9/6/11

- **Theorem:** All evolutionary systems can be written in the form

$$v'(t) = Av(t), \quad v(0) = x,$$

where A is a linear operator on a vector space X.

- Vector space: a field where numbers, functions, vectors, etc can be operated on by addition, subtraction, scalar multiplication, etc.
- $A: X \to Y$ is a linear operator if:
  - $A(x+y) = A(x) + A(y)$
  - $A(\lambda x) = \lambda A(x)$

In engineering, called Principle of Superposition. When recording drums and guitar, doesn't matter if record playing separately or record playing together.

- $A: \mathbb{R}^2 \to \mathbb{R}^2$

$A(x+y) = A(x) + A(y)$
$A(\lambda x) = \lambda A(x)$ is linear

$$A(x_1, x_2) = A((x_1, 0) + (0, x_2))$$
$$= A(x_1, 0) + A(0, x_2)$$
$$= x_1 \underbrace{A(1,0)}_{ac} + x_2 \underbrace{A(0,1)}_{bd}$$
$$= (ax_1 + bx_2, cx_1 + dx_2)$$
$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $A: \mathbb{R}^2 \to \mathbb{R}^3$

$$\to x_1 \underbrace{A(1,0)}_{ace} + x_2 \underbrace{A(0,1)}_{bdf}$$
$$= (ax_1 + bx_2, cx_1 + dx_2, ex_1 + fx_2)$$
$$= \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Use matrices to map one space into another
- In algebra, do math with numbers. In linear algebra, do math with linear maps.
- Use easy functions to model linear maps.
- Linear operators: limits (think limit laws), derivatives, integrals

"If it works for numbers, it works for linear maps."

- Solution:

$$v(t) = e^{tA}x$$
$$v'(t) = Ax\, e^{tA}x = Ax\, v(t)$$

$$e^{tA} = \lim_{n\to\infty}\left(I + \frac{tA}{n}\right)^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$
$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$
$$e^{tA}x = \lim_{n\to\infty}\left(I + \frac{tA}{n}\right)^n x = \lim_{n\to\infty}\left(I - \frac{tA}{n}\right)^{-n} x$$

converges much faster

- Homework:
  - Let $v(0) = v'(0) = 1$, then $\lim_{n\to\infty} v\left(\frac{x}{n}\right)^n = e^x$

Proof: $\ln\left(v\left(\frac{x}{n}\right)^n\right) = \ln(e^x)$

$$n\ln\left(v\left(\frac{x}{n}\right)\right) = x$$

$$\frac{\ln\left(v\left(\frac{x}{n}\right)\right)}{\frac{1}{n}} \overset{L'H}{\Rightarrow} \frac{\frac{v'\left(\frac{x}{n}\right)}{v\left(\frac{x}{n}\right)}\left(-\frac{x}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= \frac{v'\left(\frac{x}{n}\right)}{v\left(\frac{x}{n}\right)} x \to x \text{ as } n \to \infty$$

Our goal is to use this formula to show faster convergence than Mathematica

Ex: $v(x) = \frac{2+x}{2-x}$

$v'(x) = \frac{2-x-(-1)(2+x)}{(2-x)^2} = \frac{4}{(2-x)^2}$ $v(0) = v'(0) = 1$

- Homework: $|v(\frac{x}{n})^n - e^x| \to$ estimation of convergence

  $= |v(\frac{x}{n})^n - (e^{\frac{x}{n}})^n| \le C_m \cdot \frac{t^m}{n^{m-1}} \|A^m x\|$ The more complicated/higher m B, the less n it takes to converge (faster convergence)

- Is it more effective to have $n=1$ with complicated m, or less complicated m with high n?

  (Neubrander) (others)

Goal: Compute e^{tA}, where A is 100×100 matrix
We are finding a method to beat that of Mathematica and MATLAB

Step 1: Compute e^{tz}

Lemma: $r(0)=r'(0)=1 \rightarrow r(\frac{tz}{n})^n \rightarrow e^{tz}$
Proof: L'Hopital's Rule
Ex: While $10^{10^{10}}(\frac{1}{\ln n})$ goes to 0 as $n\to\infty$, it does so very slowly

Lemma: If $r(0)=r'(0)=1$, $R_z=0$, $|r(z)|\le 1$ if $R_z\le 0$, then $r(z)\to e^z$ (see back)
Proof: $|r(\frac{tz}{n})^n - e^{tz}| = |r(\frac{tz}{n})^n - (e^{\frac{tz}{n}})^n|$
$= |r(\frac{tz}{n}) - e^{\frac{tz}{n}}|\,|\sum_{j=0}^{n-1}(r(\frac{tz}{n}))^{n-1-j}(e^{\frac{tz}{n}})^j|$
$\le n\cdot|r(\frac{tz}{n}) - e^{\frac{tz}{n}}| \le n\cdot M(\frac{tz}{n})^{m+1}$

Binomial Theorem:
$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\dots+b^{n-1})$

$e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\frac{z^n}{n!}$
$r(z)=a_0+a_1z+a_2z^2+\dots = 1+z+\frac{z^2}{2!}+\dots+a_nz^n$

e^z and $r(z)$ match up for first $\underline{m}$ terms
We want large m and $n=1$

Theorem (Thomée, Kato, Hersh 1979)

Matlab / Mathematica: "bad r" / large n

$*\ |r(z)|\le 1$ for $R_z\le 0$ (if-stable) — R_z: real part of z
$**\ |r(z)-e^z|\le M\cdot|z|^{m+1}$ if $|z|$ is small (approximation of the exponential of order m)
$=\|r(\frac{tA}{n})^n - e^{tA}\| \le M_r t^{m+1}\cdot\frac{1}{n^m}\|A^{m+1}\|$

Our approach: $m=21$ / $n=1$

Theresa Sandmaier said that growth in t^{m+1} and $\|A^{m+1}\|$ is small compared to extreme decrease in M_r if $n=1$.
$\|r(tA)-e^{tA}\|\le M_r\cdot t^{m+1}\cdot\|A^{m+1}\|$

$r(z)=\frac{P(z)}{Q(z)}=\frac{b_1}{\lambda_1-z}+\dots+\frac{b_a}{\lambda_a-z}$

Partial fraction decomposition

$m=2a-1$
$\Rightarrow M_r \le \frac{1}{2^{m-1}}\cdot\frac{\sqrt{\pi}}{(a-1)!}\cdot\frac{1}{(a-2)^{a-1}}$

$r(tA)=\frac{b_1}{\lambda_1-tA}+\dots+\frac{b_a}{\lambda_a-tA}$
$=\frac{b_1}{t}(\frac{\lambda_1}{t}-A)^{-1}+\dots+\frac{b_a}{t}(\frac{\lambda_a}{t}-A)^{-1}$

Inverse matrix

Must learn how to program partial fraction decomposition and inverse matrix in Mathematica

Lemma: if $r(0)=r'(0)=1$, $R_z\le 0$, $|r(z)|\le 1$, then $|e^z - r(z)| \le M \cdot t^{m+1} \cdot \frac{1}{n^m} |z|^{m+1}$

$x_1' = a_1 x + b_1 y$
$x_2' = a_2 x + b_2 y$

We solve this like we did with systems of ordinary equations in high school. We can prove that what we do with algebra we can do with linear operators

Solution: $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}^{-1} = A \longrightarrow e^{tA}$

9/27/11

Lemma (Motivation)

$r(0) = r'(0) = 1 \Rightarrow r(\frac{ta}{n})^n \to e^{ta}$ as $n \to \infty$

Lemma (Padé)

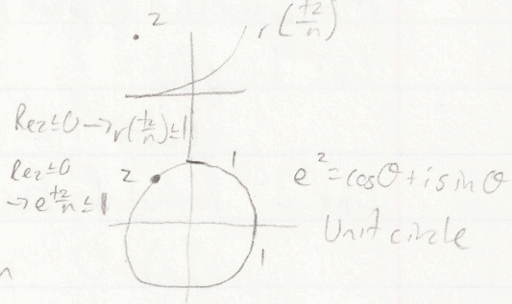
Numbers

* $|r(z)| \le 1$ for $\text{Re}\, z \le 0$

** $|r(z) - e^z| \le C \cdot |z|^{m+1}$ for $|z|$ small (radius of convergence for e^z)

$\Rightarrow |r(\frac{tz}{n})^n - e^{tz}| = |r(\frac{tz}{n})^n - (e^{\frac{tz}{n}})^n|$

$\le |r(\frac{tz}{n}) - e^{\frac{tz}{n}}| \sum_{j=0}^{n-1} \underbrace{|r(\frac{tz}{n})^j|}_{\le 1} \underbrace{|(e^{\frac{tz}{n}})^{n-1-j}|}_{\le 1} \le n$



Now apply this to $\frac{tz}{n}$

$|r(\frac{tz}{n})^n - e^{tz}| \le C \cdot t^{m+1} \cdot |z|^{m+1} \cdot \frac{1}{n^{m+1}} \cdot n$

$|r(\frac{tz}{n})^n - e^{tz}| \le C \cdot t^{m+1} \cdot |z|^{m+1} \cdot \frac{1}{n^m}$

Linear operators

Theorem (Kato) If A is a linear operator, then * and **

$\Rightarrow \|r(\frac{tA}{n})^n x - e^{tA} x\| \le C t^{m+1} \|A^{m+1} x\| \cdot \frac{1}{n^m}$

Lemma (Padé)

$r(z) = \frac{p(z)}{q(z)}$ and ** holds, then $\underset{\text{degree of } r}{m} \le \underset{\text{degree of } p}{p} + q$ ← degree of q

Want to choose p and q so that m is maximum → $m = p+q$

If $p(z) = \sum_{j=0}^{p} \frac{(m-j)!}{j!(p-j)!} z^j$, $q(z) = \sum_{j=0}^{q} \frac{(m-j)!\,q!}{p!\,j!\,(p-j)!}(-z)^j \Rightarrow m = p+q$

Best possible rational approximation

Found pattern after 30 values

Lemma (Ehle)

If $r()$ is a rational Padé approximation of order m:

$q - 2 \le p \le q$ "degree of p is at most 2 more than q"

$\Rightarrow$ ** holds

Definition: subdiagonal Padé approximation: arbitrary q and $p = q-1$

Idea: $n=1$

$$\|r(tA)x - e^{tA}x\| \leq C_r t^{m+1} \|A^{m+1}x\|$$

Lemma (Neubrander/Sandmauer)

C_r is sau (German → so) small

Mathematica Program:

Line 1: $p(z) =$

$q(z) =$

Line 2: Partial fraction decomposition:

$q(z) = 0$ $\qquad \frac{p(z)}{q(z)} = \frac{B_1}{\lambda_1 - z} + \ldots + \frac{B_a}{\lambda_a - z}$

$q(\lambda) = 0$ $\qquad q(z) = C(\lambda_1 - z)\ldots(\lambda_a - z)$

$\lambda_1, \ldots, \lambda_a$ $\qquad p(z) = B_1((\lambda_2 - z)\ldots(\lambda_a - z)) + \ldots + B_a((\lambda_1 - z)\ldots(\lambda_{a-1} - z))$

Plug in for each λ_i so $B_i = \frac{p(\lambda_i)}{\prod_{i \neq l}(\lambda_i - \lambda_l)}$

$$r(z) = \frac{p(z)}{q(z)} = \frac{B_1}{\lambda_1 - z} + \ldots + \frac{B_a}{\lambda_a - z}$$

$$e^{tA} \approx r(tA) = \frac{B_1}{\lambda_1 - tA} + \ldots + \frac{B_a}{\lambda_a - tA} = \frac{B_1}{t}\left(\frac{\lambda_1}{t}I - A\right)^{-1} + \ldots + \frac{B_a}{t}\left(\frac{\lambda_a}{t}I - A\right)^{-1}$$

↑ built into machine now

10/4/11 Best way to find inverse of matrix is by Gaussian elimination.

Ex: $\begin{bmatrix} 2&1&0\\2&2&1\\2&3&0\end{bmatrix}^{-1} \longrightarrow \left[\begin{array}{ccc:ccc} 2&1&0&1&0&0\\2&2&1&0&1&0\\2&3&0&0&0&1\end{array}\right]$ You can add, subtract rows and multiply rows by constants

Make this, this Result is inverse

$$\left[\begin{array}{ccc:ccc} 2&1&0&1&0&0\\2&2&1&0&1&0\\0&1&-1&0&-1&1\end{array}\right] \longrightarrow \left[\begin{array}{ccc:ccc} 2&1&0&1&0&0\\0&1&1&-1&1&0\\0&1&-1&0&-1&1\end{array}\right] \longrightarrow \left[\begin{array}{ccc:ccc} 2&1&0&1&0&0\\0&2&0&-1&0&1\\0&1&-1&0&-1&1\end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc:ccc} 2&0&1&1&1&-1\\0&1&0&-\frac{1}{2}&0&\frac{1}{2}\\0&1&-1&0&-1&1\end{array}\right] \longrightarrow \left[\begin{array}{ccc:ccc} 2&0&1&1&1&-1\\0&1&0&-\frac{1}{2}&0&\frac{1}{2}\\0&0&-1&\frac{1}{2}&-1&\frac{1}{2}\end{array}\right] \longrightarrow \left[\begin{array}{ccc:ccc} 1&0&0&\frac{3}{4}&0&-\frac{1}{4}\\0&1&0&-\frac{1}{2}&0&\frac{1}{2}\\0&0&1&-\frac{1}{2}&1&-\frac{1}{2}\end{array}\right]$$

$$\begin{bmatrix} 2&1&0\\2&2&1\\2&3&0\end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{4}&0&-\frac{1}{4}\\-\frac{1}{2}&0&\frac{1}{2}\\-\frac{1}{2}&1&-\frac{1}{2}\end{bmatrix}$$

Inverse Laplace Transforms

Russian way of finding inverse matrices:

$$R(\lambda, A) = \frac{1}{c(\lambda)} \sum_{k=0}^{n-1} \lambda^{n-k-1} A_k, \text{ where } c(\lambda) = \det(\lambda I - A) = \lambda^n - \sum_{k=0}^{n-1} c_k \lambda^k$$

"characteristic polynomial of A"

where $c_{n-1} = \text{Spur}(A) = \text{Trace}(A) =$ (sum of diagonals of A)

$A_0 = I$

$A_1 = A - c_{n-1} \cdot I$ → adds sum of diagonals back to diagonals of A

$c_{n-2} = \text{Trace}\left(\frac{A_1 \circ A}{2}\right)$

$A_2 = A_1 \circ A - c_{n-2} \cdot I$

$c_{n-k} = \frac{\text{Trace}(A_{k-1} A)}{k}$

$c_{n-3} = \text{Trace}\left(\frac{A_2 \circ A}{3}\right)$

$A_3 = A_2 \circ A - c_{n-3} \cdot I$

$A_k = A_{k-1} A - c_{n-k} I$

$\vdots$

$c_0 = \text{Trace}\left(\frac{A_{n-1} \circ A}{n}\right)$

$A_n = A_{n-1} \circ A - c_0 \cdot I$

Programming problem: how do we get the machine to stay in symbolic form (fractions) instead of rounding to decimal form, which results in less accurate and slower computations

*Homework: program this procedure for 2x2, 3x3, 4x4 matrices with integer values
- Check if Trace is correct ✓
- Do by hand first

We can then extend this to irrationals by multiplying and dividing matrix by some scalars

What is the procedure/algorithm Mathematica and MATHLAB use?

$$\therefore e^{tA} = \sum_{1}^{q} \frac{B_i}{t} R\left(\frac{\lambda_i}{t}, A\right) = \frac{B_1}{t}\left(\frac{\lambda_1}{t} I - A\right)^{-1} + \ldots + \frac{B_q}{t}\left(\frac{\lambda_q}{t} I - A\right)^{-1}$$

10/18/11

Testing for a General 2x2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad R(\lambda, A) = (\lambda I - A)^{-1} = \frac{1}{c(\lambda)} \begin{bmatrix} \lambda - d & b \\ c & \lambda - a \end{bmatrix}$$

$$R(\lambda, A) = \frac{1}{c(\lambda)} \sum_{k=0}^{n-1} \lambda^{n-k-1} A_k = \frac{1}{c(\lambda)} \sum_{k=0}^{1} \lambda^{1-k} A_k = \frac{1}{c(\lambda)} (\lambda I + A_1)$$

$$c(\lambda) = \det(\lambda I - A) = \lambda^n - \sum_{k=0}^{n-1} c_k \lambda^k = \lambda^2 - \sum_{k=0}^{1} c_k \lambda^k = \lambda^2 - c_0 - c_1 \lambda$$

$$c_1 = \mathrm{trace}(A) = a + d \qquad A_1 = A - c_1 I = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$$

$$c_0 = \mathrm{trace}\left(\frac{A_1 \cdot A}{2}\right) = \mathrm{trace} \begin{bmatrix} \frac{-ad+bc}{2} & 0 \\ 0 & \frac{-ad+bc}{2} \end{bmatrix} = -ad + bc$$

$$c(\lambda) = \lambda^2 - (-ab + bc) - (a+d)\lambda = \lambda^2 + ab - bc - \lambda a - \lambda d$$

$$R(\lambda, A) = \frac{1}{\lambda^2 + ab - bc - \lambda a - \lambda d} \left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} -d & b \\ c & -a \end{bmatrix} \right) = \frac{1}{\lambda^2 + ab - bc - \lambda a - \lambda d} \begin{bmatrix} \lambda - d & b \\ c & \lambda - a \end{bmatrix}$$

$$c_{n-k} = \frac{\mathrm{Trace}(A_{k-1} A)}{k} \qquad A_k = A_{k-1} A - c_{n-k} I$$

$n=2$
$k=1,2$

$$c_1 = \mathrm{trace}(IA) = a + d \qquad A_1 = IA - c_1 I = \begin{bmatrix} -d & 0 \\ 0 & -a \end{bmatrix}$$

$$c_0 = \mathrm{trace}(A_1 A)/2 = -2ad \qquad A_2 = A_1 A - c_0 I = \begin{bmatrix} -ad & -bd \\ -ac & -ad \end{bmatrix} - \begin{bmatrix} -2ad & 0 \\ 0 & -2ad \end{bmatrix} = \begin{bmatrix} ad & -bd \\ -ac & ad \end{bmatrix}$$

$$c(\lambda) = \lambda^2 + 2ad - (a+d)\lambda = \lambda^2 - (a+d)\lambda + 2ad \rightarrow \lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(2ad)}}{2} = \frac{a+d \pm \sqrt{a^2 - 6ad + d^2}}{2}$$

$$c(\lambda) = \lambda^2 - a - d + 2ad\lambda = \lambda^2 + 2ad\lambda - a - d \rightarrow \lambda = \frac{-2ad \pm \sqrt{4a^2d^2 - 4a - 4d}}{2}$$

$$e^{tA} \sim \sum_{1}^{a} \frac{b_i}{t} R\left(\frac{a_i}{t}, A\right) = \sum_{k=0}^{n-1} \mathcal{L}^{-1}\left(\frac{\lambda^{n-k-1}}{c(\lambda)}\right) A_k$$

Is it better to store symbolic notation on hard drive space (cheaper) or run it every time and use up RAM (expensive)?

Make small changes to speed up process? Ex: $\frac{\mathrm{Trace}(A_{k-1} A)}{k}$ or $\mathrm{Trace}\left(\frac{A_{k-1} A}{k}\right)$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ | $(\lambda I - A)^{-1} = \begin{bmatrix} \frac{1}{\lambda-1} & 0 \\ 0 & \frac{1}{\lambda-2} \end{bmatrix} = \frac{1}{(\lambda-1)(\lambda-2)} \begin{bmatrix} \lambda-2 & 0 \\ 0 & \lambda-1 \end{bmatrix}$

$\det(\lambda I - A) = c(\lambda)$ ✓

$R(\lambda, A) = (\lambda I - A)^{-1}$ ✓

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad c_0 = 3 \quad A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \quad c_1 = -2 \quad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ $\quad n = 2$

$$c(\lambda) = \lambda^2 - \sum_{k=0}^{1} c_k \lambda^k = \lambda^2 - c_0 - c_1 k = \lambda^2 - 3 + 4\lambda = 0 \to \lambda = -2 \pm \sqrt{7}$$

$$R(\lambda, A) = \frac{1}{c(\lambda)} \sum_{k=0}^{1} \lambda^{1-k} A_k = \frac{1}{-2\pm\sqrt{7}} (\lambda I + A_1) = \frac{1}{-2\pm\sqrt{7}} \begin{bmatrix} -2+\lambda & 0 \\ 0 & -1+\lambda \end{bmatrix}$$

$$* \; c(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2) = \lambda^2 - 3\lambda + 2 = 0 \to \lambda = 1, \lambda = 2$$

$c(\lambda) = \lambda^2 - c_0\lambda - c_1 \quad$ If c_k: $c_0 = 3, c_1 = -2 \quad c(\lambda) = \lambda^n - \sum_{k=0}^{n-1} c_k \lambda^{n-k-1}$

$$e^{tA} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} \lambda-2 & 0 \\ 0 & \lambda-1 \end{bmatrix} = \sum_{k=0}^{n-1} \lambda^{n-k-1} A_k = \sum_{k=0}^{1} \lambda^{1-k} A_k = \lambda I + A_1 \to A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

Ex:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 4 & 0 & 3 \end{bmatrix} \quad (\lambda I - A)^{-1} = \frac{1}{-12 - 5\lambda - 4\lambda^2 + \lambda^3} \begin{bmatrix} -3\lambda + \lambda^2 & -6 + 2\lambda & 6 + \lambda \\ 6 + 2\lambda & -1 - 4\lambda + \lambda^2 & -1 + 3\lambda \\ 4\lambda & 8 & -4 - \lambda + \lambda^2 \end{bmatrix}$$

$$= \sum_{k=0}^{n-1} \lambda^{n-k-1} A_k = \sum_{k=0}^{2} \lambda^{2-k} A_k = I\lambda^2 + \lambda A_1 + A_2 \to \lambda A_1 + A_2 = \begin{bmatrix} -3\lambda & -6+2\lambda & 6+\lambda \\ 6+2\lambda & -1-4\lambda & -1+3\lambda \\ 4\lambda & 8 & -4-\lambda \end{bmatrix}$$

$$\to A_1 = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -4 & 3 \\ 4 & 0 & -1 \end{bmatrix} \to A_2 = \begin{bmatrix} 0 & -6 & 6 \\ 6 & -1 & -1 \\ 0 & 8 & -4 \end{bmatrix}$$

In conclusion, the matrix multiplication symbol in Mathematica is a period.

$c(\lambda) = \det(\lambda I - A)$: characteristic polynomial

$R(\lambda, A) = (\lambda I - A)^{-1}$: resolvent of A

Worked backwards from examples with known $R(\lambda, A) = (\lambda I - A)^{-1}$ and $c(\lambda)$. Found c_k using A_k from $\sum_{k=0}^{n-1} \lambda^{n-k-1} A_k$

* We used the wrong Mathematica matrix multiplication function

11/15/11

Complex Analysis (U.S.) - Theory of functions (England)
Can use linear algebra or complex analysis to solve equations

Linear Algebra: All matrices can be decomposed, just like numbers can be decomposed

$A = uju^{-1}$

$A^2 = (uju^{-1})(uju^{-1}) = (uju^{-1})j = uj^2u^{-1}$ $\quad \lambda$: eigenvalues $\quad Ax = \lambda x$

$A^n = uj^nu^{-1}$ $\quad x$: eigenvectors

Complex Analysis: $e^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{tz} R(z, A)\,dz$ We've developed a clever way of evaluating this

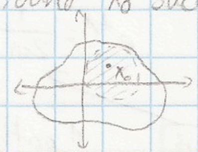
Well... cheating definition:

Analytic (holomorphic) functions: A function is called analytic in a region $U \in \mathbb{C}$ (simply connected set in complex number plane) if for all $x_0 \in U$ there exists a disk around x_0 such that

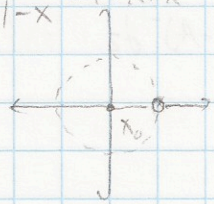
$\boxed{n \geq 0 \text{ For } n<0, a_n = 0.}$

$$f(x) = \sum_0^\infty a_n (x - x_0)^n \quad \text{for all } x \in D$$

"power series"



Ex: $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ for $|x| < 1$ for real and imaginary



basically, complex analysis is the study of power series in the complex plane.

Meromorphic functions: (similar to analytic functions)

$\boxed{-\infty < n < \infty}$

$$f(x) = \sum_{-\infty}^{\infty} a_n (x - x_0)^n$$

Cauchy's Theorem:
Let Γ be a closed curve around x_0. Then $\frac{1}{2\pi i}\int_\Gamma f(x)\,dx = a_{-1}$

Proof: $\Gamma = x_0 + r\langle \cos t, \sin t\rangle \quad t \in [0, 2\pi]$

$= x_0 + r(\cos t + i \sin t) \quad (x, y) = x(1,0) + y(0,1) = x + iy = z$

$= x_0 + re^{it} \quad e^{it} = 1 + it - \frac{1}{2}t^2 - \frac{1}{3!}t^3 + \frac{1}{4}t^4 + \ldots$

$= (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots) + i(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots) = \cos t + i \sin t$

Operations with complex numbers:

Ex: $-2 \cdot -3 = 6$

Multiply lengths: $2 \cdot 3 = 6$

Add angles: $180° + 180° = 360° \quad \cos(360°) = 1$

This extends to the entire complex plane

Ex: $(1, 1) \cdot (-1, 1) = [\sqrt{2}, 45°] \cdot [\sqrt{2}, 135°] = [2, 180°] = (-2, 0)$

Ex: $i^2 = (0,1)^2 = (0,1) \cdot (0,1) = [1, 90°] \cdot [1, 90°] = [1, 180°] = (-1, 0) = -1$

$\frac{1}{2\pi i}\int_\Gamma f(x)dx$ No special application (area, mean value, etc.), just a computation engine

$$\frac{1}{2\pi i}\int_0^{2\pi} f(x_0+re^{it})ire^{it}dt = \frac{1}{2\pi}\sum\int_0^{2\pi} a_n r^{n+1} e^{(n+1)it}dt = a_{-1}$$

For $n \neq -1$, $\int e^{(n+1)it}dt = \int(\cos(n+1)t + i\sin(n+1)t)dt = 0$

This works even if the path is not a circle.

Cauchy's Integral Theorem: Given f analytic, Γ closed curve around x_0:

i. $\frac{1}{2\pi i}\int_\Gamma f(x)dx = 0$ ii. $\frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x-x_0}dx = f(x_0) = a_0$

iii. $\frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{(x-x_0)^{n+1}}dx = \frac{f^{(n)}(x_0)}{n!} = a_n$

$$f(z) = e^{tz} = \frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x-z}dx = \frac{1}{2\pi i}\int_\Gamma \frac{e^{tx}}{x-z}dx$$

Functional Analysis

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma \frac{e^{tx}}{x-A}dx = \frac{1}{2\pi i}\int_\Gamma e^{tx}(xI-A)^{-1}dx = \frac{1}{2\pi i}\int_\Gamma e^{tx}R(x,A)dx$$

Complex Analysis

A path around A would be a path around its eigenvalues

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma (e^{tz}-r(tz))(zI-A)^{-1}dz + \frac{1}{2\pi i}\int_\Gamma r(tz)(zI-A)^{-1}dz$$

$$= \text{"small if good approximation"} + \sum \frac{b_i}{t}R\left(\frac{b_i}{t}, A\right)$$

Do e^{tA} for 2x2 and 3x3

Oxford - Dr. Trefethen and Dr. Weideman

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

Trace(A) = 1

$$A - \text{Trace}(A)I = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

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$f(z)=\sum_0^\infty a_n(z-z_0)^n$ Try to plug in matrix A (linear operator)

Dunford Functional Calculus: much easier to do $f(A)$

$$f(A):=\frac{1}{2\pi i}\int_\gamma f(z)(zI-A)^{-1}\,dz$$ Output is another matrix

$$A=uju^{-1}=\begin{bmatrix}2&1\\1&1\end{bmatrix}\cdot\begin{bmatrix}-1&0\\0&2\end{bmatrix}\cdot\begin{bmatrix}1&-1\\-1&2\end{bmatrix}$$ This is going backwards; this is the matrix decomposition

$$=\begin{bmatrix}-2&2\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&-1\\-1&2\end{bmatrix}=\begin{bmatrix}-4&6\\-3&5\end{bmatrix}$$

$\Psi: f\to f(A)$ is an algebra homomorphism

Ex: $A=\begin{bmatrix}-4&6\\-3&5\end{bmatrix}$ $\quad zI-A=\begin{bmatrix}z+4&-6\\3&z-5\end{bmatrix}$

$\det(zI-A)=$ characteristic polynomial
$=(z+4)(z-5)+18=z^2-z-20+18=z^2-z-2$
$=(z+1)(z-2)$
$z=-1,\ 2$ Eigenvalues

$$(zI-A)^{-1}=\frac{1}{(z+1)(z-2)}\begin{bmatrix}z-5&6\\-3&z+4\end{bmatrix}$$

$$f(z)=e^{tz}=\sum_0^\infty\frac{t^n}{n!}z^n=e^{tz_0}e^{t(z-z_0)}=\sum_0^\infty\left(e^{tz_0}\frac{t^n}{n!}\right)(z-z_0)^n$$

$$e^{tA}=\frac{1}{2\pi i}\int_\gamma\frac{e^{tz}}{(z+1)(z-2)}\begin{bmatrix}z-5&6\\-3&z+4\end{bmatrix}dz$$

Break into components

$$=\begin{bmatrix}\frac{1}{2\pi i}\int_\gamma e^{tz}\frac{z-5}{(z+1)(z-2)}dz & \frac{1}{2\pi i}\int_\gamma e^{tz}\frac{6}{(z+1)(z-2)}dz\\ \frac{1}{2\pi i}\int_\gamma e^{tz}\frac{-3}{(z+1)(z-2)}dz & \frac{1}{2\pi i}\int_\gamma e^{tz}\frac{z+4}{(z+1)(z-2)}dz\end{bmatrix}$$

Must break up into partial fractions so we can evaluate with the form

$$\frac{1}{2\pi i}\int_\gamma\frac{f(z)}{z-z_0}dz=f(z_0)=a_0$$ Birth of factoring

$$\frac{z-5}{(z+1)(z-2)}=\frac{A}{z+1}+\frac{B}{z-2}$$

$z-5=A(z-2)+B(z+1)$

$z=2\quad -3=3B\quad B=-1$ $\qquad z=-1\quad -6=-3A\quad A=2$

$$\frac{1}{2\pi i}\int_\gamma e^{tz}\frac{z-5}{(z+1)(z-2)}dz=\frac{2}{2\pi i}\int_\gamma\frac{e^{tz}}{z-(-1)}dz-\frac{1}{2\pi i}\int_\gamma\frac{e^{tz}}{z-2}dz$$
$$=2e^{-t}-e^{2t}$$

Generally, $$\frac{1}{2\pi i}\int_\gamma\frac{az+b}{(z+1)(z-2)}dz=Ae^{-t}+Be^{2t}$$
$A=\frac{a-b}{3}\quad B=\frac{2a+b}{3}$

Algebra, derivatives still work

$$=\begin{bmatrix}2e^{-t}-e^{2t}&-2e^{-t}+2e^{2t}\\e^{-t}-e^{2t}&-e^{-t}+2e^{2t}\end{bmatrix}$$

$t=0\to I$
$(e^{tA})'=Ae^{tA}$
$t=0\to A$

This is nearly impossible with 100×100 matrices because A is very difficult to factor and do partial fraction decomposition with 100-degree polynomials. If A does work, this is the fastest way to do A.

This is used to solve differential equations.

Concrete Problem

$x'(t) = -4x(t) + 6y(t), \; x(0) = x_0$

$y'(t) = -3x(t) + 5y(t), \; y(0) = y_0$

Abstract Cauchy Problem

$v'(t) = Av(t), \; v(0) = v_0$

$$A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} \quad v(t) = e^{tA}v_0 = (x(t), y(t)) = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Ex: Heat problem

Concrete Problem

$w_t(t,x) = w_{xx}(t,x)$

$w(0,x) = f(x)$

Abstract Cauchy Problem

$v'(t) = Av(t), \; v(0) = f$

$v(t) = w(t, \cdot), \; A = \frac{d^2}{dx^2}$

function for each t

Big Picture

Neubrander argues that anything in the world that changes with time can be reduced to an abstract Cauchy problem.

$$v(t) = e^{tA}f = \sum_0^\infty \frac{t^n}{n!} f^{(2n)}(\cdot)$$

$$e^{tA} = \frac{1}{2\pi i}\int_\gamma e^{tz}(zI - A)^{-1}\,dz$$ Most efficient method

$$= \sum_0^\infty \frac{t^n}{n!}A^n = u\left[\sum_0^\infty \frac{t^n}{n!}J^n\right]u^{-1}, \text{ where } A = uJu^{-1}$$

$$= \lim_{n\to\infty}\left(I - \frac{tA}{n}\right)^{-n}$$

$$= \lim_{n\to\infty}\left(I + \frac{tA}{n}\right)^n$$

* Transference theory?

11/29/11 <u>Dunford Calculus</u> (everyone uses this)

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{tz}R(z,A)\,dz = \frac{1}{2\pi i}\int_\Gamma \underbrace{(e^{tz}-r(tz))}_{\text{small}}R(z,A)\,dz + \frac{1}{2\pi i}\int_\Gamma r(tz)R(z,A)\,dz$$

$b_1, \dots = \lambda \dots$

Choose $r(z)=\frac{B_1}{b_1-z}+\dots+\frac{B_q}{b_q-z}$ such that $|e^z-r(z)|$ is small for $Re\, z \le 0$

$$r(tz)=\frac{B_1}{b_1-tz}+\dots+\frac{B_q}{b_q-tz} = -\frac{B_1}{t}\frac{1}{z-\frac{b_1}{t}} - \dots - \frac{B_q}{t}\frac{1}{z-\frac{b_q}{t}}$$

<u>Cauchy's Integral Theorem</u>:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)$$

$$\frac{1}{2\pi i}\int_\Gamma r(tz)R(z,A)\,dz = \frac{1}{2\pi i}\int_\Gamma\left(-\frac{B_1}{t}\frac{1}{z-\frac{b_1}{t}} - \dots - \frac{B_q}{t}\frac{1}{z-\frac{b_q}{t}}\right)R(z,A)\,dz$$

$$= -\left(\frac{B_1}{t}R\left(\frac{b_1}{t},A\right)+\dots+\frac{B_q}{t}R\left(\frac{b_q}{t},A\right)\right) \overset{?}{\sim} e^{tA}$$

where does the negative sign come from?

<u>Hille-Philips Calculus</u>: (Neubrander uses this)

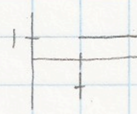
$$r(z)=\frac{B_1}{b_1-z}+\dots+\frac{B_q}{b_q-z}$$

$$\frac{B_1}{b_1-z} = \int_0^\infty e^{zs}(B_1 e^{-b_1 s})\,ds \Rightarrow r(z)=\int_0^\infty e^{zs}f(s)\,ds, \text{ where } f(s)=B_1e^{-b_1 s}+\dots+B_q e^{b_q s}$$

$$\Rightarrow r(tz) = \int_0^\infty e^{zs}\frac{1}{t} f\left(\frac{s}{t}\right)ds$$

$$\frac{1}{t} f\left(\frac{s}{t}\right) = \frac{B_1}{t}e^{-\frac{b_1 s}{t}}+\dots+\frac{B_q}{t}e^{-\frac{b_q s}{t}}$$

$$\frac{1}{z} e^{tz} = -\int_t^\infty e^{zs}\,ds = -\int_0^\infty e^{zs}H_t(s)\,ds, \text{ where } H_t(s) \text{ is 0 up to } t \text{ and 1 after } t$$



$$\Rightarrow e^{tz} = -z\int_0^\infty e^{zs}H_t(s)\,ds$$

(?)

Integration by parts

$$r(z) = \left[e^{zs}f(s)\right]_0^\infty - z\int_0^\infty e^{zs}\alpha(s)\,ds = -f(0) - z\int_0^\infty e^{zs}\alpha(s)\,ds$$

$$\alpha(s)=\int_0^s f(r)\,dr = B_1\int_0^s e^{-b_1 r}\,dr+\dots = B_1\left[-\frac{1}{b_1}e^{-b_1 r}\right]_0^s = \frac{B_1}{b_1}(1-e^{-b_1 s})$$

$$r(tz) = -f(0) - tz\int_0^\infty e^{tzs}\alpha(s)\,ds = -f(0) - z\int_0^\infty e^{zs}\alpha\left(\frac{s}{t}\right)ds$$

$$e^{tz}-r(tz) = -z\int_0^\infty e^{zs}\left(H_t(s)-\alpha\left(\frac{s}{t}\right)\right)ds + \underline{f(0)} \quad (?)$$

$$\alpha\left(\frac{s}{t}\right)=\sum_{i=1}^q \frac{B_i}{b_i}\left(1-e^{\frac{b_i s}{t}}\right)$$

$$|e^{tA}-r(tA)| = \left|\int_0^\infty(-Ae^{sA})\left(H_t(s)-\alpha\left(\frac{s}{t}\right)\right)ds\right| \le |A|\cdot\int_0^\infty\left|H_t(s)-\alpha\left(\frac{s}{t}\right)\right|ds$$

The others are trying to make $|e^{tA}-r(tA)|$ small. We are trying to make $|H_t(s)-\alpha(\frac{s}{t})|$ small. Must find function $\alpha(\frac{s}{t})$ that models $H_t(s)$. Right now our best bet is the Padé approximation.

<u>Homework</u>: $r(z)=\frac{B_1}{b_1-z}$ $\quad r(0)=1 \to r(z)=\frac{b_1}{b_1-z}$

Choose b such that

Optimize it

$$\int_0^\infty \Big|H_t(s) - \overbrace{\left(1-e^{-\frac{bs}{t}}\right)}^{\alpha(\frac{s}{t})}\Big|\,ds \text{ is small} \Rightarrow \int_0^t \left(1-e^{-\frac{bs}{t}}\right)ds + \int_t^\infty e^{-\frac{bs}{t}}\,ds \sim (0.6\dots)t? \text{ for } b=1.6?$$