MATH 455 (Honours Analysis 4)

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The following notes are adapted from lectures given by Dmitry Jakobson in Winter 2024. All errors in interpretation, reasoning, coherence, and articulation are my own.

This document's source code, located at `https://github.com/brunefig/math455/blob/main/notes.org`, can be converted into Anki flashcards with the `anki-editor` package for GNU Emacs. Flashcard cloze deletions are typeset in magenta.

preliminaries

topological spaces

A topological space (X, τ) is a set X endowed with a topology τ.

A topology τ on a set X is a collection of sets that contains $\emptyset$ and X and is closed under finite intersections and arbitrary unions. Membership in τ defines an open set in the topological space (X, τ).

2024-01-05 topology & metric spaces

connectedness

A topological space (X, τ) is a connected space if and only if $\neg\exists U, U' \in \tau \setminus \{\emptyset\} : U \cap U' = \emptyset \wedge X = U \cup U'$.

For a topological space (X, τ), $A \subseteq X$ is a connected set if and only if $\neg\exists U, U' \in \tau : A \cap U \neq \emptyset, A \cap U' \neq \emptyset, U \cap U' = \emptyset, A = (U \cap A) \cup (U' \cap A)$.

A topological space X is a path connected space if and only if $\forall x, x' \in X : \exists$ continuous $f : [0,1] \to X : [f(0) = x \wedge f(1) = x']$.

All path connected spaces are connected spaces.

For open sets in $\mathbb{R}^n$, connectedness is equivalent to path connectedness.

A topological space X can be expressed as the disjoint union of maximal connected subsets, where a connected subset is called maximal if and only if it has no connected superset in X. These subsets are the connected components of X.

A topological space X can be expressed as the disjoint union of maximal path connected subsets, where a path connected subset is called maximal if and only if it has no path connected superset in X. These subsets are the path components of X.

A path connected space has exactly one path component.

$\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \,\middle|\, x \in (0,1]\right\} \cup \{(0,0)\} \subseteq \mathbb{R}^2$ is connected but not path connected because it has two path components.

If $f : X \to \mathbb{R}$ is continuous, $A \subseteq X$ is connected, and $x, x' \in f(A)$ such that $x < x'$, then $[x, x'] \subseteq f(A)$.

Proof. Suppose $\exists c \in \big(f(x'), f(x)\big) : c \notin f(A)$. Since $f^{-1}\big((-\infty, c)\big)$ and $f^{-1}\big((c, \infty)\big)$ are open, $A \subseteq f^{-1}\big((-\infty, c)\big) \cup f^{-1}\big((c, \infty)\big)$ is not connected. ☐

examples of metric spaces

Any normed vector space is a metric space with the induced metric $d(x, x') := \|x - x'\|$.

For $p \in (0, \infty)$, $l_p := \left\{(x_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \,\middle|\, \sum_{n \in \mathbb{N}} |x_n|^p < \infty\right\}$ is a normed vector space with $\|x\|_p := \left(\sum_{n \in \mathbb{N}} |x_n|^p\right)^{1/p}$.

The sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{Z}_+}$ is a member of l_p if and only if $p > 1$.

Proof. $\left(\frac{1}{n}\right)_{n \in \mathbb{Z}_+} \in l_p \iff \sum_{n \in \mathbb{Z}_+} \left(\frac{1}{n}\right)^p < \infty \iff p > 1.$ ☐

For $p \in [1, \infty)$, $L^p([a,b]) := \left\{f(x) \,\middle|\, \int_a^b |f(x)|^p \, dx < \infty\right\}$ is a normed vector space with $\|f\|_p := \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}$.

$d(A, A') := \mathrm{vol}_n(A \triangle A')$ is a possible metric on $\mathbb{R}^n$.

For a metric space (X, d), a set $A \subseteq X$, and $\epsilon > 0$, let $A_\epsilon := \bigcup_{x \in A} B(x, \epsilon)$. Then the Hausdorff metric is $d_H(A, A') := \inf\{\epsilon > 0 \mid A' \subseteq A_\epsilon \wedge A \subseteq A'_\epsilon\}$.

p-adic numbers

Given a fixed prime p,

$$\forall q \in \mathbb{Q} \setminus \{0\} : \exists (a, b, n) \in \mathbb{Z}^3 : \left[q = p^n \cdot \frac{a}{b} \wedge \gcd(a, p) = \gcd(b, p) = 1 \right]$$

and $\mathbb{Q}$ is a normed vector space with $\|q\|_p := \begin{cases} p^{-n} & q \neq 0 \\ 0 & q = 0 \end{cases}$.

The 2-adic norm of $\frac{96}{7}$ is $\frac{1}{32}$.

The 3-adic norm of $3^{-2024} \cdot \frac{8}{13}$ is 3^{2024}.

The p-adic norm $\|q\|_p$ is small if q is divisible by a large power of p.

The p-adic norm of 0 is 0 because 0 is divisible by any power of p.

(p-adic product formula.) If $q \in \mathbb{Q} \setminus \{0\}$, then $|q| \cdot \prod_{p \text{ prime}} \|q\|_p = 1$.

convexness

A set X is convex if and only if the line segment joining any two points in X lies within X.

2024-01-10 contraction mappings & product topologies

$\alpha \in I$ henceforth refers to members of a possibly uncountable index set I.

> *If $\{A_\alpha\}_{\alpha \in I}$ is a family of connected sets in a topological space X such that $\forall \alpha, \alpha' \in I : A_\alpha \cap A_{\alpha'} \neq \emptyset$, then $\bigcup_{\alpha \in I} A_\alpha$ is connected.*

Proof. Suppose $A := \bigcup_{\alpha \in I} A_\alpha$ is not connected. Let $U, U' \subseteq X$ be open and nonempty relative to A such that $U \cap U' = \emptyset$ and $A = U \cup U'$. Since each A_α is connected, there is no α for which $A_\alpha \cap U \neq \emptyset \neq A_\alpha \cap U'$. But then $\exists \alpha, \alpha' \in I : A_\alpha \subseteq U \wedge A_{\alpha'} \subseteq U'$, which contradicts that $A_\alpha \cap A_{\alpha'} \neq \emptyset$. $\square$

total disconnectedness

A topological space X is totally disconnected if and only if every connected subset of X is a singleton.

A totally disconnected space in $\mathbb{R}$ contains only points and no intervals.

The Cantor set is a totally disconnected space in $\mathbb{R}$.

contraction mappings

For a metric space (X, d), $T : X \to X$ is a contraction mapping if and only if $\exists c \in [0,1) : \forall x, x' \in X : d\big(T(x), T(x')\big) \leq c \cdot d(x, x')$.

All contraction mappings are continuous.

(Orbit lemma.) For $x \in X$, the orbit $\left(T^n(x)\right)_{n\in\mathbb{N}}$ of a contraction mapping T on X is a Cauchy sequence.

Proof. For $n \in \mathbb{Z}_+$,

$$\begin{aligned} d\left(T^n(x), T^{n+1}(x)\right) &< c \cdot d\left(T^{n-1}(x), T^n(x)\right) \\ &< c^2 \cdot d\left(T^{n-2}(x), T^{n-1}(x)\right) \\ \cdots &< c^n \cdot d\big(x, T(x)\big). \end{aligned}$$

Let $m \geq n$. By the triangle inequality,

$$\begin{aligned} d\left(T^n(x), T^m(x)\right) &\leq \sum_{k=n}^{m-1} d\left(T^k(x), T^{k+1}(x)\right) \\ &\leq d\big(x, T(x)\big) \sum_{k=n}^{m-1} c^k \\ &= c^n \cdot d\big(x, T(x)\big) \sum_{k=0}^{m-n-1} c^k \\ &\leq \frac{c^n \cdot d\big(x, T(x)\big)}{1-c}. \end{aligned}$$

For $\epsilon > 0$, choosing $n > \log_c \left(\frac{\epsilon}{2} \cdot \frac{1-c}{d\left(x, T(x)\right)}\right)$ and $m, m' \geq n$ guarantees

$$\begin{gathered} d\big(T(m), T(m')\big) \leq \\ d\big(T(n), T(m)\big) + d\big(T(n), T(m')\big) < \epsilon. \end{gathered}$$ ☐

(Contraction mapping theorem.) If (X, d) is a nonempty and complete metric space and T is a contraction mapping on X, then

$$\exists! z \in X : T(z) = z,$$

i.e. z is the unique fixed point, and

$$\forall x \in X : \lim_{n\to\infty} T^n(x) = z,$$

where $\forall n \in \mathbb{N} : T^n(x) := \underbrace{T(T(\cdots T(x)))}_{n \text{ times}}$.

Proof. For $x \in X$, $T(x)$ is Cauchy by the orbit lemma, and since (X,d) is complete it converges to some point $z \in X$. Then z is a fixed point of T because

$$\begin{aligned} T^n(x) \to z &\implies T(T^n(x)) \to T(z) \\ &\implies T(T^n(x)) = T^{n+1}(x) \to z = T(z), \end{aligned}$$

and is its unique fixed point because

$$T(z) = z \wedge T(z') = z' \implies d(z,z') \leq c \cdot d(z,z') \iff d(z,z') = 0. \qquad \square$$

$x \mapsto \frac{x}{2}$ is a contraction mapping on $(\mathbb{R}, |\cdot|)$ with fixed point $z = 0$.

iterated function systems and fixed point sets

$\mathscr{K}(X)$ henceforth denotes the set of compact subsets of a set X.

If $m \in \mathbb{Z}_+$, $\{T_n : \mathbb{R}^d \to \mathbb{R}^d\}_{n \in [m]}$ are contraction mappings, and

$$F : \left(\mathscr{K}(\mathbb{R}^d), d_H\right) \to \left(\mathscr{K}(\mathbb{R}^d), d_H\right) : A \mapsto \bigcup_{n \in [m]} T_n(A),$$

then $\exists! A \in \mathscr{K}(\mathbb{R}^d) : F(A) = A$. This is the fixed point set of F.

Proof. F is a contraction mapping (to be demonstrated in problem set 2), and $\left(\mathscr{K}(X), d_H\right)$ is a complete metric space (???). A unique fixed point set thus exists by the contraction mapping theorem. $\square$

Let $T_0, T_1 : \mathbb{R} \to \mathbb{R}$ such that $T_0(x) := \frac{x}{3}$ and $T_1(x) := \frac{x+2}{3}$ and $F(A) : \mathscr{K}(\mathbb{R}) \to \mathscr{K}(\mathbb{R})$ such that $F(A) := T_0(A) \cup T_1(A)$. Then $F\left([0,1]\right) = T_0\left([0,1]\right) \cup T_1\left([0,1]\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, and the fixed point set of F is the middle-thirds Cantor set.

If $T_0, T_1 : \mathbb{R}^d \to \mathbb{R}^d$ are contraction mappings and $F : \left(\mathscr{K}(\mathbb{R}^d), d_H\right) \to \left(\mathscr{K}(\mathbb{R}^d), d_H\right)$ such that $F(A) = T_0(A) \cup T_1(A)$, then the composition of F with itself is

$$F^2(A) = T_0\left(T_0(A)\right) \cup T_0\left(T_1(A)\right) \cup T_1\left(T_0(A)\right) \cup T_1\left(T_1(A)\right).$$

basis for a topology

A basis $\mathscr{B}$ for a topology τ is a subset of τ such that $\forall A \in \tau : \exists \{A_\alpha\}_{\alpha \in I} \subseteq \mathscr{B} : A = \bigcup_{\alpha \in I} A_\alpha$.

The open intervals form a basis for the standard topology on $\mathbb{R}$.

For a metric space X, $\{B(x,\epsilon) \mid x \in X, \epsilon > 0\}$ is a basis for the open sets in X.

If $\mathscr{B}$ is a basis for the topology on X, then $\forall x \in X : \exists U \in \mathscr{B} : x \in U$.

If $\mathscr{B}$ is a basis for the topology τ on X, then $[U, U' \in \mathscr{B} \wedge x \in U \cap U'] \implies \exists U'' \in \tau : x \in U'' \subseteq U \cap U'$.

If $f : X \to Y$ for topological spaces (X, τ_X) and (Y, τ_Y) and $\mathscr{B}$ is a basis for τ_Y, then f is continuous if and only if $\forall U \in \mathscr{B} : f^{-1}(U) \in \tau_X$.

product topologies

For topological spaces $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in I}$, $\prod_{\alpha \in I} A_\alpha$ is a cylinder set in $X := \prod_{\alpha \in I} X_\alpha$ if and only if $\forall \alpha \in I : A_\alpha \in \tau_\alpha$.

If $X := \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, then $\forall$ open intervals $(a,b),(a',b') \subseteq \mathbb{R}$: $(a,b) \times (a',b')$ is a cylinder set in X.

For topological spaces $\{X_\alpha\}_{\alpha \in I}$, consider a cylinder set $\prod_{\alpha \in I} A_\alpha$ in $X := \prod_{\alpha \in I} X_\alpha$ such that $\exists I' \subseteq I : [|I'| \in \mathbb{N} \wedge \forall \alpha \in I \setminus |I'| : A_\alpha = X_\alpha]$. These base cylinder sets form a basis for the product topology on X.

projection maps

For a vector $x := (x_\alpha)_{\alpha \in I}$ in possibly uncountable dimensions, the function $\pi_\alpha(x) := x_\alpha$ is a projection map.

> *For topological spaces $\{X_\alpha\}_{\alpha \in I}$ let $f : Y \to \prod_{\alpha \in I} X_\alpha$, and for $\alpha \in I$ let $f_\alpha : Y \to X_\alpha : y \mapsto \pi_\alpha(f(y))$. Then f is continuous if and only if $\forall \alpha \in I : f_\alpha$ is continuous.*

Proof. Suppose f is continuous. Then, for $\alpha' \in I$ and an open set $U \subseteq X_{\alpha'}$,

$$f_{\alpha'}^{-1}(U) = f^{-1}\left(U \times \prod_{\alpha \in I \setminus \{\alpha'\}} X_\alpha\right)$$

is open as the preimage of a base cylinder set in $\prod_{\alpha \in I} X_\alpha$.

Suppose $\forall \alpha \in I : f_\alpha$ is continuous. It suffices to verify that the preimage of f for a base cylinder set U is open. Let $I' \subseteq I$ be a finite index subset for which

$$U = \prod_{\alpha \in I'} A_\alpha \times \prod_{\alpha \in I \setminus I'} X_\alpha.$$

Then

$$
\begin{aligned}
f^{-1}(U) &= \left(\bigcap_{\alpha\in I'} f_\alpha^{-1}(A_\alpha)\right) \cap \left(\bigcap_{\alpha\in I\setminus I'} f_\alpha^{-1}(X_\alpha)\right) \\
&= \bigcap_{\alpha\in I'} f_\alpha^{-1}(A_\alpha)
\end{aligned}
$$

is open in Y as the finite intersection of open sets in Y. □

2024-01-12 Hilbert spaces & Hausdorff spaces

inner product spaces

The closed unit ball in l_p is not compact because it contains $(x_n)_{n\in\mathbb{N}}$ where

$$
\forall n,k\in\mathbb{N} : \pi_k(x_n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases},
$$

which has no convergent subsequence.

An inner product space V is a normed vector space with an inner product operation $\langle v, v'\rangle$ for $v, v' \in V$ that defines its norm $\|v\|^2 = \langle v, v\rangle$.

(Parallelogram law.) The norm in a vector space V is induced by an inner product if and only if

$$
\forall v, v' \in V : \|v+v'\|^2 + \|v-v'\|^2 = 2\left(\|v\|^2 + \|v'\|^2\right)
$$

holds.

(Hanner's inequality.) For a measure space X, $p \in [1,\infty)\setminus\{2\}$, and $f, g \in L^p(X)$,

$$
\|f+g\|_p^p + \|f-g\|_p^p * \left(\|f\|_p + \|g\|_p\right)^p + \left|\|f\|_p - \|g\|_p\right|^p
$$

where $* := \begin{cases} \geq & p < 2 \\ \leq & p > 2 \end{cases}$. If $p = 2$, then this becomes the parallelogram law.

Hilbert spaces

A Hilbert space is a complete inner product space.

l_2 is a Hilbert space with $\left\langle (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \right\rangle = \sum_{n\in\mathbb{N}} |a_n \overline{b_n}|$.

For a measure space X, $L^2(X)$ is a Hilbert space with $\langle f,g\rangle := \int_X f(x)\overline{g(x)}dx$.

(Parseval's identity.) For $f \in L^2([0,2\pi])$,

$$\|f\|_2^2 = \int_0^{2\pi} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2 = \langle f,f\rangle$$

where $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$.

(Consequence of Riesz's lemma.) The closed unit ball in a Hilbert space V is compact if and only if V is finite-dimensional.

compact sets & continuity

Continuous functions map compact sets to compact sets.

Proof. Let f be continuous and A a compact set in its domain. The preimages of an open cover U of $f(A)$ form an open cover U' of A. Then a finite subcover of U' exists, and the images of sets in U' form a finite subcover of U. □

(Tychonoff.) If $\forall \alpha \in I : X_\alpha$ is compact, then $\prod_{\alpha\in I} X_\alpha$ is compact in the product topology.

For topological spaces X and Y, a bijection $f : X \to Y$ is a homeomorphism if and only if f and f^{-1} are both continuous.

For a subset A of a metric space (X,d) and $x \in X$, $d(x,A) := \inf_{x'\in A} d(x,x')$ is continuous.

Proof. Let $\epsilon,\delta > 0$. Then $\exists a \in A : d(x,A)+\delta$, so for $x' \in B(x,\epsilon)$

$$\begin{aligned} d(x',A) &\le d(x',a) \\ &\le d(x,a)+d(x,x') \\ &< d(x,A)+\delta+\epsilon. \end{aligned}$$

Taking the limit as $\delta \to 0$ gives $d(x',A) \le d(x,A)+\epsilon$. By parallel reasoning, $d(x,A) \le d(x',A)+\epsilon$, hence $|d(x,A)-d(x',A)| \le \epsilon$. □

Hausdorff spaces

A topological space (X,τ) a Hausdorff space if and only if $\forall x \neq x' \in X : \exists U,U' \in \tau : \left[U \cap U' = \varnothing \wedge (x,x') \in U \times U'\right]$.

All metric spaces are Hausdorff spaces.

Singletons in a Hausdorff space X with topology τ are closed.

Proof. Let $x' \in X \implies$. Then

$$\forall x \in X : \exists U_x \in \tau : \left[x \in U_x \wedge x' \notin U_x\right] \implies X \setminus \{x'\} = \bigcup_{x' \in X \setminus \{x'\}} U_x \in \tau.$$

□

$\tau := \{\emptyset, X\}$ is the trivial topology on X.

If τ is the trivial topology and $|X| > 1$, then (X, τ) is not a Hausdorff space.

The subspace topology on a subspace Y of a topological space (X, τ) is $\tau_Y := \left\{U \cap Y \,\middle|\, U \in \tau\right\}$.

Any subspace of a Hausdorff space is a Hausdorff space under the subspace topology.

For Hausdorff spaces X and Y, $X \times Y$ is a Hausdorff space.

Proof. Let $\mathscr{B}$ be the cylinder set basis for the product topology and let $(x_0, x_0') \neq (x_1, x_1') \in X \times Y$, assuming without loss of generality that $x_0 \neq x_1$. Then $\exists U, U' \in \tau : \left[U \cap U' = \emptyset \wedge (x_0, x_1) \in U \times U'\right]$, so $U \times X$ and $U' \times X'$ are disjoint open sets that contain (x_0, x_0') and (x_1, x_1') respectively. □

Any compact subspace of a Hausdorff space is closed.

Proof. Let (X, τ) be a Hausdorff space, $A \subseteq X$ a compact subset, and $x' \in X \setminus A$. For $x \in A$, let $U_x, U_x' \in \tau$ such that $U_x \cap U_x' = \emptyset$, $x \in U_x$, and $x' \in U_x'$. $\{U_x\}_{x \in A}$ is an open cover of A, so let $\{x_n\}_{n \in [m]} \subseteq A$ such that $\{U_{x_n}\}_{n \in [m]}$ covers A. It follows that $x' \in \bigcap_{n \in [m]} U_{x_n}' \subseteq X \setminus A$ is open and disjoint to $\bigcap_{n \in [m]} U_{x_n}$. □

A continuous bijection $f : X \to Y$ is a homeomorphism if X is compact and Y is a Hausdorff space.

Proof. Let $A \subseteq X$ be closed. Then A is compact, so $\left(f^{-1}\right)^{-1}(A) = f(A)$ is closed as a compact subset of a Hausdorff space. It follows that f^{-1} is continuous. □

normal spaces

A Hausdorff space (X,τ) is a normal space if and only if

$$\exists U,U' \in \tau : \left[U \cap U \neq \emptyset \wedge A \subseteq U \wedge A' \subseteq U'\right]$$

for any disjoint closed sets $A, A' \subseteq X$.

> *All compact Hausdorff space are normal spaces.*

Proof. Let (X,τ) be a compact Hausdorff space and $A, A' \subseteq X$ closed sets with $A \cap A' = \emptyset$. For $x \in A$, let $U_x, U_x' \in \tau$ such that $U_x \cap U_x' = \emptyset$, $x \in U_x$, and $A' \subseteq U_x'$. $\{U_x\}_{x \in A}$ is an open cover of A, so since A is compact as the closed subset of a compact set let $\{x_n\}_{n\in[m]} \subseteq A$ such that $\{U_{x_n}\}_{n\in[m]}$ covers A. It follows that $\bigcup_{n\in[m]} U_{x_n}$ and $\bigcap_{n\in[m]} U_{x_n}' \supseteq A'$ are open and disjoint. □

> *All metric spaces are normal spaces.*

Proof. Let $A, A' \subseteq X$ be closed. Then

$$U_A := \left\{x \mid d(x,A) < d(x,A')\right\}, \qquad U_A' := \left\{x \mid d(x,A') < d(x,A)\right\},$$

are disjoint, and they are respective open neighborhoods of A and A' because

$$U_A = f^{-1}\big((-\infty,0)\big), \qquad U_A' = f^{-1}\big((0,\infty)\big)$$

where $f : x \mapsto d(x,A) - d(x,A')$ is continuous as the difference between continuous functions. □

Lebesgue number

A Lebesgue number of an open cover $\{A_\alpha\}_{\alpha\in I}$ of X is $\epsilon > 0$ such that $\forall x \in X : \exists \alpha \in I : B(x,\epsilon) \subseteq A_\alpha$.

> *Any open cover of a compact metric space has a Lebesgue number.*

Proof. Let X be a compact metric space and $\{U_\alpha\}_{\alpha\in I}$ an open cover of X with finite subcover $\{U_{\alpha_n}\}_{n\in[m]}$. For $n \in [m]$ and $x \in X$, let

$$d_n(x) := d(x, X \setminus U_n),$$

noting that $B(x,\epsilon) \subseteq U_n \iff d_n(x) \geq \epsilon$ and that $\exists n \in [m] : x \in U_n \implies d_n(x) > 0$.

Let $f(x) := \max_{n\in[m]} d_n(x)$, which is continuous as the maximum of continuous functions. Then $f(X) \subseteq (0,\infty)$ is compact hence closed and bounded. It follows that $\exists\delta > 0 : f(X) \subseteq [\delta,\infty)$, and δ is a Lebesgue number for $\{U_\alpha\}_{\alpha\in I}$. □

If $f : X \to Y$ is continuous, X and Y are metric spaces, and X is compact, then f is uniformly continuous.

Proof. For $\epsilon > 0$, let δ be a Lebesgue number for $\left\{f^{-1}\big(B(y,\epsilon)\big)\right\}_{y\in Y}$. Then $\forall x \in X : \exists y \in Y : f\big(B(x,\delta)\big) = B(y,\epsilon)$, so for $x' \in B(x,\delta)$

$$d\big(f(x), f(x')\big) \leq$$
$$d\big(f(x), y\big) + d\big(f(x'), y\big) < 2\epsilon.$$