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Reconstructing Digital Signals Using Shannon's Sampling Theorem

Joseph Hamill, Graham E. Caldwell, and Timothy R. Derrick

Researchers must be cognizant of the frequency content of analog signals that they are collecting. Knowing the frequency content allows the researcher to determine the minimum sampling frequency of the data (Nyquist critical frequency), ensuring that the digital data will have all of the frequency characteristics of the original signal. The Nyquist critical frequency is 2 times greater than the highest frequency in the signal. When sampled at a rate above the Nyquist, the digital data will contain all of the frequency characteristics of the original signal but may not present a correct time–series representation of the signal. In this paper, an algorithm known as Shannon's Sampling Theorem is presented that correctly reconstructs the time–series profile of any signal sampled above the Nyquist critical frequency. This method is superior to polynomial or spline interpolation techniques in that it can reconstruct peak values found in the original signal but missing from the sampled data time–series.

In biomechanical studies, many types of experiments are conducted using a myriad of data collection devices. In most cases, signals to be collected are in analog form. These analog signals include variation in ground reaction forces, change in kinematic parameters, and fluctuations in the electrical activity of muscles. To collect such analog signals for further analysis, researchers usually convert the signal into digital form with a constant time interval between data samples. The conversion from analog to digital form may create problems for the biomechanist because details about signal characteristics are generally unknown a priori.

When conducting an experiment, scientists must know the characteristics of the signal they are collecting, particularly the frequency content of the signal. This includes knowing the amplitude, frequency, and phases of all frequency components of the signal. Knowledge of the frequency content of the signal, and in particular the highest frequency in the signal, is necessary to determine the frequency at which the analog data must be sampled and to ensure faithful conversion to digital form.

The knowledge of the highest frequency in the signal and the appropriate sampling rate is not always compatible with the equipment available for data collection. For example, in the last several years, many biomechanics laboratories have switched from using high-speed cinematography to videography. The high-speed cine cameras usually had a variable setting for the frame rate, and thus the sampling rate could be changed to meet the requirements of the signal under inspection in the experiment. However, video cameras and recorders generally have a fixed sampling frequency or frame rate of 60 Hz

The authors are with the Biomechanics Laboratory, Department of Exercise Science, University of Massachusetts, Amherst, MA 01003.

(50 Hz in Europe). High-speed video cameras and recorders with sampling frequencies of 120 to 500 Hz are significantly more expensive and thus not as prominent in biomechanics laboratories as the 60 Hz cameras. In many cases, the 60 Hz cameras are sufficient for human movement analysis. However, there are situations, such as in the analysis of rearfoot movement, for which 60 Hz cameras may be inadequate.

In this paper, we revisit the well-known sampling theorem as it pertains to biomechanical data. We then examine a common problem in analog-to-digital signal representation and present a solution based on a less commonly known numerical reconstruction equation. An algorithm based on this equation is applied to several common examples in the realm of biomechanics to demonstrate its efficacy.

Theory

Most signals are sampled at equal time intervals (δ). The reciprocal of this time interval is the sampling rate (f_s). For example, if $\delta = 0.001$ s, then f_s is 1,000 samples per second or $1/\delta = 1000$ Hz. For any biological signal, there is a sampling frequency referred to as the Nyquist critical frequency that can be represented by

$$f_c = 2f_h$$

where f_c is the Nyquist critical frequency and f_h is the highest frequency contained in the signal itself (Press, Flannery, Teukolsky, & Vetterling, 1990). The Nyquist critical frequency is the basis of the sampling theorem. The sampling theorem states that as long as the original signal is sampled at a rate greater than 2 times the highest frequency in the signal, then the sampled points completely determine the signal. Thus, if the highest frequency in a signal is 12 Hz, the minimum sampling frequency for these data would be greater than 24 Hz ($2f_h$).

If the signal is collected at a frequency greater than f_c, then the digital signal has all of the information contained in the analog signal. If the sampling frequency is below f_c, then the signal suffers an *aliasing* problem (Winter, 1990). In this case, the information content of the digitized signal not only is incomplete but can be distorted. Frequencies between $\frac{1}{2}f_s$ and f_h fold back into the spectrum, causing aliasing. In biomechanical data, aliasing is generally not a problem because many researchers choose sampling frequencies well in excess of the f_c.

While the majority of biomechanical studies adhere strictly to the sampling theorem, a problem in digital representation may still exist. Although sampling at rates above f_c ensures that the digital data contain all information found in the original analog signal, this information is not apparent in the time–series representations of signals collected just above f_c. In the following paragraphs we will illustrate this problem and present a reconstruction algorithm that can alleviate the problem.

Methods

Three types of data signals will be used to demonstrate the reconstruction application presented in this paper. In the first application, a 20 Hz sine wave was generated with a Dynascan Corporation 3010 Function Generator and sampled using a 12-bit analog-to-digital converter at a sampling rate of 400 Hz. The 400 Hz signal was then resampled at 50 Hz for comparison to the original signal. Both the 400 and 50 Hz versions of the signal were examined in the frequency domain using Fast Fourier Transform (FFT) techniques to produce a power spectral density (PSD).

For the second application, an AMTI force platform and amplifier were interfaced to a microcomputer via a 12-bit analog-to-digital converter. The vertical ground reaction force component of a single running trial of a young, healthy, adult male was collected. Data were sampled at a rate of 1000 Hz. We subjected the 1000 Hz signal to a spectral analysis using an FFT to determine its power spectral density. Based on the spectral analysis, the 1000 Hz signal was then resampled at 100 Hz.

A rearfoot motion trial was used for the third application. Four retro-reflective markers were placed on the rearfoot of a healthy, young, adult male runner according to the protocol of Clarke, Frederick, and Hamill (1983). The trial was recorded using a 200 Hz NAC high-speed camera and recorder. The videotape of the trial was digitized using a Motion Analysis VP110 video processor, first at a sampling rate of 200 Hz and then at a rate of 50 Hz. The paths of the markers were smoothed at 18 Hz using a digital filter, and the rearfoot angles of each of the 50 and 200 Hz trials were computed (Hamill, Milliron, & Healy, 1994).

Prior to implementation of Shannon's algorithm, each data signal must meet the requirements of circular continuity. The first step is to de-trend the signal. This is accomplished by calculating the slope between the first and last data points and removing this slope from the data set. This ensures that the first and last data points have the same value. A further step is to remove the bias by subtracting the mean value from all data points so that the signal oscillates about zero. In some cases, signals exhibit different slopes at the start and end and, therefore, still do not meet the requirements of circular continuity. In these cases, further processing is needed to ensure that the slopes are similar at the beginning and end of the signal. After reconstruction of the signal, these processing steps must be reversed.

Results and Discussion

Problem

The digital signal representation problem outlined above is clearly illustrated when the frequency content of a sine wave is determined. For a 20 Hz sine wave, a frequency analysis would show all of the power at 20 Hz. Therefore, it should be possible to sample the signal at a frequency of 50 Hz and reproduce the exact signal from the digital data. Figure 1a represents the original signal sampled at 400 samples per second. This is a close representation of the analog signal because of the extremely high sampling rate relative to the frequency content of the signal. Indeed, the PSD of this 400 Hz signal (Figure 1b) demonstrates that all of the power in the signal lies at 20 Hz. Figure 1c represents the signal sampled at 50 Hz, that is, at a rate slightly greater than f_c. Clearly, the representations of the signal in Figures 1a and 1c are not the same. However, in Figure 1d, a frequency analysis of the signal collected at 50 Hz illustrates the same power spectrum as in Figure 1b. This illustrates that sampling at f_c captures all of the frequency information from the original signal.

A more relevant example using biomechanical data can be demonstrated using the vertical ground reaction force (GRF) component during running. These data are often sampled at 1000 Hz (Cavanagh & Lafortune, 1980; Hamill, Bates, Knutzen, & Sawhill, 1983), well above the Nyquist limit for this signal. During a heel–toe footfall pattern in running, there is a high frequency peak often called the impact or passive peak (Nigg, Denoth, & Neukomm, 1981). Figure 2a illustrates this impact peak in the signal sampled at 1000 Hz. In this case, the impact peak has a magnitude of 1,345 N. Figure 2b represents the frequency content of this signal with significant components all below 50 Hz. For this

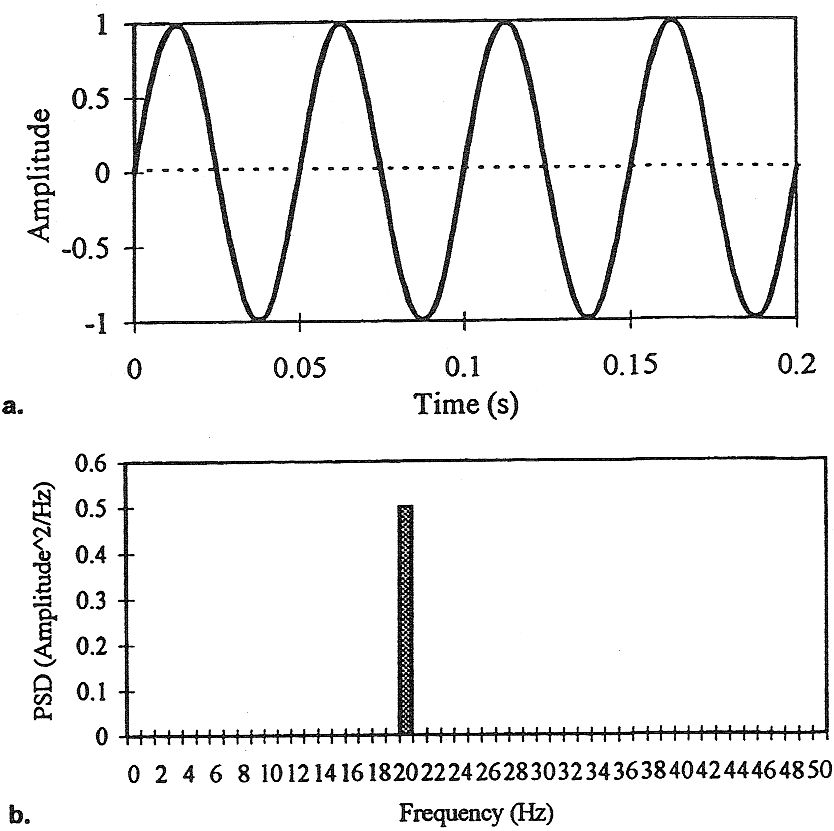
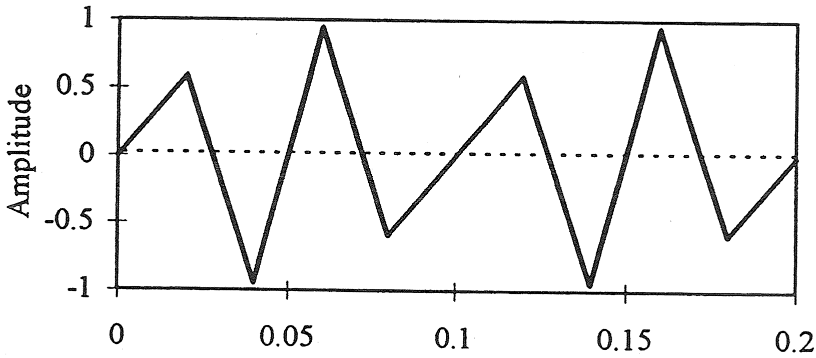


Figure 1 — (a) A 20 Hz sine wave digitally sampled at a rate of 400 Hz; (b) a frequency analysis of the sine wave presented in (a). *(continued)*

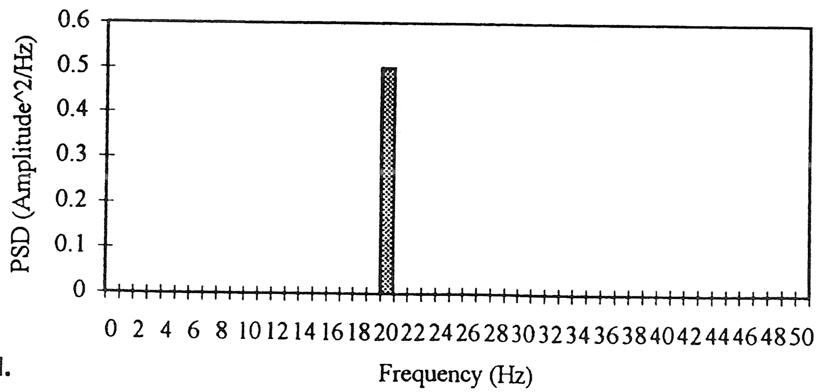
sample, 95% of the power of the signal lies below 30 Hz. At a sampling rate at the Nyquist limit (100 Hz), this peak is either missing completely or severely attenuated and may also be phase-shifted (Figure 2c). The impact peak, while present, has been attenuated such that the magnitude is 1,315 N, and it occurs 6 ms later than the peak in Figure 2a. In Figure 2d, a frequency analysis of the same GRF signal resampled at 100 Hz illustrates the same power spectrum as in Figure 2b.

Solution

We have seen that one solution to the problem of inaccurate time–series representation is to sample at a rate much greater than the f_c. Generally, it is recommended that the sampling rate be 5 to 10 times the highest frequency in the signal (Oppenheim, Willsky, & Young, 1983). However, frequency analysis shows, and the sampling theorem states, that all necessary information is captured by the digital record if the original analog signal is sampled just above f_c (twice the highest frequency in the signal). This fact leads to another solution that does not require higher sampling rates. If an analog or continuous signal, $c(t)$, is sampled at discrete intervals and has frequencies less than $\frac{1}{2} f_c$, then the signal is



c.



d.

Figure 1 — (c) A 20 Hz sine wave digitally sampled at slightly greater than the Nyquist critical frequency (50 Hz); (d) a frequency analysis of the sine wave presented in (c).

completely determined by its samples, *d(n)*. The signal can be reconstructed using the following formula:

$$s(t) = \delta \sum_{+\infty}^{n=-\infty} d(n) \frac{\sin[2\pi f_c(t - n\delta)]}{\pi(t - n\delta)}$$

where *s(t)* is the reconstructed signal at time *t*, *t* is the new sampling time, δ is the original sampling interval, *d(n)* is the n^{th} point of the sampled signal, and *n* is the point number in the sampled signal (Press et al., 1990). This equation is referred to as Shannon's Sampling Theorem (Marks, 1993) or the reconstruction formula (Proakis & Manolakis, 1988). When a signal is sampled as a discrete set of points, a linear transformation is applied to the signal to convert it into a digital signal (Zayed, 1993). However, providing that there are no frequencies greater than $\frac{1}{2} f_c$ in the original signal, Shannon's Sampling Theorem can be viewed as a way to convert the digital signal back into the original signal. The discretely sampled signal is limited by the fact that this digital signal will not truly represent the continuous analog signal (as seen in Figures 1c and 2c) until it is reconstructed using the above equation.

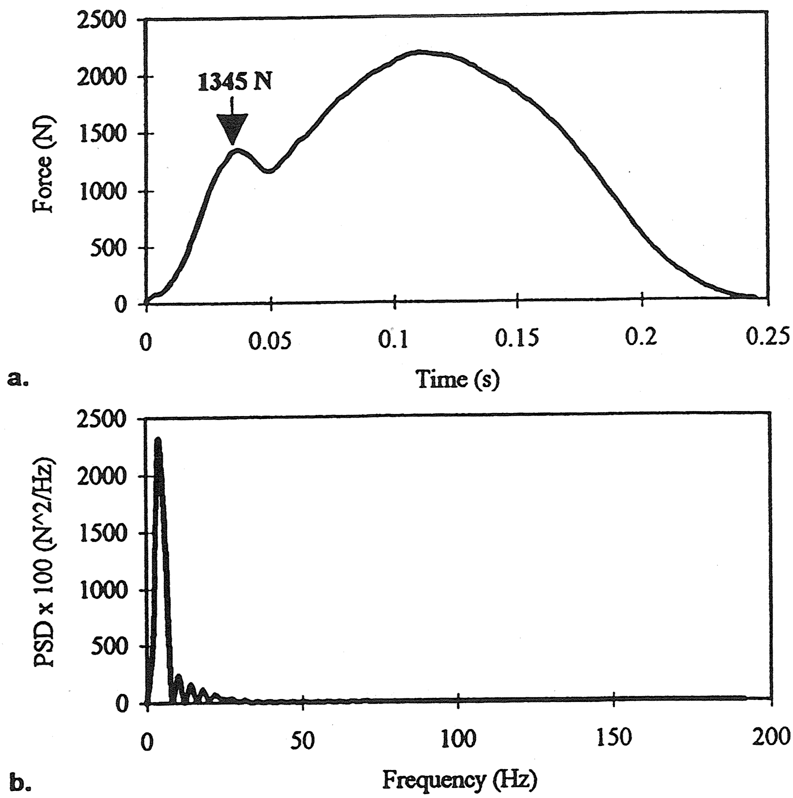
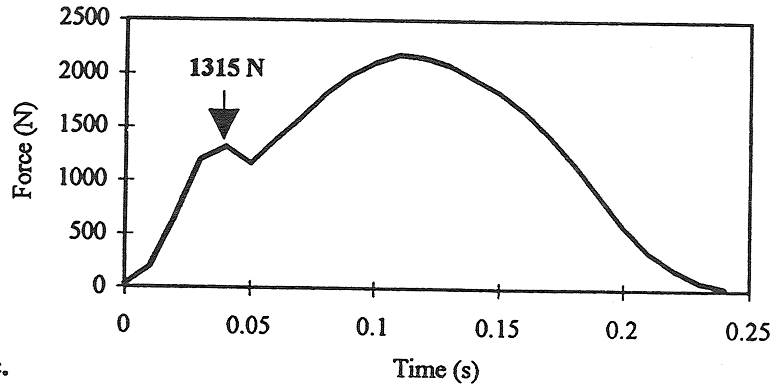


Figure 2 — (a) A vertical GRF component sampled at a rate of 1000 Hz; (b) a frequency analysis of the vertical GRF component presented in (a). ***(continued)***

The use of Shannon's Sampling Theorem can be illustrated using the previously discussed sine wave example. Based on the frequency analysis of the original sine wave, f_c was 50 Hz (Figure 1b). Figure 3a illustrates the original sine wave sampled at 400 Hz, while Figure 3b shows the time–series profile from sampling at slightly above f_c (50 Hz). Figure 3c demonstrates Shannon's Sampling Theorem in which the digital data in Figure 3b have been reconstructed at 400 Hz. It is evident that the reconstructed signal is very close to the original signal, with a mean absolute error amplitude of 0.045 for each data point.

Shannon's Sampling Theorem does not function in the same manner as a polynomial or a spline interpolation algorithm. These interpolation techniques do not rely on the frequency content of the analog signal. A polynomial or a spline can be administered to a function even if the Nyquist frequency for that signal has been violated. However, if the sampling frequency is much greater than f_c, then the spline will approximate the reconstructed signal. On the other hand, poor approximations will result if the signal is sampled just greater than f_c. In Figure 4, the original sine wave sampled at 50 Hz, or just greater than f_c, has been interpolated using a cubic spline and reconstructed using Shannon's Sampling Theorem. It is clear that the reconstruction is much closer to the original signal than is the spline interpolation.



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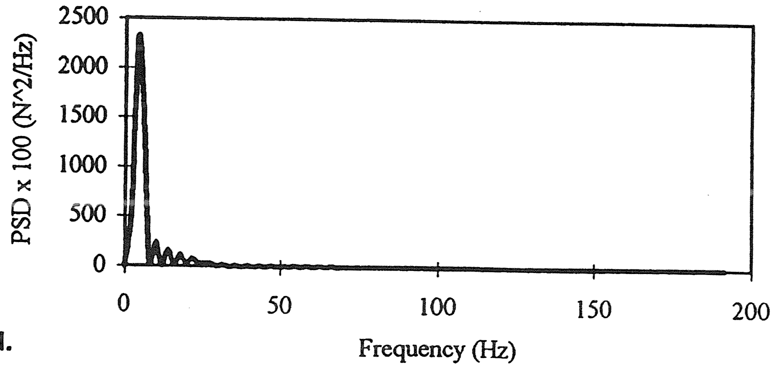

d.

Figure 2 — (c) The signal presented in (a) sampled at a rate slightly greater than the Nyquist critical frequency (100 Hz); (d) a frequency analysis of the signal presented in (c).

Figure 5 illustrates the use of the reconstruction theorem with the vertical GRF component previously discussed, using the same format as Figure 3. The GRF signal was reconstructed at 1000 Hz from the 100 Hz data using the reconstruction equation (Figure 5c). The mean absolute error between the original and reconstructed signals was 6.0 N, with the majority of the error occurring at the endpoints of the curves. Note that from Figure 5c, the reconstructed signal is able to register the impact peak much closer to the 1000 Hz signal (1,344 N vs. 1,345 N) and correct the time phase–shift of the peak.

One area in the study of human motion where aliasing has been a problem is investigation of rearfoot motion. It has been determined that the highest frequency in the Cartesian coordinate data, from which the rearfoot angle is constructed, is 16 to 18 Hz (Hamill, Bates, & Holt, 1992; Williams, Cavanagh, & Ziff, 1991). Thus, according to the sampling theorem, a sampling frequency of 36 Hz should be sufficient. However, Hamill et al. (1994) illustrated that such is not the case. Data sampled at 50 Hz lead to rearfoot angles that are not the same as those sampled at 100 Hz. A true rearfoot angle profile often has an initial high frequency peak followed by a lower frequency peak when the data are sampled at 100 Hz, that is, greater than 5 times f_h (Figure 6a). The initial peak in the rearfoot angle generated from the 50 Hz signal is not clearly present (Figure 6b). In addition, the magnitude of the peak rearfoot values and the time of

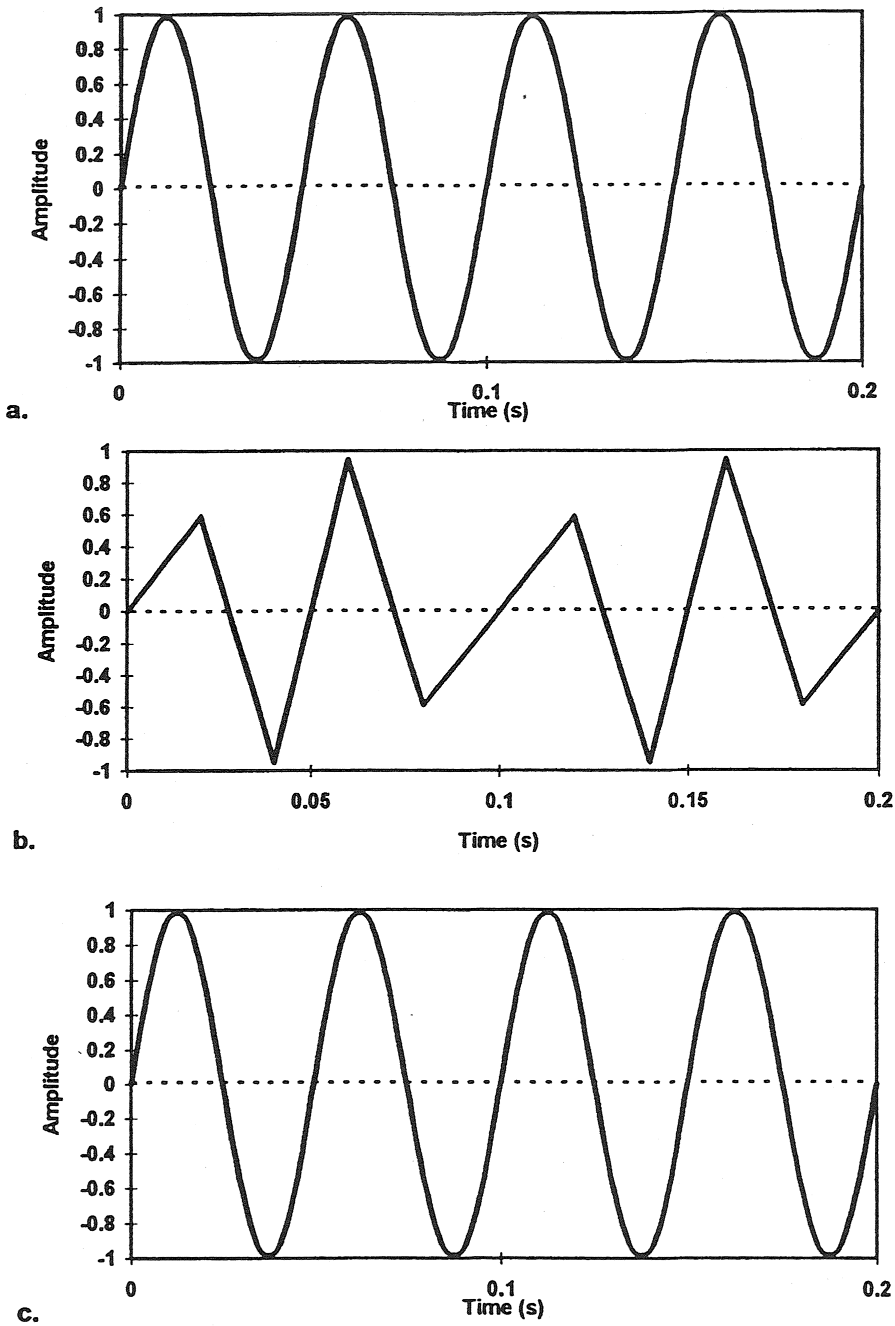

Figure 3 — (a) A 20 Hz sine wave digitally sampled at a rate of 400 Hz; (b) a 20 Hz sine wave digitally sampled at a rate slightly greater than the Nyquist critical frequency (50 Hz); (c) A 20 Hz sine wave reconstructed from the sine wave presented in (b) using the interpolating algorithm.

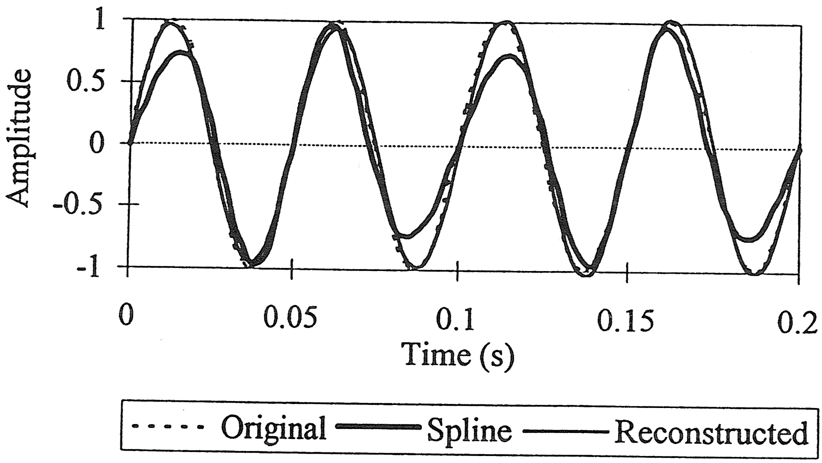

Figure 4 — A 20 Hz sine wave sampled at 400 Hz, the same sine wave resampled at 50 Hz, and a 400 Hz cubic spline interpolation of the sine wave sampled at 50 Hz.

occurrence of those peaks are not the same for the 100 Hz and 50 Hz signals (–6.97° vs. –6.34° and 110 ms vs. 100 ms). Hamill et al. (1994) suggested that researchers studying rearfoot motion must sample at least 5 times the highest frequency in the signal to represent time domain data correctly. They reported that sampling rearfoot data at 100 Hz resulted in an appropriate signal.

Since the highest frequency in the rearfoot signals is 18 Hz, the researcher has two options: either collect the data at approximately 100 Hz (Hamill et al., 1994), or collect the data at 60 Hz and reconstruct the signal using Shannon's Sampling Theorem. Note that 60 Hz is greater than the f_c and thus contains all of the information to reconstruct the signal. Figure 6c presents the reconstructed signal using Shannon's equation, resampled at 100 Hz, applied to the 60 Hz data. The 60 Hz signal presented in Figure 6b does not have the high frequency peak, while the reconstructed signal in Figure 6c does. Also, the magnitude of the peak rearfoot angle and the time of occurrence of the peak angle are the same in Figures 6a and 6c (–6.97° vs. –7.01° and 110 ms vs. 110 ms, respectively). In fact, the reconstructed rearfoot angle profile presented in Figure 6c is essentially identical to the profile generated from the 100 Hz data with a mean absolute error of 0.119° between the curves (Figure 6a). The peak values differ by only 0.04°, and the peaks occur at the same time.

Shannon's Sampling Theorem is appealing because it enables reconstruction of points based on frequencies present in the original signal. In practice, it should be used with care. The highest frequency present in the signal must be known a priori. The signal in this sense means the original biological signal plus any noise that is recorded. If one ignores the noise component, the power at these frequencies will fold back with the frequencies below f_s, causing distortion in the reconstruction. In addition, the reconstruction works best on signals that exhibit circular continuity, that is, signals whose end points have an equal magnitude and slope. If the signals do not exhibit circular continuity, then oscillations in the time domain may occur. In theory, the signal needs to be sampled at f_c, but in practice it should be sampled slightly higher than f_c to result in a good reconstruction.

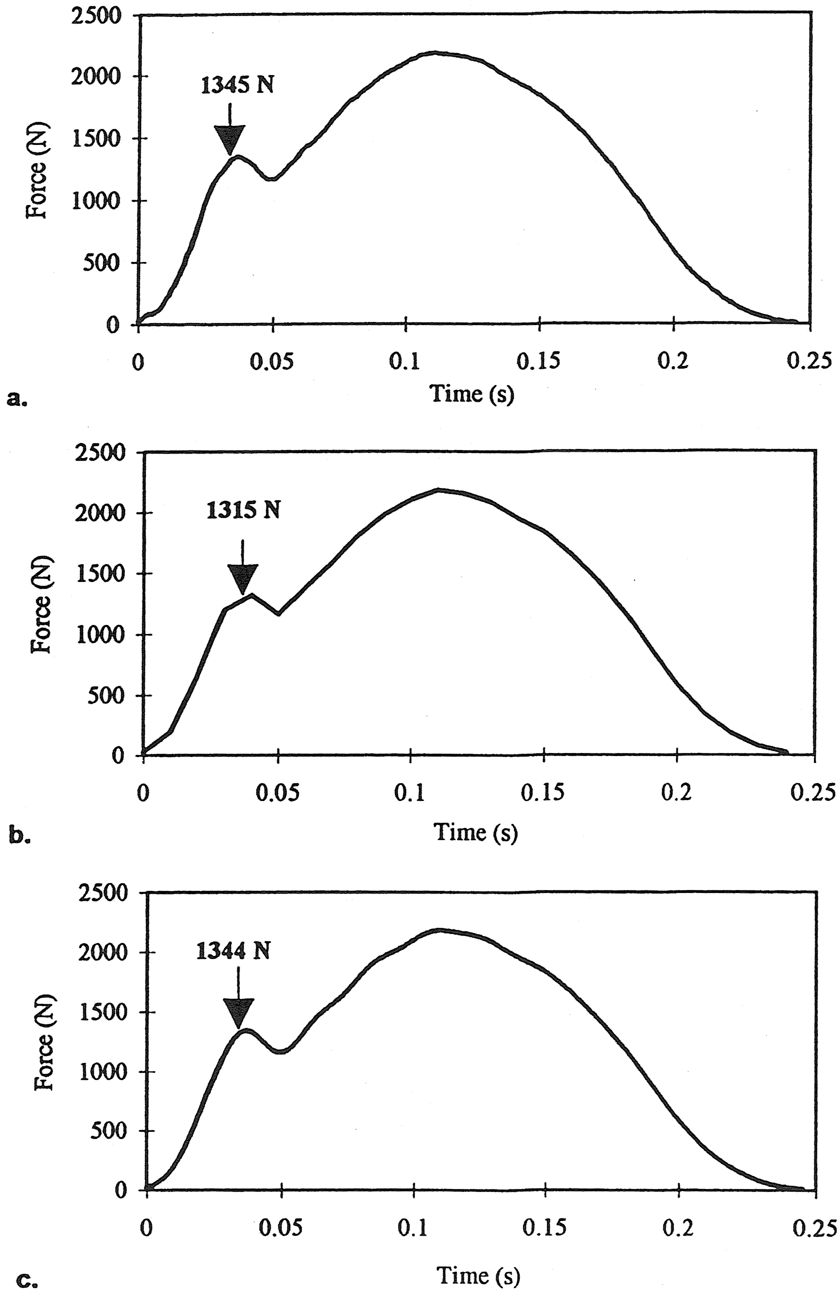

Figure 5 — (a) A vertical GRF component sampled at a rate of 1000 Hz, (b) the signal presented in (a) sampled at a rate greater than the Nyquist critical frequency (100 Hz), and (c) the vertical GRF component reconstructed from the signal presented in (b) using the interpolating algorithm.

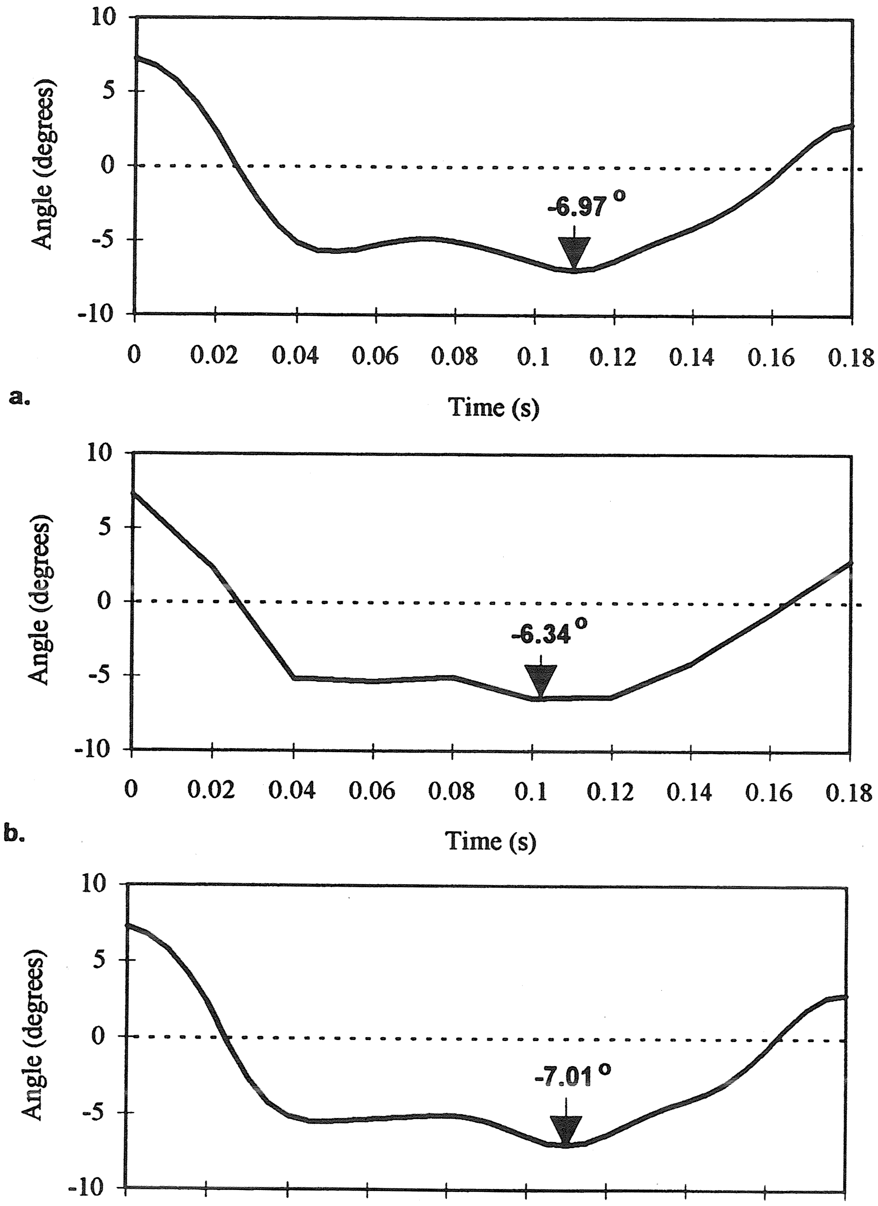

Figure 6 — (a) A rearfoot angle calculated from coordinate data sampled at a rate of 100 Hz, (b) the signal presented in (a) sampled at greater than the Nyquist critical frequency (50 Hz), and (c) the rearfoot angle constructed from the signal presented in (b) using the interpolating algorithm.

Summary

Researchers should know the frequency content of any discretely sampled data of a continuous signal. The frequency content of the continuous signal determines the minimum sampling rate referred to as the Nyquist critical frequency. Shannon's Sampling Theorem states that the frequency content of the original analog signal can be completely determined by the discretely sampled signal by adhering to the Nyquist sampling rate. However, the time–series of the discretely sampled signal by itself will not convey this information.

Researchers have generally oversampled at a rate of 5 to 10 times the highest frequency in the analog signal, and thus aliasing is not a problem. However, in certain situations, such as the study of rearfoot motion, equipment standards preclude sampling at 5 to 10 times the highest frequency in the signal. In lieu of collecting the data at 5 times the highest frequency (about 100 Hz), the researcher can still sample video data at 60 Hz, thus adhering to the sampling theorem. However, the discretely sampled data cannot be used in further analysis unless they are reconstructed using the reconstruction equation presented in this paper.

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Appendix

The following source code is written in QuickBasic and represents the interpolating algorithm. All variable names are consistent with those in the manuscript. This code was written for clarity and not necessarily for efficiency. Note that before this code is executed, the data must meet the requirements of circular continuity.

```
' newdelta = new sampling rate
' samptime = duration of the trial
' fc = Nyquist frequency
' delta = original sampling rate
' newpoints% = number of points in the reconstructed signal
' d!(n) = nth point of the original signal
' s!(I) = ith point in the reconstructed signal
pi = 3.14159265359#
samptime = (numpoints – 1) * delta
newpoints% = samptime/newdelta
fc = 1/(2 * delta)
FOR I=1 to newpoints%
   t = (I–1) * newdelta
   FOR n = 1 to numpoints%
      IF (t – (n – 1) * delta) <> 0 then
         m = SIN (2 * pi * fc * (t – (n – 1) * delta)) / (pi * (t – (n – 1) * delta))
      ELSE
         m = 1/delta
      END IF
      s!(I) = s!(I) + d!(n) * m
   NEXT n
   s!(I) = s!(I) * delta
NEXT I
```