Derivation:

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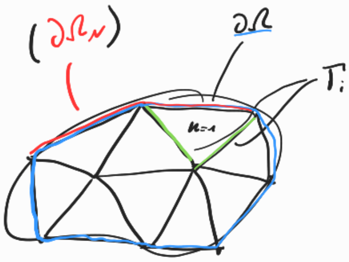
$[v] = v^+ - v^-$

$\langle v \rangle = \frac{v^+ + v^-}{2}$

$$\boxed{\begin{aligned} -\Delta u &= f \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}}$$



$\Gamma_u = \partial\Omega \cup T_i$

weak form:

$$-\int \Delta u \cdot v \, d\Omega_u = \int f \cdot v \, d\Omega_u$$

integration by parts + Gauss theorem:

$$+\int \nabla u \cdot \nabla v \, d\Omega_u - \int_{\Gamma_u} \nabla u \cdot n \cdot v \, d\Gamma_u = \int f v \, d\Omega_u$$

add zero normal flux:

$$+\int \nabla u \cdot \nabla v \, d\Omega_u - \int_{T_i} \underbrace{\nabla u \cdot n}_{=0,\ \text{because continuous in analy. sol}} [v] \, dT_i - \int_{\partial\Omega} \nabla u \cdot n \cdot v \, d\Gamma_{\partial\Omega} = \int f v \, d\Omega_u$$

then normal flux $[n \cdot \nabla u] = 0 \rightarrow$ discrete flux $\langle n \cdot \nabla u \rangle - \frac{\beta}{h}[u]$

positive penalty parameter

$$+\int \nabla u \cdot \nabla v \, d\Omega_u - \int_{T_i} \left(\langle n \cdot \nabla u \rangle - \frac{\beta}{h}[u] \right) [v] \, dT_i - \int_{\partial\Omega} \nabla u \cdot n \cdot v \, d\Gamma_{\partial\Omega} + \int_{\partial\Omega} \frac{\beta}{h} u v \, d\Gamma_{\partial\Omega} = \int f v \, d\Omega_u$$

symmetric interior penalty method:

$$\int_{T_i} [u] \langle n \cdot \nabla v \rangle \, d\Gamma_i + \int_{\partial\Omega} u \, n \cdot \nabla v \, d\Gamma_{\partial\Omega} \Big) = 0$$

when u is the exact solution

subtract this zero

$$\underbrace{\underbrace{+\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega_u}_{I}}_{\text{symmetric}} \underbrace{\underbrace{- \int_{T_i} \langle n \cdot \nabla u \rangle [v] \, dT_i}_{II} \underbrace{- \int_{T_i} [u] \langle n \cdot \nabla v \rangle \, dT_i}_{III}}_{\text{symmetric}} \underbrace{\underbrace{+ \int_{T_i} \frac{\beta}{h} [u][v] \, dT_i}_{IV}}_{\text{symmetric}} \underbrace{\underbrace{+ \int_{\partial\Omega} \frac{\beta}{h} u v \, d\Gamma_{\partial\Omega}}_{V}}_{\text{symmetric}}$$

$$\underbrace{-\int_{\partial\Omega} u \, n \cdot \nabla v \, d\Gamma_{\partial\Omega} \underbrace{- \int_{\partial\Omega} \nabla u \cdot n \cdot v \, d\Gamma_{\partial\Omega}}_{VI}}_{\text{symmetric}} = \underbrace{\int f v \, d\Omega}_{VII}$$

(center of gravity integration)

Linear Shape Functions:

$$\varphi_i = a_i + b_i x_1 + c_i x_2 \qquad i = 1,2,3$$

$$\varphi^{Edge} = \begin{bmatrix} \varphi_1^+ \\ \varphi_2^+ \\ \varphi_3^+ \\ \varphi_1^- \\ \varphi_2^- \\ \varphi_3^- \end{bmatrix} \rightarrow [\varphi^{Edge}] = \begin{bmatrix} \varphi_1^+ \\ \varphi_2^+ \\ \varphi_3^+ \\ -\varphi_1^- \\ -\varphi_2^- \\ -\varphi_3^- \end{bmatrix}$$

$$\langle n \cdot \nabla \phi^{Edge} \rangle = \frac{1}{2} n_{Edge} \cdot \nabla \varphi^E$$

$$a_i = \frac{x^j y^k - x^k y^j}{2|K|}$$

$$b_i = \frac{y^j - y^k}{|2K|}$$

$$c_i = \frac{x^k - x^j}{2|K|}$$

$$A_{ij}^k = \int \nabla \varphi_i \cdot \nabla \varphi_j \; dx$$

$$= \int \begin{pmatrix} b_i \\ c_i \end{pmatrix} \cdot \begin{pmatrix} b_j \\ c_j \end{pmatrix} dx$$

$$= (b_i b_j + c_i c_j) \int dx$$

$$1 = a_1 + b_1 x_1 + c_1 y_1$$

$$0 = a_1 + b_1 x_2 + c_1 y_2$$

$$0 = a_1 + b_1 x_3 + c_1 y_3$$

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\nabla \varphi_i = \begin{pmatrix} b_i \\ c_i \end{pmatrix}$$

$$\nabla \varphi_j = \begin{pmatrix} b_j \\ c_j \end{pmatrix}$$

$$\int f \varphi \, dA = f \int \varphi \, dA$$

$$= f \int \varphi_i \, dA$$