Gödel's First Incompleteness Theorem

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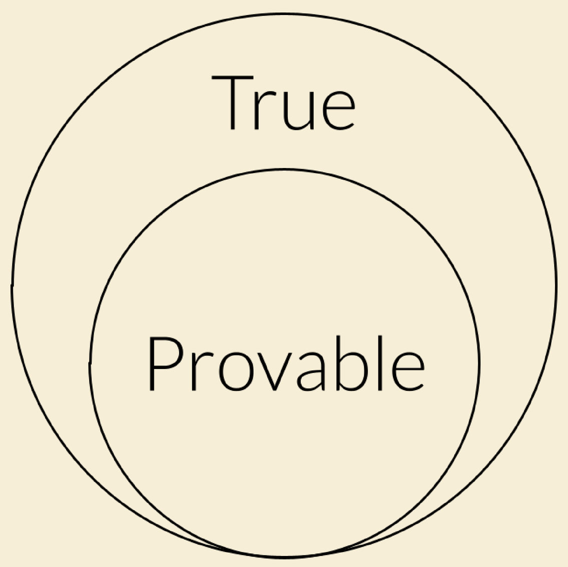
Theorem

To every ω-consistent recursive class K of formulae there correspond recursive class-signs r, such that neither $vGen(r)$ nor $\neg vGen(r)$ belongs to $Flg(k)$.



In other words, every reasonable recursive axiomatic proposition of number theory will always have propositions that cannot be proven nor disproven.

Background

- Before Gödel, metamathematicians expected math to eventually be **complete**, i.e., to be able to prove everything given the right amount of axioms.
- In the early 20th century, set theory paradoxes like those proposed by Bertrand Russell raised questions about the **consistency** of math.
- Gödel was trying to solve Hilbert's Second problem — he wanted to know if math had any inherent contradictions and if truth was self-evident.

Proposition

Statements *of* number theory could also be *about* number theory.

Gödel Numbering

To investigate the proposition, Gödel needed math to be self-referential. He created his own $Encode(G)$ function to turn mathematical statements into unique natural numbers. To do so, he would first need to convert each mathematical symbol into a number. Thus, he created a numbering system where each symbol has its own unique natural number to be used for encoding.

Constant Sign	Gödel Number
$\neg$	1
$\vee$	2
$\supset$	3
$\exists$	4
$=$	5
$\vdots$	$\vdots$

In theory, the symbols have no meaning, the axioms and formulas constructed from them are what give them their meaning.

Encoding

Given a sequence of Gödel numbers $(x_1, x_2, \ldots, x_n)$, the encoding is the product of the first n prime numbers raised to the values in the sequence.

$$Encode(x_1, x_2, \ldots, x_n) = 2^{x_1} \times 3^{x_2} \times \ldots \times p_n^{x_n}$$

This way, any given mathematical expression can be encoded algebraically. Besides, the statements can be decoded through prime factorization.

Note: $Encode(A)$ is often written as $\ulcorner A \urcorner$.

Provability

Since statement A can be proved through an axiom B, and $\ulcorner A \urcorner$) and $\ulcorner B \urcorner$ are unique numbers, Gödel proposed that there must be a mathematical relation between the two.

- We can express this relation as a function $Provability(\ulcorner A \urcorner)$ that determines whether a statement A is provable within the formal system.
- This function is essentially a binary predicate that determines if A can be proved with the current axioms.

Self-reference by diagonalization

Enumerate all formulas in the formal system F with exactly one free variable:

	$n=1$	$n=2$	$\cdots$	$n=j$
$F_1(n)$	$F_1(1)$	$F_1(2)$	$\cdots$	$F_1(j)$
$F_2(n)$	$F_2(1)$	$F_2(2)$	$\cdots$	$F_2(j)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$F_j(n)$	$F_j(1)$	$F_j(2)$	$\cdots$	$F_j(j)$

Each entry represents a formula $F_i(n)$, where i represents the formula number and n represents the parameter.

Gödel Statement

Construct a new formula G, asserting the negation of provability for each formula $F_j(j)$ in the table; this is the Gödel statement:

$$G \equiv \neg Provability(F_j(j))$$

Now, consider the truth value of G. If G were false, then by its own definition, each $F_j(j)$ would be provable and thus true. But the definition of G implies the opposite; since math is consistent, G must be true. Since G cannot be consistently proven or disproven within the system, it follows that G is true but unprovable within F.

Axiomatization

Although the proof works, one might argue that the Gödel statement could be made into an axiom to trivialize the problem.
However, doing so would only create a new system where the current G could be proved; it would change the nature of the new system, leading to further contradictions.

Implications in Math

- Forced meta-mathematics past Russell's *Principia Mathematica*.
- Established a mutual exclusivity between consistency and completeness of recursive formal systems.
- Showed that formal systems cannot capture all mathematical truths, showing the inherent incompleteness of axiomatic systems of logic.
- Used for proof in Tarski's Undefinability Theorem, where arithmetical truth cannot be defined in arithmetic.

Further reach

- Computer Science
  - Popularized the arithmetization of syntax in the years leading up to the first computers.
  - Inspired Turing, and by consequence the field of computability theory.
  - Furthered the complexity of the discussions on the limitations of computers and artificial intelligence.
- Philosophy
  - Directly challenges the idea of determinism and a fully knowable universe.
  - Furthers debate on the nature of knowledge and the transcendence of human intuition.

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