MATH 455 (Honours Analysis 4)

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The following notes are adapted from lectures given by Dmitry Jakobson in Winter 2024. All errors in interpretation, reasoning, coherence, and articulation are my own.

This document's source code, located at `https://github.com/brunefig/math455/blob/main/notes.org`, can be converted into Anki flashcards with the `anki-editor` package for GNU Emacs. Flashcard cloze deletions are typeset in magenta.

2024-01-05 topology & metric spaces

connectedness

A topological space X is a connected space if and only if there are no open sets $U, U' \in \mathscr{P}(X) \setminus \{\emptyset\}$ such that $U \cap U' = \emptyset$ and $X = U \cup U'$.

For a topological space X, $A \subseteq X$ is a connected set if and only if there are no open sets $U, U' \subseteq X$ such that $A \cap U \neq \emptyset$, $A \cap U' \neq \emptyset$, $U \cap U' = \emptyset$, and $A = (U \cap A) \cup (U' \cap A)$.

A topological space X is a path connected space if and only if $\forall x, x' \in X : \exists$ continuous $f : [0,1] \to X : [f(0) = x \wedge f(1) = x']$.

All path connected spaces are connected spaces.

For open sets in $\mathbb{R}^n$, connectedness is equivalent to path connectedness.

A topological space X can be expressed as the disjoint union of maximal connected subsets, where a connected subset is called maximal if and only if it has no connected superset in X. These subsets are the connected components of X.

A topological space X can be expressed as the disjoint union of maximal path connected subsets, where a path connected subset is called maximal if and only if it has no path connected superset in X. These subsets are the path components of X.

A path connected space has exactly one path component.

$\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \,\middle|\, x \in (0,1]\right\} \cup \{(0,0)\} \subseteq \mathbb{R}^2$ is connected but not path connected because it has two path components.

If $f : X \to \mathbb{R}$ is continuous, $A \subseteq X$ is connected, and $x, x' \in f(A)$ such that $x < x'$, then $[x, x'] \subseteq f(A)$.

Proof. Suppose $\exists c \in \big(f(x'), f(x)\big) : c \notin f(A)$. Since $f^{-1}\big((-\infty, c)\big)$ and $f^{-1}\big((c, \infty)\big)$ are open, $A \subseteq f^{-1}\big((-\infty, c)\big) \cup f^{-1}\big((c, \infty)\big)$ is not connected. $\square$

examples of metric spaces

Any normed vector space is a metric space with the induced metric $d(x, x') := ||x - x'||$.

For $p \in (0, \infty)$, $l_p := \left\{(x_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \,\middle|\, \sum_{n \in \mathbb{N}} |x_n|^p < \infty\right\}$ is a normed vector space with $||x||_p := \left(\sum_{n \in \mathbb{N}} |x_n|^p\right)^{1/p}$.

The sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{Z}_+}$ is a member of l_p if and only if $p > 1$.

Proof. $\left(\frac{1}{n}\right)_{n\in\mathbb{Z}_+} \in l_p \iff \sum_{n\in\mathbb{Z}_+}\left(\frac{1}{n}\right)^p < \infty \iff p > 1.$ $\square$

For $p \in [1,\infty)$, $L^p([a,b]) := \left\{f(x) \,\middle|\, \int_a^b |f(x)|^p\,dx < \infty\right\}$ is a normed vector space with $||f||_p := \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}$.

$d(A,A') := \mathrm{vol}_n(A \triangle A')$ is a possible metric on $\mathbb{R}^n$.

For a metric space (X,d), a set $A \subseteq X$, and $\epsilon > 0$, let $A_\epsilon := \bigcup_{x\in A} B(x,\epsilon)$. Then the Hausdorff metric is $d_H(A,A') := \inf\{\epsilon > 0 \mid A' \subseteq A_\epsilon \wedge A \subseteq A'_\epsilon\}$.

p-adic numbers

Given a fixed prime p,

$$\forall q \in \mathbb{Q}\setminus\{0\} : \exists(a,b,n) \in \mathbb{Z}^3 : \left[q = p^n \cdot \frac{a}{b} \wedge \gcd(a,p) = \gcd(b,p) = 1\right]$$

and $\mathbb{Q}$ is a normed vector space with $||q||_p := \begin{cases} p^{-n} & q \neq 0 \\ 0 & q = 0 \end{cases}$.

The 2-adic norm of $\frac{96}{7}$ is $\frac{1}{32}$.

The 3-adic norm of $3^{-2024}\cdot\frac{8}{13}$ is 3^{2024}.

The p-adic norm $||q||_p$ is small if q is divisible by a large power of p.

The p-adic norm of 0 is 0 because 0 is divisible by any power of p.

(p-adic product formula.) If $q \in \mathbb{Q}\setminus\{0\}$, then $|q|\cdot\prod_{p\text{ prime}} ||q||_p = 1$.

convexness

A set X is convex if and only if the line segment joining any two points in X lies within X.

2024-01-10 contraction mappings & product topologies

$\alpha \in I$ henceforth refers to members of a possibly uncountable index set I.

If $\{A_\alpha\}_{\alpha\in I}$ is a family of connected sets such that $\forall \alpha,\alpha' \in I : A_\alpha \cap A_{\alpha'} \neq \emptyset$, then $\bigcup_{\alpha\in I} A_\alpha$ is connected.

Proof. Suppose $A := \bigcup_{\alpha \in I} A_\alpha$ is not connected. Let $U, U' \subseteq X$ be open and nonempty relative to A such that $U \cap U' = \emptyset$ and $A = U \cup U'$. Since each A_α is connected, there is no α for which $A_\alpha \cap U \neq \emptyset \neq A_\alpha \cap U'$. But then $\exists \alpha, \alpha' \in I : A_\alpha \subseteq U \wedge A_{\alpha'} \subseteq U'$, which contradicts that $A_\alpha \cap A_{\alpha'} \neq \emptyset$. $\square$

total disconnectedness

A topological space X is totally disconnected if and only if every connected subset of X is a singleton.

A totally disconnected space in $\mathbb{R}$ contains only points and no intervals.

The Cantor set is a totally disconnected space in $\mathbb{R}$.

contraction mappings

For a metric space (X, d), $T : X \to X$ is a contraction mapping if and only if $\exists c \in [0,1) : \forall x, x' \in X : d\big(T(x), T(x')\big) \leq c \cdot d(x, x')$.

All contraction mappings are continuous.

(Orbit lemma.) For $x \in X$, the orbit $\big(T^n(x)\big)_{n \in \mathbb{N}}$ of a contraction mapping T on X is a Cauchy sequence.

Proof. For $n \in \mathbb{Z}_+$,

$$\begin{aligned} d\left(T^n(x), T^{n+1}(x)\right) &< c \cdot d\left(T^{n-1}(x), T^n(x)\right) \\ &< c^2 \cdot d\left(T^{n-2}(x), T^{n-1}(x)\right) \\ \cdots &< c^n \cdot d\big(x, T(x)\big). \end{aligned}$$

Let $m \geq n$. By the triangle inequality,

$$\begin{aligned} d\left(T^n(x), T^m(x)\right) &\leq \sum_{k=n}^{m-1} d\left(T^k(x), T^{k+1}(x)\right) \\ &\leq d\big(x, T(x)\big) \sum_{k=n}^{m-1} c^k \\ &= c^n \cdot d\big(x, T(x)\big) \sum_{k=0}^{m-n-1} c^k \\ &\leq \frac{c^n \cdot d\big(x, T(x)\big)}{1-c}. \end{aligned}$$

For $\epsilon > 0$, choosing $n > \log_c\left(\frac{\epsilon}{2} \cdot \frac{1-c}{d\left(x,T(x)\right)}\right)$ and $m, m' \geq n$ guarantees

$$d\left(T(m), T(m')\right) \leq$$
$$d\left(T(n), T(m)\right) + d\left(T(n), T(m')\right) < \epsilon. \qquad \square$$

(Contraction mapping theorem.) If (X,d) is a nonempty and complete metric space and T is a contraction mapping on X, then

$$\exists! z \in X : T(z) = z,$$

i.e. z is the unique fixed point, and

$$\forall x \in X : \lim_{n \to \infty} T^n(x) = z,$$

where $\forall n \in \mathbb{N} : T^n(x) := \underbrace{T(T(\cdots T(x)))}_{n \text{ times}}$.

Proof. For $x \in X$, $T(x)$ is Cauchy by the orbit lemma, and since (X,d) is complete it converges to some point $z \in X$. Then z is a fixed point of T because

$$\begin{aligned} T^n(x) \to z &\implies T(T^n(x)) \to T(z) \\ &\implies T(T^n(x)) = T^{n+1}(x) \to z = T(z), \end{aligned}$$

and is its unique fixed point because

$$T(z) = z \wedge T(z') = z' \implies d(z,z') \leq c \cdot d(z,z') \iff d(z,z') = 0. \qquad \square$$

$x \mapsto \frac{x}{2}$ is a contraction mapping on $(\mathbb{R}, |\cdot|)$ with fixed point $z = 0$.

iterated function systems and fixed point sets

$\mathcal{K}(X)$ henceforth denotes the set of compact subsets of a set X.

If $m \in \mathbb{Z}_+$, $\left\{T_n : \mathbb{R}^d \to \mathbb{R}^d\right\}_{n \in [m]}$ are contraction mappings, and

$$F : \left(\mathcal{K}(\mathbb{R}^d), d_H\right) \to \left(\mathcal{K}(\mathbb{R}^d), d_H\right) : A \mapsto= \bigcup_{n \in [m]} T_n(A),$$

then

$$\exists! A \in \mathcal{K}(\mathbb{R}^d) : F(A) = A.$$

This set is the fixed point set of F.

Proof. F is a contraction mapping (to be demonstrated in problem set 2), and $\big(\mathscr{K}(X), d_H\big)$ is a complete metric space (???). A unique fixed point set thus exists by the contraction mapping theorem. □

Let $T_0, T_1 : \mathbb{R} \to \mathbb{R}$ such that $T_0(x) := \frac{x}{3}$ and $T_1(x) := \frac{x+2}{3}$ and $F(A) : \mathscr{K}(\mathbb{R}) \to \mathscr{K}(\mathbb{R})$ such that $F(A) := T_0(A) \cup T_1(A)$. Then $F\big([0,1]\big) = T_0\big([0,1]\big) \cup T_1\big([0,1]\big) = \big[0, \frac{1}{3}\big] \cup \big[\frac{2}{3}, 1\big]$, and the fixed point set of F is the middle-thirds Cantor set.

If $T_0, T_1 : \mathbb{R}^d \to \mathbb{R}^d$ are contraction mappings and $F : \big(\mathscr{K}(\mathbb{R}^d), d_H\big) \to \big(\mathscr{K}(\mathbb{R}^d), d_H\big)$ such that $F(A) = T_0(A) \cup T_1(A)$, then the composition of F with itself is

$$F^2(A) = T_0\big(T_0(A)\big) \cup T_0\big(T_1(A)\big) \cup T_1\big(T_0(A)\big) \cup T_1\big(T_1(A)\big).$$

basis for a topology

A basis $\mathscr{B}$ for a topology $\mathscr{T}$ is a collection of open sets such that $\forall A \in \mathscr{T} : \exists \{A_\alpha\}_{\alpha \in I} \subseteq \mathscr{B} : A = \bigcup_{\alpha \in I} A_\alpha$.

The open intervals form a basis for the standard topology on $\mathbb{R}$.

For a metric space (X, d), $\big\{B(x, \epsilon) \,\big|\, x \in X, \epsilon > 0\big\}$ forms a basis for the open sets in X.

If $\mathscr{B}$ is a basis for the topology in X, then $\forall x \in X : \exists U \in \mathscr{B} : x \in U$.

If $\mathscr{B}$ is a basis for the topology $\mathscr{T}$ in X, then $\big[U, U' \in \mathscr{B} \land x \in U \cap U'\big] \implies \exists U'' \in \mathscr{T} : x \in U'' \subseteq U \cap U'$.

If $f : X \to Y$ for topological spaces X and Y and $\mathscr{B}$ is a basis for the topology in Y, then f is continuous if and only if $\forall U \in \mathscr{B} : f^{-1}(U)$ is open in X.

product topologies

For topological spaces $\{X_\alpha\}_{\alpha \in I}$, $\prod_{\alpha \in I} A_\alpha$ is a cylinder set in $X := \prod_{\alpha \in I} X_\alpha$ if and only if $\forall \alpha \in I : A_\alpha$ is open in X_α.

If $X := \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, then $\forall$ open intervals $(a, b), (a', b') \subseteq \mathbb{R}$: $(a, b) \times (a', b')$ is a cylinder set in X.

For topological spaces $\{X_\alpha\}_{\alpha \in I}$, consider a cylinder set $\prod_{\alpha \in I} A_\alpha$ in $X := \prod_{\alpha \in I} X_\alpha$ such that $\exists I' \subseteq I : \big[|I'| \in \mathbb{N} \land \forall \alpha \in I \setminus |I'| : A_\alpha = X_\alpha\big]$. These base cylinder sets form a basis for the product topology on X.

projection maps

For a (possibly uncountable) sequence $x := (x_\alpha)_{\alpha \in I}$, the function $\pi_\alpha(x) := x_\alpha$ is a projection map.

For topological spaces $\{X_\alpha\}_{\alpha \in I}$ let

$$f : Y \to \prod_{\alpha \in I} X_\alpha,$$

and for $\alpha \in I$ let

$$f_\alpha : Y \to X_\alpha : y \mapsto \pi_\alpha\big(f(y)\big).$$

Then f is continuous if and only if $\forall \alpha \in I : f_\alpha$ is continuous.

Proof. Suppose f is continuous. Then, for $\alpha' \in I$ and an open set $U \subseteq X_{\alpha'}$,

$$f_{\alpha'}^{-1}(U) = f^{-1}\left(U \times \prod_{\alpha \in I \setminus \{\alpha'\}} X_\alpha\right)$$

is open as the preimage of a base cylinder set in $\prod_{\alpha \in I} X_\alpha$.

Suppose $\forall \alpha \in I : f_\alpha$ is continuous. It suffices to verify that the preimage of f for base a cylinder set U is open. Let $I' \subseteq I$ be a finite index subset for which

$$U = \prod_{\alpha \in I'} A_\alpha \times \prod_{\alpha \in I \setminus I'} X_\alpha.$$

Then

$$\begin{aligned} f^{-1}(U) &= \left(\bigcap_{\alpha \in I'} f_\alpha^{-1}(A_\alpha)\right) \cap \left(\bigcap_{\alpha \in I \setminus I'} f_\alpha^{-1}(X_\alpha)\right) \\ &= \bigcap_{\alpha \in I'} f_\alpha^{-1}(A_\alpha) \end{aligned}$$

is open in Y as the finite intersection of open sets in Y.