Analysis of linear resisted projectile motion using the Lambert W function

Article *in* Acta Mechanica · February 2012

DOI: 10.1007/s00707-011-0571-2

H. Hu
Hunan University of Science and Technology

Minyi Zheng
Hefei University of Technology

Acta Mech 223, 441–447 (2012)
DOI 10.1007/s00707-011-0571-2

NOTE

H. Hu · Y. P. Zhao · Y. J. Guo · M. Y. Zheng

Analysis of linear resisted projectile motion using the Lambert W function

Received: 22 March 2011 / Revised: 7 October 2011 / Published online: 10 November 2011

Abstract We solve the linear resisted projectile motion problem using the Lambert W function. In the problem, the launching point is higher than the landing point. The explicit solutions for the range and optimal angle of elevation are expressed in terms of the Lambert W function. The two conclusions can be made for the same projectile: (i) The higher the launching point, the larger the range is. (ii) The higher the launching point, the smaller the optimal angle is.

1 Introduction

The determination of the projectile angle that maximizes the range has been a problem of theoretical and practical importance for several centuries. The dependence of the horizontal range on the firing angle of the projectile in the absence of air resistance is a topic addressed by many introductory courses in mechanics. Since, in practice, some resistance is inherently present, it would be of some interest to develop the solution of projectile motion including an appropriate drag force. The drag force is typically modeled as proportional to the projectile velocity (the linear drag model) or the projectile speed squared (the quadratic drag model) [1]. Although the linear drag model is a crude approximation to the real motion [2], this model has led to a number of interesting questions that have attracted attention [2] and sometimes gives predictions that agree with observations [3]. Morales [2], Stewart [4], Packel and Yuen [5], and Warburton and Wang [6] each analyzed linear resisted projectile motion using the Lambert W function. However, each only considered the case where the launching and landing points are at the same height.

In the following, we will consider a situation where the launching point is higher than the landing point. In this case, we show that the explicit solutions for the range and optimal angle of elevation can be also expressed in terms of the Lambert W function.

2 Overview of the Lambert W function

The Lambert $W(z)$ function is a many-valued complex function satisfying [7]

$$W(z)e^{W(z)} = z, \tag{1}$$

where, in general, z is a complex number. If z is real and $z < -1/e$, then $W(z)$ is many-valued complex. If z is real and $-1 \le z < 0$, there are two possible real values of $W(z)$: The branch satisfying $-1 \le W(z)$ is denoted by $W_0(z)$ and called the principal branch of the W function, and the other branch satisfying $W(z) \le -1$ is

H. Hu (✉) · Y. P. Zhao · Y. J. Guo · M. Y. Zheng
School of Civil Engineering, Hunan University of Science and Technology,
Xiangtan, Hunan 411201, People's Republic of China
E-mail: huihuxt@yahoo.com.cn

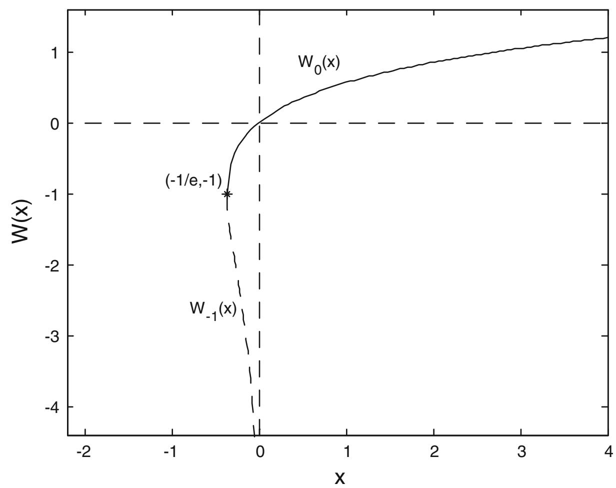

Fig. 1 Plot of the Lambert W function for real arguments. The *solid line* shows $W_0(x)$ while the *dashed line* shows $W_{-1}(x)$. The two branches meet at point $(-1/e, -1)$

denoted by $W_{-1}(z)$. When z is real and $z \geq 0$, there is a single real value for $W(z)$, which also belongs to the principal branch $W_0(z)$. Both real branches $W_0(z)$ and $W_{-1}(z)$, for z real, are presented in Fig. 1. It is the principal branch $W_0(z)$ that will be useful to us in what follows.

3 Linear resisted projectile motion

Consider that a projectile of mass m is launched from a height H of the vertical. In a linear resisting medium, the equations of motion of the projectile are [2]

$$m\ddot{x} = -k\dot{x}, \tag{2}$$
$$m\ddot{y} = -mg - k\dot{y}, \tag{3}$$

where k is the positive resistance constant. These equations can be solved with the initial conditions

$$\dot{x}(0) = v_0 \cos\theta, \quad x(0) = 0, \tag{4}$$
$$\dot{y}(0) = v_0 \sin\theta, \quad y(0) = 0, \tag{5}$$

where v_0 and θ are the initial speed and the elevation angle, respectively. The solutions of theses equations are [2]

$$x = \frac{v_0 \cos\theta}{\gamma}\left(1 - e^{-\gamma t}\right), \tag{6}$$
$$y = \left(\frac{g}{\gamma^2} + \frac{v_0 \sin\theta}{\gamma}\right)\left(1 - e^{-\gamma t}\right) - \frac{gt}{\gamma}, \tag{7}$$

where $\gamma = k/m$.

3.1 Range

Solving Eq. (6) for the time t in terms of x and substituting into Eq. (7) gives

$$y = \left(\frac{g}{v_0 \gamma \cos\theta} + \tan\theta\right) x + \frac{g}{\gamma^2} \ln\left(1 - \frac{\gamma x}{v_0 \cos\theta}\right). \tag{8}$$

On impact with the ground $y = -H$ and $x = R$, which is the horizontal range that we seek to maximize. Therefore, from Eq. (8) we have

$$R(\theta) = \frac{\cos\theta}{a} (1 - \exp[-bH - A(\theta)R(\theta)]) \tag{9}$$

where

$$A(\theta) = a \sec\theta + b \tan\theta, \tag{10}$$

$$a = \frac{\gamma}{v_0}, \ b = \frac{\gamma^2}{g}. \tag{11}$$

Equation (9) is a transcendental equation for $R(\theta)$. This equation can be rewritten as

$$A(\theta) \left[R(\theta) - \frac{\cos\theta}{a}\right] \exp\left\{A(\theta)\left[R(\theta) - \frac{\cos\theta}{a}\right]\right\} = -\frac{A(\theta)\cos\theta}{a} \exp\left[-bH - \frac{A(\theta)}{a}\cos\theta\right]. \tag{12}$$

If we compare Eq. (12) with Eq. (1), we can obtain

$$A(\theta) \left[R(\theta) - \frac{\cos\theta}{a}\right] = W_0(B_H(\theta)) \tag{13}$$

where

$$B_H(\theta) = -(1 + c \sin\theta) \exp[-bH - (1 + c \sin\theta)], \tag{14}$$

and $c = b/a = \gamma v_0/g$. From Eq. (13), we have

$$R(\theta) = \frac{\cos\theta}{a} \left(1 + \frac{W_0(B_H(\theta))}{1 + c \sin\theta}\right). \tag{15}$$

Equation (15) allows us to obtain immediately plots of the range as a function of the inclination angle, since the Lambert W function is already implemented in many computer algebra systems including Maple and Mathematica [2].

3.2 Optimal angle

Now we derive an explicit expression for the optimal angle of elevation that maximizes the range of the projectile for any chosen initial velocity. The maximum range occurs at an angle $\theta_{\max}$ satisfying $R'(\theta)\big|_{\theta_{\max}} = 0$. Differentiating Eq. (9) with respect to θ and setting $R'(\theta)\big| = 0$ yields

$$0 = -\frac{\sin\theta}{a} (1 - \exp[-bH - A(\theta)R(\theta)]) + \frac{\cos\theta}{a} A'(\theta)R(\theta) \exp[-bH - A(\theta)R(\theta)]. \tag{16}$$

From Eq. (9) we have

$$\exp[-bH - A(\theta)R(\theta)] = 1 - \frac{aR(\theta)}{\cos\theta}, \tag{17}$$

and the derivative of $A(\theta)$ is

$$A'(\theta) = \frac{a \sin\theta + b}{\cos^2\theta}. \tag{18}$$

Substituting Eqs. (17) and (18) into Eq. (16) and simplifying leads to the following expression for the maximum range $R_{\max}$ at the angle $\theta_{\max}$:

$$R_{\max} = \frac{c \cos\theta_{\max}}{a(\sin\theta_{\max} + c)}. \tag{19}$$

Hence it follows that

$$A(\theta_{\max})R_{\max} = \frac{c(1 + c\sin\theta_{\max})}{\sin\theta_{\max} + c}. \tag{20}$$

Substituting Eqs. (19) and (20) back into Eq. (9) and setting $u = \sin\theta_{\max}$ gives

$$\frac{c}{u + c} = 1 - \exp(-bH)\exp\left[-\frac{c(1 + cu)}{u + c}\right]. \tag{21}$$

After some manipulations, this equation can be written as

$$\left(\frac{ue}{u + c}\frac{c^2 - 1}{e}\right)\exp\left(\frac{ue}{u + c}\frac{c^2 - 1}{e}\right) = \frac{c^2 - 1}{e}\exp(-bH). \tag{22}$$

A comparison of Eq. (22) with Eq. (1) leads to the following relation:

$$\frac{ue}{u + c}\frac{c^2 - 1}{e} = W_0\left(\frac{c^2 - 1}{e}e^{-bH}\right). \tag{23}$$

Solving for u and setting $\theta_{\max} = \sin^{-1} u$ gives the final results:

$$\theta_{\max} = \sin^{-1}\left[\frac{cW_0(\frac{c^2-1}{e}e^{-bH})}{c^2 - 1 - W_0(\frac{c^2-1}{e}e^{-bH})}\right],\ c \neq 1. \tag{24}$$

If $c = 1$, then from Eq. (21) we have

$$\theta_{\max} = \sin^{-1}\left[\frac{1}{\exp(1 + bH) - 1}\right]. \tag{25}$$

Note that for the special case of $H = 0$, Eqs. (15), (24), and (25) become, respectively,

$$R(\theta) = \frac{\cos\theta}{a}\left(1 + \frac{W_0(B_0(\theta))}{1 + c\sin\theta}\right),\ B_0(\theta) = -(1 + c\sin\theta)\exp[-(1 + c\sin\theta)], \tag{26}$$

$$\theta_{\max} = \sin^{-1}\left[\frac{cW_0(\frac{c^2-1}{e})}{c^2 - 1 - W_0(\frac{c^2-1}{e})}\right],\ c \neq 1, \tag{27}$$

$$\theta_{\max} = \sin^{-1}\left(\frac{1}{e - 1}\right), \tag{28}$$

which are in agreement with the results obtained by Morales [2] and Stewart [4]. Equation (24) can be also rewritten into its equivalent form using the defining equation for the Lambert W function, namely

$$\theta_{\max} = \sin^{-1}\left[\frac{c}{\exp[W_0(\frac{c^2-1}{e}e^{-bH}) + bH + 1] - 1}\right]. \tag{29}$$

This form has the advantage that it is valid for all positive c including $c = 1$. If $H = 0$, Eq. (29) becomes [4]

$$\theta_{\max} = \sin^{-1}\left[\frac{c}{\exp[W_0(\frac{c^2-1}{e}) + 1] - 1}\right]. \tag{30}$$

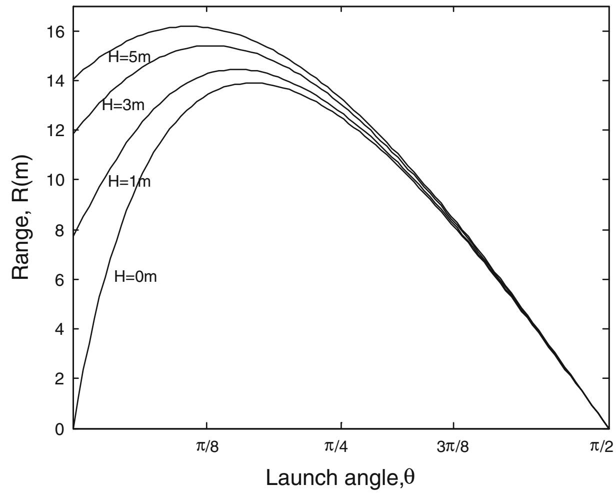

Fig. 2 Range versus angle of projection for $v_0 = 20\,\mathrm{ms}^{-1}$; $\gamma = 1.0\,\mathrm{s}^{-1}$; and $H = 0, 1, 3$, and $5\,\mathrm{m}$

4 Discussion

It is interesting to compare Eq. (15) with Eq. (26). Let $f(x) = -xe^{-x}$. Then, we have $f'(x) = \mathrm{d}f/\mathrm{d}x = (x-1)e^{-x}$ and $f''(x) = (2-x)e^{-x}$. Since $f'(1) = 0$ and $f''(1) = e^{-1} > 0$, the minimum value of $f(x)$ on $[1, \infty]$ is $f(1) = -e^{-1}$. Therefore,

$$-1/e \leq B_0(\theta) = -(1 + c\sin\theta)e^{-(1+c\sin\theta)} \leq 0, \quad 0^0 \leq \theta \leq 90^0. \tag{31}$$

Obviously, we also have

$$-1/e \leq B_0(\theta) \leq B_H(\theta) = B_0(\theta)/e^{bH} \leq 0, \quad 0^0 \leq \theta \leq 90^0. \tag{32}$$

Note that the function $W_0(z)$ is increasing for $z \geq -1/e$ (see Fig. 1). Then, from Eqs. (15), (26), and (32) we have

$$\frac{\cos\theta}{a}\left(1 + \frac{W_0(B_H(\theta))}{1 + c\sin\theta}\right) \geq \frac{\cos\theta}{a}\left(1 + \frac{W_0(B_0(\theta))}{1 + c\sin\theta}\right), \quad 0^0 \leq \theta \leq 90^0. \tag{33}$$

This inequality indicates that the range for $H = 0$ is always less than the corresponding range for $H > 0$. Figure 2 shows the range as a function of the initial launch angle for $v_0 = 20$ ms^{-1} and $\gamma = 1.0$ s^{-1} and for various heights $H(H = 0, 1, 3, 5$ m). Figure 3 demonstrates the range as a function of the initial launch angle for $v_0 = 20$ ms^{-1} and $H = 50$ m and for various resistive constants ($\gamma = 0.01, 0.1, 0.5, 1.0$ s^{-1}).

Now we compare Eq. (29) with Eq. (30). Note that the derivative of the function $W(z)$ is [7]

$$\frac{\mathrm{d}W(z)}{\mathrm{d}z} = \frac{1}{e^{W(Z)}[1 + W(z)]} = \frac{W(z)}{z[1 + W(z)]}. \tag{34}$$

Let $g(x) = W_0(\frac{c^2-1}{e}e^{-x}) + x$, then the derivative of $g(x)$ is

$$\begin{aligned} g'(x) &= \frac{W_0(\frac{c^2-1}{e}e^{-x})}{\frac{c^2-1}{e}e^{-x}\left[1 + W_0(\frac{c^2-1}{e}e^{-x})\right]}\left(\frac{c^2-1}{e}e^{-x}\right)' + 1 \\ &= \frac{1}{1 + W_0(\frac{c^2-1}{e}e^{-x})}. \end{aligned} \tag{35}$$

Obviously, if $x \geq 0$, then

$$\frac{c^2-1}{e}e^{-x} \geq \frac{-1}{e}e^{-x} \geq -\frac{1}{e}. \tag{36}$$

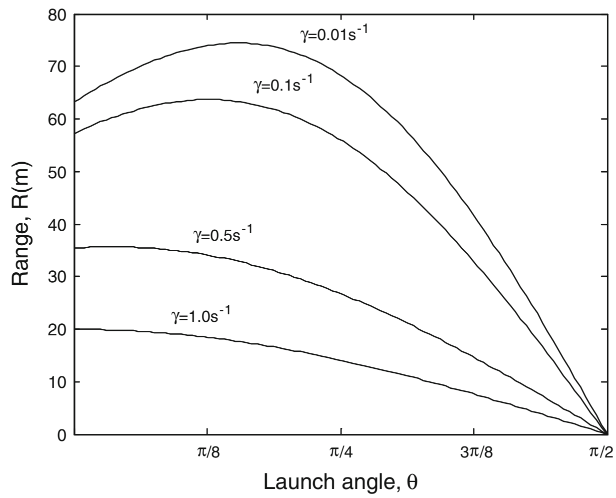

Fig. 3 Range versus angle of projection for $v = 20\,\mathrm{ms}^{-1}$; $H = 50\,\mathrm{m}$; and $\gamma = 0.01, 0.1, 0.5$, and $1.0\,\mathrm{s}^{-1}$

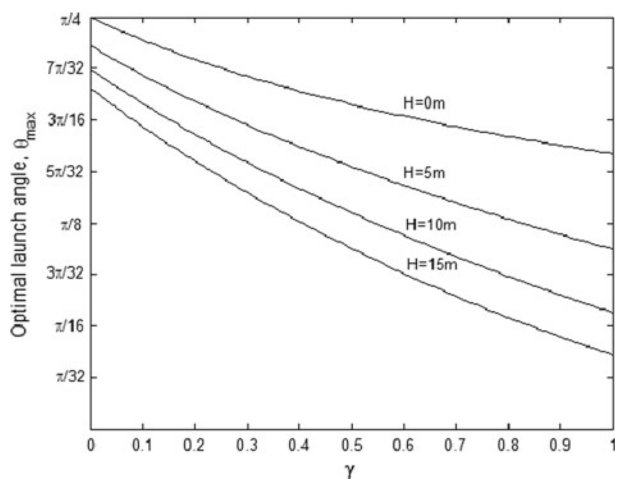

Fig. 4 Optimal angle as a function of the parameter γ for $v = 20\,\mathrm{ms}^{-1}$ and $H = 0, 5, 10$, and $15\,\mathrm{m}$

Since the function $W_0(z)$ is increasing for $z \geq -1/e$, from Eq. (36) we have

$$1 + W_0\left(\frac{c^2-1}{e}e^{-x}\right) \geq 1 + W_0\left(-\frac{1}{e}\right) \geq 0. \tag{37}$$

Therefore,

$$g'(x) = \frac{1}{1 + W_0\left(\frac{c^2-1}{e}e^{-x}\right)} \geq 0, \tag{38}$$

which leads to the conclusion that the function $g(x)$ is increasing for $x \geq 0$. Therefore, we have

$$\exp[g(bH) + 1] \geq \exp[g(0) + 1], \tag{39}$$

or

$$\exp\left[W_0\left(\frac{c^2-1}{e}e^{-bH}\right) + bH + 1\right] \geq \exp\left[W_0\left(\frac{c^2-1}{e}\right) + 1\right]. \tag{40}$$

From Eqs. (29), (30), and (40), we have

$$\sin^{-1}\left[\frac{c}{\exp[W_0(\frac{c^2-1}{e}e^{-bH})+bH+1]-1}\right]\leq\sin^{-1}\left[\frac{c}{\exp[W_0(\frac{c^2-1}{e})+1]-1}\right]. \tag{41}$$

This inequality indicates that the optimal angle for $H > 0$ is always smaller than the corresponding optimal angle for $H = 0$. Figure 4 shows the optimal angle as a function of the parameter γ for $v = 20\,\mathrm{ms}^{-1}$ and $H = 0$, 5, 10, and 15 m.

5 Conclusions

The linear resisted projectile motion problem where the launching point is higher than the landing point has been analyzed using the Lambert W function. The exact solutions for the range and optimal angle of elevation are expressed in terms of this function, and the following conclusions can be made for the same projectile:

(i) The higher the launching point, the larger the range is.
(ii) The higher the launching point, the smaller the optimal angle is.

We believe these results may find applications in sports such as shot put.

Acknowledgments The authors are grateful to the anonymous reviewer for helpful comments and suggestions in preparing the revised manuscript. The work reported here was supported by the National Natural Science Foundation of China (Grant No. 50775071) and the University Teaching Research Fund of Hunan Province of China (Grant No. G20924).

References

1. Hayen, J.C.: Projectile motion in a resistant medium Part I: exact solution and properties. Int. J. Non-Linear Mech. **38**, 357–369 (2003)
2. Morales, D.A.: Exact expressions for the range and the optimal angle of a projectile with linear drag. Can. J. Phys. **83**, 67–83 (2005)
3. Erlichson, H.: Maximum projectile range with drag and lift, with particular application to golf. Am. J. Phys. **51**, 357–362 (1983)
4. Stewart, S.M.: An analytic approach to projectile motion in a linear resisting medium. Int. J. Math. Educ. Sci. Tech. **37**, 411–431 (2006)
5. Packel, E.W., Yuen, D.S.: Projectile motion with resistance and the Lambert W function. College Math. J. **35**, 337–350 (2004)
6. Warburton, R.D.H., Wang, J.: Analysis of asymptotic projectile motion with air resistance using the Lambert W function. Am. J. Phys. **72**, 1404–1407 (2004)
7. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST Handbook of Mathematical Functions. p. 111, Cambridge University Press, Cambridge (2010)