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A Hierarchical Approach to a Tri-Objective Portfolio Optimization Problem Considering an ESG Index

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Abstract: Traditional portfolio construction primarily revolves around a bi-objective approach, focusing on minimizing portfolio variance while maximizing expected returns. However, this approach leaves out other objectives that could interest decision makers. In this work, we incorporate an extra objective, namely the environmental, social, and governance index (ESG), as a secondary objective. This addition empowers investors to customize their portfolios by defining explicit trade-off thresholds between expected returns and risk, considering the ESG index. To achieve this goal, we make use of external archiving techniques and evolutionary algorithms. In particular, we first find approximate solutions to the bi-objective problem; then, we look for efficient solutions for ESG. We tested our approach with data on the Dow Jones, S&P500, and Nasdaq100 from Yahoo Finance. The results show that the proposed methodology can identify portfolios with good returns and risks considering ESG.

Keywords: ESG score; portfolio optimization; multi-objective optimization; archiving techniques

MSC: 90C29

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1. Introduction

Portfolio optimization typically adheres to the foundational framework introduced by Harry Markowitz [1]. This framework primarily revolves around a two-objective paradigm that involves maximizing a portfolio's expected returns while concurrently minimizing the associated risk, often quantified by variance. This classical approach endeavours to reconcile the pursuit of superior returns with the need to curtail financial risk exposure [2].

Extensive research has unveiled the inherent multi-objective complexity of the classical portfolio optimization model originally conceived by Markowitz [2–4]. Consequently, this model often proves insufficient in accommodating modern investors' diverse and evolving interests. This transformation stems from the realization that investors are inclined to choose portfolios that align closely with their personal preferences rather than rigidly adhering to the concept of the most efficient portfolio [5]. Several alternative approaches have surfaced in response to this shift, each striving to integrate a broader spectrum of objectives beyond the conventional considerations of risk and return. These supplementary attributes encompass elements such as financial stability; dividend yields; future growth prospects; and sustainability, which is the central focus of our current study [2,4].

Our specific focus on sustainability, in conjunction with financial criteria, derives from the concept of Socially Responsible Investment (SRI), an investment strategy advocating for the allocation of funds to companies and initiatives that champion environmentally and socially responsible practices, as supported by [6]. As indicated in prior research [4], the substantial investments already channelled into SRI mutual funds underscore the burgeoning demand for such products.

Within the broader domain of SRI, a distinct subset strongly emphasises portfolio construction guided by ESG indices. Under this approach, each company undergoes a comprehensive evaluation across the following three essential categories: environmental, social, and governance [6]. These evaluations encompass pivotal factors such as employment quality, health and safety, human rights, product responsibility, emissions, board composition, and other relevant criteria [4]. The assessment of each company is based on the metrics within each category, ultimately culminating in the calculation of a total ESG score for that company. This aggregate score is derived from the combination of the three category scores, thereby furnishing a valuable sustainability ranking among firms [4,6].

In portfolio optimization within the SRI context, various frameworks have emerged to find optimal portfolios with ESG considerations. These frameworks tackle the tri-criterion portfolio selection problem by employing a spectrum of methods, including both exact techniques and heuristics like Multi-Objective Evolutionary Algorithms (MOEAs) [7]. Furthermore, these approaches can be categorized into the following two fundamental strategies: a priori and posteriori methods. A priori methods incorporate decision-maker preferences into an optimization problem to generate a recommended portfolio. In contrast, a posteriori methods focus on approximating the Pareto set/front, allowing the decision maker to select their most preferred solution [3]. For a more detailed exploration of recent multi-criteria decision-making (MCDM) approaches in tri-criterion portfolio selection within the SRI context, please refer to Section 2.

Note that although maximizing an ESG index is highly desirable, in our opinion, a decision maker might not be interested in the maximum for the ESG objective, since it might be a solution with poor performance and high risk. Thus, we propose first dealing with a relaxed version of the classical problem using epsilon dominance [8–11]. This dominance has been widely used in the literature, for instance, to find approximate solutions and later perform post-processing for a specific purpose [12–16].

The primary contributions of this paper can be summarized as follows: (i) We introduce a hierarchical approach for incorporating the ESG score as a secondary objective, building upon an extension of classical optimal solutions (see Section 3.1). This method aligns with the a priori approach, allowing investors to define an acceptable trade-off deterioration level of the classical portfolio objectives (return and risk). (ii) We apply and compare various sampling techniques to generate an approximate set of optimal portfolio solutions. (iii) We present a straightforward visual representation of the nondominated surface projected in 2D, showcasing the proposed solutions. This approach, while rooted in an a priori method, offers investors a user-friendly option for decision making akin to the Markowitz approach.

The subsequent sections of the paper are organized as follows: In Section 2, we thoroughly review the existing literature on the tri-criterion portfolio selection problem within the realm of SRI. Section 3 provides a comprehensive explanation of our hierarchical approach, supported by the necessary background on MCDM and MOEAs. Next, Section 4 presents the multi-objective algorithms that were used to address the problem. In Section 5, we showcase the empirical results obtained through the application of our proposed approach to portfolios created based on stock market indices such as the S&P500, Dow Jones, and Nasdaq100, selecting a promising subset of the component assets. Finally, Section 6 offers our concluding remarks and key insights from this study.

2. Literature Review

The concept of determining the Pareto-efficient frontier in portfolio selection through Mean-Variance (M-V) optimization was initially introduced by Markowitz [1]. Over time, Markowitz's model has evolved, incorporating more intricate risk measures; additional constraints; and, more recently, the ability to accommodate supplementary objectives. Within this framework, two primary approaches for addressing the expanded portfolio optimization problem are evident, namely (i) exact methods and (ii) heuristic methodologies. In the subsequent sections, we offer a comprehensive review of theoretical and practical endeavours that have utilized exact methods or heuristic approaches in the context of extended M-V portfolio selection, focusing on SRI.

As previously surveyed by Steuer and Na (2003) [17], a considerable body of literature has explored various methods for solving portfolio selection problems with additional criteria. Notably, Ehrgott et al. (2004) employed an MCDM approach to expand the problem to encompass five objectives [2].

In the SRI context, Hirschberger et al. [3] introduced a comprehensive framework for calculating the non-dominated surface in tri-criterion portfolio selection. This extension builds upon Markowitz's portfolio selection approach by incorporating an additional linear criterion, such as dividends, liquidity, or sustainability. By solving a quad–lin–lin program, they offer an exact method for computing the non-dominated surface, which has the potential to outperform conventional portfolio strategies for multi-criteria decision makers [3,7]. An empirical application that integrates sustainability as the third criterion was presented to demonstrate the composition of the non-dominated surface [3,7].

In a study by Utz et al. [4], sustainability was integrated as the third criterion to establish an efficient variance-expected return–sustainability frontier. This exploration aimed to elucidate how the sustainable mutual fund industry can enhance its sustainability levels. The tri-criterion non-dominated surface is derived using the Quadratic Constrained Linear Program (QCLP) approach. The experimental findings suggest an opportunity to augment sustainability levels without compromising risk and return levels [4].

Steuer and Utz (2023) [18] explored a tri-criterion approach that calculates efficient surfaces and non-contour curves, referred to as NC-efficient fronts, which span the efficient surface to identify points with optimal ESG integration. These portfolios are not confined to the M-V efficient frontier, as they reside within the interior of the M-V-ESG-efficient surface. Steuer and Utz's work characterizes this efficient surface and provides a method for identifying points that achieve the ideal balance between risk, return, and ESG considerations. Future research may involve assessing the feasibility and performance of NC-efficient portfolios with real-world data.

A paper by Lauria et al. (2022) [6] presented an ESG-valued framework that integrates ESG scores into dynamic pricing theory. This framework introduces the concept of the *ESG-valued return*, derived from a combination of the ESG score of an asset and its return. It uniquely incorporates an ESG affinity parameter, allowing for flexible weighting of ESG considerations in valuation, alongside the traditional financial risk aversion parameter. The model's applications extend to portfolio optimization, risk assessment, option pricing, and the computation of shadow riskless rates. It addresses the dynamic nature of ESG ratings and emphasises the necessity of standardization in ESG score valuation while recognizing challenges in parameter identification and the need for further research in various facets of the framework.

Hilario-Caballero et al. (2020) [7] also extended the classical Markowitz mean-variance approach with a tri-criterion portfolio selection model, considering investor preferences for portfolio carbon risk exposure. Their method employs the efficient multi-objective genetic algorithm ev-MOGA, based on $\epsilon-$dominance, to approximate the 3D Pareto front. Furthermore, they introduced an a posteriori approach to incorporate investors' climate change-related preferences (quantified by carbon risk exposure) and their risk-aversion attitudes into the solution process.

Previous works have explored various methods to incorporate ESG scores as an additional objective in the Markowitz model. In contrast, our proposed approach leverages the $-\epsilon$ dominance concept to ensure approximately optimal portfolios in the classical approach. This approach includes a priori consideration of investor preferences concerning the ESG score, allowing for a tailored exploration of optimal portfolios on the M-V-ESG surface.

3. The Hierarchical Tri-Objective Approach

MCDM is an inherent tool for modelling the problem, and this section provides readers with the necessary concepts and algorithms relevant to our proposed framework and solution techniques. Furthermore, this section offers a detailed explanation of our model, ensuring a thorough comprehension of its intricacies.

3.1. Multi-Objective Optimization

A multi-objective optimization problem (MOP) [19] can be defined as follows:

$$\min_{x \in Q} F(x), \tag{1}$$

where $Q \subset \mathbb{R}^n$ is the set of feasible solutions and $F : Q \to \mathbb{R}^k, F(x) = (f_1(x), \ldots, f_k(x))^T$ is defined by the individual objective functions. In general, Q is expressed by equality and inequality constraints.

$$Q = \{x \in \mathbb{R}^n | g_i(x) \leq 0, i = 1, \ldots, l, \text{ and } h_j(x) = 0, j = 1, \ldots, m\}, \tag{2}$$

where l is the number of inequality constraints and m is the number of equality constraints. To define optimality of MOPs, the concept of dominance is typically used.

Definition 1. *Pareto Dominance [20]*

1. *Let $v, w \in \mathbb{R}^k$. Then, vector v is less than w (in short, $(v <_p w)$) if $\forall i \in \{1, \ldots, k\}, v_i < w_i$. The relation $\leq_p$ is analogous.*
2. *A vector $(y \in Q)$ is called dominated by a vector $(x \in Q\ (x \prec y))$ with respect to an MOP as in (1) if*

$$F(x) \leq_p F(y) \quad \text{and} \quad F(x) \neq F(y)$$

otherwise, y is called non-dominated by x.

Definition 2. *Pareto Set/Front*

1. *A point $(x \in Q)$ is called (Pareto) optimal or a Pareto point of an MOP as in (1) if there exists no $y \in Q$ that dominates x.*
2. *The set of all Pareto-optimal solutions is called the Pareto set, i.e.,*

$$P_Q := \{x \in Q |\ \nexists y \in Q, y \prec x\}.$$

3. *The image $(F(P_Q))$ of P_Q is called the Pareto front.*

In the context of our model, the concept of $-\epsilon$ dominance is particularly relevant. This concept can be viewed as a more stringent form of dominance [21].

Definition 3. *$-\epsilon$ Dominance [22]. Let $\epsilon \in \mathbb{R}^k_+$ and $x, y \in \mathbb{R}^n$. x is said to $-\epsilon$ dominate y $x \prec_{-\epsilon} y$ with respect to an MOP as in (1) if*

$$F(x) + \epsilon \leq_p F(y) \quad \text{and} \quad F(x) + \epsilon \neq F(y).$$

Definition 4. *The set of approximate solutions $P_{Q,\epsilon}$. $P_{Q,\epsilon}$ denotes the set of points in $Q \subset \mathbb{R}^n$ that are not $-\epsilon$-dominated by any other point in Q, i.e.,*

$$P_{Q,\epsilon} : \{x \in Q |\ \nexists y \in Q, y \prec_{-\epsilon} x\}.$$

3.2. Classical Portfolio Optimization Problem

Within the broader context of portfolio optimization, we now present the classical mean-variance problem originally formulated by Markowitz [1–3], which can be expressed as follows:

$$\max\left\{ z_1'(x) = \mu^T x = \sum_{i=1}^{n} \mu_i x_i \right\} \tag{3}$$

$$\min\left\{ z_2'(x) = x^T \Sigma x = \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} x_i x_j \right\} \tag{4}$$

$$s.a. \sum_{i}^{n} x_i = 1, \quad \forall i \in \{1, \ldots, n\}, \quad x_i \geq 0. \tag{5}$$

Here, n denotes the number of available assets, $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is a vector representing a portfolio specifying the proportions of capital invested in each asset (i.e., x_i is the investment portion in asset $i \in \{1, \ldots, n\}$), $\mu = (\mu_1, \ldots, \mu_n)^T$ is a vector of the expected returns of each asset, and $\Sigma = (\sigma_{ij})_{i=1\ldots,n, j=1,\ldots,n}$ is a $n \times n$ matrix where σ_{ij} is the covariance between the returns of assets i and j.

3.3. Portfolio Optimization with ESG

To introduce the third objective, we draw inspiration from previous work, such as [3], which expanded the classical portfolio optimization problem to incorporate three distinct objectives. This extended framework integrates the ESG index as a fundamental component. Mathematically, this augmentation can be represented as

$$\max\left\{ z_1'(x) = \mu^T x = \sum_{i=1}^{n} \mu_i x_i \right\} \tag{6}$$

$$\min\left\{ z_2'(x) = x^T \Sigma x = \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} x_i x_j \right\} \tag{7}$$

$$\min\left\{ z_3'(x) = \eta^T x = \sum_{i=1}^{n} \eta_i \right\} \tag{8}$$

$$s.a. \sum_{i}^{n} x_i = 1, \quad \forall i \in \{1, \ldots, n\}, \quad x_i \geq 0,$$

where the notation retains the characteristics of the previous model and a third condition is added where $\eta = (\eta_1, \ldots, \eta_n)^T \in \mathbb{R}^n$ is a vector of ESG score risk and η_i represents the ESG score risk to which asset $i \in \{1, \ldots, M\}$ (i.e., a metric representing the risk of an asset causing environmental, social, or governance problems based on the company's ESG index).

Another important consideration is that while portfolio selection theory is predominantly built around the variance criterion, according to [3], investors typically think in terms of standard deviation, since it shares the same units (percentage return) as expected return. Therefore, we also explore the following alternative formulation for the second objective:

$$\min\left\{ \sqrt{z_2'(x)} = \sqrt{x^T \Sigma x} = \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} x_i x_j} \right\}. \tag{9}$$

Finally, to standardize the problem, the first objective can be reformulated by minimizing the objective in the following alternative manner:

$$\max\left\{ z_1'(x) = \mu^T x = \sum_{i=1}^{n} \mu_i x_i \right\} = -\min\left\{ -z_1'(x) = -\mu x = -\sum_{i=1}^{n} \mu_i x_i \right\}. \tag{10}$$

The method developed in the next subsection takes Models (3) and (4) and (6)–(8) into consideration to derive optimal portfolio solutions.

3.4. Hierarchical Approach

It is crucial to acknowledge that while the ESG score holds significant importance, it remains a secondary objective because portfolios with low expected returns or high risk are generally not considered acceptable, even if they excel in ESG performance. The hierarchical approach we propose can be summarized as follows:

Initially, the investor specifies an acceptable level of deterioration for the expected return and risk—essentially, the percentage of points the investor is willing to forgo in the pursuit of solutions with superior ESG performance. This deterioration concept aligns theoretically with epsilon dominance. Utilizing this information, we compute the set of approximate solutions, denoted as $P_{Q,\epsilon}$, by focusing solely on the classical bi-objective problems ((3) and (4)). Subsequently, we introduce the third objective, which is the ESG index, and filter out any dominated solutions using all three objectives, as detailed in Models (6)–(8).

Assuming that we already have the population of feasible solutions denoted as Q and an epsilon value (ϵ), we can compute the approximately optimal portfolios, referred to as $P_{Q,\epsilon}$, for Equations (3) and (4) by employing Algorithm 1, and it is possible to apply Algorithm 2 over $P_{Q,\epsilon}$ to filter any dominated solution considering Equations (6)–(8).

Algorithm 1 A: ArchiveUpdate$P_{Q,\epsilon}$, (P, A_0) [21].

Require: P: population (portfolios) A_0: current archive, $\epsilon \in \mathbb{R}^k_+$
Ensure: updated archive A, the set $P_{Q,\epsilon}$ considering only P.
```
A = A_0
for all p ∈ P do
    if ∄a ∈ A : a ≺_{-ε} p then
        A = A ∪ {p}
    end if
    for all a ∈ A do
        if p ≺_{-ε} a then A = A \ {a}
        end if
    end for
end for
```

Algorithm 2 A: ArchiveUpdate$P_Q(P, A_0)$ based on [21].

Require: P: population (portfolios) A_0: current archive
Ensure: updated archive A, the set P_Q considering only P.
```
A = A_0
for all p ∈ P do
    if ∄a ∈ A : a ≺ p then
        A = A ∪ {p}
    end if
    for all a ∈ A do
        if p ≺ a then A = A \ {a}
        end if
    end for
end for
```

The right side of Figure 1 visually depicts the configuration of the exact Pareto front for classical Problems (3) and (4). This illustration also showcases approximate solutions P_Q and $P_{Q,\epsilon}$ using Algorithms 2 and 1, respectively. For this example, $\epsilon = (0.01, 0.01)$ (i.e., the investor allows 1% of deterioration in the annual expected return and the annual expected risk). The image on the left side presents the projection of optimal portfolio solutions, considering the model given by Equations (6)–(8) and utilizing the $P_{Q,\epsilon}$ set generated with the classical problem. These portfolios were constructed using four generated assets with returns following a normal random distribution and assigning random uniform weights for the investment allocation in each asset.

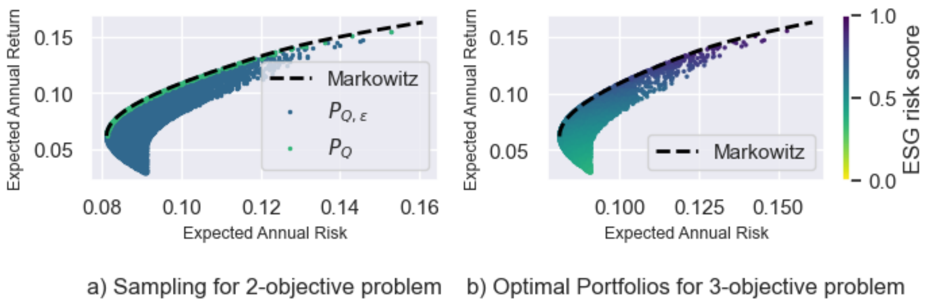

Figure 1. Optimal portfolio solutions for four generated assets with returns distributed normally (mean = 0, std = 0.01). (**a**) The exact solutions for the Markowitz problem and approximations of P_Q and $P_{Q,\epsilon}$ for two objectives, considering $\epsilon = (0.01, 0.01)$. (**b**) The optimal portfolio solutions for three objectives using the $P_{Q,\epsilon}$ set for two objectives.

As a final remark, according to Hirschberger, in numerous finance textbooks, the parameters μ and Σ are commonly defined as the sample-based mean and variance of historical asset returns, respectively. It is important to note that the hierarchical approach we propose exhibits the flexibility to encompass a range of extensions of the classical model designed to enhance the performance of the standard sample-based mean-variance model. These extensions typically involve the derivation of more robust estimators for μ and Σ aimed at reducing estimation errors or the implementation of constraints on the set of feasible portfolios (S′) to improve portfolio optimization [3].

4. Methods to Solve the Portfolio Problem

The principal challenge in this approach lies in obtaining a high-quality approximation of $P_{Q,\epsilon}$ within the classical problem, as outlined in Equations (3) and (4). The constraints inherent in classical portfolio optimization lead to the creation of a geometric object known as the unit simplex (S) [23].

$$S := \left\{ x \in \mathbb{R}^n | \forall i = 1, \ldots, n, x_i \geq 0 \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}.$$

To address this challenge effectively, it becomes essential to employ sampling techniques within the unit simplex. These techniques generate portfolios that are sufficiently close to the Pareto set and offer a thorough exploration of the simplex, ensuring a more accurate representation of the solution space.

Four distinct heuristics are explored to tackle this challenge. First, we utilize points generated by the Das–Dennis approach, providing a uniform and extensive exploration of the simplex. Secondly and thirdy, we employ MOEAs like NSGA-II and SMS-EMOA, which navigate the simplex to approximate solutions close to the Pareto front. Lastly, we implement a directed search strategy to leverage our knowledge of the exact solutions to the classical problem. These heuristics collectively enhance our ability to efficiently generate well-approximated solutions within the unit simplex [21,23–26].

4.1. Das–Dennis Approach

Das and Dennis [24] introduced the simplex-lattice approach for multi-objective optimization, which was subsequently named after them. This structured approach relies on an integer gap parameter (p, where $p \geq 1$), resulting in the creation of $\binom{M+p-1}{p}$ structured points. While it may not facilitate the creation of a well-spaced distribution with an arbitrary number of points, previous results [23] indicate that this method can be a viable option for comprehensive exploration within the entire simplex, particularly when 'p' is sufficiently large.

4.2. Multi-Objective Evolutionary Algorithm Approach

Another pertinent technique for addressing multi-objective problems (MOPs) is evolutionary multi-objective optimization [27,28]. Within this domain, a wide array of methodologies has been developed, collectively known as multi-objective evolutionary algorithms (MOEAs). These methods offer distinct advantages, relying on stochastic search procedures and providing approximations of the entire Pareto set and Pareto front in a single execution. Furthermore, it was suggested in [13,21] that algorithms such as Algorithms 2 and 1 can be effectively combined with MOEAs to generate acceptable solutions. In the following, we delve into a review of two of the most representative methodologies for designing a MOEAs and introduce their most notable exponent.

Dominance-based methods are a category of approaches that directly utilize the dominance relationship. These methods typically consist of the following two key components: a dominance ranking and a diversity estimator. The dominance ranking arranges solutions based on the partial order established by Pareto dominance, thereby emphasising the Pareto front. To enable comparisons among solutions that are mutually non-dominated, a diversity estimator is introduced, creating a total order for the solutions. This estimator is rooted in the notion that decision makers prefer a well-distributed set of solutions along the front, emphasising diversity.

One of the most widely adopted methods within this category is the Non-Dominated Sorting Genetic Algorithm II (NSGA-II), introduced in [25]. NSGA-II operates through two key mechanisms. First, it employs non-dominated sorting to rank the current solutions based on the Pareto dominance relationship, effectively quantifying how many solutions dominate the current one. This ranking aids in identifying optimal values and utilizing them to generate new solutions.

The second mechanism is the crowding distance, which assesses the proximity of a point to its neighbours. This mechanism helps identify solutions with less distance, signifying the presence of more solutions in that particular region. These components serve the objectives of convergence and spread, respectively. As a result of these mechanisms, NSGA-II exhibits strong overall performance and has gained prominence, particularly when dealing with two or three objectives.

On the other hand, indicator-based methods rely on performance indicators to determine an individual's contribution to the solution generated by the algorithm. These methods often eliminate the need for a density estimator, as one is inherently incorporated into the indicator. One prominently studied method in this category is SMS-EMOA [26], which employs the hypervolume indicator to guide the search process. It computes the contribution of each solution to the hypervolume and subsequently removes the solution with the minimum contribution from the population. SMS-EMOA is occasionally integrated with dominance-based methods to enhance algorithm efficiency. In such cases, the indicator is employed to rank solutions when the entire population is non-dominated.

The general concept of harnessing the power of MOEAs, such as NSGA-II and SMS-EMOA, is to utilize the entire population generated throughout the process of solving the classical problem. This involves considering all the generations leading up to the generation of the approximate Pareto set.

4.3. Directed Search Approach

Leveraging the knowledge of the Pareto front in the context of the bi-objective classical problem, one effective exploration method is *directed search*. This method powers the ability to guide the search from a specific point ($x \in Q$) towards a desired direction ($d \in \mathbb{R}^k$). A directional vector ($v \in \mathbb{R}^n$) can be computed to facilitate this process, such as that proposed in [29].

$$\lim_{h \to 0} \frac{F(x+hv) - F(x)}{h} = J(x)v = d, \tag{11}$$

where $F : \mathbb{R}^n \to \mathbb{R}^k$, as described in MOP (1), and $J(F)(x)$ represents the Jacobian matrix, which is defined as follows:

$$J(F)(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \dots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k(x)}{\partial x_1} & \dots & \frac{\partial f_k(x)}{\partial x_n} \end{bmatrix}. \tag{12}$$

In particular, the method employed to build upon this concept is the descent method. The general idea involves solving the system of linear equations ($J(F)(x)v = d$). However, in many cases, the number of parameters ((n)) exceeds the number of objectives ((k)), causing the system of equations to be undetermined and lacking a unique solution. To address this, the problem is reformulated as $v = J(F)(x)^+ = d$, where $J(F)(x)^+$ denotes the pseudo-inverse of the Jacobian matrix ($J(F)(x)$).

With this formulation, a search direction (v) in the decision space is determined, and it becomes possible to generate a new point ($y = x + tv$, where $t \in R_k$ is a positive step size, i.e., y represents a movement from x in the direction of v). This method enhances the efficiency and purposefulness of exploring the solution space, and it is summarized in Algorithm 3.

Algorithm 3 DirectedSearch: Descent Method based on [29].

Require: $x \in \mathbb{R}^n$: initial point, $F : \mathbb{R}^n \to \mathbb{R}^k$,$d \in \mathbb{R}^k$: desired direction, t: step size, L: stop criteria (could be a maximum number of iterations).
Ensure: $x_{new} \in \mathbb{R}^n$ resulting point in the desired direction.
  $x_{new} = x$, compute $F(x)$
  repeat
    Compute $J(F)(x_{new})$
    Compute $J(F)(x_{new})^+$
    Compute $x_{new} = x_{new} + tJ(F)(x_{new})^+ d$
  until L

Specifically, for our purposes, the desired direction corresponds to the acceptable deterioration in risk and return (i.e., $d = \epsilon$). The Jacobian matrix for Models (3) and (4) is $J(F)(x) = (\mu, 2\Sigma x)^T$. The initial point for searching has to belong to P_Q. To approximate $P_{Q,\epsilon}$, we begin by selecting m points ($x_1, \dots, x_m$) from P_Q and applying Algorithm 3 to generate $x'_1, \dots, x'_m$ new points. Subsequently, with an integer parameter (s), we generate s additional points by sampling within the segment ($[x_i, x'_i] \subset Q, \forall i \in \{1, \dots, m\}$). This process yields a population of $m \cdot s$ points.

4.4. Performance Assessment

The forthcoming section compares the proposed methods through computational results. In this endeavour, the metric known as *hypervolume* (HV), also referred to as the S metric, holds particular significance. It quantifies the volume of the objective space enclosed by the approximation of the Pareto front (S; in this case, $S = P_{Q,\epsilon}$) and bounded from above by a reference point ($r \in \mathbb{R}^n$) such that $\forall x \in P_{Q,\epsilon}, x \prec r$ [30]. The hypervolume indicator is expressed as follows:

$$HV(S, r) = \lambda_m \left(\bigcup_{x \in S} [z, r] \right),$$

where λ_m is the m-dimensional Lebesgue measure. Hypervolume is a well-suited measure for comparing approximate Pareto fronts because it provides a comprehensive assessment of the quality of the approximation. By quantifying the volume of the objective space that is covered by the solutions in the approximation, hypervolume effectively captures both convergence (how close the solutions are to the true Pareto front) and diversity (how well the solutions spread across the front).

However, it is worth noting that when the exact Pareto front for the bi-objective problem is known, other metrics, such as generational distance (GD), GDplus, inverted generational distance (IGD), become applicable. For additional information and details about these metrics, please refer to [30].

5. Computational Results

The performance evaluation of the hierarchical approach outlined in Section 3.4 was conducted using open data sourced from Yahoo Finance. The experiments utilized two years of historical adjusted prices (from 1 November 2021 to 27 October 2023). Additionally, the ESG score data were collected from Yahoo Finance, with the latest available data as of October 2023 being used. It is important to note that the ESG data on the website are sourced from Sustainalytics, Inc., Boston, MA, USA, and presented as an ESG risk score measure, where a higher ESG risk score indicates a less favourable rating for the company.

The assets used in the portfolios were selected from the components of prominent market indices, like the Dow Jones, S&P500, and Nasdaq100. Specifically, three portfolios were constructed using the non-dominated assets from each of these indices. In this context, the dominance of assets refers to their dominance within single-asset portfolios [2]. Therefore, a non-dominated asset is not outperformed by any other asset concerning the two classical criteria, namely the mean of its returns and its variance as a measure of risk.

Unfortunately, not all assets had available data for the considered time period or publicly accessible ESG scores. We exclusively considered assets for which we had complete access to information. Specifically, we recovered 29 assets from the Dow Jones, 76 from the Nasdaq100, and 443 from the S&P500. Among these, we selected six non-dominated assets for the Dow Jones, six for the Nasdaq100, and nine for the S&P500 to construct portfolios to test our approach.

The selected non-dominated assets for the portfolios were Procter & Gamble Co. (PG) from Cincinnati, OH, USA; Eli Lilly and Co. (LLY) from Indianapolis, IN, USA; Johnson & Johnson (JNJ) from New Brunswick, NJ, USA; McKesson Corp. (MCK) from Irving, TX, USA; McDonald's Corp. (MCD) from Chicago, IL, USA; Berkshire Hathaway Inc. (BRK-B) from Omaha, NE, USA; Amgen Inc. (AMGN) from Thousand Oaks, CA, USA; International Business Machines Corp. (IBM) from Armonk, NY, USA; Consolidated Edison Inc. (ED) from New York, NY, USA; Mondelez International Inc. (MDLZ) from Chicago, IL, USA; O'Reilly Automotive Inc. (ORLY) from Springfield, MO, USA; Vertex Pharmaceuticals Inc. (VRTX) from Boston, MA, USA; PepsiCo Inc. (PEP) from Purchase, NY, USA; PACCAR Inc. (PCAR) from Bellevue, WA, USA; and Chevron Corp. (CVX) from San Ramon, CA, USA. The S&P500 portfolio consists of PG, LLY, JNJ, MCK, MCD, BRK-B, AMGN, IBM, and ED, while the Dow & Jones portfolio includes PG, MCD, JNJ, IBM, CVX, and AMGN. In contrast, the Nasdaq100 portfolio is composed of MDLZ, ORLY, VRTX, PEP, PCAR, and AMGN. Although the Dow & Jones and S&P500 portfolios share several assets, differences in the number and composition of assets lead to distinct optimal portfolios, whereas the Nasdaq100 portfolio stands out due to its distinct asset mix. This does not limit the ability to showcase the performance of the proposed hierarchical schema.

Figure 2 provides a visual representation of the non-dominated assets that were incorporated into the portfolios. To obtain the approximate Pareto front ($P_{Q,\epsilon}$) (optimal portfolios) and allow for a 1% deterioration in the expected annual return and risk (i.e., $\epsilon = (0.01, 0.01)^T$), the four sampling techniques discussed in Section 4 were employed to generate a population of $25,000$ feasible portfolios for each technique (the most similar number of individuals was generated in the case of the Das–Dennis approach). Table 1 offers a summary of the characteristics of each portfolio and presents the key parameters utilized in each of the sampling techniques.

Table 1. Summary of the characteristics of each portfolio considered in testing our model.

	Dow Jones	Nasdaq100	S&P 500
# of recovery assets	29	76	443
# of non-dominated assets	6	6	6
epsilon	(0.01, 0.01)	(0.01, 0.01)	(0.01, 0.01)
$\|P\|$ [1] Das–Dennis	26,334	26,334	24,310
$\|P\|$ NSGA-II	25,000	25,000	25,000
$\|P\|$ SMS-EMOA	25,000	25,000	25,000
$\|P\|$ Directed Search	25,000	25,000	25,000
Parameters for Das–Dennis	$p = 17$	$p = 17$	$p = 9$
Parameters for NSGA-II	$pop\_size = 100$, $generations = 250$	$pop\_size = 100$, $generations = 250$	$pop\_size = 100$, $generations = 250$
Parameters for SMS-EMOA	$pop\_size = 100$, $generations = 250$	$pop\_size = 100$, $generations = 250$	$pop\_size = 100$, $generations = 250$
Parameters for Directed Search	$m = 500$, $s = 50$, $step\_size = 0.001$	$m = 500$, $s = 50$, $step\_size = 0.001$	$m = 500$, $s = 50$, $step\_size = 0.001$

[1] Population generated by one particular sampling technique.

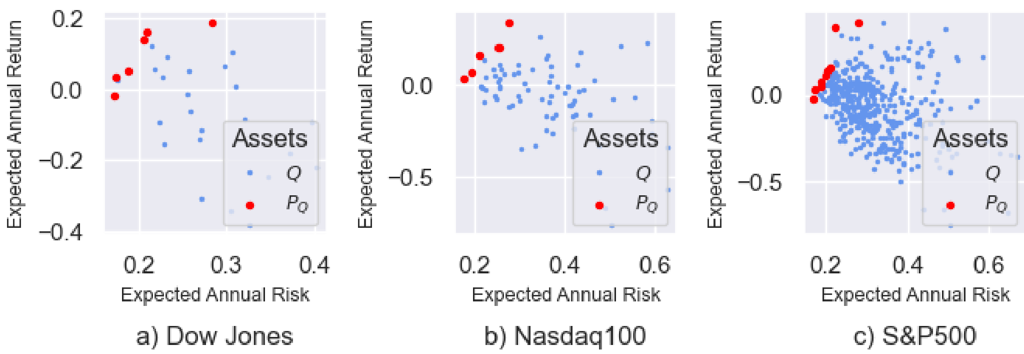

Figure 2. Non-dominated assets.(**a**) Dow Jones; (**b**) Nasdaq100; (**c**) S&P500

In the following, we summarize the results obtained by each of the sampling techniques for the three test portfolio problems. Each run was repeated twenty times. Below, we report the results of our tests.

Table 2 illustrates that SMS-EMOA consistently outperforms other methods in sampling approximately optimal portfolios. Nevertheless, directed search and NSGA-II deliver similar results, with directed search showing better performance in portfolios generated from the S&P500 index. Figures 3–5 depict the shapes of the approximate Pareto fronts obtained using each technique, showcasing that MOEAs generally provide a well-spaced space, at least from a visual perspective. However, Figure 6 shows the critical difference test, where SMS-EMOA achieves, in general, the best performance of all sampling methods.

Finally, Figure 7 presents a two-dimensional projection of the proposed optimal portfolios for the tri-criterion portfolio problem. These portfolios are exclusively considered with the allowed deterioration rate in return and risk. Generally, it can be observed that the ESG risk score is lower when the overall risk is lower, and the ESG risk score is also lower when the proposed portfolios are situated further away from the Pareto set in two dimensions, aligning with expectations.

Table 2. Hypervolume results for each of the sampling techniques utilized in the generation of $P_{Q,\epsilon}$ with $\epsilon = (0.01, 0.01)$, in bold we show the best results according to each indicator and market index.

Market Index	Sampling Method	GD	GDplus	HV	IGD	IGDplus
Dow Jones	Das–Dennis	0.009862 (0.00000000)	0.009859 (0.00000000)	1.101261 (0.00000000)	0.000892 (0.00000000)	0.000566 (0.00000000)
	Directed Search	**0.001989** (0.00000000)	**0.001835** (0.00000000)	1.107397 (0.00000000)	0.000080 (0.00000000)	0.000040 (0.00000000)
	NSGA-II	0.002574 (0.00007422)	0.002264 (0.00005940)	1.107681 (0.00002490)	0.000068 (0.00000551)	0.000053 (0.00000520)
	SMS-EMOA	0.002082 (0.00012400)	0.001859 (0.00009062)	**1.107742** (0.00036524)	**0.000047** (0.00003871)	**0.000023** (0.00000561)
Nasdaq100	Das–Dennis	0.009138 (0.00000000)	0.009133 (0.00000000)	0.979601 (0.00000000)	0.001259 (0.00000000)	0.001046 (0.00000000)
	Directed Search	0.002831 (0.00000000)	0.002651 (0.00000000)	0.991440 (0.00000000)	0.000085 (0.00000000)	0.000059 (0.00000000)
	NSGA-II	0.002420 (0.00003775)	0.002332 (0.00003928)	0.991479 (0.00003289)	0.000089 (0.00000547)	0.000073 (0.00000492)
	SMS-EMOA	**0.001976** (0.00006546)	**0.001855** (0.00006726)	**0.991793** (0.00006992)	**0.000046** (0.00000534)	**0.000030** (0.00000271)
S&P500	Das–Dennis	0.011090 (0.00000000)	0.011075 (0.00000000)	1.052160 (0.00000000)	0.002590 (0.00000000)	0.002182 (0.00000000)
	Directed Search	**0.001637** (0.00000000)	**0.001177** (0.00000000)	**1.069357** (0.00000000)	**0.000083** (0.00000000)	**0.000052** (0.00000000)
	NSGA-II	0.002920 (0.00011953)	0.002572 (0.00010768)	1.069147 (0.00007539)	0.000166 (0.00001529)	0.000128 (0.00001156)
	SMS-EMOA	0.001975 (0.00009921)	0.001693 (0.00010119)	1.065334 (0.00408483)	0.000534 (0.00032642)	0.000169 (0.00013042)

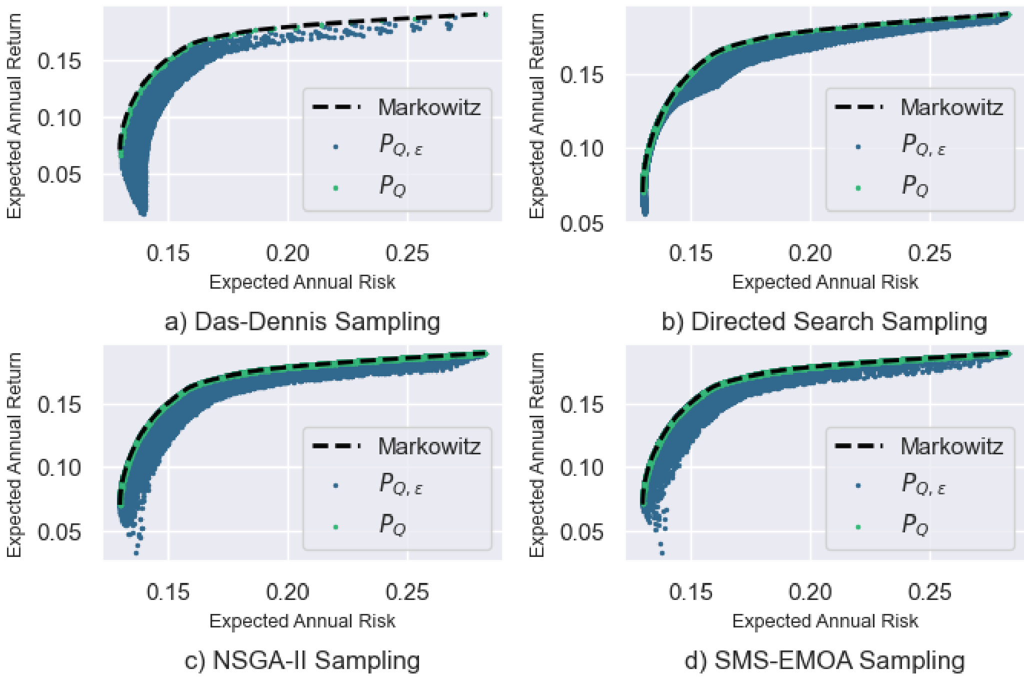

Figure 3. Approximate Pareto front for non-dominated assets selected from the Dow Jones market index with $\epsilon = (0.01, 0.01)$. (**a**) Das–Dennis sampling; (**b**) directed search sampling; (**c**) NSGA-II sampling; (**d**) SMS-EMOA sampling.

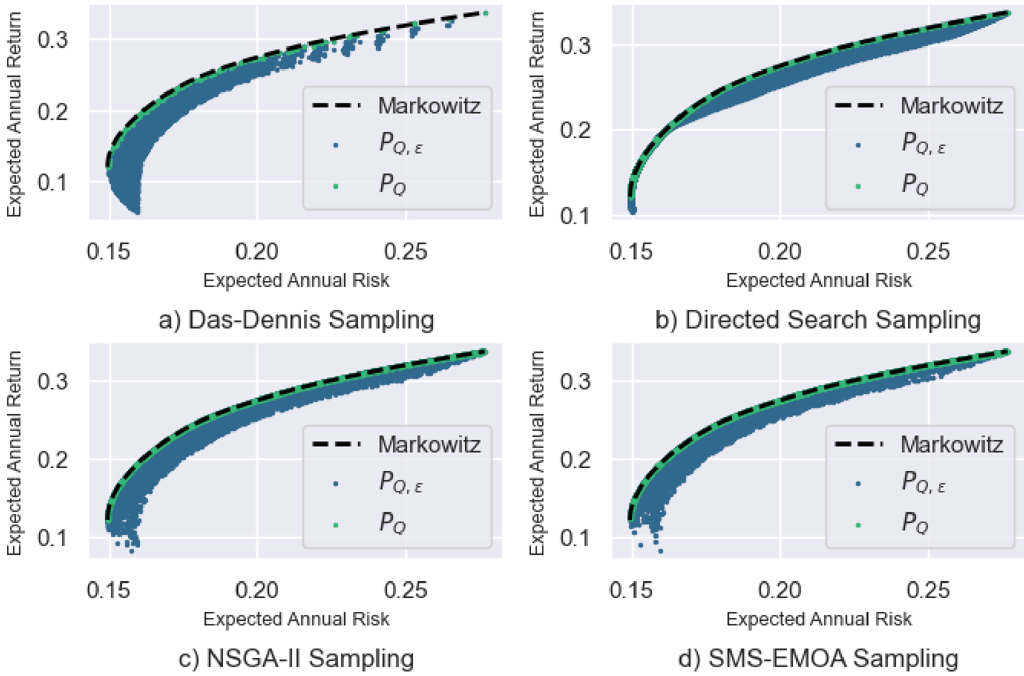

Figure 4. Approximate Pareto front for non-dominated assets selected from the Nasdaq100 market index with $\epsilon = (0.01, 0.01)$. (**a**) Das–Dennis sampling; (**b**) directed search sampling; (**c**) NSGA-II sampling; (**d**) SMS-EMOA sampling.

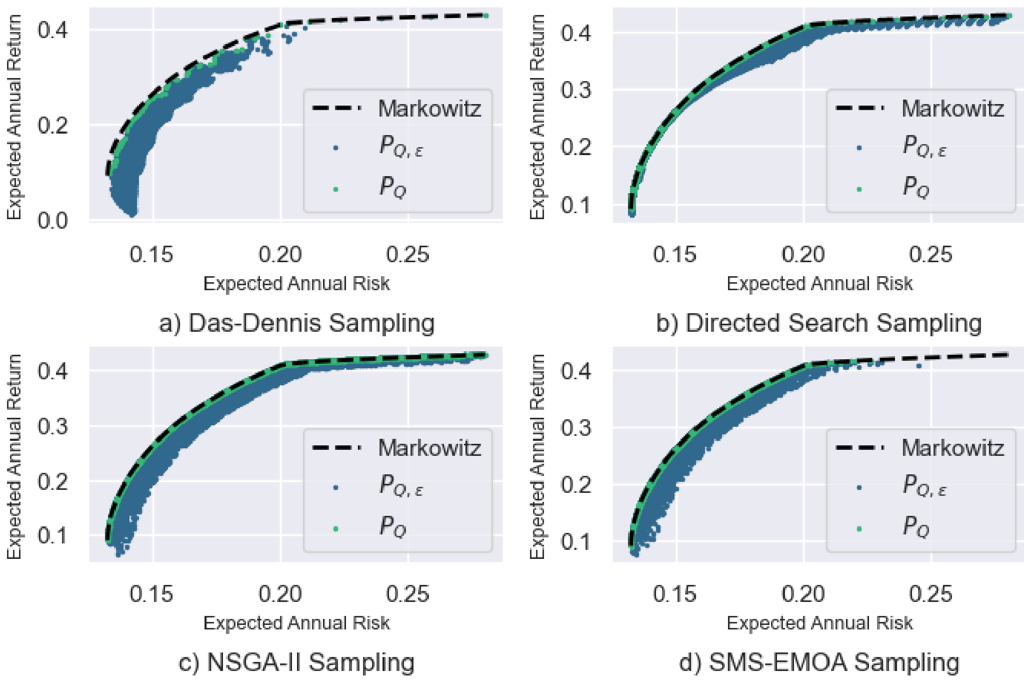

Figure 5. Approximate Pareto front for non-dominated assets selected from the S&P500 market index with $\epsilon = (0.01, 0.01)$. (**a**) Das–Dennis sampling; (**b**) directed search sampling; (**c**) NSGA-II sampling; (**d**) SMS-EMOA sampling.

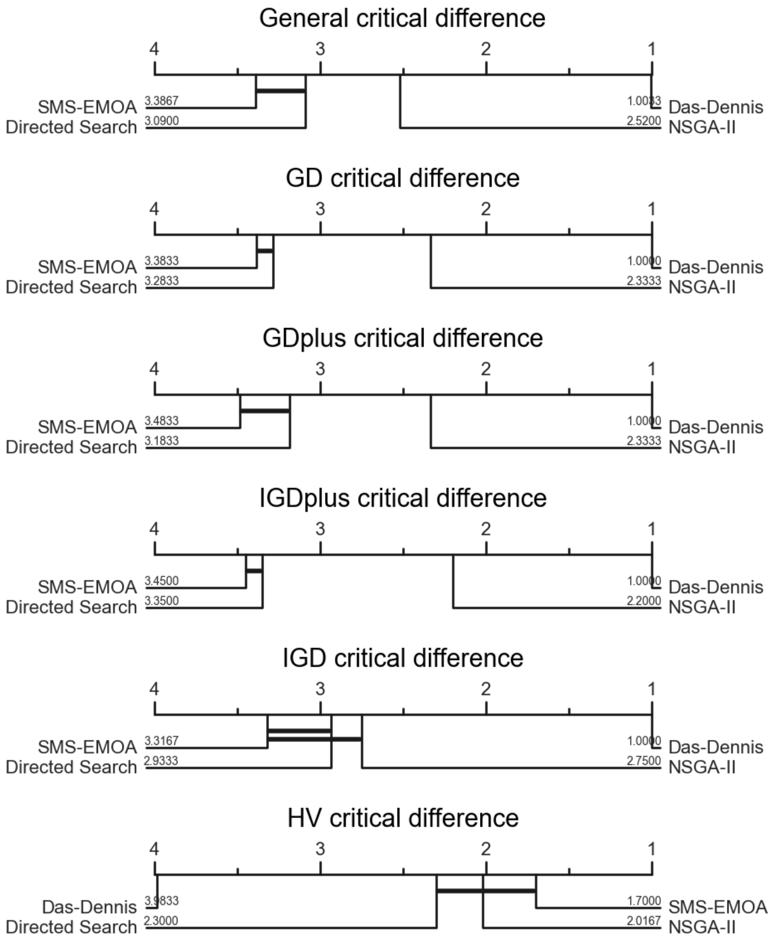

Figure 6. Critical difference for the different measures (GD, GDplus, IGD, IGDplus, and HV).

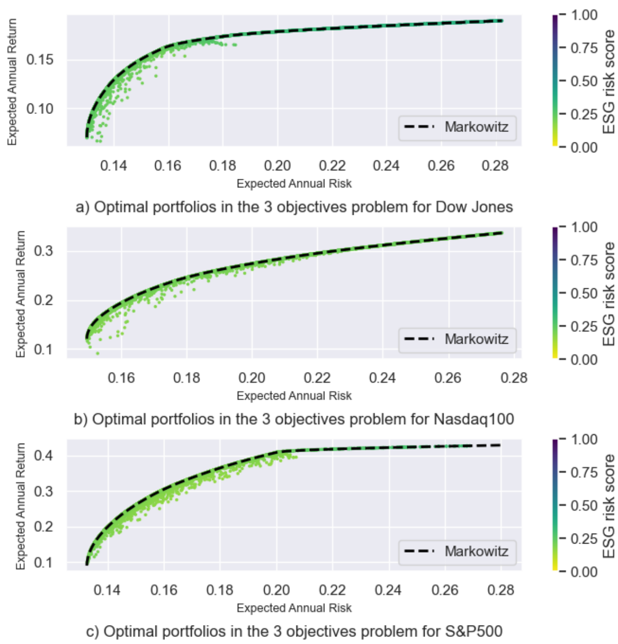

Figure 7. Two-dimensional projection of the optimal portfolios in the tri-criterion problem considering the approximate Pareto set generated in the classical problem with $\epsilon = (0.01, 0.01)$ using SMS-EMOA. (**a**) Portfolios constructed with Dow Jones assets; (**b**) portfolios constructed with Nasdaq100 assets; (**c**) portfolios constructed with S&P500 assets.

6. Conclusions

In this study, we introduced a novel framework for portfolio optimization that extends the classical Markowitz model by incorporating a tri-objective approach. Traditional portfolio construction has long been focused on a bi-objective paradigm, aiming to strike a balance between maximizing expected returns and minimizing portfolio variance. However, the world of finance is evolving, and investors increasingly seek portfolios that align with their individual preferences, going beyond the traditional risk–return trade-off. Our proposed model addresses this shift by introducing the ESG index, which evaluates companies based on environmental, social, and governance criteria, as a secondary objective.

The importance of this study lies in its ability to empower investors to create portfolios that not only meet their financial objectives but also reflect their values and concerns related to sustainability and responsible investing. By allowing investors to define explicit trade-off thresholds between expected returns and risk while considering the ESG index, our framework provides a holistic and socially responsible approach to portfolio management.

We rigorously tested our approach using the following four distinct sampling techniques: Das–Dennis, NSGA-II, SMS-EMOA, and directed search. Each technique offers unique advantages in terms of generating approximate optimal portfolios, and the results shed light on their comparative performance. Our analysis demonstrates that SMS-EMOA generally excels in the sampling of approximately optimal portfolios, although directed search and NSGA-II also deliver competitive results.

Furthermore, our hierarchical approach, which seamlessly integrates the ESG index as a secondary objective, provides investors with a user-friendly method to navigate the complex landscape of tri-objective portfolio optimization. This approach enables investors to systematically identify portfolios that balance their financial objectives, risk tolerance, and ESG preferences.

In conclusion, our study presents a valuable contribution to the field of portfolio optimization, particularly in the context of socially responsible investing. By extending the classical Markowitz model to accommodate the ESG index as a secondary objective and employing various sampling techniques, we offer investors a more comprehensive and customizable approach to building portfolios that align with their financial and ethical priorities. This research opens the door to a new era of portfolio management, where investors can effectively balance their financial goals with sustainability and responsible investment principles.

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Data Availability Statement: The data that support the findings of this study are available in [MO-PortafoliosInversion] at [https://github.com/yeudiellm/MO-PortafoliosInversion] (1 July 2024), reference number [reference number]. These data were derived from the following resources available in the public domain: [https://finance.yahoo.com/URL/DOI, https://www.slickcharts.com/] (1 July 2024).

Conflicts of Interest: The authors declare no conflicts of interest.

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