

MATH 455 (Honours Analysis 4)

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preliminaries

topological spaces

A **topological space** (X, τ) is a **set** X endowed with a **topology** τ .

A **topology** τ on a set X is a collection of sets that contains \emptyset and X and is closed under **finite intersections and arbitrary unions**. Membership in τ defines an **open set** in the topological space (X, τ) .

2024-01-05 topology & metric spaces

connectedness

A topological space (X, τ) is a **connected space** if and only if $\neg \exists U, U' \in \tau \setminus \{\emptyset\} : U \cap U' = \emptyset \wedge X = U \cup U'$.

For a topological space (X, τ) , $A \subseteq X$ is a **connected set** if and only if $\neg \exists U, U' \in \tau : A \cap U \neq \emptyset, A \cap U' \neq \emptyset, U \cap U' = \emptyset, A = (U \cap A) \cup (U' \cap A)$.

A topological space X is a **path connected space** if and only if $\forall x, x' \in X : \exists$ continuous $f : [0, 1] \rightarrow X : [f(0) = x \wedge f(1) = x']$.

All path connected spaces are **connected spaces**.

For **open** sets in \mathbb{R}^n , **connectedness** is equivalent to **path connectedness**.

A topological space X can be expressed as the **disjoint union of maximal connected subsets**, where a connected subset is called maximal if and only if it has no connected superset in X . These subsets are the **connected components** of X .

A topological space X can be expressed as the disjoint union of maximal path connected subsets, where a path connected subset is called maximal if and only if it has no path connected superset in X . These subsets are the path components of X .

A path connected space has exactly one path component.

$\left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x \in (0, 1] \right\} \cup \{(0, 0)\} \subseteq \mathbb{R}^2$ is connected but not path connected because it has two path components.

If $f : X \rightarrow \mathbb{R}$ is continuous, $A \subseteq X$ is connected, and $x, x' \in f(A)$ such that $x < x'$, then $[x, x'] \subseteq f(A)$.

Proof. Suppose $\exists c \in (f(x'), f(x)) : c \notin f(A)$. Since $f^{-1}((-\infty, c))$ and $f^{-1}((c, \infty))$ are open, $A \subseteq f^{-1}((-\infty, c)) \cup f^{-1}((c, \infty))$ is not connected. \square

examples of metric spaces

Any normed vector space is a metric space with the induced metric $d(x, x') := \|x - x'\|$.

For $p \in (0, \infty)$, $l_p := \{(x_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \sum_{n \in \mathbb{N}} |x_n|^p < \infty\}$ is a normed vector space with $\|x\|_p := \left(\sum_{n \in \mathbb{N}} |x_n|^p\right)^{1/p}$.

The sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{Z}_+}$ is a member of l_p if and only if $p > 1$.

Proof. $\left(\frac{1}{n}\right)_{n \in \mathbb{Z}_+} \in l_p \iff \sum_{n \in \mathbb{Z}_+} \left(\frac{1}{n}\right)^p < \infty \iff p > 1$. \square

For $p \in [1, \infty)$, $L^p([a, b]) := \left\{ f(x) \mid \int_a^b |f(x)|^p dx < \infty \right\}$ is a normed vector space with $\|f\|_p := \left(\int_a^b |f(x)|^p dx\right)^{1/p}$.

$d(A, A') := \text{vol}_n(A \Delta A')$ is a possible metric on \mathbb{R}^n .

For a metric space (X, d) , a set $A \subseteq X$, and $\epsilon > 0$, let $A_\epsilon := \bigcup_{x \in A} B(x, \epsilon)$. Then the Hausdorff metric is $d_H(A, A') := \inf\{\epsilon > 0 \mid A' \subseteq A_\epsilon \wedge A \subseteq A'_\epsilon\}$.

p -adic numbers

Given a fixed prime p ,

$$\forall q \in \mathbb{Q} \setminus \{0\} : \exists (a, b, n) \in \mathbb{Z}^3 : \left[q = p^n \cdot \frac{a}{b} \wedge \gcd(a, p) = \gcd(b, p) = 1 \right]$$

and \mathbb{Q} is a normed vector space with $\|q\|_p := \begin{cases} p^{-n} & q \neq 0 \\ 0 & q = 0 \end{cases}$.

The 2-adic norm of $\frac{96}{7}$ is $\frac{1}{32}$.

The 3-adic norm of $3^{-2024} \cdot \frac{8}{13}$ is 3^{2024} .

The p -adic norm $\|q\|_p$ is small if q is divisible by a large power of p .

The p -adic norm of 0 is 0 because 0 is divisible by any power of p .

(p -adic product formula.) If $q \in \mathbb{Q} \setminus \{0\}$, then $|q| \cdot \prod_{p \text{ prime}} \|q\|_p = 1$.

convexness —————

A set X is **convex** if and only if the line segment joining any two points in X lies within X .

2024-01-10 contraction mappings & product topology

$\alpha \in I$ henceforth refers to members of a possibly uncountable index set I .

If $\{A_\alpha\}_{\alpha \in I}$ is a family of **connected** sets in a topological space X such that $\forall \alpha, \alpha' \in I : A_\alpha \cap A_{\alpha'} \neq \emptyset$, then $\bigcup_{\alpha \in I} A_\alpha$ is **connected**.

Proof. Suppose $A := \bigcup_{\alpha \in I} A_\alpha$ is not connected. Let $U, U' \subseteq X$ be open and nonempty relative to A such that $U \cap U' = \emptyset$ and $A = U \cup U'$. Since each A_α is connected, there is no α for which $A_\alpha \cap U \neq \emptyset \neq A_\alpha \cap U'$. But then $\exists \alpha, \alpha' \in I : A_\alpha \subseteq U \wedge A_{\alpha'} \subseteq U'$, which contradicts that $A_\alpha \cap A_{\alpha'} \neq \emptyset$. \square

total disconnectedness —————

A topological space X is **totally disconnected** if and only if every connected subset of X is a singleton.

A **totally disconnected** space in \mathbb{R} contains **only points** and **no intervals**.

The **Cantor set** is a totally disconnected space in \mathbb{R} .

contraction mappings

For a metric space (X, d) , $T : X \rightarrow X$ is a **contraction mapping** if and only if $\exists c \in [0, 1) : \forall x, x' \in X : d(T(x), T(x')) \leq c \cdot d(x, x')$.

All contraction mappings are **continuous**.

(Orbit lemma.) For $x \in X$, the **orbit** $(T^n(x))_{n \in \mathbb{N}}$ of a **contraction mapping** T on X is a **Cauchy sequence**.

Proof. For $n \in \mathbb{Z}_+$,

$$\begin{aligned} d(T^n(x), T^{n+1}(x)) &< c \cdot d(T^{n-1}(x), T^n(x)) \\ &< c^2 \cdot d(T^{n-2}(x), T^{n-1}(x)) \\ &\dots < c^n \cdot d(x, T(x)). \end{aligned}$$

Let $m \geq n$. By the triangle inequality,

$$\begin{aligned} d(T^n(x), T^m(x)) &\leq \sum_{k=n}^{m-1} d(T^k(x), T^{k+1}(x)) \\ &\leq d(x, T(x)) \sum_{k=n}^{m-1} c^k \\ &= c^n \cdot d(x, T(x)) \sum_{k=0}^{m-n-1} c^k \\ &\leq \frac{c^n \cdot d(x, T(x))}{1 - c}. \end{aligned}$$

For $\epsilon > 0$, choosing $n > \log_c \left(\frac{\epsilon}{2} \cdot \frac{1-c}{d(x, T(x))} \right)$ and $m, m' \geq n$ guarantees

$$\begin{aligned} d(T(m), T(m')) &\leq \\ d(T(n), T(m)) + d(T(n), T(m')) &< \epsilon. \end{aligned}$$

□

(**Contraction mapping theorem.**) If (X, d) is a **nonempty** and **complete** metric space and T is a **contraction mapping on X** , then

$$\exists! z \in X : T(z) = z,$$

i.e. z is *the unique fixed point*, and

$$\forall x \in X : \lim_{n \rightarrow \infty} T^n(x) = z,$$

where $\forall n \in \mathbb{N} : T^n(x) := T(T(\cdots T(x)))$.
 n times

Proof. For $x \in X$, $T(x)$ is Cauchy by the orbit lemma, and since (X, d) is complete it converges to some point $z \in X$. Then z is a fixed point of T because

$$\begin{aligned} T^n(x) \rightarrow z &\implies T(T^n(x)) \rightarrow T(z) \\ &\implies T(T^n(x)) = T^{n+1}(x) \rightarrow z = T(z), \end{aligned}$$

and is its unique fixed point because

$$T(z) = z \wedge T(z') = z' \implies d(z, z') \leq c \cdot d(z, z') \iff d(z, z') = 0. \quad \square$$

$x \mapsto \frac{x}{2}$ is a contraction mapping on $(\mathbb{R}, |\cdot|)$ with fixed point $z = 0$.

iterated function systems and fixed point sets —————

$\mathcal{K}(X)$ henceforth denotes the set of compact subsets of a set X .

If $m \in \mathbb{Z}_+$, $\{T_n : \mathbb{R}^d \rightarrow \mathbb{R}^d\}_{n \in [m]}$ are contraction mappings, and

$$F : (\mathcal{K}(\mathbb{R}^d), d_H) \rightarrow (\mathcal{K}(\mathbb{R}^d), d_H) : A \mapsto \bigcup_{n \in [m]} T_n(A),$$

then $\exists! A \in \mathcal{K}(\mathbb{R}^d) : F(A) = A$. This is the *fixed point set* of F .

Proof. F is a contraction mapping (to be demonstrated in problem set 2), and $(\mathcal{K}(X), d_H)$ is a complete metric space (???). A unique fixed point set thus exists by the contraction mapping theorem. \square

Let $T_0, T_1 : \mathbb{R} \rightarrow \mathbb{R}$ such that $T_0(x) := \frac{x}{3}$ and $T_1(x) := \frac{x+2}{3}$ and $F(A) : \mathcal{K}(\mathbb{R}) \rightarrow \mathcal{K}(\mathbb{R})$ such that $F(A) := T_0(A) \cup T_1(A)$. Then $F([0, 1]) = T_0([0, 1]) \cup T_1([0, 1]) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and the *fixed point set* of F is *the middle-thirds Cantor set*.

If $T_0, T_1 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are *contraction mappings* and $F : (\mathcal{K}(\mathbb{R}^d), d_H) \rightarrow (\mathcal{K}(\mathbb{R}^d), d_H)$ such that $F(A) = T_0(A) \cup T_1(A)$, then the composition of F with itself is

$$F^2(A) = T_0(T_0(A)) \cup T_0(T_1(A)) \cup T_1(T_0(A)) \cup T_1(T_1(A)).$$

basis for a topology

A **basis** \mathcal{B} for a topology τ is a subset of τ such that $\forall A \in \tau : \exists \{A_\alpha\}_{\alpha \in I} \subseteq \mathcal{B} : A = \bigcup_{\alpha \in I} A_\alpha$.

The open intervals form a basis for the standard topology on \mathbb{R} .

For a metric space X , $\{B(x, \epsilon) \mid x \in X, \epsilon > 0\}$ is a basis for the open sets in X .

If \mathcal{B} is a basis for the topology on X , then $\forall x \in X : \exists U \in \mathcal{B} : x \in U$.

If \mathcal{B} is a basis for the topology τ on X , then $[U, U' \in \mathcal{B} \wedge x \in U \cap U'] \implies \exists U'' \in \tau : x \in U'' \subseteq U \cap U'$.

If $f : X \rightarrow Y$ for topological spaces (X, τ_X) and (Y, τ_Y) and \mathcal{B} is a basis for τ_Y , then f is continuous if and only if $\forall U \in \mathcal{B} : f^{-1}(U) \in \tau_X$.

product topology

For topological spaces $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in I}$, $\prod_{\alpha \in I} A_\alpha$ is a cylinder set in $X := \prod_{\alpha \in I} X_\alpha$ if and only if $\forall \alpha \in I : A_\alpha \in \tau_\alpha$.

If $X := \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, then \forall open intervals $(a, b), (a', b') \subseteq \mathbb{R} : (a, b) \times (a', b')$ is a cylinder set in X .

For topological spaces $\{X_\alpha\}_{\alpha \in I}$, consider a cylinder set $\prod_{\alpha \in I} A_\alpha$ in $X := \prod_{\alpha \in I} X_\alpha$ such that $\exists I' \subseteq I : [I' \in \mathbb{N} \wedge \forall \alpha \in I \setminus I' : A_\alpha = X_\alpha]$. These base cylinder sets form a basis for the product topology on X .

projection maps

For a vector $x := (x_\alpha)_{\alpha \in I}$ in possibly uncountable dimensions, the function $\pi_\alpha(x) := x_\alpha$ is a projection map.

For topological spaces $\{X_\alpha\}_{\alpha \in I}$ let $f : Y \rightarrow \prod_{\alpha \in I} X_\alpha$, and for $\alpha \in I$ let $f_\alpha : Y \rightarrow X_\alpha : y \mapsto \pi_\alpha(f(y))$. Then f is continuous if and only if $\forall \alpha \in I : f_\alpha$ is continuous.

Proof. Suppose f is continuous. Then, for $\alpha' \in I$ and an open set $U \subseteq X_{\alpha'}$,

$$f_{\alpha'}^{-1}(U) = f^{-1}\left(U \times \prod_{\alpha \in I \setminus \{\alpha'\}} X_\alpha\right)$$

is open as the preimage of a base cylinder set in $\prod_{\alpha \in I} X_\alpha$.

Suppose $\forall \alpha \in I : f_\alpha$ is continuous. It suffices to verify that the preimage of f for a base cylinder set U is open. Let $I' \subseteq I$ be a finite index subset for which

$$U = \prod_{\alpha \in I'} A_\alpha \times \prod_{\alpha \in I \setminus I'} X_\alpha.$$

Then

$$\begin{aligned} f^{-1}(U) &= \left(\bigcap_{\alpha \in I'} f_\alpha^{-1}(A_\alpha) \right) \cap \left(\bigcap_{\alpha \in I \setminus I'} f_\alpha^{-1}(X_\alpha) \right) \\ &= \bigcap_{\alpha \in I'} f_\alpha^{-1}(A_\alpha) \end{aligned}$$

is open in Y as the finite intersection of open sets in Y . □

2024-01-12 Hilbert spaces & Hausdorff spaces

inner product spaces

The closed unit ball in l_p is not **compact** because it contains $(x_n)_{n \in \mathbb{N}}$ where

$$\forall n, k \in \mathbb{N} : \pi_k(x_n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases},$$

which has no convergent subsequence.

An **inner product space** V is a **normed vector space** with an **inner product operation** $\langle v, v' \rangle$ for $v, v' \in V$ that induces its **norm** $\|v\|^2 =: \langle v, v \rangle$.

(**Parallelogram law**.) The **norm** in a vector space V is induced by an **inner product** if and only if

$$\forall v, v' \in V : \|v + v'\|^2 + \|v - v'\|^2 = 2(\|v\|^2 + \|v'\|^2)$$

holds.

(**Hanner's inequality**.) For a measure space X , $p \in [1, \infty) \setminus \{2\}$, and $f, g \in L^p(X)$,

$$\|f + g\|_p^p + \|f - g\|_p^p * (\|f\|_p + \|g\|_p)^p + |\|f\|_p - \|g\|_p|^p$$

where $*$:= $\begin{cases} \geq & p < 2 \\ \leq & p > 2 \end{cases}$. If $p = 2$, then this becomes **the parallelogram law**.

Hilbert spaces

A Hilbert space is a complete inner product space.

l_2 is a Hilbert space with $\langle (a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \rangle = \sum_{n \in \mathbb{N}} |a_n \overline{b_n}|$.

For a measure space X , $L^2(X)$ is a Hilbert space with $\langle f, g \rangle := \int_X f(x) \overline{g(x)} dx$.

(Parseval's identity.) For $f \in L^2([0, 2\pi])$, which has a basis $\{e^{inx}\}_{n \in \mathbb{Z}}$,

$$\begin{aligned} \|f\|_2^2 &= \int_0^{2\pi} |f(x)|^2 dx \\ &= 2\pi \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 = \langle f, f \rangle \end{aligned}$$

where $\hat{f}(n) = \langle f, e^{inx} \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$.

(Consequence of Riesz's lemma.) The closed unit ball in a Hilbert space V is compact if and only if V is finite-dimensional.

compact sets & continuity

Continuous functions map compact sets to compact sets.

Proof. Let f be continuous and A a compact set in its domain. The preimages of an open cover U of $f(A)$ form an open cover U' of A . Then a finite subcover of U' exists, and the images of sets in U' form a finite subcover of U . \square

(Tychonoff.) If $\forall \alpha \in I : X_\alpha$ is compact, then $\prod_{\alpha \in I} X_\alpha$ is compact in the product topology.

For topological spaces X and Y , a bijection $f : X \rightarrow Y$ is a homeomorphism if and only if f and f^{-1} are both continuous.

For a subset A of a metric space (X, d) and $x \in X$, $d(x, A) := \inf_{x' \in A} d(x, x')$ is continuous.

Proof. Let $\epsilon, \delta > 0$. Then $\exists a \in A : d(x, a) < \delta$, so for $x' \in B(x, \epsilon)$

$$\begin{aligned} d(x', A) &\leq d(x', a) \\ &\leq d(x, a) + d(x, x') \\ &< d(x, A) + \delta + \epsilon. \end{aligned}$$

Taking the limit as $\delta \rightarrow 0$ gives $d(x', A) \leq d(x, A) + \epsilon$. By parallel reasoning, $d(x, A) \leq d(x', A) + \epsilon$, hence $|d(x, A) - d(x', A)| \leq \epsilon$. \square

Hausdorff spaces

A topological space (X, τ) is a **Hausdorff space** if and only if $\forall x \neq x' \in X : \exists U, U' \in \tau : [U \cap U' = \emptyset \wedge (x, x') \in U \times U']$.

All **metric spaces** are Hausdorff spaces.

*Singletons in a **Hausdorff space** X with topology τ are **closed**.*

Proof. Let $x' \in X$. Then

$$\forall x \in X : \exists U_x \in \tau : [x \in U_x \wedge x' \notin U_x] \implies X \setminus \{x'\} = \bigcup_{x' \in X \setminus \{x'\}} U_x \in \tau. \quad \square$$

$\tau := \{\emptyset, X\}$ is the **trivial topology** on X .

If τ is the **trivial topology** and $|X| > 1$, then (X, τ) is **not a Hausdorff space**.

The **subspace topology** on a subspace Y of a topological space (X, τ) is $\tau_Y := \{U \cap Y \mid U \in \tau\}$.

Any subspace of a **Hausdorff space** is a **Hausdorff space** under the **subspace topology**.

*For Hausdorff spaces (X, τ) and (X', τ') , $X \times X'$ under the product topology is a **Hausdorff space**.*

Proof. Let \mathcal{B} be the cylinder set basis for the product topology and let $(x_0, x'_0) \neq (x_1, x'_1) \in X \times Y$, assuming without loss of generality that $x_0 \neq x_1$. Then

$$\exists U_0, U_1 \in \tau : [U_0 \cap U_1 = \emptyset \wedge (x_0, x_1) \in U_0 \times U_1],$$

so $U_0 \times X'$ and $U_1 \times X'$ are disjoint open sets that contain (x_0, x'_0) and (x_1, x'_1) respectively. \square

*Any **compact** subspace of a Hausdorff space is **closed**.*

Proof. Let (X, τ) be a Hausdorff space, $A \subseteq X$ a compact subset, and $x' \in X \setminus A$. For $x \in A$, let $U_x, U'_x \in \tau$ such that $U_x \cap U'_x = \emptyset$ and $(x, x') \in U_x \times U'_x$.

Since $\{U_x\}_{x \in A}$ is an open cover of A , let $\{x_n\}_{n \in [m]} \subseteq A$ such that $\{U_{x_n}\}_{n \in [m]}$ covers A . Then $x' \in \bigcap_{n \in [m]} U'_{x_n} \subseteq X \setminus A$, and this intersection is open and disjoint to $\bigcup_{n \in [m]} U_{x_n}$. Thus $X \setminus A$ is open. \square

A *continuous bijection* $f : X \rightarrow Y$ is a *homeomorphism* if X is *compact* and Y is a *Hausdorff space*.

Proof. Let $A \subseteq X$ be closed. Then A is compact, so $(f^{-1})^{-1}(A) = f(A)$ is closed as a compact subset of a Hausdorff space. It follows that f^{-1} is continuous. \square

normal spaces

A Hausdorff space (X, τ) is a *normal space* if and only if

$$\exists U, U' \in \tau : [U \cap U' \neq \emptyset \wedge A \subseteq U \wedge A' \subseteq U']$$

for any *disjoint closed sets* $A, A' \subseteq X$.

All *compact Hausdorff space* are *normal spaces*.

Proof. Let (X, τ) be a compact Hausdorff space and $A, A' \subseteq X$ closed sets with $A \cap A' = \emptyset$. For $x \in A$, let $U_x, U'_x \in \tau$ such that $U_x \cap U'_x = \emptyset$, $x \in U_x$, and $A' \subseteq U'_x$.

Since $\{U_x\}_{x \in A}$ is an open cover of A and A is compact as the closed subset of a compact set, let $\{x_n\}_{n \in [m]} \subseteq A$ such that $\{U_{x_n}\}_{n \in [m]}$ covers A . It follows that $\bigcup_{n \in [m]} U_{x_n}$ and $\bigcap_{n \in [m]} U'_{x_n} \supseteq A'$ are open and disjoint. \square

All *metric spaces* are *normal spaces*.

Proof. Let $A, A' \subseteq X$ be disjoint and closed. Then

$$U_A := \{x \mid d(x, A) < d(x, A')\}, \quad U'_A := \{x \mid d(x, A') < d(x, A)\},$$

are disjoint, and they are respective open supersets of A and A' because $x \in A \implies d(x, A) = 0 < d(x, A')$ (and vice versa) and

$$U_A = f^{-1}((-\infty, 0)), \quad U'_A = f^{-1}((0, \infty))$$

where $f : x \mapsto d(x, A) - d(x, A')$ is continuous as the difference between continuous functions. \square

Lebesgue number

A *Lebesgue number* of an open cover $\{A_\alpha\}_{\alpha \in I}$ of X is $\epsilon > 0$ such that $\forall x \in X : \exists \alpha \in I : B(x, \epsilon) \subseteq A_\alpha$.

Any open cover of a compact metric space has a Lebesgue number.

Proof. Let X be a compact metric space and $\{U_\alpha\}_{\alpha \in I}$ an open cover of X with finite subcover $\{U_{\alpha_n}\}_{n \in [m]}$. For $n \in [m]$ and $x \in X$, let

$$d_n(x) := d(x, X \setminus U_n),$$

noting that $B(x, \epsilon) \subseteq U_n \iff d_n(x) \geq \epsilon$ and that $\exists n \in [m] : x \in U_n \implies d_n(x) > 0$.

Let $f(x) := \max_{n \in [m]} d_n(x)$, which is continuous as the maximum of continuous functions. Then $f(X) \subseteq (0, \infty)$ is compact hence closed and bounded. It follows that $\exists \delta > 0 : f(X) \subseteq [\delta, \infty)$ and δ is a Lebesgue number for $\{U_\alpha\}_{\alpha \in I}$. \square

If $f : X \rightarrow Y$ is continuous, X and Y are metric spaces, and X is compact, then f is uniformly continuous.

Proof. For $\epsilon > 0$, let δ be a Lebesgue number for $\{f^{-1}(B(y, \epsilon))\}_{y \in Y}$. Then $\forall x \in X : \exists y \in Y : f(B(x, \delta)) \subseteq B(y, \epsilon)$, so for $x' \in B(x, \delta)$

$$\begin{aligned} d(f(x), f(x')) &\leq \\ d(f(x), y) + d(f(x'), y) &< 2\epsilon. \end{aligned}$$

\square

2024-01-17 product spaces & separable spaces

metrizability

A topological space (X, τ) is metrizable if and only if there exists a metric d on X that induces τ . In other words, the set of open balls with respect to d is a basis for τ .

Any topological space that is not normal is non-metrizable.

If $\forall n \in \mathbb{N} : (X_n, d_n)$ are metric spaces, then $X := \prod_{n \in \mathbb{N}} X_n$ is metrizable under the product topology.

Proof. For $n \in \mathbb{N}$ and $x_n, y_n \in X_n$ let $\tilde{d}(x_n, y_n) := \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} \leq 1$, noting that this is an equivalent metric to d_n . Then define

$$D : X \rightarrow \mathbb{R} : (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} \left(\frac{\tilde{d}_n(x_n, y_n)}{2^n} \right).$$

As shown in problem set 1, D induces the product topology on X . \square

countable product of compact metric spaces

$\tau := \mathcal{P}(X)$ is the discrete topology on X .

$X := \prod_{n \in \mathbb{N}} \{0, 1\}$ is not compact under the discrete topology because the set of singletons in X is an open cover of X with no finite subcover.

(Tychonoff for countable products.) If $\forall n \in \mathbb{N} : (X_n, \tau_n)$ is a compact topological space, then $X := \prod_{n \in \mathbb{N}} X_n$ is compact under the product topology.

Proof. Suppose X is not compact and let $C := \{C_\alpha\}_{\alpha \in I}$ be an open cover of X with no finite subcover.

Suppose for every $x_0 \in X_0$ that there is a set $U_{x_0} \in \tau_0$ such that

$$x_0 \in U_{x_0} \wedge \exists I' \subseteq I : \left[|I'| \in \mathbb{N} \wedge \left(U_{x_0} \times \prod_{n \in \mathbb{Z}_+} X_n \right) \subseteq \bigcup_{\alpha \in I'} C_\alpha \right]$$

Then

$$X \subseteq \bigcup_{x_0 \in X_0} \left(U_{x_0} \times \prod_{n \in \mathbb{Z}_+} X_n \right).$$

Since X_0 is compact, this open cover has a finite subcover and contradicts that C has no finite subcover. Thus there is some $x_0 \in X_0$ for which no such U_{x_0} exists.

By similar reasoning, there exists $x_1 \in X_1$ such that no cylinder set of the form $U_{x_0} \times U_{x_1} \times \prod_{n \in \mathbb{Z}_+ \setminus \{1\}} X_n$ both satisfies $(x_0, x_1) \in U_{x_0} \times U_{x_1}$ and has a finite open cover in C ; otherwise,

$$\left(U_{x_0} \times \prod_{n \in \mathbb{Z}_+} X_n \right) \subseteq \bigcup_{x_1 \in X_1} \left(U_{x_0} \times U_{x_1} \times \prod_{n \in \mathbb{Z}_+ \setminus \{1\}} X_n \right)$$

would have a finite subcover by compactness of X_1 and thus $(U_{x_0} \times \prod_{n \in \mathbb{Z}_+} X_n)$ would have a finite open cover in C .

Selecting $x_n \in X_n$ in this fashion for every $n \in \mathbb{N}$ provides a point $x := (x_n)_{n \in \mathbb{N}} \in X$ such that, for any $m \in \mathbb{N}$, every cylinder set of the form

$$\prod_{n=0}^m U_n \times \prod_{n=m+1}^{\infty} X_n,$$

where $(x_n)_{n=0}^m \in \prod_{n=0}^m U_n$, has no finite open cover in C . But this cannot be; $\exists C_\alpha \in C : x \in C_\alpha$, implying that x lies in some cylinder set $V := \prod_{n \in \mathbb{N}} V_n \subseteq C_\alpha$ in

the product topology basis. Taking m to be the maximum index n for which $V_n \neq X_n$ gives

$$(x_n)_{n \in \mathbb{N}} \in \left(\prod_{n=0}^m V_n \times \prod_{n=m+1}^{\infty} X_n \right) \subseteq C_\alpha,$$

which is certainly a finite cover. □

separability

A metric space is **separable** if and only if it has a countable dense subset.

*All **compact** metric spaces are separable.*

Proof. Let X be a compact metric space. For $n \in \mathbb{Z}_+$, $X \subseteq \bigcup_{x \in X} B(x, \frac{1}{n})$, and by compactness, let $N_n \subseteq X$ be a finite subset such that $X \subseteq \bigcup_{x \in N_n} B(x, \frac{1}{n})$. Thus $\bigcup_{n \in \mathbb{Z}_+} N_n \subseteq X$ is dense and countable as the countable union of finite sets. □

A metric space X with topology τ is **separable** if and only if $\exists \{U_n\}_{n \in \mathbb{N}} \subseteq \tau : \forall U \in \tau : \exists N \subseteq \mathbb{N} : U = \bigcup_{n \in N} U_n$; in other words, τ has a countable basis.

Any **subspace** of a separable metric space is **separable**.

\mathbb{R}^n , $C([a, b])$, and $L^p([a, b])$ for $p \in [1, \infty)$ are examples of separable spaces.

$L^\infty([a, b])$ is an example of a **non-separable** space.