MATH 455 (Honours Analysis 4)

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The following notes are adapted from lectures given by Dmitry Jakobson in Winter 2024. All errors in interpretation, reasoning, coherence, and articulation are my own.

This document's source code, located at `https://github.com/brunefig/math455/blob/main/notes.org`, can be converted into Anki flashcards with the `anki-editor` package for GNU Emacs. Flashcard cloze deletions are typeset in magenta.

preliminaries

topological spaces

A topological space (X,τ) is a set X endowed with a topology τ.

A topology τ on a set X is a collection of sets that contains $\emptyset$ and X and is closed under finite intersections and arbitrary unions. Membership in τ defines an open set in the topological space (X,τ).

2024-01-05 topology & metric spaces

connectedness

A topological space (X,τ) is a connected space if and only if $\neg\exists U,U' \in \tau\setminus\{\emptyset\} : U \cap U' = \emptyset \wedge X = U \cup U'$.

For a topological space (X,τ), $A \subseteq X$ is a connected set if and only if $\neg\exists U,U' \in \tau : A \cap U \neq \emptyset, A \cap U' \neq \emptyset, U \cap U' = \emptyset, A = (U \cap A) \cup (U' \cap A)$.

A topological space X is a path connected space if and only if $\forall x,x' \in X : \exists$ continuous $f : [0,1] \to X : [f(0) = x \wedge f(1) = x']$.

All path connected spaces are connected spaces.

For open sets in $\mathbb{R}^n$, connectedness is equivalent to path connectedness.

A topological space X can be expressed as the disjoint union of maximal connected subsets, where a connected subset is called maximal if and only if it has no connected superset in X. These subsets are the connected components of X.

A topological space X can be expressed as the disjoint union of maximal path connected subsets, where a path connected subset is called maximal if and only if it has no path connected superset in X. These subsets are the path components of X.

A path connected space has exactly one path component.

$\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \,\middle|\, x \in (0,1]\right\} \cup \{(0,0)\} \subseteq \mathbb{R}^2$ is connected but not path connected because it has two path components.

> *If $f : X \to \mathbb{R}$ is continuous, $A \subseteq X$ is connected, and $x, x' \in f(A)$ such that $x < x'$, then $[x, x'] \subseteq f(A)$.*

Proof. Suppose $\exists c \in \big(f(x'), f(x)\big) : c \notin f(A)$. Since $f^{-1}\big((-\infty, c)\big)$ and $f^{-1}\big((c, \infty)\big)$ are open, $A \subseteq f^{-1}\big((-\infty, c)\big) \cup f^{-1}\big((c, \infty)\big)$ is not connected. $\square$

examples of metric spaces

Any normed vector space is a metric space with the induced metric $d(x, x') := \|x - x'\|$.

For $p \in (0, \infty)$, $l_p := \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \,\middle|\, \sum_{n\in\mathbb{N}} |x_n|^p < \infty\right\}$ is a normed vector space with $\|x\|_p := \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{1/p}$.

> *The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{Z}_+}$ is a member of l_p if and only if $p > 1$.*

Proof. $\left(\frac{1}{n}\right)_{n\in\mathbb{Z}_+} \in l_p \iff \sum_{n\in\mathbb{Z}_+} \left(\frac{1}{n}\right)^p < \infty \iff p > 1$. $\square$

For $p \in [1, \infty)$, $L^p([a,b]) := \left\{f(x) \,\middle|\, \int_a^b |f(x)|^p \, dx < \infty\right\}$ is a normed vector space with $\|f\|_p := \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}$.

$d(A, A') := \mathrm{vol}_n(A \triangle A')$ is a possible metric on $\mathbb{R}^n$.

For a metric space (X, d), a set $A \subseteq X$, and $\epsilon > 0$, let $A_\epsilon := \bigcup_{x\in A} B(x, \epsilon)$. Then the Hausdorff metric is $d_H(A, A') := \inf\{\epsilon > 0 \mid A' \subseteq A_\epsilon \wedge A \subseteq A'_\epsilon\}$.

p-adic numbers

Given a fixed prime p,

$$\forall q \in \mathbb{Q} \setminus \{0\} : \exists (a, b, n) \in \mathbb{Z}^3 : \left[q = p^n \cdot \frac{a}{b} \wedge \gcd(a, p) = \gcd(b, p) = 1 \right]$$

and $\mathbb{Q}$ is a normed vector space with $\|q\|_p := \begin{cases} p^{-n} & q \neq 0 \\ 0 & q = 0 \end{cases}$.

The 2-adic norm of $\frac{96}{7}$ is $\frac{1}{32}$.

The 3-adic norm of $3^{-2024} \cdot \frac{8}{13}$ is 3^{2024}.

The p-adic norm $\|q\|_p$ is small if q is divisible by a large power of p.

The p-adic norm of 0 is 0 because 0 is divisible by any power of p.

(p-adic product formula.) If $q \in \mathbb{Q} \setminus \{0\}$, then $|q| \cdot \prod_{p \text{ prime}} \|q\|_p = 1$.

convexness

A set X is convex when the line segment joining any two points in X lies within X.

2024-01-10 contraction mappings & product topology

$\alpha \in I$ henceforth refers to members of a possibly uncountable index set I.

If $\{A_\alpha\}_{\alpha \in I}$ is a family of connected sets in a topological space X such that $\forall \alpha, \alpha' \in I : A_\alpha \cap A_{\alpha'} \neq \emptyset$, then $\bigcup_{\alpha \in I} A_\alpha$ is connected.

Proof. Suppose $A := \bigcup_{\alpha \in I} A_\alpha$ is not connected. Let $U, U' \subseteq X$ be open and nonempty relative to A such that $U \cap U' = \emptyset$ and $A = U \cup U'$. Since each A_α is connected, there is no α for which $A_\alpha \cap U \neq \emptyset \neq A_\alpha \cap U'$. But then $\exists \alpha, \alpha' \in I : A_\alpha \subseteq U \wedge A_{\alpha'} \subseteq U'$, which contradicts that $A_\alpha \cap A_{\alpha'} \neq \emptyset$. □

total disconnectedness

A topological space X is totally disconnected when every connected subset of X is a singleton.

A totally disconnected space in $\mathbb{R}$ contains only points and no intervals.

The Cantor set is a totally disconnected space in $\mathbb{R}$.

contraction mappings

For a metric space (X,d), $T : X \to X$ is a contraction mapping when $\exists c \in [0,1) : \forall x, x' \in X : d\big(T(x), T(x')\big) \le c \cdot d(x, x')$.

All contraction mappings are continuous.

> *(Orbit lemma.) For $x \in X$, the orbit $\big(T^n(x)\big)_{n \in \mathbb{N}}$ of a contraction mapping T on X is a Cauchy sequence.*

Proof. For $n \in \mathbb{Z}_+$,

$$\begin{aligned} d\left(T^n(x), T^{n+1}(x)\right) &< c \cdot d\left(T^{n-1}(x), T^n(x)\right) \\ &< c^2 \cdot d\left(T^{n-2}(x), T^{n-1}(x)\right) \\ \cdots &< c^n \cdot d\big(x, T(x)\big). \end{aligned}$$

Let $m \ge n$. By the triangle inequality,

$$\begin{aligned} d\left(T^n(x), T^m(x)\right) &\le \sum_{k=n}^{m-1} d\left(T^k(x), T^{k+1}(x)\right) \\ &\le d\big(x, T(x)\big) \sum_{k=n}^{m-1} c^k \\ &= c^n \cdot d\big(x, T(x)\big) \sum_{k=0}^{m-n-1} c^k \\ &\le \frac{c^n \cdot d\big(x, T(x)\big)}{1-c}. \end{aligned}$$

For $\epsilon > 0$, choosing $n > \log_c\left(\frac{\epsilon}{2} \cdot \frac{1-c}{d\left(x, T(x)\right)}\right)$ and $m, m' \ge n$ guarantees

$$\begin{gathered} d\big(T(m), T(m')\big) \le \\ d\big(T(n), T(m)\big) + d\big(T(n), T(m')\big) < \epsilon. \end{gathered}$$

□

> *(Contraction mapping theorem.) If (X,d) is a nonempty and complete metric space and T is a contraction mapping on X, then*
>
> $$\exists! z \in X : T(z) = z,$$

i.e. z is the unique fixed point, and

$$\forall x \in X : \lim_{n\to\infty} T^n(x) = z,$$

where $\forall n \in \mathbb{N} : T^n(x) := \underbrace{T(T(\cdots T(x)))}_{n \text{ times}}$.

Proof. For $x \in X$, $T(x)$ is Cauchy by the orbit lemma, and since (X,d) is complete it converges to some point $z \in X$. Then z is a fixed point of T because

$$\begin{aligned} T^n(x) \to z &\implies T(T^n(x)) \to T(z) \\ &\implies T(T^n(x)) = T^{n+1}(x) \to z = T(z), \end{aligned}$$

and is its unique fixed point because

$$T(z) = z \wedge T(z') = z' \implies d(z,z') \leq c \cdot d(z,z') \iff d(z,z') = 0. \qquad \square$$

$x \mapsto \frac{x}{2}$ is a contraction mapping on $(\mathbb{R}, |\cdot|)$ with fixed point $z = 0$.

iterated function systems and fixed point sets

$\mathscr{K}(X)$ henceforth denotes the set of compact subsets of a set X.

If $m \in \mathbb{Z}_+$, $\{T_n : \mathbb{R}^d \to \mathbb{R}^d\}_{n\in[m]}$ are contraction mappings, and

$$F : \left(\mathscr{K}(\mathbb{R}^d), d_H\right) \to \left(\mathscr{K}(\mathbb{R}^d), d_H\right) : A \mapsto \bigcup_{n\in[m]} T_n(A),$$

then $\exists! A \in \mathscr{K}(\mathbb{R}^d) : F(A) = A$. This is the fixed point set of F.

Proof. F is a contraction mapping (to be demonstrated in problem set 2), and $\left(\mathscr{K}(X), d_H\right)$ is a complete metric space (???). A unique fixed point set thus exists by the contraction mapping theorem. $\square$

Let $T_0, T_1 : \mathbb{R} \to \mathbb{R}$ such that $T_0(x) := \frac{x}{3}$ and $T_1(x) := \frac{x+2}{3}$ and $F(A) : \mathscr{K}(\mathbb{R}) \to \mathscr{K}(\mathbb{R})$ such that $F(A) := T_0(A) \cup T_1(A)$. Then $F\big([0,1]\big) = T_0\big([0,1]\big) \cup T_1\big([0,1]\big) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, and the fixed point set of F is the middle-thirds Cantor set.

If $T_0, T_1 : \mathbb{R}^d \to \mathbb{R}^d$ are contraction mappings and $F : \left(\mathscr{K}(\mathbb{R}^d), d_H\right) \to \left(\mathscr{K}(\mathbb{R}^d), d_H\right)$ such that $F(A) = T_0(A) \cup T_1(A)$, then the composition of F with itself is

$$F^2(A) = T_0\big(T_0(A)\big) \cup T_0\big(T_1(A)\big) \cup T_1\big(T_0(A)\big) \cup T_1\big(T_1(A)\big).$$

basis for a topology

A basis $\mathscr{B}$ for a topology τ is a subset of τ such that $\forall A \in \tau : \exists \{A_\alpha\}_{\alpha \in I} \subseteq \mathscr{B} : A = \bigcup_{\alpha \in I} A_\alpha$.

The open intervals form a basis for the standard topology on $\mathbb{R}$.

For a metric space X, $\{B(x,\epsilon) \mid x \in X, \epsilon > 0\}$ is a basis for the open sets in X.

If $\mathscr{B}$ is a basis for the topology on X, then $\forall x \in X : \exists U \in \mathscr{B} : x \in U$.

If $\mathscr{B}$ is a basis for the topology τ on X, then $[U, U' \in \mathscr{B} \wedge x \in U \cap U'] \implies \exists U'' \in \tau : x \in U'' \subseteq U \cap U'$.

If $f : X \to Y$ for topological spaces (X, τ_X) and (Y, τ_Y) and $\mathscr{B}$ is a basis for τ_Y, then f is continuous if and only if $\forall U \in \mathscr{B} : f^{-1}(U) \in \tau_X$.

product topology

For topological spaces $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in I}$, $\prod_{\alpha \in I} A_\alpha$ is a cylinder set in $X := \prod_{\alpha \in I} X_\alpha$ if and only if $\forall \alpha \in I : A_\alpha \in \tau_\alpha$.

If $X := \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, then $\forall$ open intervals $(a,b), (a',b') \subseteq \mathbb{R}$: $(a,b) \times (a',b')$ is a cylinder set in X.

For topological spaces $\{X_\alpha\}_{\alpha \in I}$, consider a cylinder set $\prod_{\alpha \in I} A_\alpha$ in $X := \prod_{\alpha \in I} X_\alpha$ such that $\exists I' \subseteq I : [|I'| \in \mathbb{N} \wedge \forall \alpha \in I \setminus |I'| : A_\alpha = X_\alpha]$. These base cylinder sets form a basis for the product topology on X.

projection maps

For a vector $x := (x_\alpha)_{\alpha \in I}$ in possibly uncountable dimensions, the function $\pi_\alpha(x) := x_\alpha$ is a projection map.

> *For topological spaces $\{X_\alpha\}_{\alpha \in I}$ let $f : Y \to \prod_{\alpha \in I} X_\alpha$, and for $\alpha \in I$ let $f_\alpha : Y \to X_\alpha : y \mapsto \pi_\alpha(f(y))$. Then f is continuous if and only if $\forall \alpha \in I : f_\alpha$ is continuous.*

Proof. Suppose f is continuous. Then, for $\alpha' \in I$ and an open set $U \subseteq X_{\alpha'}$,

$$f_{\alpha'}^{-1}(U) = f^{-1}\left(U \times \prod_{\alpha \in I \setminus \{\alpha'\}} X_\alpha\right)$$

is open as the preimage of a base cylinder set in $\prod_{\alpha \in I} X_\alpha$.

Suppose $\forall \alpha \in I : f_\alpha$ is continuous. It suffices to verify that the preimage of f for a base cylinder set U is open. Let $I' \subseteq I$ be a finite index subset for which

$$U = \prod_{\alpha \in I'} A_\alpha \times \prod_{\alpha \in I \setminus I'} X_\alpha .$$

Then

$$\begin{aligned} f^{-1}(U) &= \left(\bigcap_{\alpha \in I'} f_\alpha^{-1}(A_\alpha) \right) \cap \left(\bigcap_{\alpha \in I \setminus I'} f_\alpha^{-1}(X_\alpha) \right) \\ &= \bigcap_{\alpha \in I'} f_\alpha^{-1}(A_\alpha) \end{aligned}$$

is open in Y as the finite intersection of open sets in Y. □

2024-01-12 Hilbert spaces & Hausdorff spaces

inner product spaces

The closed unit ball in l_p is not compact because it contains $(x_n)_{n \in \mathbb{N}}$ where

$$\forall n,k \in \mathbb{N} : \pi_k(x_n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases},$$

which has no convergent subsequence.

An inner product space V is a normed vector space with an inner product operation $\langle v, v' \rangle$ for $v, v' \in V$ that induces its norm $\|v\|^2 =: \langle v, v \rangle$.

(Parallelogram law.) The norm in a vector space V is induced by an inner product if and only if

$$\forall v, v' \in V : \|v + v'\|^2 + \|v - v'\|^2 = 2\left(\|v\|^2 + \|v'\|^2\right)$$

holds.

(Hanner's inequality.) For a measure space X, $p \in [1, \infty) \setminus \{2\}$, and $f, g \in L^p(X)$,

$$\|f+g\|_p^p + \|f-g\|_p^p * \left(\|f\|_p + \|g\|_p\right)^p + \left|\|f\|_p - \|g\|_p\right|^p$$

where $* := \begin{cases} \geq & p < 2 \\ \leq & p > 2 \end{cases}$. If $p = 2$, then this becomes the parallelogram law.

Hilbert spaces

A Hilbert space is a complete inner product space.

l_2 is a Hilbert space with $\langle (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \rangle = \sum_{n\in\mathbb{N}} |a_n \overline{b_n}|$.

For a measure space X, $L^2(X)$ is a Hilbert space with $\langle f, g\rangle := \int_X f(x)\overline{g(x)}dx$.

(Parseval's identity.) For $f \in L^2([0,2\pi])$, which has a basis $\{e^{inx}\}_{n\in\mathbb{Z}}$,

$$\begin{aligned}\|f\|_2^2 &= \int_0^{2\pi} |f(x)|^2 dx \\ &= 2\pi \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2 = \langle f, f\rangle\end{aligned}$$

where $\hat{f}(n) = \langle f, e^{inx}\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$.

(Consequence of Riesz's lemma.) The closed unit ball in a Hilbert space V is compact if and only if V is finite-dimensional.

compact sets & continuity

Continuous functions map compact sets to compact sets.

Proof. Let f be continuous and A a compact set in its domain. The preimages of an open cover U of $f(A)$ form an open cover U' of A. Then a finite subcover of U' exists, and the images of sets in U' form a finite subcover of U. $\square$

(Tychonoff.) If $\forall \alpha \in I : X_\alpha$ is compact, then $\prod_{\alpha\in I} X_\alpha$ is compact in the product topology.

For topological spaces X and Y, a bijection $f : X \to Y$ is a homeomorphism if and only if f and f^{-1} are both continuous.

For a subset A of a metric space (X,d) and $x \in X$, $d(x,A) := \inf_{x'\in A} d(x,x')$ is continuous.

Proof. Let $\epsilon, \delta > 0$. Then $\exists a \in A : d(x,A)+\delta$, so for $x' \in B(x,\epsilon)$

$$\begin{aligned}d(x',A) &\leq d(x',a) \\ &\leq d(x,a) + d(x,x') \\ &< d(x,A) + \delta + \epsilon.\end{aligned}$$

Taking the limit as $\delta \to 0$ gives $d(x',A) \leq d(x,A)+\epsilon$. By parallel reasoning, $d(x,A) \leq d(x',A)+\epsilon$, hence $|d(x,A) - d(x',A)| \leq \epsilon$. $\square$

Hausdorff spaces

A topological space (X,τ) is a Hausdorff space when $\forall x \neq x' \in X : \exists U, U' \in \tau : \left[U \cap U' = \emptyset \wedge (x,x') \in U \times U'\right]$.

All metric spaces are Hausdorff spaces.

> *Singletons in a Hausdorff space X with topology τ are closed.*

Proof. Let $x' \in X$. Then

$$\forall x \in X : \exists U_x \in \tau : \left[x \in U_x \wedge x' \notin U_x\right] \implies X \setminus \{x'\} = \bigcup_{x' \in X \setminus \{x'\}} U_x \in \tau.$$

□

$\tau := \{\emptyset, X\}$ is the trivial topology on X.

If τ is the trivial topology and $|X| > 1$, then (X,τ) is not a Hausdorff space.

The subspace topology on a subspace Y of a topological space (X,τ) is $\tau_Y := \left\{U \cap Y \;\middle|\; U \in \tau\right\}$.

Any subspace of a Hausdorff space is a Hausdorff space under the subspace topology.

> *For Hausdorff spaces (X,τ) and (X',τ'), $X \times X'$ under the product topology is a Hausdorff space.*

Proof. Let $\mathscr{B}$ be the cylinder set basis for the product topology and let $(x_0, x_0') \neq (x_1, x_1') \in X \times Y$, assuming without loss of generality that $x_0 \neq x_1$. Then

$$\exists U_0, U_1 \in \tau : \left[U_0 \cap U_1 = \emptyset \wedge (x_0, x_1) \in U_0 \times U_1\right],$$

so $U_0 \times X'$ and $U_1 \times X'$ are disjoint open sets that contain (x_0, x_0') and (x_1, x_1') respectively. □

> *Any compact subspace of a Hausdorff space is closed.*

Proof. Let (X,τ) be a Hausdorff space, $A \subseteq X$ a compact subset, and $x' \in X \setminus A$. For $x \in A$, let $U_x, U_x' \in \tau$ such that $U_x \cap U_x' = \emptyset$ and $(x,x') \in U_x \times U_x'$.

Since $\{U_x\}_{x \in A}$ is an open cover of A, let $\{x_n\}_{n \in [m]} \subseteq A$ such that $\{U_{x_n}\}_{n \in [m]}$ covers A. Then $x' \in \bigcap_{n \in [m]} U_{x_n}' \subseteq X \setminus A$, and this intersection is open and disjoint to $\bigcup_{n \in [m]} U_{x_n}$. Thus $X \setminus A$ is open. □

A continuous bijection $f : X \to Y$ is a homeomorphism if X is compact and Y is a Hausdorff space.

Proof. Let $A \subseteq X$ be closed. Then A is compact, so $\left(f^{-1}\right)^{-1}(A) = f(A)$ is closed as a compact subset of a Hausdorff space. It follows that f^{-1} is continuous. □

normal spaces

A Hausdorff space (X, τ) is a normal space when

$$\exists U, U' \in \tau : [U \cap U \neq \emptyset \wedge A \subseteq U \wedge A' \subseteq U']$$

for any disjoint closed sets $A, A' \subseteq X$.

All compact Hausdorff space are normal spaces.

Proof. Let (X, τ) be a compact Hausdorff space and $A, A' \subseteq X$ closed sets with $A \cap A' = \emptyset$. For $x \in A$, let $U_x, U'_x \in \tau$ such that $U_x \cap U'_x = \emptyset$, $x \in U_x$, and $A' \subseteq U'_x$.

Since $\{U_x\}_{x \in A}$ is an open cover of A and A is compact as the closed subset of a compact set, let $\{x_n\}_{n \in [m]} \subseteq A$ such that $\{U_{x_n}\}_{n \in [m]}$ covers A. It follows that $\bigcup_{n \in [m]} U_{x_n}$ and $\bigcap_{n \in [m]} U'_{x_n} \supseteq A'$ are open and disjoint. □

All metric spaces are normal spaces.

Proof. Let $A, A' \subseteq X$ be disjoint and closed. Then

$$U_A := \{x \mid d(x, A) < d(x, A')\}, \qquad U'_A := \{x \mid d(x, A') < d(x, A)\},$$

are disjoint, and they are respective open supersets of A and A' because $x \in A \implies d(x, A) = 0 < d(x, A')$ (and vice versa) and

$$U_A = f^{-1}\big((-\infty, 0)\big), \qquad U'_A = f^{-1}\big((0, \infty)\big)$$

where $f : x \mapsto d(x, A) - d(x, A')$ is continuous as the difference between continuous functions. □

Lebesgue number

A Lebesgue number of an open cover $\{A_\alpha\}_{\alpha \in I}$ of X is $\epsilon > 0$ such that $\forall x \in X : \exists \alpha \in I : B(x, \epsilon) \subseteq A_\alpha$.

Any open cover of a compact metric space has a Lebesgue number.

Proof. Let X be a compact metric space and $\{U_\alpha\}_{\alpha\in I}$ an open cover of X with finite subcover $\{U_{\alpha_n}\}_{n\in[m]}$. For $n\in[m]$ and $x\in X$, let

$$d_n(x) := d(x, X\setminus U_n),$$

noting that $B(x,\epsilon)\subseteq U_n \iff d_n(x)\geq\epsilon$ and that $\exists n\in[m]: x\in U_n \implies d_n(x)>0$.

Let $f(x) := \max_{n\in[m]} d_n(x)$, which is continuous as the maximum of continuous functions. Then $f(X)\subseteq(0,\infty)$ is compact hence closed and bounded. It follows that $\exists\delta>0: f(X)\subseteq[\delta,\infty)$ and δ is a Lebesgue number for $\{U_\alpha\}_{\alpha\in I}$. □

If $f : X\to Y$ is continuous, X and Y are metric spaces, and X is compact, then f is uniformly continuous.

Proof. For $\epsilon>0$, let δ be a Lebesgue number for $\left\{f^{-1}\big(B(y,\epsilon)\big)\right\}_{y\in Y}$. Then $\forall x\in X : \exists y\in Y : f\big(B(x,\delta)\big) = B(y,\epsilon)$, so for $x'\in B(x,\delta)$

$$d\big(f(x),f(x')\big)\leq$$
$$d\big(f(x),y\big)+d\big(f(x'),y\big)<2\epsilon.$$ □

2024-01-17 product spaces & separable spaces

metrizability

A topological space (X,τ) is metrizable when there exists a metric d on X that induces τ. In other words, the set of open balls with respect to d is a basis for τ.

Any topological space that is not normal is non-metrizable.

If $\forall n\in\mathbb{N} : (X_n,d_n)$ are metric spaces, then $X := \prod_{n\in\mathbb{N}} X_n$ is metrizable under the product topology.

Proof. For $n\in\mathbb{N}$ and $x_n,y_n\in X_n$ let $\tilde{d}(x_n,y_n) := \frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}\leq 1$, noting that this is an equivalent metric to d_n. Then define

$$D : X\to\mathbb{R} : (x_n)_{n\in\mathbb{N}},(y_n)_{n\in\mathbb{N}}\mapsto\sum_{n\in\mathbb{N}}\left(\frac{\tilde{d}_n(x_n,y_n)}{2^n}\right).$$

As shown in problem set 1, D induces the product topology on X. □

countable product of compact metric spaces

$\tau := \mathscr{P}(X)$ is the discrete topology on X.

$X := \prod_{n\in\mathbb{N}}\{0,1\}$ is not compact under the discrete topology because the set of singletons in X is an open cover of X with no finite subcover.

(Tychonoff for countable products.) If $\forall n \in \mathbb{N} : (X_n, \tau_n)$ is a compact topological space, then $X := \prod_{n\in\mathbb{N}} X_\alpha$ is compact under the product topology.

Proof. Suppose X is not compact and let $C := \{C_\alpha\}_{\alpha\in I}$ be an open cover of X with no finite subcover.

Suppose for every $x_0 \in X_0$ that there is a set $U_{x_0} \in \tau_0$ such that

$$x_0 \in U_{x_0} \wedge \exists I' \subseteq I : \left[|I'| \in \mathbb{N} \wedge \left(U_{x_0} \times \prod_{n\in\mathbb{Z}_+} X_n \right) \subseteq \bigcup_{\alpha\in I'} C_\alpha . \right]$$

Then

$$X \subseteq \bigcup_{x_0\in X_0} \left(U_{x_0} \times \prod_{n\in\mathbb{Z}_+} X_n \right).$$

Since X_0 is compact, this open cover has a finite subcover and contradicts that C has no finite subcover. Thus there is some $x_0 \in X_0$ for which no such U_{x_0} exists.

By similar reasoning, there exists $x_1 \in X_1$ such that no cylinder set of the form $U_{x_0} \times U_{x_1} \times \prod_{n\in\mathbb{Z}_+\setminus\{1\}} X_n$ both satisfies $(x_0, x_1) \in U_{x_0} \times U_{x_1}$ and has a finite open cover in C; otherwise,

$$\left(U_{x_0} \times \prod_{n\in\mathbb{Z}_+} X_n \right) \subseteq \bigcup_{x_1\in X_1} \left(U_{x_0} \times U_{x_1} \times \prod_{n\in\mathbb{Z}_+\setminus\{1\}} X_n \right)$$

would have a finite subcover by compactness of X_1 and thus $\left(U_{x_0} \times \prod_{n\in\mathbb{Z}_+} X_n\right)$ would have a finite open cover in C.

Selecting $x_n \in X_n$ in this fashion for every $n \in \mathbb{N}$ provides a point $x := (x_n)_{n\in\mathbb{N}} \in X$ such that, for any $m \in \mathbb{N}$, every cylinder set of the form

$$\prod_{n=0}^{m} U_n \times \prod_{n=m+1}^{\infty} X_n,$$

where $(x_n)_{n=0}^m \in \prod_{n=0}^m U_n$, has no finite open cover in C. But this cannot be; $\exists C_\alpha \in C : x \in C_\alpha$, implying that x lies in some cylinder set $V := \prod_{n\in\mathbb{N}} V_n \subseteq C_\alpha$ in the product topology basis. Taking m to be the maximum index n for which $V_n \neq X_n$ gives

$$(x_n)_{n\in\mathbb{N}} \in \left(\prod_{n=0}^{m} V_n \times \prod_{n=m+1}^{\infty} X_n\right) \subseteq C_\alpha,$$

which is certainly a finite cover. □

separability

A metric space is separable when it has a countable dense subset.

> *All compact metric spaces are separable.*

Proof. Let X be a compact metric space. For $n \in \mathbb{Z}_+$, $X \subseteq \bigcup_{x\in X} B\left(x, \frac{1}{n}\right)$, and by compactness, let $N_n \subseteq X$ be a finite subset such that $X \subseteq \bigcup_{x\in N_n} B\left(x, \frac{1}{n}\right)$. Thus $\bigcup_{n\in\mathbb{Z}_+} N_n \subseteq X$ is dense and countable as the countable union of finite sets. □

A metric space X with topology τ is separable if and only if $\exists \{U_n\}_{n\in\mathbb{N}} \subseteq \tau : \forall U \in \tau : \exists N \subseteq \mathbb{N} : U = \bigcup_{n\in N} U_n$; in other words, τ has a countable basis.

Any subspace of a separable metric space is separable.

$\mathbb{R}^n$, $C([a,b])$, and $L^p([a,b])$ for $p \in [1,\infty)$ are examples of separable spaces.

$L^\infty([a,b])$ is an example of a non-separable space.

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total boundedness

A metric space X is totally bounded when $\forall \epsilon > 0 : \exists X' \subseteq X : |X'| \in \mathbb{N} \wedge X \subseteq \bigcup_{x\in X'} B(x,\epsilon)$.

(Cantor Intersection Theorem.) A metric space X is complete if and only if every collection $\{A_n\}_{n\in\mathbb{N}}$ of nonempty closed subsets of X such that

$$\forall n \in \mathbb{N} : A_n \subseteq A_{n+1} \quad \text{and} \quad \lim_{n\to\infty} \operatorname{diam}(A_n) = 0$$

satisfies $\exists x \in X : \bigcap_{n\in\mathbb{N}} A_n = \{x\}$.

A metric space X is (1) complete and totally bounded $\iff$ (2) X is compact $\iff$ (3) X is sequentially compact.

continuous functions on a compact metric space

For a compact metric space X, $C(X) := \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$ is a normed vector space with $\|f\|_\infty := \max_{x\in X}|f(x)|$.

If a metric space (X,d) is compact, then $(C(X), ||\cdot||_\infty)$ is complete.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C(X)$ and let $(f_{n_k})_{k\in\mathbb{N}}$ be a subsequence such that $\forall k \in \mathbb{N} : \left\|f_{n_{k+1}} - f_{n_k}\right\|_\infty \le 2^k$. For $x \in X$ and $j,k \in \mathbb{N}$,

$$\left|f_{n_{k+j}}(x) - f_{n_k}(x)\right| \le \left\|f_{n_{k+j}} - f_{n_k}\right\|_\infty \le \sum_{i=k}^{k+j-1} \left\|f_{n_{i+1}} - f_{n_i}\right\|_\infty \le \sum_{i=k}^{\infty} 2^i,$$

hence $\left(f_{n_k}(x)\right)$ is a Cauchy sequence and converges in $\mathbb{R}$. Let f be the pointwise limit of $\left(f_{n_k}(x)\right)_{k\in\mathbb{N}}$. Taking the limit of the previous inequality as $j \to \infty$ gives

$$\forall k \in \mathbb{N} : \left\|f - f_{n_k}\right\|_\infty = \max_{x\in X}\left|f(x) - f_{n_k}(x)\right| \le \sum_{i=k}^{\infty} 2^i,$$

so the convergence is uniform. Thus $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with a convergent subsequence in $C(X)$ and itself converges. $\square$

boundedness of sets of functions

If X is a metric space and $\forall n \in \mathbb{N} : f_n : X \to \mathbb{R}$, then $\{f_n\}_{n\in\mathbb{N}}$ is pointwise bounded when $\forall x \in X : \{f_n(x)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is a bounded set.

If X is a metric space and $\forall n \in \mathbb{N} : f_n : X \to \mathbb{R}$, then $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded when $\exists m > 0 : \forall (x,n) \in X \times \mathbb{N} : |f_n(x)| \le m$.

equicontinuity

A set $\mathscr{F}$ of real-valued functions on a metric space X is equicontinuous at $x \in X$ when $\forall \epsilon > 0 : \exists \delta > 0 : \forall f \in \mathscr{F} : \forall x' \in X : d(x,x') < \delta \implies |f(x) - f(x')| < \epsilon$.

A set $\mathscr{F}$ of real-valued functions on a topological space X is pointwise equicontinuous, or simply equicontinuous, when $\forall x \in X : \mathscr{F}$ is equicontinuous at x.

Every finite set of continuous functions is equicontinuous.

For $a,b,m \in \mathbb{R}_+$ such that $a < b$,

$$\left\{f \in C([a,b]) \,\middle|\, \forall x \in [a,b] : \left|f'(x)\right| \le m\right\}$$

is equicontinuous by the Mean Value Theorem, since

$$\left|f(x)-f(x')\right| \le \left|x-x'\right| \sup_{t\in[x,x']} \left|f'(t)\right| \le \left|x-x'\right| m < \epsilon$$

for $x, x' \in [a,b]$ such that $x < x' \wedge |x-x'| < \frac{\epsilon}{m}$.

(Arzelà–Ascoli lemma.) If a metric space (X,d) is separable and $(f_n)_{n\in\mathbb{N}}$ is an equicontinuous and pointwise bounded sequence of functions in $C(X)$, then c2::a subsequence of $(f_n)_{n\in\mathbb{N}}$ converges pointwise to some function $f : X \to \mathbb{R}$.

Proof. Let $X' := \{x_k\}_{k\in\mathbb{N}} \subseteq X$ be dense in X. Since $\left\{f_n(x_0)\right\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded, $\left(f_n(x_0)\right)_{n\in\mathbb{N}}$ has a convergent subsequence. Let $(n_{0,i})_{i\in\mathbb{N}}$ be a sequence of indices that defines such a subsequence, and let $l_0 := \lim_{i\to\infty} f_{n_{0,i}}(x_0)$.

For $k \in \mathbb{Z}_+$, let $(n_{k,i})_{i\in\mathbb{N}}$ be a subsequence of $(n_{k-1,i})_{i\in\mathbb{N}}$ such that $f_{n_{k,i}}(x_k) \to l_k$. This is always possible because $\left(f_{n_{k-1,i}}(x_k)\right)_{i\in\mathbb{N}}$ is bounded and thus has a convergent subsequence. To avoid producing an "empty" sequence from this infinite construction, take the diagonal sequence $(n_{i,i})_{i\in\mathbb{N}}$ and let

$$f(x_k) := l_k = \lim_{i\to\infty} f_{n_{i,i}}$$

for $k \in \mathbb{N}$. Then f is the pointwise limit of $\left\{f_{n_{i,i}}\right\}$ in X'.

Now let $x \in X$ and $\epsilon > 0$. Since $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous, $\exists \delta > 0 : \forall n \in \mathbb{N} : \forall x' \in X : d(x,x') < \delta \implies |f_n(x) - f_n(x')| < \epsilon$, and since X' is dense $\exists x' \in X' : d(x,x') < \delta$. Let $m \in \mathbb{N}$ such that

$$\forall i \ge j \ge m : \left|f_{n_{i,i}}(x') - f_{n_{j,j}}(x')\right| < \frac{\epsilon}{3}.$$

By the triangle inequality,

$$\begin{aligned} \forall i \ge j \ge m : \left|f_{n_{i,i}}(x) - f_{n_{j,j}}(x)\right| &\le \left|f_{n_{i,i}}(x) - f_{n_{i,i}}(x')\right| \\ &+ \left|f_{n_{i,i}}(x') - f_{n_{j,j}}(x')\right| \\ &+ \left|f_{n_{j,j}}(x') - f_{n_{j,j}}(x)\right| < 3 \cdot \frac{\epsilon}{3}. \end{aligned}$$

It follows that $\left(f_{n_{i,i}}(x)\right)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, so it converges; setting $f(x)$ to its limit makes f converge pointwise in X.