plant
Output
Feedback
Elements
Feedback
Signal

Polynomial matrix descriptions (PMDs)

$$\begin{aligned} A(\rho)x(t) &= B(\rho)u(t) \\ y(t) &= C(\rho)x(t) + D(\rho)u(t) \end{aligned}$$

$\rho = \frac{d}{dt}$ in the continuous time case or $\rho x(t) = x(t+1)$ in the discrete time case.

Matrix Polynomial

$$T(s) = T_0 + T_1 s + T_2 s^2 + \cdots + T_d s^d \in \mathbb{R}[s]^{m\times n}, \;\; T_i \in \mathbb{R}^{m\times n}$$

If $T_d \neq 0$: $T(s)$ has degree d.

T(s) regular

$T(s)$ is square $(m = n)$ and $\det T(s) \neq 0$

If $T(s)$ is not regular then it is singular.

T(s) unimodular

$T(s)$ is square $(m = n)$ and $\det T(s) = c \neq 0 \in \mathbb{R}$

Finite zero of $T(s)$

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Outline

2. Structure of polynomial matrices

Smith canonical form in $\mathbb{C}$

Unimodular equivalence

$T_1(s), T_2(s) \in \mathbb{R}[s]^{n\times m}$ are **unimodular equivalent** in $\mathbb{C}$ if $\exists U_L(s) \in \mathbb{R}[s]^{n\times n}$ and $U_R(s) \in \mathbb{R}[s]^{m\times m}$ unimodular:

$$U_L(s)T_1(s)U_R(s) = T_2(s).$$

Theorem (Smith canonical form)

$T(s)$ is unimodular equivalent to:

$$S^{\mathbb{C}}_{T(s)} = diag\left[f_1(s), f_2(s), \ldots, f_r(s), 0_{n-r,m-r}\right]$$

$$r = rankT(s),\ f_j(s) = \prod_{i=1}^{\mu}(s-s_i)^{\sigma^{ij}},\ j = 1, \ldots, r \text{ and } \sigma_{ij} \leq \sigma_{i,j+1}$$

$s_i \in \mathbb{C},\ i = 1, \ldots, \mu$: *the distinct* **finite zeros** *of $T(s)$.*
$(s-s_i)^{\sigma^{ij}}$: *the* **finite elementary divisors** *of $T(s)$.*

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Example-Smith Form

Example

$$A(s) = \begin{bmatrix} s^3 - 3s + 5 & s + 2 \\ s^2 & 1 \end{bmatrix}$$

Then $A \approx S^{\mathbb{C}}_{A(s)}(s)$ where

$$S^{\mathbb{C}}_{A(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s-1)(s+\frac{5}{2}) \end{bmatrix}$$

The finite zeros of $A(s)$: $s = 1$ and $s = -\frac{5}{2}$
The finite elementary divisors of $A(s)$: $(s-1)$ and $(s+\frac{5}{2})$.

Dual or Reversal matrix

$$\widetilde{T}(s) = \mathbf{revT}(\mathbf{s}) = s^d T\left(\frac{1}{s}\right) = \sum_{i=0}^{d} T_{d-i} s^i.$$

Infinite elementary divisors of $T(s)$

The finite elementary divisors of $\widetilde{T}(s)$ of the form

$$s^{\sigma_i},\ 0 \leq \sigma_1 \leq \ldots \leq \sigma_n.$$

Remark

$T(s)$ has no infinite elementary divisors, iff T_d has full rank.

Structure of polynomial matrices at infinity

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Example

$$A(s) = \begin{bmatrix} s^3 - 3s + 5 & s + 2 \\ s^2 & 1 \end{bmatrix}$$

$$\widetilde{A}(s) = \begin{bmatrix} 1 - 3s^2 + 5s^3 & s^2 + 2s^3 \\ s & s^3 \end{bmatrix}$$

$$S^{\mathbb{C}}_{\widetilde{A}(s)}(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^4(s-1)(s+\frac{2}{5}) \end{bmatrix}$$

$A(s)$ has one infinite elementary divisor of order 4

Minimal Bases

$$T(s) = T_0 + T_1 s + T_2 s^2 + \cdots + T_d s^d \in \mathbb{R}[s]^{m\times n}$$

Right and left null spaces of $T(s)$:
$\mathcal{N}_r\left(T(s)\right) = \left\{v(s) \in \mathbb{R}[s]^{m\times 1} | T(s)v(s) = 0_{n\times 1}\right\}$
$\mathcal{N}_\ell\left(T(s)\right) = \left\{v(s) \in \mathbb{R}[s]^{1\times n} | v(s)T(s) = 0_{1\times m}\right\}$

Minimal bases of $T(s)$

$$\{x_1(s), x_2(s), \ldots, x_{n-r}(s)\},\ \{y_1(s), y_2(s), \ldots, y_{m-r}(s)\}$$

Minimal bases of $\mathcal{N}_r\left(T(s)\right)$ and $\mathcal{N}_\ell\left(T(s)\right)$ respectively

$$\eta_1 = \deg(x_1(s)) \le \eta_2 = \deg(x_2(s)) \le \ldots \le \eta_{n-r} = \deg(x_{n-r}(s))$$

$$\epsilon_1 = \deg(y_1(s)) \le \epsilon_2 = \deg(y_2(s)) \le \ldots \le \epsilon_{m-r} = \deg(y_{m-r}(s))$$

$\{\eta_1, \ldots, \eta_{n-r}\}$, $\{\epsilon_1, \ldots, \epsilon_{m-r}\}$: **Minimal indices** of $T(s)$

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$\{\eta_1, \ldots, \eta_{n-r}\}, \ \{\epsilon_1, \ldots, \epsilon_{m-r}\}$: **Minimal indices** of $T(s)$

Row (column) degrees

$T(s) \in \mathbb{R}[s]^{n\times m}$, d_i:the maximum degree of the entries of the ith row of $T(s)$, $i = 1, \ldots, n$.
$d_1, d_2, \ldots, d_n$: **row degrees** of $T(s)$.
The **column degrees** of $T(s)$ are defined similarly.

Highest row (column) degree coefficient matrix

$T(s) \in \mathbb{R}[s]^{n\times m}$ with row degrees $d_1, \ldots d_n$.
The **highest row degree coefficient matrix** of $T(s)$, $T_{hr} \in \mathbb{R}^{n\times m}$ is the matrix whose jth row consists of the coefficients of s^{d_j} in the jth row of $T(s)$ for $j = 1, \ldots, n$.
The **highest column degree coefficient matrix** of $T(s)$, $T_{hc} \in \mathbb{R}^{n\times m}$ is defined similarly.

$T(s) \in \mathbb{R}[s]^{n\times m}$ is **row (column) reduced** if $T_{hr}(T_{hc})$ has full row (column) rank.

Row and Column Degrees

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Dual minimal bases

Minimal basis

$T(s) \in \mathbb{R}[s]^{n\times m}$ with $n \leq m$ $(n \geq m)$
$T(s)$ minimal basis if it has full row (column) rank $\forall s \in \mathbb{C}$ and is row (column) reduced.

Dual Minimal Bases

$N_1(s) \in \mathbb{R}[s]^{m_1\times n}$, $N_2(s) \in \mathbb{R}[s]^{m_2\times n}$ are both minimal bases and also satisfy:

$$m_1 + m_2 = n, \quad and \quad N_1(s)N_2(s)^T = 0$$

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Dual minimal bases

Theorem

Let $N_1(s) \in \mathbb{R}[s]^{m_1 \times n}, N_2(s) \in \mathbb{R}[s]^{m_2 \times n}$ *be dual minimal bases with row degrees* $(\eta_1, \eta_2, \ldots, \eta_{m_1})$ *and* $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{m_2})$ *respectively, then*

$$\sum_{i=1}^{m_1} \eta_i = \sum_{j=1}^{m_2} \epsilon_j .$$

3. Constructing strong quadratifications from dual minimal bases

Strong quadratification

Definition

$Q(s)$ of degree 2 is a **quadratification** of $T(s)$ if $\exists p, q \geq 0$ and $U(s), V(s)$ unimodular such that

$$U(s)\begin{bmatrix} I_p & \\ & Q(s) \end{bmatrix} V(s) = \begin{bmatrix} I_q & \\ & T(s) \end{bmatrix}.$$

If $\operatorname{rev} Q(s)$ is a quadratification of $\operatorname{rev} T(s)$, then $Q(s)$ is a **strong quadratification** of $T(s)$.

$T(s) \in \mathbb{R}[s]^{m \times n}$ and $Q(s)$ a strong quadratification of $T(s)$. The size of $Q(s)$ will be $(\hat{n}+m) \times (\hat{n}+n)$ with $\hat{n} \geq 0$, so that

$$M(s)Q(s)N(s)^T = \begin{bmatrix} I_{\hat{n}} & \\ & T(s) \end{bmatrix}$$

where M(s),N(s) are unimodular.

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Theorem

Let $T(s) \in \mathbb{R}[s]^{m \times n}$ of degree d and $Q(s)$ strong quadratification of $T(s)$. Then

1. $dim\left(\mathcal{N}_r\left(T(s)\right)\right) = dim\left(\mathcal{N}_r\left(Q(s)\right)\right)$ *and* $dim\left(\mathcal{N}_l\left(T(s)\right)\right) = dim\left(\mathcal{N}_l\left(Q(s)\right)\right)$,
2. $T(s)$ *and* $Q(s)$ *have the same finite elementary divisors,*
3. $T(s)$ *and* $Q(s)$ *have the same infinite elementary divisors.*

If $T(s)$ is regular then $Q(s)$ is also regular.

Zigzag polynomial matrices

Forward-zigag polynomial matrix

$Z(s) \in \mathbb{R}[s]^{m\times n}$ with $m < n$ is a **forward-zigzag** matrix if

a) each row of $Z(s)$ is of the form

$$\begin{bmatrix} \underbrace{0\ldots0}_{\text{maybe none}} & 1 & s^{p_1} & s^{p_2} & \ldots & s^{p_k} & \underbrace{0\ldots0}_{\text{maybe none}} \end{bmatrix}$$

with at least two nonzero entries in each row. The nonzero entries of each row lie in consecutive adjacent columns, with

$$0 < p_1 < p_2 < \cdots < p_k, \; k \geq 1$$

b) For $i = 2, \ldots, m$, the last non zero entry of the $(i-1)$-th row and the first non zero entry of the i-th row are in the same column.

c) $Z(s)$ has no zero columns.

Example

The matrix

$$Z(s) = \begin{bmatrix} 1 & s^2 & s^3 & s^7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & s & s^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & s^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & s^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & s^2 & s^6 \end{bmatrix}$$

is a forward-zigzag matrix.

Theorem

Any forward-zigzag matrix is a minimal basis.

Unit column sequence of a forward-zigzag matrix $Z(s)$

The symbol string of length n

$$S_1, S_2, \ldots, S_n$$

consisting of U's and N's where "U" indicates a **unit column** and "N" indicates a **non-unit column** of $Z(s)$.

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Degree-gap sequence of a forward-zigzag matrix $Z(s)$

Let C_j, C_{j+1} be two adjacent columns in $Z(s)$. Then there is a unique row R_i having two nonzero entries in these columns. The degree-gap sequence of $Z(s)$ is the ordered list

$$\delta_1, \delta_2, \ldots, \delta_{n-1}$$

where

$$\delta_j := \deg Z_{i,j+1} - \deg Z_{i,j} \geq 1.$$

Structure sequence of a forward-zigzag matrix $Z(s)$

The sequence S of length $2n-1$ obtained by interleaving the unit column sequence and the degree-gap sequence of $Z(s)$.

$$S = \begin{bmatrix} S_1 & \delta_1 & S_2 & \delta_2 & \ldots & S_{n-1} & \delta_{n-1} & S_n \end{bmatrix}$$

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Example

$$Z(s) = \begin{bmatrix} 1 & s^2 & s^3 & s^7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & s & s^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & s^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & s^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & s^2 & s^6 \end{bmatrix}$$

Unit column sequence of $Z(s)$:

$$U, N, N, U, N, U, U, U, N, N$$

Degree gap sequence of $Z(s)$:

$$2, 1, 4, 1, 4, 2, 3, 2, 4$$

Structure sequence of $Z(s)$:

$$S = [U\ 2\ N\ 1\ N\ 4\ U\ 1\ N\ 4\ U\ 2\ U\ 3\ U\ 2\ N\ 4\ N]$$

Backward-zigag polynomial matrix

$\widehat{Z}(s) \in \mathbb{R}[s]^{m\times n}$ with $m < n$ is a **backward-zigzag** matrix if

a) each row of $\widehat{Z}(s)$ is of the form

$$\begin{bmatrix} \underbrace{0\dots0}_{\text{maybe none}} & s^{q_l} & \dots & s^{q_2} & s^{q_1} & 1 & \underbrace{0\dots0}_{\text{maybe none}} \end{bmatrix}$$

with at least two nonzero entries in each row. The nonzero entries of each row lie in consecutive adjacent columns, with

$$q_l > \cdots > q_2 > q_1 > 0, \; l \geq 1$$

b) For $i = 2, \dots, m$, the last non zero entry of the $(i-1)$-th row and the first non zero entry of the i-th row are in the same column.

c) $\widehat{Z}(s)$ has no zero columns.

Example

$$\widehat{Z}(s) = \begin{bmatrix} s^5 & s^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s^4 & s^3 & s & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s^4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s^6 & s & 1 \end{bmatrix}$$

Unit column sequence of $\widehat{Z}(s)$:

$$N, N, U, N, N, U, U, U, N, U$$

Degree gap sequence of $\widehat{Z}(s)$:

$$3, 2, 1, 2, 1, 4, 2, 5, 1$$

Structure sequence of $\widehat{Z}(s)$:

$$S = [N\ 3\ N\ 2\ U\ 1\ N\ 2 N\ 1\ U\ 4\ U\ 2\ U\ 5\ N\ 1\ U]$$

Dual Zigzag matrices

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$Z(s) \in \mathbb{R}[s]^{m_1 \times n}$ and $\widehat{Z}(s) \in \mathbb{R}[s]^{m_2 \times n}$ are said to be **dual zigzag matrices** or to form a dual zigzag pair, if they have

a) the same degree-gap sequence,

b) complementary unit column sequences.

The number of rows of a zigzag matrix is equal to the number of its unit columns, hence:

Corollary

If $Z(s) \in \mathbb{R}[s]^{m_1 \times n}$ and $\widehat{Z}(s) \in \mathbb{R}[s]^{m_2 \times n}$ is a dual zigzag pair, then $m_1 + m_2 = n$.

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a) the same degree-gap sequence,

b) complementary unit column sequences.

The number of rows of a zigzag matrix is equal to the number of its unit columns, hence:

Corollary

If $Z(s) \in \mathbb{R}[s]^{m_1 \times n}$ and $\widehat{Z}(s) \in \mathbb{R}[s]^{m_2 \times n}$ is a dual zigzag pair, then $m_1 + m_2 = n$.

Example

$$Z(s)=\begin{bmatrix}1 & s & s^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & s^2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & s^4 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & s & s^3 & s^4 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & s^2 & s^5\end{bmatrix}$$

with degree-gap sequence $1,5,2,4,1,2,1,2,3$ and unit column sequence U,N,U,U,U,N,N,U,N,N and

$$\widehat{Z}(s)=\begin{bmatrix}s & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & s^{12} & s^7 & s^5 & s & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & s^2 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & s^3 & s^2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s^3 & 1\end{bmatrix}$$

with degree-gap sequence $1,5,2,4,1,2,1,2,3$ and unit column sequence N,U,N,N,N,U,U,N,U,U

Proposition

A forward-zigzag matrix is uniquely determined by its structure sequence. The same is true for a backward-zigzag matrix.

$$S = \begin{bmatrix} S_1 & \delta_1 & S_2 & \delta_2 & \dots & S_{n-1} & \delta_{n-1} & S_n \end{bmatrix}, \; S_i \in \{N, U\}$$

the structure sequence of $Z(s)$.
The first row of $Z(s)$ will be formed according to the subsequence of S between $S_1 = U$ and the next U in S:

$$S_{init} = \begin{bmatrix} S_1 & \delta_1 & S_2 & \delta_2 & \dots & S_{n-1} & \delta_{k-1} & S_k, \; 2 \leq k < n \end{bmatrix}, \; 2 \leq k < n$$

$$Z_1(s) = \begin{bmatrix} 1 & s^{\delta_1} & s^{\delta_1+\delta_2} & \dots & s^{\delta_1+\cdots+\delta_{k-1}} & 0 & \dots & 0 \end{bmatrix}$$

$$Z(s) = \left[\begin{array}{cccc|cccc} 1 & s^{\delta_1} & \dots & s^{\delta_1+\cdots+\delta_{k-2}} & s^{\delta_1+\cdots+\delta_{k-1}} & 0 & \dots & 0 \\ \hline & & 0 & & & & \widetilde{Z}(s) & \end{array}\right]$$

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Example

$$S = [U \;\; 2 \;\; U \;\; 3 \;\; N \;\; 1 \;\; N \;\; 2 \;\; U \;\; 1 \;\; N \;\; 4 \;\; U \;\; 1 \;\; N]$$

Structure sequence of the first row of $Z(s)$:

$$S_{Zinit} = \begin{bmatrix} U & 2 & U \end{bmatrix} \Rightarrow Z_1(s) = \begin{bmatrix} 1 & s^2 & 0 & \dots & 0 \end{bmatrix}$$

$$Z(s) = \begin{bmatrix} 1 & s^2 \, 0 \dots 0 \\ 0 & \widetilde{Z}(s) \end{bmatrix}$$

Structure sequence of the first row of $\widetilde{Z}(s)$:

$$S_{\widetilde{Z}init} = \begin{bmatrix} U & 3 & N & 1 & N & 2 & U \end{bmatrix}$$

$$\Rightarrow \widetilde{Z}_1(s) = \begin{bmatrix} 1 & s^3 & s^4 & s^6 & 0 & \dots & 0 \end{bmatrix}$$

$$Z(s) = \begin{bmatrix} 1 & s^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & s^3 & s^4 & s^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & s & s^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & s \end{bmatrix}$$

Existance of dual minimal matrices

For every forward-zigzag matrix $Z(s)$, there exists a unique backward-zigzag matrix that is dual to $Z(s)$, denoted by $Z^{\diamond}(s)$. Similar for any backward-zigzag matrix.

Corollary (Row degree sums)

Let $Z(s) \in \mathbb{R}[s]^{m \times n}$ and $Z^{\diamond}(s) \in \mathbb{R}[s]^{k \times n}$ be dual zigzag matrices with row degrees $\{\eta_1, \eta_2, \ldots \eta_m\}$ and $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ respectively. Then

a. $\displaystyle\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{k} \epsilon_i$ and
b. $\displaystyle\sum_{i=1}^{\alpha} \eta_i \neq \sum_{i=1}^{\beta} \epsilon_i$ whenever $(\alpha, \beta) \neq (m, k)$ with $1 \leq \alpha \leq m$ and $1 \leq \beta \leq k$.

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Construction of dual minimal bases using zigzag matrices

Alternating signs matrix

The $n \times n$ diagonal **alternating signs matrix** is defined by

$$\Sigma_n = diag\left[1, -1, 1, -1 \ldots, (-1)^{n-1}\right]$$

Lemma

$Z(s) \in \mathbb{R}[s]^{m \times n}$ zigzag matrix (forward or backward), and $Z^{\diamond}(s) \in \mathbb{R}[s]^{(n-m) \times n}$its dual. Then

$$Z(s) \cdot \Sigma_n \cdot \left(Z^{\diamond}(s)\right)^T = 0_{m \times (n-m)}$$

and $Z(s)$ and $Z^{\diamond}(s) \cdot \Sigma_n$ are dual minimal bases.

Theorem

Let $\{\eta_1, \eta_2, \ldots, \eta_m\}$ and $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ be any two sets of positive integers such that

1. $\displaystyle\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{k} \epsilon_i$ *and*
2. $\displaystyle\sum_{i=1}^{\alpha} \eta_i \neq \sum_{i=1}^{\beta} \epsilon_i$, *whenever* $(\alpha, \beta) \neq (m, k)$, $1 \leq \alpha \leq m$, $1 \leq \beta \leq k$.

Then $\exists!\ Z(s) \in \mathbb{R}[s]^{m\times(m+k)}$ *forward-zigzag matrix with row degrees* $\{\eta_1, \eta_2, \ldots, \eta_m\}$ *such that* $Z^{\diamond}(s) \in \mathbb{R}[s]^{k\times(m+k)}$ *has row degrees* $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$.
The structure sequence of $Z(s)$ *is constructed as follows:*

1. Define $\ell_0 := 0$ and compute the partial sums $\ell_\alpha := \sum_{i=1}^{\alpha} \eta_i$ for $\alpha = 1, \ldots, m-1$ and $r_\beta = \sum_{i=1}^{\beta} \epsilon_i$ for $\beta = 1, \ldots, k$.
2. Merge the lists $\ell_0 < \ell_1 < \cdots < \ell_{m-1}$ and $r_1 < r_2 < \cdots < r_k$ into a single ordered list of length $n = m + k$

$$\ell_0 < \cdots < \ell_\alpha < \cdots < r_\beta < \cdots < \ell_\gamma < \cdots < r_k. \qquad (1)$$

3. For the degree-gap sequence of $Z(s)$: compute the $n-1$ differences between adjacent entries in (1).
4. For the unit column sequence of $Z(s)$: $\ell_\alpha \to$ U and $r_\beta \to$ N.
5. Interleave the unit column sequence with the degree-gap sequence to get the structure sequence of $Z(s)$.

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Example

Consider

$\{\eta_1, \eta_2, \ldots, \eta_5\} = \{2, 5, 1, 3, 3\}$ and $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_4\} = \{5, 4, 1, 4\}$

$$\sum_{i=1}^{5} \eta_i = 14 = \sum_{i=1}^{4} \epsilon_i \quad \text{and} \quad \sum_{i=1}^{\alpha} \eta_i \neq \sum_{i=1}^{\beta} \epsilon_i$$

whenever $(\alpha, \beta) \neq (5, 4)$, $1 \leq \alpha \leq 5$, $1 \leq \beta \leq 4$.
Then $\exists Z(s) \in \mathbb{R}[s]^{5\times 9}$ with row degrees $\{\eta_1, \eta_2, \ldots, \eta_5\}$ and its dual $Z^{\diamond}(s) \in \mathbb{R}[s]^{4\times 9}$ with row degrees $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_4\}$.

1. Consider the partial sums:
   $\ell_0 = 0,\ \ell_1 = 2,\ \ell_2 = 7,\ \ell_3 = 8,\ \ell_4 = 11$ and
   $r_1 = 5,\ r_2 = 9,\ r_3 = 10,\ r_4 = 14.$
2. Merge the two lists of partial sums into a single ordered list:

$$\{\ell_0, \ell_1, r_1, \ell_2, \ell_3, r_2, r_3, \ell_4, r_4\} = \{0, 2, 5, 7, 8, 9, 10, 11, 14\}$$

Example

3. The degree-gap sequence: $2, 3, 2, 1, 1, 1, 1, 3$
4. The unit column sequence: $U, U, N, U, U, N, N, U, N$
5. The unit column sequence of $Z^{\diamond}(s)$: $N, N, U, N, N, U, U, N, U$
6. $S_{Z(s)} = [U, 2, U, 3, N, 2, U, 1, U, 1, N, 1, N, 1, U, 3, N]$
   $S_{Z^{\diamond}(s)} = [N, 2, N, 3, U, 2, N, 1, N, 1, U, 1, U, 1, N, 3, U]$.

Hence $Z(s) = \begin{bmatrix} 1 & s^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & s^3 & s^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & s & s^2 & s^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & s^3 \end{bmatrix}$

and $Z^{\diamond}(s) = \begin{bmatrix} s^5 & s^3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s^4 & s^2 & s & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^4 & s^3 & 1 \end{bmatrix}$

Theorem

Let $\{\eta_1, \eta_2, \ldots, \eta_m\}$ and $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ be any two sets of nonnegative integers such that

$$\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{k} \epsilon_i .$$

Then $\exists N_1(s) \in \mathbb{R}[s]^{m\times(m+k)}$ and $N_2(s) \in \mathbb{R}[s]^{k\times(m+k)}$ dual minimal bases with row degrees $\{\eta_1, \eta_2, \ldots, \eta_m\}$ and $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ respectively. There are infinitely many such pairs. Let the lists $\{\eta_1, \eta_2, \ldots, \eta_m\}$ and $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ be ordered so that

$$0 = \eta_1 = \cdots = \eta_{m_0},\ 0 < \eta_i \text{ if } m_0 < i \text{ and}$$

$$0 = \epsilon_1 = \cdots = \epsilon_{k_0},\ 0 < \epsilon_j \text{ if } k_0 < j,$$

and define the set

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and define the set

$$\{(m_1, k_1), \ldots, (m_t, k_t)\} =$$

$$\left\{(\gamma, \rho) : \sum_{i=m_0+1}^{\gamma} \eta_i = \sum_{i=k_0+1}^{\rho} \epsilon_i,\ m_0 + 1 \leq \gamma \leq m,\ k_0 + 1 \leq \rho \leq k\right\}$$

where $m_1 < \cdots < m_t = m$ and $k_1 < \cdots < k_t = k$. Then:

(a) $\forall i = 1, \ldots, t$, $\exists Z_i(s)$ forward-zigzag matrix with row degrees $\left\{\eta_{m_{i-1}+1}, \ldots, \eta_{m_i}\right\}$ such that $Z_i^{\diamond}(s)$ has row degrees $\left\{\epsilon_{k_{i-1}+1}, \ldots, \epsilon_{k_i}\right\}$.

(b) Define the matrices

$$N_1(s) := \left[\begin{array}{c|c} I_{m_0}\ 0_{m_0 \times k_0} & \\ \hline & \begin{matrix} Z_1(s) & & \\ & \ddots & \\ & & Z_t(s) \end{matrix} \end{array}\right]$$ and

$\{(m_1, k_1), \ldots, (m_t, k_t)\} =$

$$\left\{ (\gamma, \rho) : \sum_{i=m_0+1}^{\gamma} \eta_i = \sum_{i=k_0+1}^{\rho} \epsilon_i, \; m_0 + 1 \leq \gamma \leq m, \; k_0 + 1 \leq \rho \leq k \right\}$$

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$$
N_2(s) := \left[\begin{array}{c|ccc}
0_{k_0\times m_0} \quad I_{k_0} & & & \\
\hline
 & Z_1^{\diamond}(s)\cdot\Sigma^{(1)} & & \\
 & & \ddots & \\
 & & & Z_t^{\diamond}(s)\cdot\Sigma^{(t)}
\end{array}\right]
$$

where $\Sigma^{(1)}, \ldots, \Sigma^{(t)}$ are alternating signs matrices of appropriate sizes. The matrices $N_1(s)$ and $N_2(s)$ are dual minimal bases with row degrees $\{\eta_1, \eta_2, \ldots, \eta_m\}$ and $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ respectively, and sizes $m \times (m+k)$ and $k \times (m+k)$ respectively.

Theorem

There exists a pair of dual minimal bases $N_1(s) \in \mathbb{R}[s]^{m\times(m+k)}$ *and* $N_2(s) \in \mathbb{R}[s]^{k\times(m+k)}$ *with row degrees* $\{\eta_1, \eta_2, \ldots, \eta_m\}$ *and* $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ *respectively, if and only if*

$$\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{k} \epsilon_i .$$

Construction of strong quadratifications

$$T(s) = T_0 + T_1 s + T_2 s^2 + \cdots + T_d s^d.$$

$T(s) \in \mathbb{R}[s]^{m \times n}$, $rank T(s) = r$, degree $d > 2$
Goal: find a quadratification of $T(s)$ i.e.,
$Q(s) \in \mathbb{R}[s]^{(\hat{n}+m)\times(\hat{n}+n)}$, with $\hat{n} > 0$, and $rank Q(s) = r + \hat{n}$, such that

$$M(s)Q(s)N(s)^T = \begin{bmatrix} I_{\hat{n}} & \\ & T(s) \end{bmatrix}$$

Theorem

Let $T(s) \in \mathbb{R}[s]^{m \times n}$*, degree* $d > 2$ *and assume that* $\exists \widehat{Q}(s) \in \mathbb{R}[s]^{\hat{n} \times (\hat{n}+n)}$, $\widetilde{Q}(s) \in \mathbb{R}[s]^{m \times (\hat{n}+n)}$:

(a) $\widehat{Q}(s)$ *is a minimal basis and has degree* 2.

(b) $\widetilde{Q}(s)$ *has degree less than or equal to* 2 *and satisfies* $\widetilde{Q}(s)\widehat{N}(s)^T = T(s)$ *where* $\widehat{N}(s) \in \mathbb{R}[s]^{n \times (\hat{n}+n)}$ *is a minimal basis dual to* $\widehat{Q}(s)$.

Then:

1. *The matrix polynomial of degree 2*

$$Q(s) = \begin{bmatrix} \widehat{Q}(s) \\ \widetilde{Q}(s) \end{bmatrix} \in \mathbb{R}[s]^{(\hat{n}+m) \times (\hat{n}+n)}$$

is a **quadratification** *of* $T(s)$.

2. *If the row degrees of* $\widehat{Q}(s)$ *are all equal to 2 and the row degrees of* $\widehat{N}(s)$ *are all equal to* $d-2$*, then* $Q(s)$ *is a* **strong quadratification** *of* $T(s)$.

Theorem

Let $T(s) \in \mathbb{R}[s]^{m\times n}$, *degree* $d > 2$ *and assume that* $\exists \widehat{Q}(s) \in \mathbb{R}[s]^{\hat{n}\times(\hat{n}+n)}$, $\widetilde{Q}(s) \in \mathbb{R}[s]^{m\times(\hat{n}+n)}$:

a. $\widehat{Q}(s)$ *is a minimal basis and has degree* 2.
b. $\widetilde{Q}(s)$ *has degree less than or equal to* 2 *and satisfies* $\widetilde{Q}(s)\widehat{N}(s)^T = T(s)$ *where* $\widehat{N}(s) \in \mathbb{R}[s]^{n\times(\hat{n}+n)}$ *is a minimal basis dual to* $\widehat{Q}(s)$.

Then:

1. *The matrix polynomial of degree 2*

$$Q(s) = \begin{bmatrix} \widehat{Q}(s) \\ \widetilde{Q}(s) \end{bmatrix} \in \mathbb{R}[s]^{(\hat{n}+m)\times(\hat{n}+n)}$$

   is a **quadratification** *of* $T(s)$.

2. *If the row degrees of* $\widehat{Q}(s)$ *are all equal to* 2 *and the row degrees of* $\widehat{N}(s)$ *are all equal to* $d-2$, *then* $Q(s)$ *is a* **strong quadratification** *of* $T(s)$.

Construction procedure for the case where 2 divides nd

There are two steps to the construction procedure:

1. Choose a pair of dual minimal bases $\widehat{Q}(s) \in \mathbb{R}[s]^{\hat{n}\times(\hat{n}+n)}$ and $\widehat{N}(s) \in \mathbb{R}[s]^{n\times(\hat{n}+n)}$ having row degrees all equal to 2 and $d-2=\hat{d}$ respectively.
   2k = nd for some integer $k>n$. So, $\exists \hat{n}>0$ such that $k=\hat{n}+n$. Then

   $$2(\hat{n}+n)=nd \Leftrightarrow 2\hat{n}=n\hat{d} \Leftrightarrow \sum_{i=1}^{\hat{n}} 2 = \sum_{i=1}^{n} \hat{d}$$

   They can be constructed with the help of dual zigzag matrices.
2. Solve for $\widetilde{Q}(s)$ in $\widetilde{Q}(s)\widehat{N}(s)^T = T(s)$.
   Let $\widetilde{Q}(s)=\widetilde{Q}_0+\widetilde{Q}_1 s+\widetilde{Q}_2 s^2$, $\widehat{N}(s)=\widehat{N}_0+\widehat{N}_1 s+\cdots+\widehat{N}_{\hat{d}} s^{\hat{d}}$ and $T(s)$ having the usual form

   $$T(s)=T_0+T_1 s+T_2 s^2+\cdots+T_d s^d.$$

Construction procedure for the case where 2 divides nd

There are two steps to the construction procedure:

1. Choose a pair of dual minimal bases $\widehat{Q}(s) \in \mathbb{R}[s]^{\hat{n}\times(\hat{n}+n)}$ and $\widehat{N}(s) \in \mathbb{R}[s]^{n\times(\hat{n}+n)}$ having row degrees all equal to 2 and $d-2=\hat{d}$ respectively.
   2k $=$ **nd** for some integer $k>n$. So, $\exists \hat{n}>0$ such that $k=\hat{n}+n$. Then

   $$2(\hat{n}+n)=nd \Leftrightarrow 2\hat{n}=n\hat{d} \Leftrightarrow \sum_{i=1}^{\hat{n}} 2=\sum_{i=1}^{n} \hat{d}$$

   They can be constructed with the help of dual zigzag matrices.
2. Solve for $\widetilde{Q}(s)$ in $\widetilde{Q}(s)\widehat{N}(s)^T=T(s)$.
   Let $\widetilde{Q}(s)=\widetilde{Q}_0+\widetilde{Q}_1 s+\widetilde{Q}_2 s^2$, $\widehat{N}(s)=\widehat{N}_0+\widehat{N}_1 s+\cdots+\widehat{N}_{\hat{d}} s^{\hat{d}}$ and $T(s)$ having the usual form

   $$T(s)=T_0+T_1 s+T_2 s^2+\cdots+T_d s^d.$$

Consider the convolution

$$\underbrace{\begin{bmatrix} \widetilde{Q}_0 & \widetilde{Q}_1 & \widetilde{Q}_2 \end{bmatrix}}_{m\times 3(\hat{n}+n)} \underbrace{\begin{bmatrix} \widehat{N}_0^T & \widehat{N}_1^T & \dots & \widehat{N}_{\hat{d}}^T & 0 & 0 \\ 0 & \widehat{N}_0^T & \widehat{N}_1^T & \dots & \widehat{N}_{\hat{d}}^T & 0 \\ 0 & 0 & \widehat{N}_0^T & \widehat{N}_1^T & \dots & \widehat{N}_{\hat{d}}^T \end{bmatrix}}_{3(\hat{n}+n)\times n(d+1)} = \underbrace{\begin{bmatrix} T_0 & T_1 & \dots & T_d \end{bmatrix}}_{m\times n(d+1)}$$

Solve for $\widetilde{Q}_2$ from

$$\widetilde{Q}_2 \widehat{N}_{\hat{d}}^T = T_d$$

since $\widehat{N}_{\hat{d}}^T$ has full column rank. Then

$$\begin{aligned} &\begin{bmatrix} \widetilde{Q}_0 & \widetilde{Q}_1 \end{bmatrix} \begin{bmatrix} \widehat{N}_0^T & \widehat{N}_1^T & \dots & \widehat{N}_{\hat{d}}^T & 0 \\ 0 & \widehat{N}_0^T & \widehat{N}_1^T & \dots & \widehat{N}_{\hat{d}}^T \end{bmatrix} \\ &= \begin{bmatrix} T_0 & \dots & T_{d-1} \end{bmatrix} - \widetilde{Q}_2 \begin{bmatrix} 0 & 0 & \widehat{N}_0^T & \dots & \widehat{N}_{\hat{d}-1}^T \end{bmatrix} \end{aligned}$$

For each possible choice of $\widetilde{Q}_2$, $\begin{bmatrix} \widetilde{Q}_0 & \widetilde{Q}_1 \end{bmatrix}$ is uniquely defined.

Example

Consider the matrix polynomial of degree $d = 5$ and size 3×2:

$$\begin{aligned} T(s) &= T_0 + T_1 s + T_2 s^2 + T_3 s^3 + T_4 s^4 + T_5 s^5 \\ &= \begin{bmatrix} 3s^5 + 2s^4 + s + 3 & 6s^3 + 2s^2 - 3s \\ 2s^5 - s^3 + 4s & s^2 + 4s \\ s^2 + 1 & s^4 - 2s \end{bmatrix} \end{aligned}$$

where

$$T_0 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \; T_1 = \begin{bmatrix} 1 & -3 \\ 4 & 4 \\ 0 & -2 \end{bmatrix}, \; T_2 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \; T_3 = \begin{bmatrix} 0 & 6 \\ -1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$T_4 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; T_5 = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$2\hat{n} = n\hat{d} \Rightarrow \hat{n} = 3$$

Construct two dual zigzag matrices $\widehat{Q}(s) \in \mathbb{R}[s]^{3\times 5}$ with all its row degrees equal to 2 and $\widehat{Q}^{\diamond}(s) \in \mathbb{R}[s]^{2\times 5}$ with all its row degrees equal to 3:

$$\widehat{Q}(s) = \begin{bmatrix} 1 & s^2 & 0 & 0 & 0 \\ 0 & 1 & s & s^2 & 0 \\ 0 & 0 & 0 & 1 & s^2 \end{bmatrix} \text{ and } \widehat{Q}^{\diamond}(s) = \begin{bmatrix} s^3 & s & 1 & 0 & 0 \\ 0 & 0 & s^3 & s^2 & 1 \end{bmatrix}.$$

The desired dual minimal bases will be the matrices $\widehat{Q}(s)$ and

$$\widehat{N}(s) = \widehat{Q}^{\diamond}(s) \cdot \Sigma_5 = \begin{bmatrix} s^3 & -s & 1 & 0 & 0 \\ 0 & 0 & s^3 & -s^2 & 1 \end{bmatrix}.$$

We are now looking for $\widetilde{Q}(s) \in \mathbb{R}[s]^{3\times 5}$. Let $\widetilde{Q}(s) = \widetilde{Q}_0 + \widetilde{Q}_1 s + \widetilde{Q}_2 s^2$, $\widehat{N}(s) = \widehat{N}_0 + \widehat{N}_1 s + \cdots + \widehat{N}_{\hat{d}} s^{\hat{d}}$. We can choose

$$\widetilde{Q}_2 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation

$$\begin{bmatrix}\widetilde{Q}_0 & \widetilde{Q}_1\end{bmatrix}\begin{bmatrix}\widehat{N}_0^T & \widehat{N}_1^T & \dots & \widehat{N}_{\hat{d}}^T & 0 \\ 0 & \widehat{N}_0^T & \widehat{N}_1^T & \dots & \widehat{N}_{\hat{d}}^T\end{bmatrix}$$

$$= \begin{bmatrix}T_0 & \dots & T_{d-1}\end{bmatrix} - \widetilde{Q}_2\begin{bmatrix}0 & 0 & \widehat{N}_0^T & \dots & \widehat{N}_{\hat{d}-1}^T\end{bmatrix}$$

gives the unique solution

$$\widetilde{Q}_0 = \begin{bmatrix}0 & -1 & 3 & -2 & 0 \\ -1 & -4 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0\end{bmatrix} \text{ and } \widetilde{Q}_1 = \begin{bmatrix}2 & 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & -1 & 1 & 1 & -2\end{bmatrix}.$$

Therefore the matrix $\widetilde{Q}(s)$ is

$$\widetilde{Q}(s) = \begin{bmatrix}3s^2+2s & -1 & 3 & -3s-2 & -3s \\ 2s^2-1 & -4 & 0 & -1 & 4s \\ 0 & -s+1 & s+1 & s & -2s\end{bmatrix}.$$

$$Q(s) = \begin{bmatrix} \widehat{Q}(s) \\ \widetilde{Q}(s) \end{bmatrix} = \begin{bmatrix} 1 & s^2 & 0 & 0 & 0 \\ 0 & 1 & s & s^2 & 0 \\ 0 & 0 & 0 & 1 & s^2 \\ 3s^2+2s & -1 & 3 & -3s-2 & -3s \\ 2s^2-1 & -4 & 0 & -1 & 4s \\ 0 & -s+1 & s+1 & s & -2s \end{bmatrix}.$$

The Smith forms of $T(s)$ and $Q(s)$ are

$$S^{\mathbb{C}}_{T(s)} = \begin{bmatrix} 1 & 0 \\ 0 & s \\ 0 & 0 \end{bmatrix}, \text{ and } S^{\mathbb{C}}_{Q(s)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and so indeed $T(s)$ and $Q(s)$ have the same finite elementary divisor structures.

Consider now the reverse matrices of $T(s)$ and $Q(s)$

$$\operatorname{rev} T(s) = \begin{bmatrix} 3+2s+s^4+3s^5 & 6s^2+2s^3-3s^4 \\ 2-s^2+4s^4 & s^3+4s^4 \\ s^3+s^5 & s-2s^4 \end{bmatrix}, \quad \text{and}$$

$$\operatorname{rev} Q(s) = \begin{bmatrix} s^2 & 1 & 0 & 0 & 0 \\ 0 & s^2 & s & 1 & 0 \\ 0 & 0 & 0 & s^2 & 1 \\ 3+2s & -s^2 & 3s^2 & -3s-2s^2 & -3s \\ 2-s^2 & -4s^2 & 0 & -s^2 & 4s \\ 0 & -s+s^2 & s+s^2 & s & -2s \end{bmatrix}$$

and their Smith forms

$$S^{\mathbb{C}}_{\operatorname{rev} T(s)} = \begin{bmatrix} 1 & 0 \\ 0 & s \\ 0 & 0 \end{bmatrix}, \text{ and } S^{\mathbb{C}}_{\operatorname{rev} Q(s)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which means that $T(s)$ and $Q(s)$ indeed have the same infinite elemetary divisors as well.

The special case where 2 divides d

$d = 2k$, $k \geq 1$. Then $2\hat{n} = n\hat{d} \Rightarrow \hat{n} = n(k-1)$. Hence $\widehat{Q}(s) \in \mathbb{R}[s]^{n(k-1)\times nk}$ and $\widehat{N}(s) \in \mathbb{R}[s]^{n\times nk}$. The row degrees of $\widehat{Q}(s)$ are all equal to 2 and the row degrees of $\widehat{N}(s)$ are all equal to $\hat{d} = d - 2 = 2(k-1)$. They can be chosen as follows:

$$\widehat{Q}(s) = \left(\begin{bmatrix} s^2 & -1 & & \\ & \ddots & \ddots & \\ & & s^2 & -1 \end{bmatrix}_{(k-1)\times k} \right) \otimes I_n \text{ and } \widehat{N}(s)^T = \begin{bmatrix} 1 \\ s^2 \\ s^4 \\ \vdots \\ s^{2(k-1)} \end{bmatrix} \otimes I_n$$

where $\otimes$ is the Kronecker product.

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B. \end{bmatrix}$$

The strong quadratification we obtain is:

$$Q(s) = \begin{bmatrix} s^2 I_n & -I_n & & \\ & \ddots & \ddots & \\ & & s^2 I_n & -I_n \\ B_0(s) & \dots & B_{k-2}(s) & B_{k-1}(s) \end{bmatrix}$$

where $B_j(s) = T_{2j} + sT_{2j+1}$ for $j = 0, \dots, k-2$, and $B_{k-1}(s) = T_{2(k-1)} + sT_{2k-1} + s^2 T_{2k}$.

Example

Consider the matrix polynomial of degree $d = 4 \Rightarrow k = 2$ and size 3×4:

$$\begin{aligned} T(s) &= T_0 + T_1 s + T_2 s^2 + T_3 s^3 + T_4 s^4 \\ &= \begin{bmatrix} s^2 + 4s - 1 & 2s^4 - 5s^3 + 2s & 4s^2 - s & s^4 - 3s^2 + 5 \\ 2s^3 + 4s^2 + 4 & s^4 - 2 & 3 & 2s^2 - 3s \\ 2 & s^2 - 2 & s - 1 & s^3 \end{bmatrix} \end{aligned}$$

where

$$T_0 = \begin{bmatrix} -1 & 0 & 0 & 5 \\ 4 & -2 & 3 & 0 \\ 2 & -2 & -1 & 0 \end{bmatrix}, \; T_1 = \begin{bmatrix} 4 & 2 & -1 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; T_2 = \begin{bmatrix} 1 & 0 & 4 & -3 \\ 4 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 0 & -5 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; T_4 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$B_0(s) = T_0 + sT_1 = \begin{bmatrix} 4s-1 & 2s & -s & 5 \\ 4 & -2 & 3 & -3s \\ 2 & -2 & s-1 & 0 \end{bmatrix} \text{ and}$$

$$B_1(s) = T_2 + sT_3 + s^2T_4 = \begin{bmatrix} 1 & 2s^2-5s & 4 & s^2-3 \\ 2s+4 & s^2 & 0 & 2 \\ 0 & 1 & 0 & s \end{bmatrix}.$$

Therefore, the proposed strong quadratification will be

$$\begin{aligned} Q(s) &= \begin{bmatrix} s^2 I_4 & -I_4 \\ 0 & s^2 I_4 \\ B_0(s) & B_1(s) \end{bmatrix} \\ &= \begin{bmatrix} s^2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & s^2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & s^2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & s^2 & 0 & 0 & 0 & -1 \\ 4s-1 & 2s & -s & 5 & 1 & 2s^2-5s & 4 & s^2-3 \\ 4 & -2 & 3 & -3s & 2s+4 & s^2 & 0 & 2 \\ 2 & -2 & s-1 & 0 & 0 & 1 & 0 & s \end{bmatrix} \end{aligned}$$

$$S^{\mathbb{C}}_{T(s)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ and } S^{\mathbb{C}}_{Q(s)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\operatorname{rev} T(s) = \begin{bmatrix} s^2+4s^3-s^4 & 2-5s+2s^3 & 4s^2-s^3 & 1-3s^2+5s^4 \\ 2s+4s^2+4s^4 & 1-2s^4 & 3s^4 & 2s^2-3s^3 \\ 2s^4 & s^2-2s^4 & s^3-s^4 & s \end{bmatrix}, \text{ and}$$

$$\operatorname{rev} Q(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & -s^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -s^2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -s^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -s^2 \\ 4s-s^2 & 2s & -s & 5s^2 & s^2 & 2-5s & 4s^2 & 1-3s^2 \\ 4s^2 & -2s^2 & 3s^2 & -3s & 2s+4s^2 & 1 & 0 & 2s^2 \\ 2s^2 & -2s^2 & s-s^2 & 0 & 0 & s^2 & 0 & s \end{bmatrix}$$

With respective Smith forms:

$$S^{\mathbb{C}}_{\operatorname{rev} T(s)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s^2 & 0 \end{bmatrix}, \text{ and } S_{\operatorname{rev} Q(s)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s^2 & 0 \end{bmatrix}$$

Therefore the proposed quadratification indeed preserves the finite and infinite divisor structure of $T(s)$.

Outline

4. Quadratification of polynomial matrices with symmetries

$$T(s) = T_0 + T_1 s + \cdots + T_n s^n, \;\; T_i \in \mathbb{R}^{p\times p}$$

Define the matrices

$$\begin{aligned}
A_0 &= diag\left\{ I_{p(n-1)} \quad -T_0 \right\} \\
A_k &= \begin{bmatrix} I_{p(n-k-1)} & 0 & 0 \\ 0 & C_k & 0 \\ 0 & 0 & I_{p(k-1)} \end{bmatrix}, \; k = 1, \ldots, n-1, \\
& \quad \text{where } C_k = \begin{bmatrix} -T_k & I_p \\ I_p & 0 \end{bmatrix} \\
A_n &= diag\left\{ T_n \quad I_{p(n-1)} \right\}
\end{aligned}$$

Consider the linearization:

$$L(s) = sA_{odd}^{-1} - A_{even}$$

where

$$A_{even} = A_0 A_2 \dots A_n^{-1} \text{ and } A_{odd} = A_1 A_3 \dots A_{n-1}, \text{ when n is even}$$

and

$$A_{even} = A_0 A_2 \dots A_{n-1} \text{ and } A_{odd} = A_1 A_3 \dots A_n^{-1}, \text{ when n is odd.}$$

$$L(s) = sA_{odd} - A_{even}$$

where

$A_{even} = A_0 A_2 \ldots A_{n-1}$ and $A_{odd} = A_n A_{n-2}^{-1} \ldots A_3^{-1} A_1^{-1}$, when n is odd.

L(s) and T(s) share the same finite and infinite elementary divisor structure.

We aim to generalize this method in order to obtain a quadratification of $T(s)$.
Distinguish two different cases:

1. n is odd
2. n is even

$$\begin{array}{rcl} P_0 & = & T_0 \\ P_1 & = & T_1 + sT_2 \\ P_i & = & sT_{i+1}, \quad \text{for } i = 2, \ldots, n-2 \\ P_{n-1} & = & 0 \\ P_n & = & T_n \end{array}$$

Using the matrices $P_j,\ j = 0, \ldots, n$ we define:

$$\begin{array}{rcl} A_0 & = & diag\left\{ I_{p(n-1)} \quad -P_0 \right\} \\ A_k & = & \begin{bmatrix} I_{p(n-k-1)} & 0 & 0 \\ 0 & C_k & 0 \\ 0 & 0 & I_{p(k-1)} \end{bmatrix}, \; k = 1, \ldots, n-1, \\ & & \text{where } C_k = \begin{bmatrix} -P_k & I_p \\ I_p & 0 \end{bmatrix} \\ A_n & = & diag\left\{ P_n \quad I_{p(n-1)} \right\} \end{array}$$

Define the matrices:

$$A_{even} = A_0 A_2 \dots A_{n-1}, \quad \text{and } A_{odd} = A_n A_{n-2}^{-1} \dots A_1^{-1}$$

and consider the quadratification:

$$Q(s) = sA_{odd} - A_{even}$$

$$= \begin{bmatrix} sT_n & -I_p & & & & & \\ -I_p & 0 & sI_p & & & & \\ & sI_p & s^2T_{n-1} + sT_{n-2} & & & & \\ & & -I_p & & & & \\ & & & \ddots & & & \\ & & & & s^2T_4 + sT_3 & -I_p & \\ & & & & -I_p & 0 & sI_p \\ & & & & & sI_p & s^2T_2 + sT_1 + T_0 \end{bmatrix}$$

Q(s) and T(s) share the same finite elementary divisor structure.

Example

Consider the symmetric matrix polynomial T(s) of degree $d = 5$ and size 3×3:

$$T(s) = \begin{bmatrix} 3s^5 + 2s^4 + s + 3 & 6s^3 + 2s^2 - 3s & s^2 + 1 \\ 6s^3 + 2s^2 - 3s & s^2 + 4s & s^4 - 2s \\ s^2 + 1 & s^4 - 2s & 4s \end{bmatrix}$$

$$T_0 = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ T_1 = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix},\ T_2 = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 0 & 6 & 0 \\ 6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ T_4 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ T_5 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the proposed quadratification is:

$$Q(s) = \begin{bmatrix} sT_5 & -I_3 & & & & \\ -I_3 & 0 & sI_3 & & & \\ & sI_3 & s^2T_4 + sT_3 & -I_3 & & \\ & & -I_3 & 0 & sI_3 & \\ & & & sI_3 & s^2T_2 + s_1^T + T_0 \end{bmatrix} =$$

$$\left[\begin{array}{ccc|c|ccc|c|ccc} 3s & 0 & 0 & & & & & & & & \\ 0 & 0 & 0 & -I_3 & & & & & & & \\ 0 & 0 & 0 & & & & & & & & \\ \hline & -I_3 & & 0_{3\times 3} & & sI_3 & & & & & \\ \hline & & & & 2s^2 & 6s & 0 & & & & \\ & & & sI_3 & 6s & 0 & s^2 & -I_3 & & & \\ & & & & 0 & s^2 & 0 & & & & \\ \hline & & & & & -I_3 & & 0_{3\times 3} & & sI_3 & \\ \hline & & & & & & & & s+3 & 2s^2-3s & s^2+1 \\ & & & & & & & sI_3 & 2s^2-3s & s^2+4s & -2s \\ & & & & & & & & s^2+1 & -2s & 4s \end{array}\right]$$

$$S^{\mathbb{C}}_{T(s)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & sp(s) \end{bmatrix}, \quad S^{\mathbb{C}}_{Q(s)} = \begin{bmatrix} I_{14} & 0 \\ 0 & sp(s) \end{bmatrix}$$

$p(s) = \frac{1}{3}(4 - 47s + 28s^2 - 38s^3 - 122s^4 + 89s^5 + 94s^6 - 13s^7 - 19s^8 - 12s^9 + 2s^{11} + 3s^{12})$

$$\operatorname{rev} T(s) = \begin{bmatrix} 3s^5 + s^4 + 2s + 3 & -3s^4 + 2s^3 + 6s^2 & s^5 + s^3 \\ 6s^2 + 2s^3 - 3s^4 & s^3 + 4s^4 & -2s^4 + s \\ s^5 + s^3 & -2s^4 + s & 4s^4 \end{bmatrix}$$

$\operatorname{rev} Q(s) =$

$$\left[\begin{array}{c|c|c|c|c} \begin{matrix} 3s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & -s^2 I_3 & & & \\ \hline -s^2 I_3 & 0_{3\times 3} & sI_3 & & \\ \hline & sI_3 & \begin{matrix} 2 & 6s & 0 \\ 6s & 0 & 1 \\ 0 & 1 & 0 \end{matrix} & -s^2 I_3 & \\ \hline & & -s^2 I_3 & 0_{3\times 3} & sI_3 \\ \hline & & & sI_3 & \begin{matrix} s + 3s^2 & 2 - 3s & s^2 + 1 \\ 2 - 3s & 1 + 4s & -2s \\ s^2 + 1 & -2s & 4s \end{matrix} \end{array}\right]$$

$$S^{\mathbb{C}}_{\mathrm{rev}\,T(s)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & sq(s) \end{bmatrix}, \; S^{\mathbb{C}}_{\mathrm{rev}\,Q(s)} = \begin{bmatrix} I_6 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s^2 I_7 & 0 \\ 0 & 0 & 0 & s^2 q(s) \end{bmatrix}$$

$p(s) = \frac{1}{4}(3 + 2s - 12s^3 - 19s^4 - 13s^5 + 94s^6 + 89s^7 - 122s^8 - 38s^9 + 28s^{10} - 47s^{11} + 4s^{12})$.

The respective infinite elementary divisors of $T(s)$ and $Q(s)$ are:

$$\{w, w\} \text{ and } \{w, w^2, w^2, w^2, w^2, w^2, w^2, w^2, w^2\}.$$

Therefore, $T(s)$ and $Q(s)$ share the same finite zero structure but not the same infinite elementary divisor structure.

Set $m = n - 1$

$$\begin{array}{lcl} P_0 & = & T_0 \\ P_1 & = & T_1 + sT_2 \\ P_i & = & sT_{i+1}, \quad \text{for } i = 2, \ldots, m-2 \\ P_{m-1} & = & 0 \\ P_m & = & T_m + sT_n \end{array}$$

Using the matrices P_j, $j = 0, \ldots, m$ we define:

$$\begin{array}{lcl} A_0 & = & diag\left\{ I_{p(m-1)} \quad -P_0 \right\} \\ A_k & = & \begin{bmatrix} I_{p(m-k-1)} & 0 & 0 \\ 0 & C_k & 0 \\ 0 & 0 & I_{p(k-1)} \end{bmatrix}, \ k = 1, \ldots, m-1, \\ & & \quad \text{where } C_k = \begin{bmatrix} -P_k & I_p \\ I_p & 0 \end{bmatrix} \\ A_m & = & diag\left\{ P_m \quad I_{p(m-1)} \right\} \end{array}$$

Define the matrices:

$$A_{even} = A_0 A_2 \dots A_{m-1}, \text{ and } A_{odd} = A_m A_{m-2}^{-1} \dots A_1^{-1}$$

and consider the quadratification:

$$Q(s) = sA_{odd} - A_{even}$$

$$= \begin{bmatrix} s^2T_n + sT_{n-1} & -I_p & & & & & \\ -I_p & 0 & sI_p & & & & \\ & sI_p & s^2T_{n-2} + sT_{n-3} & & & & \\ & & -I_p & & & & \\ & & & \ddots & & & \\ & & & & s^2T_4 + sT_3 & -I_p & \\ & & & & -I_p & 0 & sI_p \\ & & & & & sI_p & s^2T_2 + sT_1 + T_0 \end{bmatrix}$$

Q(s) and T(s) share the same finite elementary divisor structure.

Outline

5. Conclusions and Extensions

Conclusions

- We presented a stong quadratification of $T(s) \in \mathbb{R}[s]^{m \times n}$ of degree d, that preserves the finite and infinite divisor structure, is applicable to rectangular matrices, needs for 2 to divide nd or md.
- We introduced a different quadratification of $T(s) \in \mathbb{R}[s]^{m \times n}$ of degree d, that preserves the finite zero structure and potential symmetry, is easily constructed, applicable to square matrices, regardless of whether 2 divides nd or not.

Extensions

- Search for a similar construction of a quadratification (or more generally ℓ-ification) in the case that 2 (ℓ) does not divide neither nd nor md.
- Search for a modification of the second method that preserves the infinite elementary divisor structure.

Conclusions

- We presented a stong quadratification of $T(s) \in \mathbb{R}[s]^{m \times n}$ of degree d, that preserves the finite and infinite divisor structure, is applicable to rectangular matrices, needs for 2 to divide nd or md.
- We introduced a different quadratification of $T(s) \in \mathbb{R}[s]^{m \times n}$ of degree d, that preserves the finite zero structure and potential symmetry, is easily constructed, applicable to square matrices, regardless of whether 2 divides nd or not.

Extensions

- Search for a similar construction of a quadratification (or more generally ℓ-ification) in the case that 2 (ℓ) does not divide neither nd nor md.
- Search for a modification of the second method that preserves the infinite elementary divisor structure.

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Thank you for your attention!!!