MATH 455 (Honours Analysis 4)

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The following notes are adapted from lectures given by Dmitry Jakobson in Winter 2024. All errors in interpretation, reasoning, coherence, and articulation are my own.

This document's source code, located at `https://github.com/brunefig/math455/blob/main/notes.org`, can be converted into Anki flashcards with the `anki-editor` package for GNU Emacs. Flashcard cloze deletions are typeset in magenta.

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2024-01-05 topology and metric spaces

connectedness

A topological space X is a connected space if and only if there are no open sets $U, U' \in \mathcal{P}(X) \setminus \{\emptyset\}$ such that $U \cap U' = \emptyset$ and $X = U \cup U'$.

For a topological space X, $A \subseteq X$ is a connected set if and only if there are no open sets $U, U' \subseteq X$ such that $A \cap U \neq \emptyset$, $A \cap U' \neq \emptyset$, $U \cap U' = \emptyset$, and $A = (U \cap A) \cup (U' \cap A)$.

A topological space X is a path connected space if and only if $\forall x, x' \in X : \exists$ continuous $f : [0, 1] \to X : [f(0) = x \wedge f(1) = x']$.

All path connected spaces are connected spaces.

For open sets in $\mathbb{R}^n$, connectedness is equivalent to path connectedness.

A topological space X can be expressed as the disjoint union of maximal connected subsets, where a connected subset is called maximal if and only if it has no connected superset in X. These subsets are the connected components of X.

A topological space X can be expressed as the disjoint union of maximal path connected subsets, where a path connected subset is called maximal if and only if it has no path connected superset in X. These subsets are the path components of X.

A path connected space has exactly one path component.

$\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \,\middle|\, x \in (0,1]\right\} \cup \{(0,0)\} \subseteq \mathbb{R}^2$ is connected but not path connected because it has two path components.

> *If $f : X \to \mathbb{R}$ is continuous, $A \subseteq X$ is connected, and $x, x' \in f(A)$ such that $x < x'$, then $[x, x'] \subseteq f(A)$.*

Proof. Suppose $\exists c \in \big(f(x'), f(x)\big) : c \notin f(A)$. Then, since $f^{-1}\big((-\infty, c)\big)$ and $f^{-1}\big((c, \infty)\big)$ are open, $A \subseteq f^{-1}\big((-\infty, c)\big) \cup f^{-1}\big((c, \infty)\big)$ is not connected. $\square$

examples of metric spaces

Any normed vector space is a metric space with the induced metric $d(x, x') := ||x - x'||$.

For $p \in (0, \infty)$, $l_p := \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \,\middle|\, \sum_{n\in\mathbb{N}} |x_n|^p < \infty\right\}$ is a normed vector space with $||x||_p := \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{1/p}$.

> *The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{Z}_+}$ is a member of l_p if and only if $p > 1$.*

Proof. $\left(\frac{1}{n}\right)_{n\in\mathbb{Z}_+} \in l_p \iff \sum_{n\in\mathbb{Z}_+} \left(\frac{1}{n}\right)^p < \infty \iff p > 1.$ $\square$

For $p \in [1, \infty)$, $L^p([a,b]) := \left\{f(x) \,\middle|\, \int_a^b |f(x)|^p dx < \infty\right\}$ is a normed vector space with $||f||_p := (\int_a^b |f(x)|^p dx)^{1/p}$.

$d(A, A') := \text{vol}_n(A \triangle A')$ is a possible metric on $\mathbb{R}^n$.

For a metric space (X, d), a set $A \subseteq X$, and $\epsilon > 0$, let $A_\epsilon \coloneqq \bigcup_{x \in A} B(x, \epsilon)$. Then the Hausdorff metric is $d_H(A, B) \coloneqq \inf\left\{ \epsilon > 0 \;\middle|\; A' \subseteq A_\epsilon \wedge A \subseteq A'_\epsilon \right\}$.

p-adic numbers

Given a fixed prime p,

$$\forall q \in \mathbb{Q} : \exists (a, b, n) \in \mathbb{Z}^3 : \left[q = p^n \cdot \frac{a}{b} \wedge \gcd(a, p) = \gcd(b, p) = 1 \right],$$

and $\mathbb{Q}$ is a normed vector space with $||q||_p \coloneqq \begin{cases} p^{-n} & q \neq 0 \\ 0 & q = 0 \end{cases}$.

The 2-adic norm of $\frac{96}{7}$ is $\frac{1}{32}$.

The 3-adic norm of $3^{-2024} \cdot \frac{8}{13}$ is 3^{2024}.

The p-adic norm $||q||_p$ is small if q is divisible by a large power of p.

The p-adic norm of 0 is 0 because 0 is divisible by any power of p.

(p-adic product formula.) If $q \in \mathbb{Q} \setminus \{0\}$, then $|q| \cdot \prod_{p \text{ prime}} ||q||_p = 1$.

convexness

A set X is convex if and only if the line segment joining any two points in X lies within X.