Gödel's First Incompleteness Theorem

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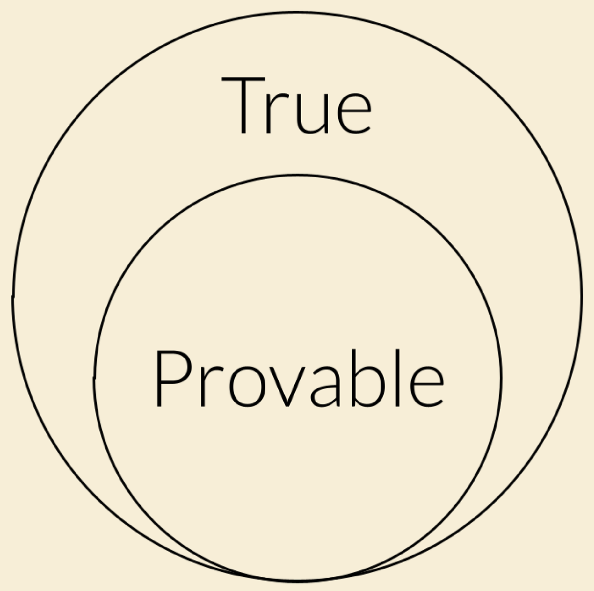
Theorem

To every ω-consistent recursive class K of formulae there correspond recursive class-signs r, such that neither $vGen(r)$ nor $\neg vGen(r)$ belongs to $Flg(k)$.



In other words, every reasonable recursive axiomatic proposition of number theory will always have propositions that cannot be proven nor disproven.

Background

- Before Gödel, metamathematicians expected math to eventually be ***complete***.
- In the early 20th century, set theory paradoxes like those by Bertrand Russell raised questions about the ***consistency*** of math.
- Gödel was trying to solve Hilbert's Second problem; he wanted to know if math had any inherent contradictions and if truth was self-evident.

Proposition

Statements *of* number theory could also be *about* number theory.

Gödel Numbering

Gödel created his own $Encode(G)$ function to turn mathematical statements into unique natural numbers. To do so, he would first need to **convert each mathematical symbol into a number**. Thus, he created a numbering system where each symbol has its own unique natural number to be used for encoding.

Constant Sign	Gödel Number
$\neg$	1
$\vee$	2
$\supset$	3
$\exists$	4
$=$	5
$\vdots$	$\vdots$

In theory, the symbols have no meaning, the axioms and formulas constructed from them are what give them their meaning.

Encoding

Given a sequence of Gödel numbers $(x_1, x_2, \ldots, x_n)$, its encoding is given by the product of the first n prime numbers raised to the values in the sequence.

$$Encode(x_1, x_2, \ldots, x_n) = 2^{x_1} \times 3^{x_2} \times \ldots \times p_n^{x_n}$$

This way, any given mathematical expression can be **encoded algebraically**. Besides, the statements can be decoded through prime factorization.

Note: $Encode(A)$ is often written as $\ulcorner A \urcorner$.

Provability

Since statement A can be proved through an axiom B, and $\ulcorner A \urcorner$, $\ulcorner B \urcorner$ are unique numbers, Gödel proposed that there must be a mathematical relation between the two.

- We can express this relation as a function $Provability(\ulcorner A \urcorner)$ that **determines whether a statement A is provable** within the formal system.
- This function is essentially a binary predicate that determines if A can be proved through any axiom B.

Self-reference by Diagonalization

Enumerate all formulas in the formal system F with exactly one free variable:

$$\begin{array}{c|c|c|c|c}
 & n=1 & n=2 & \cdots & n=j \\
\hline
F_1(n) & F_1(1) & F_1(2) & \cdots & F_1(j) \\
F_2(n) & F_2(1) & F_2(2) & \cdots & F_2(j) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
F_j(n) & F_j(1) & F_j(2) & \cdots & F_j(j)
\end{array}$$

Each entry represents a formula $F_i(n)$, where i represents the formula number and n represents the parameter.

Gödel Statement

Construct a new formula G, asserting the negation of provability for each formula $F_j(j)$ in the table:

$$G \equiv \neg Provability(\ulcorner F_j(j) \urcorner)$$

Truth Value

If G were false, then by its own definition, each $F_j(j)$ would be provable and thus true. However the definition of G states the opposite, and since math is consistent, G must be true. This means G is true but unprovable within F.

Axiomatization

One might argue that the Gödel statement could be made into an axiom to trivialize the problem. However, doing so would only create a new system where the current G could be proved; thus, it would change the nature of the system, leading to further contradictions.

Implications in Math

- Forced meta-mathematics past Russell's *Principia Mathematica*.
- Established a mutual exclusivity between consistency and completeness of recursive formal systems.
- Used for proof in Tarski's Undefinability Theorem, where arithmetical truth cannot be defined in arithmetic.

Further reach

- Computer Science
  - Popularized the arithmetization of syntax in the years leading up to the first computers.
  - Inspired Turing, and by consequence the field of computability theory.
  - Established limitations on computers and artificial intelligence.
- Philosophy
  - Directly challenged the ideas of determinism and reductionism.
  - Furthered debate on the nature of knowledge and the transcendence of human intuition.
  - Prompted a reevaluation of epistemology in light of truths outside formal systems.

Criticism

- **Applicability:** Critics question the practical relevance of Gödel's theorems outside of formal mathematical systems. The theorems might not have direct implications for everyday mathematics or scientific inquiry.
- **Assumptions:** Gödel's proofs rely on certain assumptions about mathematical reasoning. Critics debate the validity of these assumptions and whether alternative frameworks could lead to different conclusions.
- **Philosophical Interpretations:** Some philosophers argue that Gödel's theorems have been over-interpreted or misunderstood, and they contend that the implications of the theorems might not be as profound as believed.

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