MATH 455 (Honours Analysis 4)

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The following notes are adapted from lectures given by Dmitry Jakobson in Winter 2024. All errors in interpretation, reasoning, coherence, and articulation are my own.

This document's source code, located at `https://github.com/brunefig/math455/blob/main/notes.org`, can be converted into Anki flashcards with the `anki-editor` package for GNU Emacs. Flashcard cloze deletions are typeset in magenta.

2024-01-05 topology & metric spaces

connectedness

A topological space X is a connected space if and only if there are no open sets $U, U' \in \mathscr{P}(X) \setminus \{\varnothing\}$ such that $U \cap U' = \varnothing$ and $X = U \cup U'$.

For a topological space X, $A \subseteq X$ is a connected set if and only if there are no open sets $U, U' \subseteq X$ such that $A \cap U \neq \varnothing$, $A \cap U' \neq \varnothing$, $U \cap U' = \varnothing$, and $A = (U \cap A) \cup (U' \cap A)$.

A topological space X is a path connected space if and only if $\forall x, x' \in X : \exists$ continuous $f : [0,1] \to X : [f(0) = x \wedge f(1) = x']$.

All path connected spaces are connected spaces.

For open sets in $\mathbb{R}^n$, connectedness is equivalent to path connectedness.

A topological space X can be expressed as the disjoint union of maximal connected subsets, where a connected subset is called maximal if and only if it has no connected superset in X. These subsets are the connected components of X.

A topological space X can be expressed as the disjoint union of maximal path connected subsets, where a path connected subset is called maximal if and only if it has no path connected superset in X. These subsets are the path components of X.

A path connected space has exactly one path component.

$\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \middle| x \in (0,1]\right\} \cup \{(0,0)\} \subseteq \mathbb{R}^2$ is connected but not path connected because it has two path components.

If $f : X \to \mathbb{R}$ is continuous, $A \subseteq X$ is connected, and $x, x' \in f(A)$ such that $x < x'$, then $[x, x'] \subseteq f(A)$.

Proof. Suppose $\exists c \in \big(f(x'), f(x)\big) : c \notin f(A)$. Since $f^{-1}\big((-\infty, c)\big)$ and $f^{-1}\big((c, \infty)\big)$ are open, $A \subseteq f^{-1}\big((-\infty, c)\big) \cup f^{-1}\big((c, \infty)\big)$ is not connected. □

examples of metric spaces

Any normed vector space is a metric space with the induced metric $d(x, x') := ||x - x'||$.

For $p \in (0, \infty)$, $l_p := \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \middle| \sum_{n\in\mathbb{N}} |x_n|^p < \infty\right\}$ is a normed vector space with $||x||_p := \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{1/p}$.

The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{Z}_+}$ is a member of l_p if and only if $p > 1$.

Proof. $\left(\frac{1}{n}\right)_{n\in\mathbb{Z}_+} \in l_p \iff \sum_{n\in\mathbb{Z}_+} \left(\frac{1}{n}\right)^p < \infty \iff p > 1$. □

For $p \in [1, \infty)$, $L^p([a,b]) := \left\{f(x) \middle| \int_a^b |f(x)|^p \, dx < \infty\right\}$ is a normed vector space with $||f||_p := \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}$.

$d(A,A') \coloneqq \mathrm{vol}_n(A \triangle A')$ is a possible metric on $\mathbb{R}^n$.

For a metric space (X,d), a set $A \subseteq X$, and $\epsilon > 0$, let $A_\epsilon \coloneqq \bigcup_{x \in A} B(x,\epsilon)$. Then the Hausdorff metric is $d_H(A,A') \coloneqq \inf\{\epsilon > 0 \mid A' \subseteq A_\epsilon \wedge A \subseteq A'_\epsilon\}$.

p-adic numbers

Given a fixed prime p,

$$\forall q \in \mathbb{Q} \setminus \{0\} : \exists (a,b,n) \in \mathbb{Z}^3 : \left[q = p^n \cdot \frac{a}{b} \wedge \gcd(a,p) = \gcd(b,p) = 1 \right]$$

and $\mathbb{Q}$ is a normed vector space with $||q||_p \coloneqq \begin{cases} p^{-n} & q \neq 0 \\ 0 & q = 0 \end{cases}$.

The 2-adic norm of $\frac{96}{7}$ is $\frac{1}{32}$.

The 3-adic norm of $3^{-2024} \cdot \frac{8}{13}$ is 3^{2024}.

The p-adic norm $||q||_p$ is small if q is divisible by a large power of p.

The p-adic norm of 0 is 0 because 0 is divisible by any power of p.

(p-adic product formula.) If $q \in \mathbb{Q} \setminus \{0\}$, then $|q| \cdot \prod_{p \text{ prime}} ||q||_p = 1$.

convexness

A set X is convex if and only if the line segment joining any two points in X lies within X.

2024-01-10 contraction mappings & product topologies

$\alpha \in I$ henceforth refers to members of a possibly uncountable index set I.

If $\{A_\alpha\}_{\alpha \in I}$ is a family of connected sets such that $\forall \alpha, \alpha' \in I : A_\alpha \cap A_{\alpha'} \neq \emptyset$, then $\bigcup_{\alpha \in I} A_\alpha$ is connected.

Proof. Suppose $A \coloneqq \bigcup_{\alpha \in I} A_\alpha$ is not connected. Let $U, U' \subseteq X$ be open and nonempty relative to A such that $U \cap U' = \emptyset$ and $A = U \cup U'$. Since A_α is connected, there is no α for which $A_\alpha \cap U \neq \emptyset \neq A_\alpha \cap U'$. But then $\exists \alpha, \alpha' \in I : A_\alpha \subseteq U \wedge A_{\alpha'} \subseteq U'$, which contradicts that $A_\alpha \cap A_{\alpha'} \neq \emptyset$. □

total disconnectedness

A topological space X is totally disconnected if and only if every connected subset of X is a singleton.

A totally disconnected space in $\mathbb{R}$ contains only points and no intervals.

The Cantor set is a totally disconnected space in $\mathbb{R}$.

contraction mappings

For a metric space (X,d), $T : X \to X$ is a contraction mapping if and only if $\exists c \in [0,1) : \forall x, x' \in X : d\big(T(x), T(x')\big) \le c \cdot d(x,x')$.

All contraction mappings are continuous.

> *(Orbit lemma.) For $x \in X$, the orbit $(T^n(x))_{n\in\mathbb{N}}$ of a contraction mapping T on X is a Cauchy sequence.*

Proof. For $n \in \mathbb{Z}_+$,

$$\begin{aligned} d\left(T^n(x), T^{n+1}(x)\right) &< c \cdot d\left(T^{n-1}(x), T^n(x)\right) \\ &< c^2 \cdot d\left(T^{n-2}(x), T^{n-1}(x)\right) \\ \cdots &< c^n \cdot d\big(x, T(x)\big). \end{aligned}$$

Let $m \in \mathbb{Z}_+ \cap (n, \infty)$. By the triangle inequality,

$$\begin{aligned} d(T^n(x), T^m(x)) &\le \sum_{k=n}^{m-1} d(T^k(x), T^{k+1}(x)) \\ &\le d(x, T(x)) \sum_{k=n}^{m-1} c^k \\ &= c^n \cdot d(x, T(x)) \sum_{k=0}^{m-n-1} c^k \\ &\le \frac{c^n \cdot d(x, T(x))}{1-c}. \end{aligned}$$

□

> *If $\lim_{n\to\infty} T^n(x) = z$, then $T(z) = z$.*

Proof. $T^n(x) \to z \implies T(T^n(x)) \to T(z)$, so $T(T^n(x)) = T^{n+1}(x) \to z = T(z)$. □

(Contraction mapping theorem.) If (X,d) is a nonempty and complete metric space and T is a contraction mapping on X, then

$$\exists! z \in X : T(z) = z,$$

where z is called the fixed point, and

$$\forall x \in X : \lim_{n\to\infty} T^n(x) = z,$$

where $\forall n \in \mathbb{N} : T^n(x) := \underbrace{T(T(\cdots T(x)))}_{n \text{ times}}$.

Proof. For $x \in X$, $T(x)$ is Cauchy by the orbit lemma, and since (X,d) is complete it converges to some point $z \in X$. Then z is a fixed point of T, and it is its unique fixed point because.

$$T(z) = z \wedge T(z') = z' \implies d(z,z') \le c \cdot d(z,z') \iff d(z,z') = 0.$$ □

$x \mapsto \frac{x}{2}$ is a contraction mapping on $(\mathbb{R}, |\cdot|)$ with fixed point $z = 0$.

iterated function systems and fixed point sets

For a set X, let $\mathscr{K}(X)$ denote the set of compact subsets of X.

For $m \in \mathbb{Z}_+$ and contraction mappings $\{T_n : \mathbb{R}^d \to \mathbb{R}^d\}_{n\in[m]}$, let

$$F : \left(\mathscr{K}(\mathbb{R}^d), d_H\right) \to \left(\mathscr{K}(\mathbb{R}^d), d_H\right)$$

such that $\forall A \in \mathscr{K}(\mathbb{R}^d) : F(A) = \bigcup_{n\in[m]} T_n(A)$. Then

$$\exists! A \in \mathscr{K}(\mathbb{R}^)d : F(A) = A.$$

This set is the fixed point set of F.

Proof. F is a contraction mapping (PS2), and $\left(\mathscr{K}(X), d_H\right)$ is a complete metric (???). A unique fixed point set thus exists by the contraction mapping theorem. □

Let $T_0, T_1 : \mathbb{R} \to \mathbb{R}$ such that $T_0(x) := \frac{x}{3}$ and $T_1(x) := \frac{x+2}{3}$ and $F(A) : \mathscr{K}(\mathbb{R}) \to \mathscr{K}(\mathbb{R})$ such that $F(A) := T_0(A) \cup T_1(A)$. Then $F\big([0,1]\big) = T_0\big([0,1]\big) \cup T_1\big([0,1]\big) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, and the fixed point set of F is the middle-thirds Cantor set.

If $T_0, T_1 : \mathbb{R}^d \to \mathbb{R}^d$ are contraction mappings and $F : \big(\mathscr{K}(\mathbb{R}^d), d_H\big) \to \big(\mathscr{K}(\mathbb{R}^d), d_H\big)$ such that $F(A) = T_0(A) \cap T_1(A)$, then $F^2(A) = T_0\big(T_0(A)\big) \cup T_0\big(T_1(A)\big) \cup T_1\big(T_0(A)\big) \cup T_1\big(T_1(A)\big)$.