# The Stereographic Projection

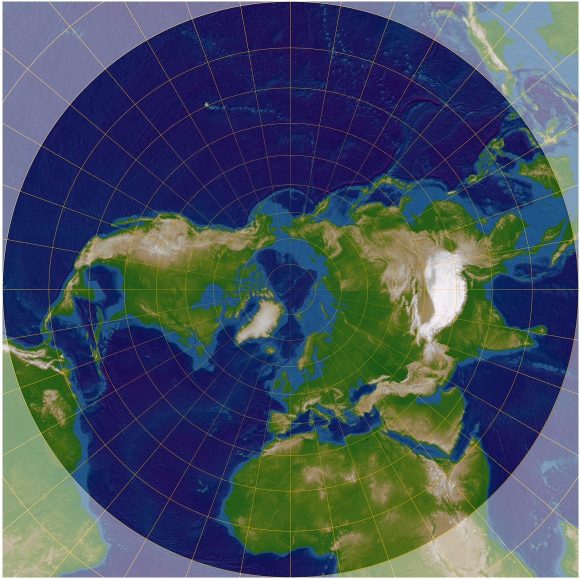
## IOHK Education Get-Together

Dr. Lars Brünjes

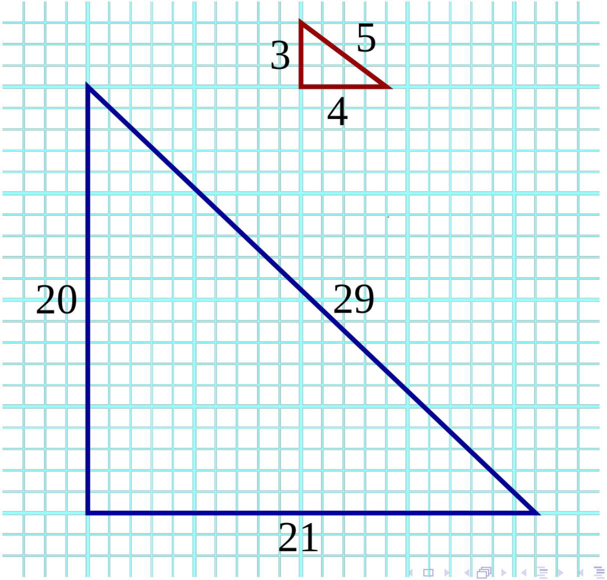
IOHK

September 16, 2019

# What do these. . .



- $3^2 + 4^2 = 5^2$
- $20^2 + 21^2 = 29^2$



- Both are examples of a mathematical construction called stereographic projection.

# Stereographic Projection

- Both are examples of a mathematical construction called stereographic projection.
- The first example deals with "classical" Euclidean geometry.

# Stereographic Projection

- Both are examples of a mathematical construction called stereographic projection.
- The first example deals with "classical" Euclidean geometry.
- The second example is taken from the domain of number theory, the branch of mathematics studying natural numbers.

# Stereographic Projection

- Both are examples of a mathematical construction called stereographic projection.
- The first example deals with "classical" Euclidean geometry.
- The second example is taken from the domain of number theory, the branch of mathematics studying natural numbers.
- Both are tied together by an area of mathematics called algebraic geometry.

# Stereographic Projection

- Both are examples of a mathematical construction called stereographic projection.
- The first example deals with "classical" Euclidean geometry.
- The second example is taken from the domain of number theory, the branch of mathematics studying natural numbers.
- Both are tied together by an area of mathematics called algebraic geometry.
- Viewed from the right angle, both examples are instances of a beautiful theorem in algebraic geometry: *Every $n$-dimensional smooth projective variety of degree two over a field $k$ with a $k$-rational point is isomorphic to the $n$-dimensional projective space.*

# Stereographic Projection

- Both are examples of a mathematical construction called stereographic projection.
- The first example deals with "classical" Euclidean geometry.
- The second example is taken from the domain of number theory, the branch of mathematics studying natural numbers.
- Both are tied together by an area of mathematics called algebraic geometry.
- Viewed from the right angle, both examples are instances of a beautiful theorem in algebraic geometry: *Every $n$-dimensional smooth projective variety of degree two over a field $k$ with a $k$-rational point is isomorphic to the $n$-dimensional projective space.*
- I won't go into technical details, but the idea can be understood with just highschool mathematics!

# What is Algebraic Geometry?

- Algebra is — contrary to what you may think — *not* "maths with letters instead of numbers". Instead it's the study of certain mathematical structures (groups, rings, fields,...). For our purposes, we can think of fields as "things" that allow us to add, subtract, multiply and divide with the usual rules.
  - One example is $\mathbb{R}$, the reals, numbers like $-7$, $\frac{1}{3}$, $\sqrt{3}$ and $\pi$.
  - Another example is $\mathbb{Q}$, the rationals, numbers like $-7$ and $\frac{1}{3}$, but *neither* $\sqrt{3}$ *nor* $\pi$.

# What is Algebraic Geometry?

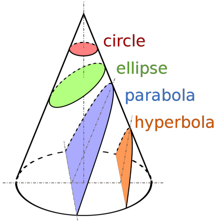
- Algebra is — contrary to what you may think — *not* “maths with letters instead of numbers”. Instead it’s the study of certain mathematical structures (groups, rings, fields,...). For our purposes, we can think of fields as “things” that allow us to add, subtract, multiply and divide with the usual rules.
  - One example is $\mathbb{R}$, the reals, numbers like $-7$, $\frac{1}{3}$, $\sqrt{3}$ and $\pi$.
  - Another example is $\mathbb{Q}$, the rationals, numbers like $-7$ and $\frac{1}{3}$, but *neither* $\sqrt{3}$ *nor* $\pi$.
- Geometry is the study of shapes and their relations in space, things like points, lines, circles, spheres and parabolas.

# What is Algebraic Geometry?

- Algebra is — contrary to what you may think — *not* "maths with letters instead of numbers". Instead it's the study of certain mathematical structures (groups, rings, fields,...). For our purposes, we can think of fields as "things" that allow us to add, subtract, multiply and divide with the usual rules.
  - One example is $\mathbb{R}$, the reals, numbers like $-7$, $\frac{1}{3}$, $\sqrt{3}$ and $\pi$.
  - Another example is $\mathbb{Q}$, the rationals, numbers like $-7$ and $\frac{1}{3}$, but *neither* $\sqrt{3}$ *nor* $\pi$.
- Geometry is the study of shapes and their relations in space, things like points, lines, circles, spheres and parabolas.
- Algebraic Geometry studies algebraic objects by using geometry. The idea is to associate algebraic structures with geometric objects and then be able to apply geometric intuition and techniques to solving algebraic problems.

# What are Varieties of Degree Two?

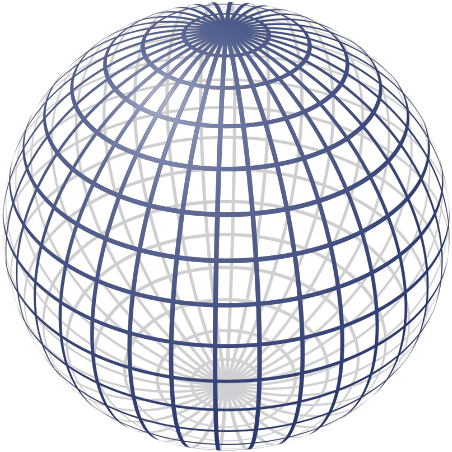
- Let's first look at curves (the one-dimensional case).
- Curves of degree two are just conic sections: circles, ellipses, parabolas and hyperbolas.



# Surfaces of Degree Two

Surfaces (the two-dimensional case) are things like

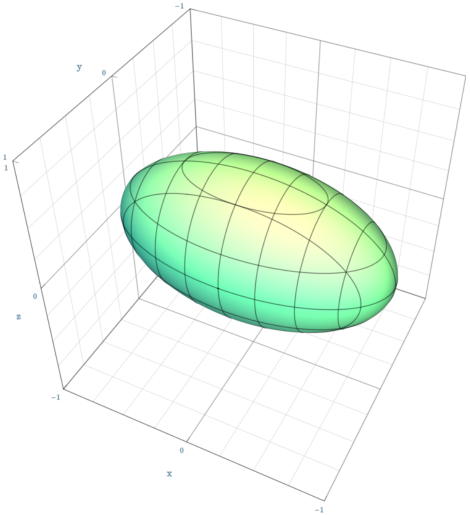
- spheres



# Surfaces of Degree Two

Surfaces (the two-dimensional case) are things like

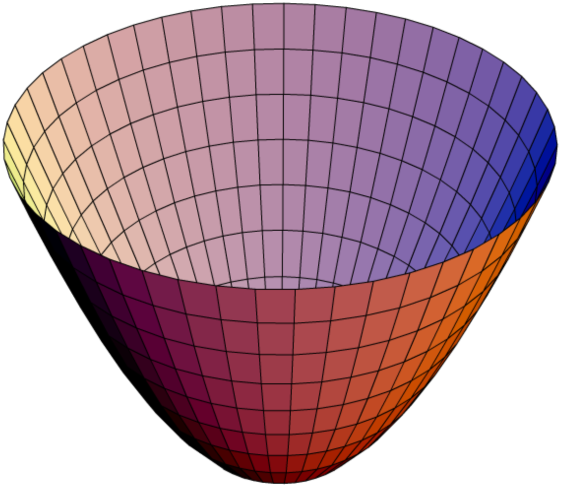
- spheres
- ellipsoids



# Surfaces of Degree Two

Surfaces (the two-dimensional case) are things like

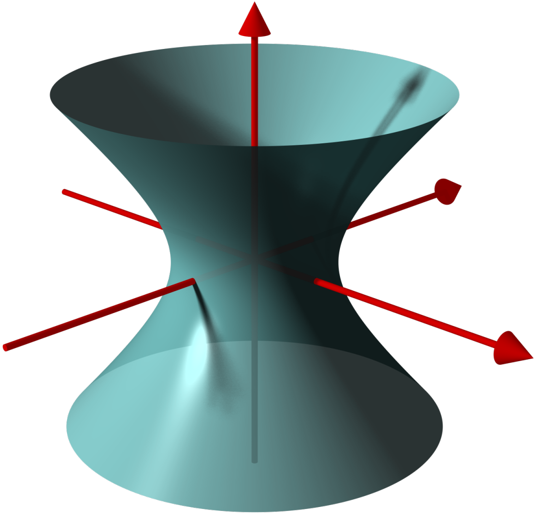
- spheres
- ellipsoids
- paraboloids



# Surfaces of Degree Two

Surfaces (the two-dimensional case) are things like

- spheres
- ellipsoids
- paraboloids
- hyperboloids



# Higher Dimensional Varieties

- There are (of course) also three-dimensional, four-dimensional and even higher-dimensional varieties.
- They are difficult to draw, though...

# Quadratic Equations

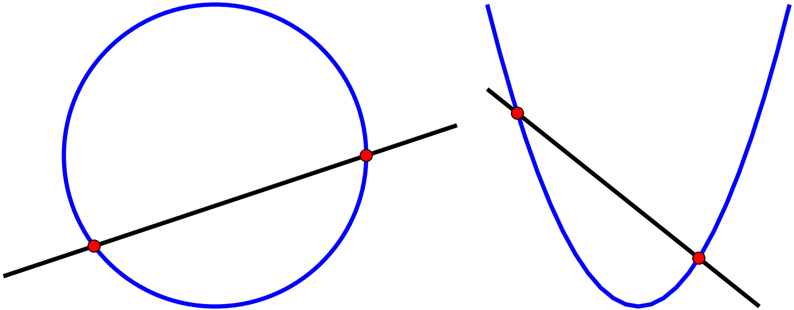
- René Descartes (1596–1650) was the first to draw a connection between algebra and geometry by introducing coordinates.
- Using coordinates, conic sections correspond to equations in two variables $x$ and $y$ of degree two:
  - $x^2 + y^2 = 1$ (circle with center $(0, 0)$ and radius 1)
  - $(\frac{x}{3})^2 + (\frac{y}{2})^2 = 1$ (ellipsis with axes 6 and 4, centered at $(0, 0)$)
  - $y = x^2$ (parabola)
  - $xy = 1$ (hyperbola)

# Quadratic Equations

- René Descartes (1596–1650) was the first to draw a connection between algebra and geometry by introducing coordinates.
- Using coordinates, surfaces of degree two correspond to equations in three variables $x$, $y$ and $z$ of degree two:
  - $x^2 + y^2 + z^2 = 1$ (sphere with center $(0, 0, 0)$ and radius 1)
  - $(\frac{x}{3})^2 + (\frac{y}{2})^2 + (\frac{z}{5}) = 1$ (ellipsoid with axes 6, 4 and 10, centered at $(0, 0, 0)$)
  - $z = x^2 + y^2$ (paraboloid)
  - $x^2 + y^2 - z^2 = 1$ (hyperboloid)

# Geometric Interpretation of Degree

- We have seen that varieties of degree two are defined by polynomial equations of degree two (i.e. quadratic equations).
- There is also a nice geometric interpretation of degree: Lines intersect a (smooth, projective) variety of degree two in two points:



(This is true for *all dimensions*, but difficult to draw for surfaces, let alone even higher dimensional varieties...)

# Degenerate/Exceptional Cases

- The curve corresponding to the quadratic equation $xy = 0$ looks like this:

  It has a so-called singularity at $(0, 0)$. We exclude such cases by insisting on smooth varieties.

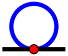
# Degenerate/Exceptional Cases

- The curve corresponding to the quadratic equation $xy = 0$ looks like this:

  It has a so-called singularity at $(0, 0)$. We exclude such cases by insisting on smooth varieties.

- Sometimes, when the line is tangential, we seem to only have *one* intersection:

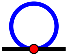
  

  This intersection actually counts *twice*, the two intersection points just happen to be the same.

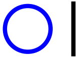
- The curve corresponding to the quadratic equation $xy = 0$ looks like this:

  It has a so-called singularity at $(0, 0)$. We exclude such cases by insisting on smooth varieties.
- Sometimes, when the line is tangential, we seem to only have *one* intersection:

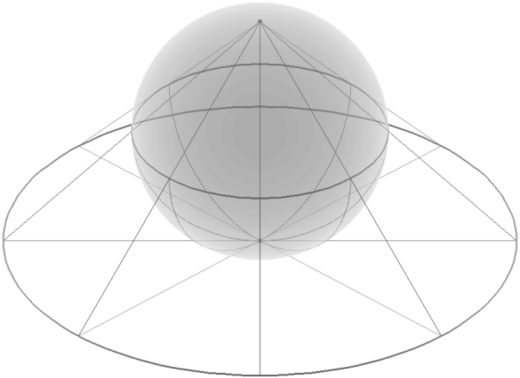
  

  This intersection actually counts *twice*, the two intersection points just happen to be the same.
- Sometimes, we have *no* intersection at all.

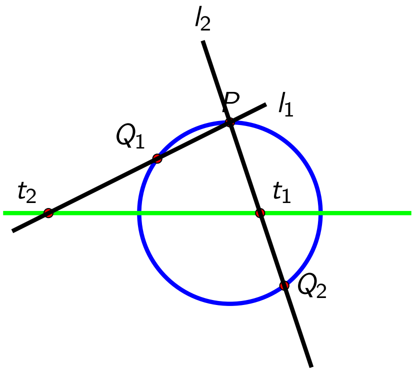
  This is taken care of by considering projective varieties (where we add stuff "at infinity"), but we simply ignore this now.

- Fix one point $P$ of the $n$-dimensional variety $X$.
- Each line $l$ through $P$ intersects $X$ in exactly one other point $Q$.
- We thus get a 1:1-mapping from lines $l$ through $P$ and points $Q$ of $X$.
- All lines through a given point are in 1:1-correspondence to the $n$-dimensional projective space.



# Stereographic Projection for the Unit Circle

- As fixed point $P$, we choose the "North Pole" $(0, 1)$.
- Each line $l$ through $P$ intersects the circle in exactly one other point $Q$.
- Each line $l$ is determined by its intersection $t$ with the $x$-axis.
- In the diagram, the stereographic projection maps $Q_1$ to $t_1$ and $Q_2$ to $t_2$.

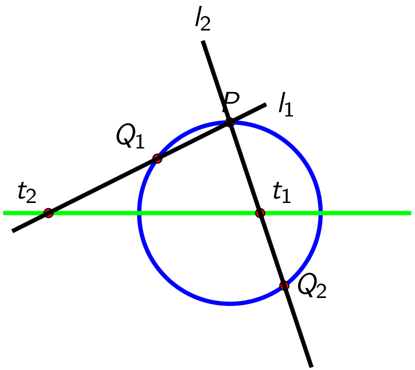


# Stereographic Projection for the Unit Circle

- The line through $(0,1)$ and $(t,0)$ is given by $(x,y) = (0,1) + \alpha\big[(t,0) - (0,1)\big] = (\alpha t, 1-\alpha)$ for $\alpha \in \mathbb{R}$.
- To lie on the circle, we must have $x^2 + y^2 = 1$, so

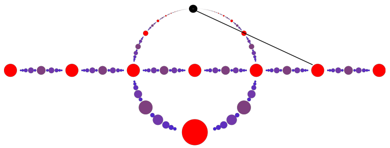
$$\begin{aligned}(\alpha t)^2 + (1-\alpha)^2 &= 1\\ \alpha^2 t^2 + 1 - 2\alpha + \alpha^2 &= 1\\ \alpha\big[\alpha t^2 - 2 + \alpha\big] = \alpha\big[(t^2+1)\alpha - 2\big] &= 0\\ \alpha = 0 \quad \text{or} \quad \alpha &= \frac{2}{t^2+1}\end{aligned}$$

- $\alpha = 0$ corresponds to $P$, so we want the other solution, $\alpha = \frac{2}{t^2+1}$, so $(x,y) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$.



- We observe that if
  - the equation defining our variety has coefficients in $\mathbb{Q}$ and
  - our fixed point $P$ has coordinates in $\mathbb{Q}$ and
  - $t \in \mathbb{Q}$,

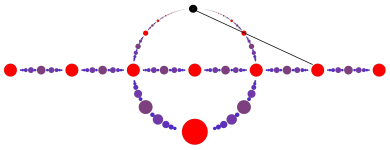
  then we get a point $(x, y)$ with $x, y \in \mathbb{Q}$.
- This has nothing to do with our example of the unit circle, it holds for all (smooth, projective) varieties of degree two.
- So stereographic projection allows us to find all points on such a variety with rational coordinates, provided we have one point with rational coordinates.



- Let's try this for the unit circle!
- Our solution was $(x,y) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$, so for rational $t$, we indeed get rational $x$ and $y$.
- Let's try $t = 2$:

$$(x,y) = \left(\frac{2\cdot 2}{2^2+1}, \frac{2^2-1}{2^2+1}\right) = \left(\frac{4}{5}, \frac{3}{5}\right).$$

- We check $\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = 1$, or (multiplying by $5^2$) $3^2 + 4^2 = 5^2$, the smallest (non-trivial) Pythagorean triple!
- We can find all Pythagorean triples this way, they exactly correspond to rational points on the unit circle!



# Summary

- Algebraic Geometry is the area of mathematics that studies polynomial equations by means of geometry.
- This allows applying geometric constructions to solve algebraic problems and in particular problems from Number Theory.
- As an example of this technique, we have seen how the geometric construction of Stereographic Projection can be used to solve the number theoretical problem of finding all Pythagorean Triples.

# Summary

- Algebraic Geometry is the area of mathematics that studies polynomial equations by means of geometry.
- This allows applying geometric constructions to solve algebraic problems and in particular problems from Number Theory.
- As an example of this technique, we have seen how the geometric construction of Stereographic Projection can be used to solve the number theoretical problem of finding all Pythagorean Triples.
- This, by the way, is far from true for degrees higher than two. Even *curves* of degree three, so-called elliptic curves, are very complicated beasts indeed, and understanding them better is one of the Millenium Problems, whose solution would earn you $1000000 and eternal fame.